10
Special Relativity Theory

THIS chapter is devoted completely to the Special Relativity Theory. The reason behind this is to motivate readers to think, because, through this theory, space coordinates and time are related. Newton's equation of motion is modified, and times are different in moving and stationary systems. Furthermore, an event that occurs in the past in one system can become a future event in another system. All those are theoretically possible. To make these into our daily lives, further research is needed. Therefore, study of this theory not only can broaden our minds, but also can lead us to the invention of some new devices that will turn the theory into practical applications.

The development of this theory is based on famous experiments carried out by Michelson and Morley.* They found that the velocity of light always has the constant value despite the relative motions of source, observer, or medium. This result cannot be explained by the Galilean transformation that has been used throughout the previous chapters.

The set of transformations derived by Hendrik Antoon Lorentz, a Dutch physicist, solves that problem. The transformation is known as the Lorentz transformation and is the basis of the Special Relativity Theory. Albert Einstein in 1905 systematically recognized the limitations of Galilean transformation. He chose to modify the concept of time from absolute scale to space dependence. He made only two assumptions:

1) The laws of dynamics, including electromagnetic phenomena, must have the same form in systems moving with uniform velocity relative to each other.

2) The speed of light c is a universal constant, independent of any relative motion of the souce and the observer.

Using these assumptions, Einstein was able to formulate logically precise theories. The Special Relativity Theory of 1905 considers reference systems that are in uniform motion with respect to one another. The more general treatment of accelerated reference systems is the subject of the General Relativity Theory that was developed in 1915.

In Section 10.1, we shall discuss the Lorentz transformation. The conditions and assumptions, which are made for this transformation, are discussed in detail. In Section 10.2, we shall study the Brehme diagram, which is a graphical representation of the Lorentz transformation. The construction and the interpretation of this diagram will be discussed in the section. Through the example of the Brehme diagram, we can see that a region for past events in a system can be a region for future events in another system. Lastly, some consequences of the Special Relativity Theory are presented in Section 10.3. We shall discuss how to change some equations of Newtonian mechanics into relativistic forms.

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*For the Michelson–Morley experiment, see Silberstein, L., *The Theory of Relativity*, Macmillan, London, 1924, p. 71.

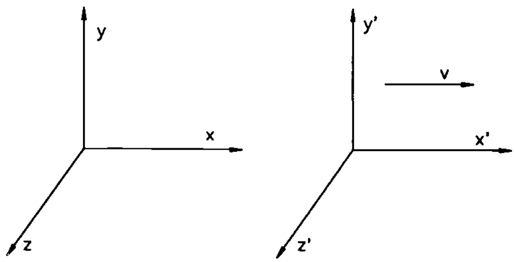

Fig. 10.1 Primed system moving with velocity v relative to the unprimed system.

10.1 Lorentz Transformation

Consider two reference systems as shown in Fig. 10.1. The primed system is moving with uniform velocity v along the x axis relative to the unprimed system. The Michelson–Morley experiments may be described as follows. A spark from a light source is emitted from the common origin of the two systems when they are coincided. As the light wave propagates spherically into the space, it is found that the spherical wave is the same in both systems regardless where the spark is released in the moving system or the stationary system. It is also found that the spherical wave front is not affected by whether or not the medium was moving.

This situation cannot be explained by the Galilean transformation that can be written as

$$x = x' + vt \qquad y = y' \qquad z = z' \qquad t = t$$

Under this transformation, the spherical wave in one system will be distorted in the other system because of the vt term in the x direction. However, if the second assumption of Einstein as stated above is observed, the equations for spherical surfaces in the both systems can be written as

$$x^2 + y^2 + z^2 = (ct)^2 \tag{10.1}$$

$$x'^2 + y'^2 + z'^2 = (ct')^2 \tag{10.2}$$

To derive the Lorentz transform, we define

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict$$
$$x_1' = x', \qquad x_2' = y', \qquad x_3' = z', \qquad x_4' = ict'$$

where $i = \sqrt{-1}$. This is known as Minkowski space, which is the complex four-dimensional space time. To establish the relationship between the two systems, we assume

$$x'_\alpha = \sum_{\beta=1}^{4} a_{\alpha\beta} x_\beta \tag{10.3}$$

and four unit vectors are orthogonal to each other. Using matrix notation, Eq. (10.3) becomes

$$X' = AX$$

where

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{14} & 0 & 0 & a_{44} \end{pmatrix}$$

