9 Vibration of Systems with Multiple Degrees of Freedom

IN Chapter 8, we studied linear vibration systems with one degree of freedom and mathematical methods that are fundamental for analyzing these problems. In this chapter, we will study linear and nonlinear systems with multiple degrees of freedom and relatively advanced mathematical techniques for dealing with them.

Section 9.1 deals with various types of vibration systems with two degrees of freedom. There are five examples to illustrate different methods of formulating and analyzing them.

In Section 9.2, we will study vibration systems with multiple degrees of freedom. Because a system with n degrees of freedom is associated with n differential equations, one way to deal with it is to apply matrix methods. With the fundamentals of matrix introduced in Chapter 5, this section may be considered as the additional application of the matrix. Readers will see that there are many advantages with this formulation.

In Section 9.3, we will present the method of lumped parameters with transfer matrices for modeling a vibration system. They may be considered as approximations for modeling the continuous system. The advantage of this approach is that the governing equation can be formulated by the method of transfer matrices, and frequencies and shapes of principal modes can be determined without solving the equations completely. Furthermore, the result of this method can be used to check the results solved from the partial differential equations for a continuous system.

Section 9.4 covers the vibration of continuous systems, which include vibrating string, beam, membrane, and sound wave. Governing equations for these systems are known as wave equations. The use of Fourier series for periodic functions is illustrated repeatedly. Wave equations for one-dimensional space in rectangular, cylindrical, and spherical coordinates are all considered. We notice that the wave form remains the same in rectangular coordinates as the wave propagates either in the positive x or negative x direction. The wave form decays in the cylindrical coordinates because of the properties of Bessel functions. In the spherically symmetric wave, the amplitude decays inversely proportionally to the distance from the center of the wave. From these, the reader can learn some fundamentals in the formulation of the equations and in the determination of solutions.

Section 9.5 is devoted specially to nonlinear systems. As we know from mathematics, a systematic method for solving nonlinear problems is the small perturbation method, which has been introduced in Chapter 5 and is not to be repeated here. Of course, many nonlinear problems can be solved with today's powerful computers. The Runge–Kutta method, which is presented in Appendix A, is a useful tool for obtaining the numerical solutions. However, the disadvantage of numerical method is that it cannot show explicitly the parameters involved in the solutions.

Stability analysis is specially important for nonlinear systems and is presented in Section 9.6. From this section, the reader will find some fundamentals for this subject.

9.1 Vibration Systems with Two Degrees of Freedom

A vibration system with two degrees of freedom requires two spatial coordinates to describe its motion. Consequently, there are two governing equations for the motion and two natural frequencies of vibration. When the system is in a force-free vibration, it vibrates, usually at the combination of two normal modes corresponding to the natural frequencies. However, under forced harmonic vibration, the system will vibrate at the frequency of the excitation in addition to natural frequencies. Resonance will take place if the exciting frequency is the same as one of two natural frequencies. Details of these different situations will be illustrated in the following examples.

Example 9.1

Consider the undamped system as shown in Fig. 9.1. Coordinates x_1 and x_2 are the displacements of m_1 and m_2 away from their equilibrium positions, respectively. Formulate the governing equations of the motion; find the natural frequencies and the steady-state solutions.

Solution. The governing equations may be obtained from the balance of forces. They can be obtained also from Lagrange's equations. Let us take Lagrange's approach. It is seen easily that for the system, kinetic energy is

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2$$

potential energy is

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}kx_2^2$$

and Lagrange's function is

$$L = T - V$$

Hence, the equation for x_1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0 \qquad (9.1)$$

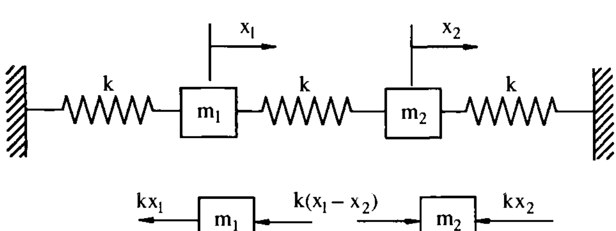

Fig. 9.1 Undamped mass-spring system with two degrees of freedom.