Using the orthogonality assumption, we have

$$AA^T = I$$

i.e.,

$$a_{11}^2 + a_{14}^2 = 1, \qquad a_{41}^2 + a_{44}^2 = 1, \qquad a_{11}a_{41} + a_{14}a_{44} = 0 \tag{10.4}$$

Here we have three equations, but there are four unknowns to be determined. One additional equation is obtained from the relationship between the origin of the primed system and the corresponding coordinate in the unprimed system:

$$x_1 = vt = -\frac{iv}{c}(ict) = -i\beta x_4 \tag{10.5}$$

where $\beta = v/c$. But for this origin, we also can write

$$x_1' = 0 = a_{11}x_1 + a_{14}x_4 = (-a_{11}i\beta + a_{14})x_4$$

Hence we have

$$a_{14} = i\beta a_{11} \tag{10.6}$$

Using the preceding equation together with the three equations in Eq. (10.4) from orthogonality, we find

$$a_{11} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{10.7a}$$

i.e.,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$a_{14} = i\beta\gamma \tag{10.7b}$$

$$a_{44} = \gamma \tag{10.7c}$$

$$a_{41} = -i\beta\gamma \tag{10.7d}$$

Therefore the Lorentz matrix is

$$A = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{10.8}$$

The transformation can then be written explicitly as

$$x_1' = \gamma(x_1 + i\beta x_4)$$

or

$$x' = \gamma(x - vt) \tag{10.9a}$$

$$y' = y \tag{10.9b}$$

$$z' = z \tag{10.9c}$$

$$x_4' = \gamma(-i\beta x_1 + x_4)$$

or

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{10.9d}$$

Not that as $v \ll c$, $\beta \to 0$, and $\gamma = 1$, the Lorentz transformation reduces to the Galilean transformation.

Because the two systems are in relative motion, the unprimed system may be considered as moving with velocity $-v$ along x' axis. The relations between the two systems can be written as

$$x = \gamma(x' + vt) \tag{10.10a}$$

$$y = y' \tag{10.10b}$$

$$z = z' \tag{10.10c}$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{10.10d}$$

Applications of the Lorentz Transformation

Length contraction of a rigid rod. Consider a rigid rod of length ℓ

$$\ell = x_2 - x_1$$

along the unprimed x axis. When this rod is carried in the moving system, we find

$$\ell = x_2 - x_1 = \gamma(x_2' + vt') - \gamma(x_1' + vt') = \gamma(x_2' - x_1')$$

or

$$x_2' - x_1' = \ell' = \sqrt{1 - \beta^2}\,\ell \tag{10.11}$$

This means that if the rigid rod is measured in the moving system, the length becomes shorter because

$$\sqrt{1 - \beta^2} < 1$$

This fact is known as the Lorentz–Fitzgerald contraction.

Moving clock. Consider a clock fixed at the origin of the moving system

$$x' = 0, \qquad x = vt$$

The conversion of time gives

$$\begin{aligned} t_2' - t_1' &= \gamma\left[\left(t_2 - \frac{v}{c^2}x_2\right) - \left(t_1 - \frac{v}{c^2}x_1\right)\right] \\ &= \gamma\left[\left(t_2 - \frac{v^2}{c^2}t_2\right) - \left(t_1 - \frac{v^2}{c^2}t_1\right)\right] \\ &= \gamma(1 - \beta^2)(t_2 - t_1) \\ &= \sqrt{1 - \beta^2}(t_2 - t_1) \end{aligned} \tag{10.12}$$

That means that the time in the moving system becomes shorter than the time in the stationary system. Just to see the dramatic effect of the result, let us consider $\beta^2 = 0.99$. We find one year in the moving system; the corresponding time in the stationary system is 10 years.

Verification of Lorentz Transformation with Light Pulse Released from Different Systems

A light pulse released at the origin of the unprimed system at the instant when the two origins are coincident. The wave front in the positive x direction is at

$$x = ct$$

Using Eq. (10.9a) and Eq. (10.10d), the corresponding position in the x' system is determined as follows:

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(ct - vt) = \gamma(c - v)t \\ &= \gamma(c - v)\left[\gamma\left(t' + \frac{v}{c^2}x'\right)\right] \\ &= \gamma^2\left[(c - v)t' + \frac{v}{c}x' - \frac{v^2}{c^2}x'\right] \end{aligned}$$

Simplifying leads to

$$x' = ct' \tag{10.13}$$

The wave front in the negative x direction is at

$$x = -ct$$

The corresponding position in the moving system is

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(-ct - vt) \\ &= -\gamma(c + v)t = -\gamma(c + v)\left[\gamma\left(t' + \frac{v}{c^2}x'\right)\right] \end{aligned}$$

Rearranging gives

$$x' = -ct' \tag{10.14}$$

Therefore, they are spherical surfaces in both systems.

A light pulse is released at the origin of the primed system moving with velocity v at the instant when the two origins are coincident.

The wave front in the positive x' direction is

$$x' = ct'$$

The corresponding position of the wave front in the unprimed system is

$$\begin{aligned} x &= \gamma(x' + vt') = \gamma(ct' + vt') \\ &= \gamma(c + v)\left[\gamma\left(t - \frac{v}{c^2}x\right)\right] \end{aligned}$$

Simplifying leads to

$$x = ct$$

The wave front in the negative x' direction is

$$x' = -ct'$$

The corresponding wave front in the negative x direction is

$$\begin{aligned} x &= \gamma(x' + ct') = \gamma(-ct' + vt') = -\gamma(c - v)t' \\ &= -\gamma(c - v)\left[\gamma\left(t - \frac{v}{c^2}x\right)\right] \end{aligned}$$

Rearranging gives

$$x = -ct$$

Therefore, we find that the wave fronts are spherical surfaces in both systems.

10.2 Brehme Diagram

The Brehme diagram is a very useful tool for visualizing the Lorentz transformation. Note that in four-dimensional space time, y and z are not changed, but x and t are transformed into x' and t'. Therefore, the Brehme diagram is designed to show the relationship of Lorentz transformation and the coordinates of an event in the both systems, moving and stationary.

First let us construct the Brehme diagram as follows:

1) From the known value of velocity v of the moving system, calculate α such that

$$\alpha = \sin^{-1}\frac{v}{c} \tag{10.15}$$

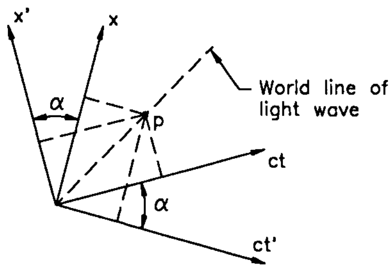

Fig. 10.2 Construction of the Brehme diagram.

2) Draw two straight axes for ct' and ct, with ct axis rotated by the angle α counterclockwise as shown in Fig. 10.2.

3) Draw x axis perpendicular to ct' axis.

4) Draw x' axis perpendicular to ct axis. Note that x, ct axes are for the unprimed system and x', ct' axes for primed system.

5) Draw a line from the origin bisecting the angle between axes of either primed or unprimed system. This line is called a world line of the light pulse. Any point P on this line, as shown in Fig. 10.2, represents the same spherical wave front in the two systems.

To see the significance of the Brehme diagram, let us take any point A as shown in Fig. 10.3.

$$\sin\alpha = \beta$$

$$\theta = 90\text{ deg} - \alpha$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\cos\alpha}$$
$$= \sec\alpha$$

At point A

$$x = OT + PT = OD\sec\alpha + PA\tan\alpha$$
$$= x'\gamma + ct'\gamma\beta = \gamma(x' + c\beta t') = \gamma(x' + vt') \qquad (10.16)$$

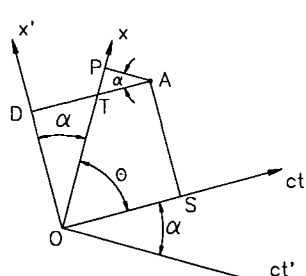

Fig. 10.3 Verification of the Brehme diagram.

This agrees with Eq. (10.10a). Also at point A

$$ct = OS = DT + TA = OD\tan\alpha + PA\sec\alpha$$
$$= x'\gamma\beta + ct'\gamma = \gamma(ct' + \beta x')$$

or

$$t = \gamma[t' + (v/c^2)x'] \tag{10.17}$$

which agrees with Eq. (10.10d). Therefore, the Brehme diagram truly reproduces the features of the Lorentz transformation.

Example 10.1

Suppose that one system is moving with a constant velocity of 2.598×10^8 m/s relative to another system. Choose the x and x' axes along the direction of the velocity. 1) Construct a Brehme diagram to relate the both systems. 2) Using the Brehme diagram constructed in step 1, indicate the regions in the diagram that represent the future in one system but the past in the other system. 3) Determine also the regions that represent the left of the reference position in one system but the right of the reference position in the other system.