Similarly, for x_2, the equation is

$$m_2\ddot{x}_2 - k(x_1 - x_2) + kx_2 = 0 \tag{9.2}$$

Equations (9.1) and (9.2) are linear second-order differential equations with constant coefficients. The steady-state solution can be assumed as

$$x_1 = A_1 e^{i\omega t}$$
$$x_2 = A_2 e^{i\omega t}$$

Substituting these into the governing equations gives

$$\begin{aligned} (2k - \omega^2 m_1)A_1 - kA_2 &= 0 \\ -kA_1 + (2k - \omega^2 m_2)A_2 &= 0 \end{aligned} \tag{9.3}$$

Because A_1 and A_2 are not zero, the determinant of the coefficients must be zero, i.e.,

$$\begin{vmatrix} (2k - \omega^2 m_1) & -k \\ -k & (2k - \omega^2 m_2) \end{vmatrix} = 0$$

To save some writing, let us change the symbol ω^2 to λ, then the preceding determinant leads to the characteristic equation

$$\lambda^2 - \frac{m_1 + m_2}{m_1 m_2} 2k\lambda + \frac{3k^2}{m_1 m_2} = 0 \tag{9.4}$$

The two roots of the equation are

$$\left.\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} = \frac{k}{m_1 m_2}\left[(m_1 + m_2) \mp \sqrt{(m_1 - m_2)^2 + m_1 m_2}\right]$$

Therefore, the natural frequencies of the system are found to be

$$\left.\begin{matrix} \omega_1 \\ \omega_2 \end{matrix}\right\} = \left\{\frac{k}{m_1 m_2}\left[(m_1 + m_2) \mp \sqrt{(m_1 - m_2)^2 + m_1 m_2}\right]\right\}^{\frac{1}{2}} \tag{9.5}$$

Because there are two natural frequencies, the steady-state solution can be written as

$$x_j = \mathrm{Re}\left[A_{j1} e^{i\omega_1 t} + A_{j2} e^{i\omega_2 t}\right] \qquad j = 1, 2$$

where A_{j1} and A_{j2} are arbitrary complex coefficients. Without losing generality, we write explicitly the steady-state solution as

$$x_j = a_j \cos\omega_1 t + b_j \sin\omega_1 t + c_j \cos\omega_2 t + d_j \sin\omega_2 t \quad j = 1, 2 \tag{9.6}$$

where a_j, b_j, c_j, and $d_j (j = 1, 2)$ are real arbitrary constants to be determined. By using the initial conditions $x_j(0)$ and $x_j(0)$, we have

$$x_j(0) = a_j + c_j \qquad j = 1, 2 \tag{9.7}$$

$$\dot{x}_j(0) = \omega_1 b_j + \omega_2 d_j \quad j = 1, 2 \tag{9.8}$$

Note that Eq. (9.3) is valid for each mode of the vibration. When Eq. (9.6) is substituted into Eqs. (9.1) and (9.2) for the first normal mode, the coefficients of $\cos \omega_1 t$ and $\sin \omega_1 t$ must be zero. Hence we find

$$\left(2 - \omega_1^2 \frac{m_1}{k}\right) a_1 - a_2 = 0$$

$$\left(2 - \omega_1^2 \frac{m_1}{k}\right) b_1 - b_2 = 0$$

or

$$k_1 a_1 - a_2 = 0, \qquad k_1 b_1 - b_2 = 0 \tag{9.9}$$

where $k_1 = 2 - \omega_1^2(m_1/k)$. Similarly, for the second normal mode we have

$$k_2 c_2 - c_1 = 0, \qquad k_2 d_2 - d_1 = 0 \tag{9.10}$$

where $k_2 = 2 - \omega_2^2(m_2/k)$. From Eqs. (9.7–9.10), a_j, b_j, c_j, and d_j are determined. The results are written as follows:

$$a_1 = \frac{1}{1 - k_1 k_2}[x_1(0) - k_2 x_2(0)] \qquad a_2 = k_1 a_1 \tag{9.11}$$