Solution. 1) Construction of a Brehme diagram

$$\sin\alpha = \beta = \frac{v}{c} = \frac{2.598}{3.000} = 0.866$$
$$\alpha = 60 \text{ deg}$$

The Brehme diagram is constructed as shown in Fig. 10.4.

2) Take point P as a reference point. Draw a line PR from P perpendicular to ct axis. Note that the region to the left of line PR represents events taking

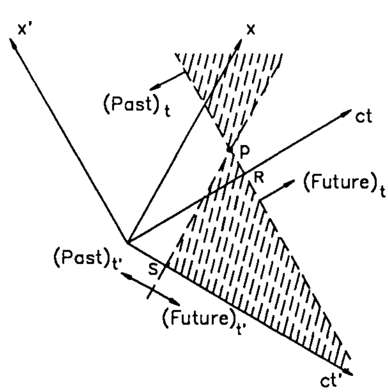

Fig. 10.4 Future and past overlapping in the shaded regions.

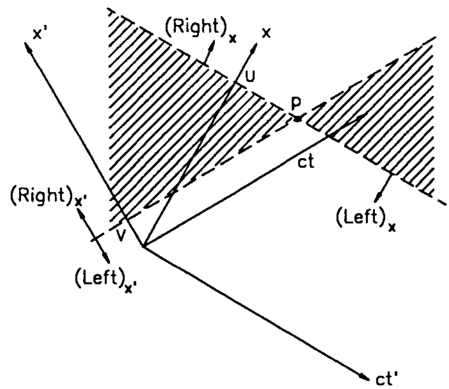

Fig. 10.5 Right and left overlapping in the shaded regions.

place in the past in the unprimed system; to the right of that line represents events that are to take place in the future. On the other hand, draw a line PS from P perpendicular to ct' axis. The region to the left of line PS represents events that have occurred in the past, and the region to the right of line PS represents events that are going to happen in the future in the primed system. The shaded regions represent the future in one system but the past in the other system as shown in Fig. 10.4.

3) From P, draw a line PU, normal to the x axis as shown in Fig. 10.5. The region below line PU means that events happen at places with small values in x coordinate as denoted by "Left"; above line PU represents events taking place at larger values of x as indicated by "Right." On the other hand, the line PV is normal to x' axis. The region below line PV represents events happen at places with smaller value in x', and the region above line PV represents that events take place at the larger value of x'. The shaded regions depict "Right" in one system and "Left" in the other system.

Example 10.2

Consider a case of twin brothers A and B. B takes a space trip traveling with relativistic velocity to another planet. After arriving on the planet, B stays briefly and then comes back. Assume that the time for acceleration in the beginning of the trip, brief stay on the planet, and for the deceleration at the end of return trip are small compared to the duration of whole trip. Determine which brother, A or B, is actually younger in age.

Solution. Construct a Brehme diagram for the two systems as shown in Fig. 10.6a for the trip of B to the planet. Consider x', ct' are the axes for the moving system and x and ct axes for the stationary system. Assume that A, B are always at the origins of space coordinates. The time spent by B is Δt_b and by A is Δt_a. From the diagram we find

$$\Delta t_b = \Delta t_a \cos\alpha$$

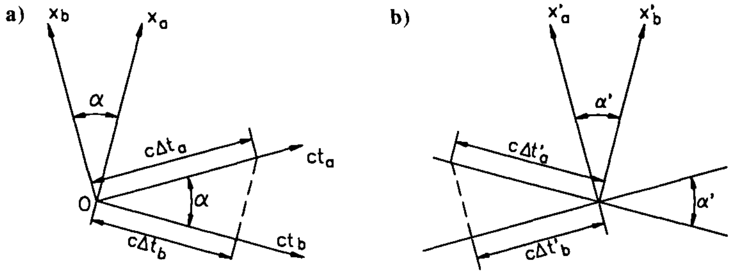

Fig. 10.6 Brehme diagrams for brother *B*.

Construct another Brehme diagram for the two systems as shown in Fig. 10.6b for the return trip of B. Again assume that A, B are at the origins of space coordinates. Converting the time, we have

$$\Delta t'_b = \gamma\left(t_2 + \frac{v}{c^2}x_2\right)_a - \gamma\left(t_1 + \frac{v}{c^2}x_1\right)_a = \gamma(t_2 - t_1)_a + \gamma\frac{v}{c^2}(x_2 - x_1)_a$$

$$= \gamma(t_2 - t_1)_a - \gamma\frac{v^2}{c^2}(t_2 - t_1)_a = \Delta t'_a \cos\alpha'$$

The prime symbol is used only to indicate the quantities of the return trip. In this way the expression includes the possibility that α' may be different from α. The total time for the round trip of B is

$$\Delta t_b + \Delta t'_b = \Delta t_a \cos\alpha + \Delta t'_a \cos\alpha'$$

Therefore, we conclude that B is younger.

10.3 Immediate Consequences in Kinematics and Dynamics

Addition of Velocities

Suppose that point P in the moving system moves with velocity u along x' axis. The velocity of the moving system is v. The velocity of P in the stationary system is no longer $u + v$ under the Lorentz transformation because

$$x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$u = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - v\,\mathrm{d}t}{\mathrm{d}t - v\,\mathrm{d}x/c^2} = \frac{\mathrm{d}x/\mathrm{d}t - v}{1 - (v/c^2)(\mathrm{d}x/\mathrm{d}t)}$$

Solving for the velocity of P in the stationary system, we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u + v}{1 + uv/c^2} \tag{10.18}$$

Note that if u and v are small compared with c, then the velocity of P reduces to the addition of u and v. We can apply this result to the superposition of two Lorentz transformations. Consider three frames of reference S, S^*, and S^{**}. S^* has the velocity v relative to S, and S^{**} has the velocity u relative to S^*. The transformation equations relating S^{**} and S are

$$x^{**} = \gamma_w(x - wt) \tag{10.19a}$$

$$y^{**} = y \tag{10.19b}$$

$$z^{**} = z \tag{10.19c}$$

$$t^{**} = \gamma_w\left(t - \frac{wx}{c^2}\right) \tag{10.19d}$$

where

$$w = \frac{u + v}{1 + uv/c^2} \tag{10.19e}$$

and

$$\gamma_w = \frac{1}{\sqrt{1 - (w/c)^2}} \tag{10.19f}$$

Equations of Motion in Relativistic Form

Time change. Consider a particle at rest in the primed system. The velocity of the particle in the unprimed system is v. The changes of time in the two systems are related by

$$(\mathrm{d}\tau)^2 = (\mathrm{d}t)^2\left[1 - \frac{v^2}{c^2}\right] = (\mathrm{d}t)^2(1 - \beta^2)$$

or

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \beta^2} \tag{10.20}$$

where $\mathrm{d}\tau$ is the change of time in the moving system and $\mathrm{d}t$ is the change of time in the stationary system.

Equation of motion. By defining the relativistic mass and momentum as

$$m \equiv \frac{m_0}{\sqrt{1 - \beta^2}} = \gamma m_0 \qquad P_i \equiv m_0\frac{\mathrm{d}x_i}{\mathrm{d}t} \tag{10.21}$$

and the four-component force as

$$\mathcal{F}_i \equiv m_0\frac{\mathrm{d}^2 x_i}{\mathrm{d}\tau^2} \tag{10.22}$$

we find that the Newtonian equation becomes

$$\mathcal{F}_i = m_0 \frac{\mathrm{d}^2 x_i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau} m_0 \frac{\mathrm{d}x_i}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} m_0 \gamma \frac{\mathrm{d}x_i}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}t} m_0 \gamma v_i = \gamma \frac{\mathrm{d}}{\mathrm{d}t} m v_i$$

According to the meaning of the term, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} m v_i = \mathcal{F}_i / \gamma = F_i \tag{10.23}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{m_0 v_i}{\sqrt{1-\beta^2}} = F_i$$

where m_0 is the rest mass, $\mathcal{F}_i$ is the ith component of the Minkowski force, and F_i is the ith component of the coordinate force as observed in the unprimed system.

Relativistic energy. By defining four vector components

$$\mu_1 \equiv \frac{\mathrm{d}x}{\mathrm{d}\tau}, \qquad \mu_2 \equiv \frac{\mathrm{d}y}{\mathrm{d}\tau}, \qquad \mu_3 \equiv \frac{\mathrm{d}z}{\mathrm{d}\tau}, \qquad \mu_4 \equiv ic \frac{\mathrm{d}t}{\mathrm{d}\tau} = ic\gamma$$

then we have

$$\begin{aligned} \sum_{i=1}^{4} \mu_i^2 &= -c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \\ &= \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 (-c^2 + v^2) = \frac{1}{1-\beta^2}(v^2 - c^2) = -c^2 \end{aligned} \tag{10.24}$$