$$b_1 = \frac{1}{(1 - k_1 k_2)\omega_1}[\dot{x}_1(0) - k_2 \dot{x}_2(0)] \qquad b_2 = k_1 b_1 \tag{9.12}$$

$$c_1 = \frac{-k_2}{1 - k_1 k_2}[k_1 x_1(0) - x_2(0)] \qquad c_2 = c_1/k_2 \tag{9.13}$$

$$d_1 = \frac{-k_2}{(1 - k_1 k_2)\omega_2}[k_1 \dot{x}_1(0) - \dot{x}_2(0)] \qquad d_2 = d_1/k_2 \tag{9.14}$$

Substituting Eqs. (9.11–9.14) into Eq. (9.6) gives the steady-state solutions of $x_1(t)$ and $x_2(t)$. From these we can see that in general the system is vibrating at the combination of two normal modes. To simplify the equations, let us consider a special case, i.e., $m_1 = m_2 = m$. Then the two natural frequencies are

$$\omega_1 = \sqrt{k/m}, \qquad \omega_2 = \sqrt{3k/m} \tag{9.15}$$

The constants are found to be

$$k_1 = 1, \qquad k_2 = -1$$

$$a_1 = \frac{1}{2}[x_1(0) + x_2(0)] = a_2$$

$$b_1 = \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)] = b_2$$

$$c_1 = \frac{1}{2}[x_1(0) - x_2(0)] = -c_2$$

$$d_1 = \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)] = -d_2$$

Hence the steady-state solutions are

$$x_1(t) = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t + \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t + \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t \qquad (9.16)$$

$$x_2(t) = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t - \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t - \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t \qquad (9.17)$$

Using these equations, we can see that it is possible for the system to oscillate at a particular frequency. If $x_1(0) = x_2(0)$ and $\dot{x}_1(0) = \dot{x}_2(0)$, the system will vibrate at the first normal mode. On the other hand if $x_1(0) = -x_2(0)$ and $\dot{x}_1(0) = -\dot{x}_2(0)$, then the system vibrates at the second normal mode. However, these conditions are hard to produce in the real world. Therefore, in general the vibration is a combination of two modes.

Through this example, a few remarks shall be made here. Note that the system can vibrate at one of the natural frequencies. The lower frequency is called the fundamental frequency, and the corresponding mode is the fundamental mode. The values of λ_i are called eigenvalues of the characteristic equation. The corresponding ratios of a_2/a_1 and c_2/c_1 or b_2/b_1 and d_2/d_1 obtained from Eqs. (9.9) and (9.10) are the component ratios of eigenvectors. In this example, ω_1 and ω_2 are different. A special case for $\omega_1 = \omega_2$ will be discussed later.

Example 9.2

Consider a torsional system with two degrees of freedom as shown in Fig. 9.2. Assume that the disks have mass moments of inertia of J_1 and J_2 with respect to the rotation axis. θ_1 and θ_2 are the angular displacements of the disks from their equilibrium positions, respectively. The torsional stiffness for the portion of

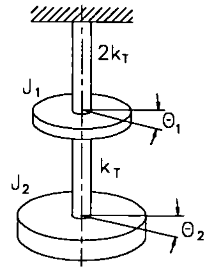

Fig. 9.2 Torsional system with two degrees of freedom.

the shaft between the disks can be expressed as $k_T = GJ/\ell$ where G is the shear modular of elasticity, J is the torsional constant of the cross section, and ℓ is length of the shaft. The torsional stiffness for the portion of the shaft between the support and the first disk is $2k_T$. Formulate the equations of motion, and determine the natural frequencies and shapes of the principal modes.

Solution. This is a conservative system. The kinetic and potential energies are written as

$$T = \tfrac{1}{2}J_1\dot{\theta}_1^2 + \tfrac{1}{2}J_2\dot{\theta}_2^2$$

$$V = \tfrac{1}{2}(2k_T)\theta_1^2 + \tfrac{1}{2}k_T(\theta_1 - \theta_2)^2$$

Lagrange's function is $L = T - V$. The equations of motion are then

$$J_1\ddot{\theta}_1 + 2k_T\theta_1 + k_T(\theta_1 - \theta_2) = 0 \tag{9.18}$$

$$J_2\ddot{\theta}_2 - k_T(\theta_1 - \theta_2) = 0 \tag{9.19}$$

By assuming

$$\theta_i = A_i e^{i\omega t}$$

we have

$$(3k_T/J_1 - \omega^2)A_1 - (k_T/J_1)A_2 = 0 \tag{9.20}$$

$$(-k_T/J_2)A_1 + (k_T/J_2 - \omega^2)A_2 = 0 \tag{9.21}$$

Because A_1 and A_2 cannot be all zero, the determinant of the coefficients must be zero, i.e.,

$$\begin{vmatrix} (3k_T/J_1 - \omega^2) & (-k_T/J_1) \\ (-k_T/J_2) & (k_T/J_2 - \omega^2) \end{vmatrix} = 0$$

Expanding the determinant leads to the characteristic equation

$$\omega^4 - k_T(3/J_1 + 1/J_2)\omega^2 + 2k_T^2/(J_1J_2) = 0 \tag{9.22}$$

The roots are

$$\omega_1^2 = \frac{k_T}{2}\left[(3/J_1 + 1/J_2) - \sqrt{(3/J_1)^2 - (2/J_1J_2) + (1/J_2)^2}\right] \quad (9.23a)$$

$$\omega_2^2 = \frac{k_T}{2}\left[(3/J_1 + 1/J_2) + \sqrt{(3/J_1)^2 - (2/J_1J_2) + (1/J_2)^2}\right] \quad (9.23b)$$

Substituting these into Eq. (9.20) gives the modes as

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k_T/J_1}{3k_T/J_1 - \omega_1^2} \quad (9.24a)$$

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k_T/J_1}{3k_T/J_1 - \omega_2^2} \quad (9.24b)$$

Example 9.3

Consider the vibration of an automobile modeled as a two-degree-of-freedom system, as shown in Fig. 9.3. The numerical values of the parameters are given as follows:

$$m = 1460\text{ kg}, \qquad \ell_1 = 1.37\text{ m}, \qquad \ell_2 = 1.68\text{ m}$$

$$k_1 = 35\text{ kN/m}, \qquad k_2 = 38\text{ kN/m}, \qquad I_C = 2170\text{ kg-m}^2$$

Determine the natural frequencies and the amplitude ratios under the normal modes of vibration.

Solution. To find the equations of motion, we take Lagrange's approach. Choose x and θ as the generalized coordinates where x is the vertical displacement of the center of mass and θ is the angular displacement of the automobile from the equilibrium position. Then the system will have kinetic energy

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I_c\dot{\theta}^2$$

and potential energy

$$V = \tfrac{1}{2}k_1(x - \ell_1\theta)^2 + \tfrac{1}{2}k_2(x + \ell_2\theta)^2$$

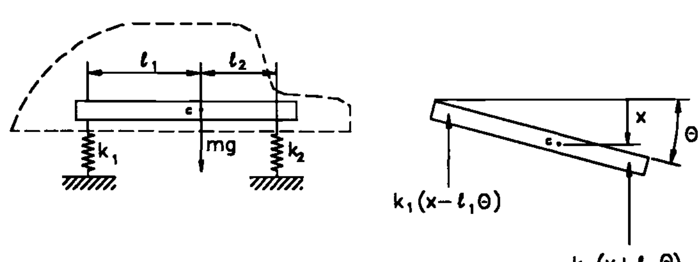

Fig. 9.3 Simplified model for a vibrating automobile.