Note that μ_i is the component of the proper velocity that is obtained by the distance traveled in the unprimed system divided by time interval from moving clock; v_i is the component of the coordinate velocity that is the distance divided by the time interval from a fixed clock. On the other hand, taking the dot product of the Minkowski force with the proper velocity gives

$$\begin{aligned} \sum_{i=1}^{4} \mathcal{F}_i \mu_i &= m_0 \sum_i \frac{\mathrm{d}\mu_i}{\mathrm{d}\tau} \mu_i = \frac{1}{2} m_0 \frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{i=1}^{4} \mu_i^2 \\ &= \frac{1}{2} m_0 \frac{\mathrm{d}}{\mathrm{d}\tau}(-c^2) = 0 \end{aligned}$$

$$\mathcal{F}_4 = \frac{-1}{\mu_4} \sum_{i=1}^{3} \mathcal{F}_i \mu_i = \frac{-1}{\mu_4} \sum_{i=1}^{3} \gamma F_i \gamma v_i = \frac{i\gamma}{c} (\boldsymbol{F} \cdot \boldsymbol{V})$$

or

$$\frac{i\gamma}{c}(\boldsymbol{F}\cdot\boldsymbol{V}) = \mathcal{F}_4 = m_0\frac{\mathrm{d}}{\mathrm{d}\tau}\mu_4 = m_0\frac{\mathrm{d}}{\mathrm{d}\tau}(ic\gamma)$$

$$(\boldsymbol{F}\cdot\boldsymbol{V}) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0c^2}{\sqrt{1-\beta^2}}$$

But the meaning of $\boldsymbol{F}\cdot\boldsymbol{V}$ is the rate change of the kinetic energy as discussed in Section 2.3; therefore, we find

$$\text{K.E.} = \frac{m_0c^2}{\sqrt{1-\beta^2}} \tag{10.25}$$

Note that Eqs. (10.23) and (10.25) are usually given in college physics books without explanation.

Example 10.3

Consider that a particle is moving on the x axis under a constant coordinate force F. Assume it starts from rest at $t = 0$. Find the maximum limiting velocity of the particle.

Solution. Using Eq. (10.23), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0v}{\sqrt{1-v^2/c^2}}\right) = F$$

Integrating leads to

$$\frac{m_0v}{\sqrt{1-v^2/c^2}} = Ft$$

Solving for v, we find

$$v = \frac{cFt}{\sqrt{(Ft)^2+(m_0c)^2}}$$

Therefore, the velocity of the particle is always less than c. The limiting value is c as t approaches infinity. This is not true in Newtonian mechanics.

Problems

10.1. Show that the wave equation

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

is invariant under a Lorentz transformation but not under a Galilean transformation.

10.2. Derive the relationship for the velocity of a particle along the x' axis in a primed system to the velocity in the unprimed system under the Lorentz transformation.

10.3. Derive the relationship for the acceleration of a particle along x' axis in the primed system compared to that in the unprimed system under the Lorentz transformation.

10.4. Do the following:

(a) Determine the velocity of a moving system such that 7 days in the moving system is equivalent to 1 year in the stationary system.

(b) Construct a Brehme diagram with scales on the axes for the systems determined in part (a).

10.5. Verify the Brehme diagram Fig. 10.6b for the return trip of B.

10.6. Do the following:

(a) Determine the velocity of a moving system so that an observer in the moving system can see an event that happened one day ago in the stationary system.

(b) Construct a Brehme diagram and mark the position of the observer in the diagram for the systems described in part (a).

10.7. Prove that if velocities u and v are less than speed of light c, the result of the addition of velocities through relativity theory can never be greater than c.

10.8. Prove that the kinetic energy expressed by Eq. (10.25) will reduce to K.E. $= \frac{1}{2}m_0v^2 + m_0c^2$ if $v \ll c$. Discuss the significance of m_0c^2.

10.9. Consider a system moving along x axis with velocity v relative to the stationary system. A sphere in the moving system is described by

$$x'^2 + y'^2 + z'^2 = a^2$$

What will be the shape as observed in the stationary system?

10.10. Two particles, with rest masses m_1, m_2, move along the x axis with velocities u_1, u_2, respectively. They collide and coalesce to form a single particle. Assuming the laws of conservation of relativistic mass and momentum, prove that the rest mass m_3 and velocity u_3 of the resulting single particle are given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\,\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

$$u_3 = \frac{m_1\gamma_1u_1 + m_2\gamma_2u_2}{m_1\gamma_1 + m_2\gamma_2}$$

where

$$\gamma_1 = \frac{1}{\sqrt{1-(u_1/c)^2}}, \qquad \gamma_2 = \frac{1}{\sqrt{1-(u_2/c)^2}}$$