With the use of Lagrange's equation, we obtain

$$m\ddot{x} + (k_1 + k_2)x + (k_2\ell_2 - k_1\ell_1)\theta = 0 \tag{9.25}$$

$$I_c\ddot{\theta} + (k_2\ell_2 - k_1\ell_1)x + \left(k_1\ell_1^2 + k_2\ell_2^2\right)\theta = 0 \tag{9.26}$$

The preceding equations are statically coupled because there are angular displacement θ terms in the equation for translational motion, Eq. (9.25), and translational displacement x terms in the equation for rotational motion, Eq. (9.26). Note that Eqs. (9.25) and (9.26) are linear ordinary differential equation with constant coefficients. The steady-state solutions can be assumed as

$$x(t) = Xe^{i\omega t}$$

$$\theta(t) = \Theta e^{i\omega t}$$

By substituting these solutions in Eqs. (9.25) and (9.26), we have

$$\begin{pmatrix} (k_1 + k_2 - \omega^2 m) & -(k_1\ell_1 - k_2\ell_2) \\ -(k_1\ell_1 - k_2\ell_2) & \left(k_1\ell_1^2 + k_2\ell_2^2 - \omega^2 I_c\right) \end{pmatrix} \begin{pmatrix} X \\ \Theta \end{pmatrix} = 0 \tag{9.27}$$

Simplifying Eq. (9.27) with the substitution of given numerical quantities gives the characteristic equation as

$$(73{,}000 - 1460\omega^2)(172{,}942.7 - 2170\omega^2) - (15{,}890)^2 = 0$$

From this we find the two natural frequencies to be

$$\omega_1 = 6.894 \text{ rad/s}, \qquad \omega_2 = 9.065 \text{ rad/s}$$

The amplitude ratios corresponding to the natural frequencies are found from Eq. (9.20) to be

$$\left(\frac{X}{\Theta}\right)_{\omega_1} = -4.401 \text{ m/rad} \qquad \left(\frac{X}{\Theta}\right)_{\omega_2} = 0.338 \text{ m/rad}$$

Example 9.4

For the system shown in Fig. 9.4, let the initial conditions $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. With the use of Laplace transform method, determine the general solution of the system when m_1 is excited by a harmonic force $F_1 \sin\omega t$. To simplify the consideration, assume $m_1 = m_2 = m$.

Solution. From Example 9.1, we can obtain the equations for the motion as

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_1 \sin\omega t \tag{9.28}$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \tag{9.29}$$

The Laplace transform is a powerful tool for solving linear differential equations as was discussed in Section 8.5. Here we illustrate how the method can be applied

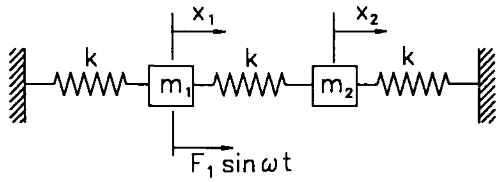

Fig. 9.4 Two-degree-of-freedom system under forced harmonic excitation.

to two equations simultaneously. Taking the Laplace transform of the preceding equations, i.e., multiplying both sides of equations by $e^{-st}\mathrm{d}t$ and integrating from zero to infinity, gives

$$ms^2X + 2kX_1 - kX_2 = F_1\frac{\omega}{s^2+\omega^2} \tag{9.30}$$

$$ms^2X_2 - kX_1 + 2kX_2 = 0 \tag{9.31}$$

where X_i is the transformed function of $x_i(t)$ and $(\omega/s^2+\omega^2) = \mathcal{L}(\sin\omega t)$ obtained from Appendix F. Rewrite the equations in matrix form as

$$\begin{pmatrix} ms^2+2k & -k \\ -k & ms^2+2k \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix} \tag{9.32}$$

or

$$Z(s)X = F$$

where $Z(s)$ is the coefficient matrix of Eq. (9.32). Premultiplying Eq. (9.32) by the inverse matrix of $Z(s)$ gives

$$X = [Z(s)]^{-1}F$$

or

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = [Z(s)]^{-1}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

$$= \frac{\mathrm{adj}[Z(s)]}{|Z(s)|}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

$$= \frac{1}{|Z(s)|}\begin{pmatrix} ms^2+2k & k \\ k & ms^2+2k \end{pmatrix}\begin{pmatrix} F_1\dfrac{\omega}{s^2+\omega^2} \\ 0 \end{pmatrix}$$

Carrying out the matrix algebra leads to

$$X_1 = \frac{(ms^2+2k)\omega F_1}{[(ms^2+2k)^2-k^2](s^2+\omega^2)} \tag{9.33}$$

$$X_2 = \frac{k\omega F_1}{[(ms^2+2k)^2-k^2](s^2+\omega^2)} \tag{9.34}$$

With the use of the partial fractions expansion given in Appendix E, we can express X_i as

$$X_1 = \frac{\omega F_1}{2(\omega^2 - \omega_1^2)m}\frac{1}{s^2+\omega_1^2} + \frac{\omega F_1}{2(\omega^2 - \omega_2^2)m}\frac{1}{s^2+\omega_2^2} + \frac{\omega_1^2(2\omega_1^2 - \omega^2)F_1}{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)k}\frac{\omega}{s^2+\omega^2} \tag{9.35}$$

$$X_2 = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\frac{\omega_1}{s^2+\omega_1^2} - \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\frac{\omega_2}{s^2+\omega_2^2} + \frac{\omega_1^2\omega_2^2 F_1}{3k(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}\frac{\omega}{s^2+\omega^2} \tag{9.36}$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Taking inverse Laplace transform, we find

$$x_1(t) = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\sin\omega_1 t + \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\sin\omega_2 t + \frac{\omega_1^2(2\omega_1^2 - \omega^2)F_1}{k(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}\sin\omega t \tag{9.37}$$

$$x_2(t) = \frac{\omega\omega_1 F_1}{2k(\omega^2 - \omega_1^2)}\sin\omega_1 t - \frac{\omega\omega_2 F_1}{6k(\omega^2 - \omega_2^2)}\sin\omega_2 t + \frac{\omega_1^2\omega_2^2 F_1}{3k(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}\sin\omega t \tag{9.38}$$

From this result we can conclude that the system will vibrate at the combination of three frequencies. Resonance will take place as ω approaches either ω_1 or ω_2. Note also that the Laplace transform method is very systematic and straightforward.

Example 9.5

Consider a damped system with two degrees of freedom as shown in Fig. 9.5. Find the equations of motion. Determine the natural frequencies and the response of principal modes. Discuss all possible cases for different roots of the characteristic equation.

Solution. From the balance of forces in the free-body diagram, we find

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - c_1\dot{x}_1 - c_2(\dot{x}_1 - \dot{x}_2) \tag{9.39}$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \tag{9.40}$$

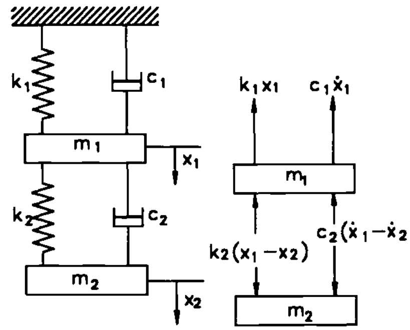

Fig. 9.5 Damped system with two degrees of freedom.

which can be rearranged as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = 0$$
$$-c_2\dot{x}_1 - k_2x_1 + m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = 0$$

Assume that the solutions are in the form of

$$x_1 = X_ie^{st}$$

Then we have

$$\begin{aligned}[m_1s^2 + (c_1 + c_2)s + (k_1 + k_2)]X_1 - (c_2s + k_2)X_2 = 0\\ -(c_2s + k_2)X_1 + (m_2s^2 + c_2s + k_2)X_2 = 0\end{aligned} \tag{9.41}$$

Because X_1 and X_2 cannot be zero, the determinant of the coefficients must be zero. Expanding the determinant gives

$$\begin{aligned}m_1m_2s^4 + [m_1c_2 + m_2(c_1 + c_2)]s^3 + [m_1k_2 + m_2(k_1 + k_2)\\ + c_1c_2]s^2 + (k_1c_2 + k_2c_1)s + k_1k_2 = 0\end{aligned} \tag{9.42}$$

From this equation, s is expected to have four roots. When these roots are substituted into Eq. (9.41), they will give four relationships between X_1 and X_2. Note that because all the physical constants m_i, k_i, and c_i are positive and all the signs are plus, there is no possibility of a positive root. Thus the following possibilities exist for the four roots: 1) all four roots are complex numbers that will be two pairs of complex conjugates; 2) all four roots are real and negative; and 3) two roots are real and negative, and the other two complex conjugates.

Now let us examine these three possible cases. For the two pairs of complex conjugates, i.e.,

$$\begin{aligned}s_1 = -p_1 + iq_1, \qquad s_2 = -p_1 - iq_1\\ s_3 = -p_2 + iq_2, \qquad s_4 = -p_2 - iq_2\end{aligned} \tag{9.43}$$

where p_1, p_2, q_1, and q_2 are real and positive. The first two roots will give the following solutions:

$$x_1 = X_{11}\exp[(-p_1 + iq_1)t] + X_{12}\exp[(-p_1 - iq_1)t]$$
$$= e^{-p_1 t}(X_{11}e^{iq_1 t} + X_{12}e^{-iq_1 t}) = A_{11}e^{-p_1 t}\sin(q_1 t + \phi_{11}) \quad (9.44)$$

and

$$x_2 = X_{21}\exp[(-p_1 + iq_1)t] + X_{22}\exp[(-p_1 - iq_1)t]$$
$$= A_{21}e^{-p_1 t}\sin(q_1 t + \phi_{21}) \quad (9.45)$$

These two solutions represent oscillatory motion with the magnitudes decaying exponentially. In a similar way, for roots s_3 and s_4, we have another two solutions. Combining all four roots, the general solutions are then

$$x_1 = A_{11}e^{-p_1 t}\sin(q_1 t + \phi_{11}) + A_{12}e^{-p_2 t}\sin(q_2 t + \phi_{12}) \quad (9.46)$$

$$x_2 = A_{21}e^{-p_1 t}\sin(q_1 t + \phi_{21}) + A_{22}e^{-p_2 t}\sin(q_2 t + \phi_{22}) \quad (9.47)$$

where $A_{11}, A_{12}, A_{21}, A_{22}, \phi_{11}, \phi_{12}, \phi_{21}$, and ϕ_{22} are to be determined. With the use of Eq. (9.41), for each root, four relationships are established. Another four relationships can be found by the four initial conditions $x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0)$. Therefore all the constants will be determined.

For the second case, four roots are real and negative, then the motion is not oscillatory; the displacements of masses are decaying exponentially. This case is similar to the overdamped case discussed in Section 8.4.

Finally, for the third case, two roots are real and negative, and the other two are a pair of complex conjugates. The general solutions are then the combination of the terms, as in Eq. (9.44), and the other terms of exponential functions:

$$x_i = A_i e^{-pt}\sin(qt + \phi_i) + c_i e^{-s_3 t} + d_i e^{-s_4 t}$$

The constants are determined through the same procedures as discussed for the first case.

9.2 Matrix Formulation for Systems with Multiple Degrees of Freedom

There are usually n ordinary differential equations for describing a system of n degrees of freedom. Solving these equations is straightforward but cumbersome and time-consuming if n is large. Fortunately, matrix methods are ideal for this purpose, and many matrix operations can be carried out by digital computers. In this section we will discuss the matrix techniques for various properties of vibrating systems.

Free Vibration of Undamped Systems

The equations of motion for an n degrees-of-freedom system expressed in matrix form are simplified to

$$M\ddot{X} + KX = 0 \quad (9.48)$$

where the mass matrix is

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

the stiffness matrix is

$$K = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}$$

and the displacement vector (a column matrix) is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Note that M and K are square symmetric matrices. Premultiplying Eq. (9.48) by M^{-1}, we find

$$I\ddot{X} + M^{-1}KX = 0$$

or

$$I\ddot{X} + AX = 0 \tag{9.49}$$

where $A = M^{-1}k$ is called the system matrix or the dynamic matrix because the dynamic properties of the system are defined by this matrix. By assuming the solution of the equation in the form of

$$X = Ce^{i\omega t}$$

we have

$$\ddot{X} = -\omega^2 X$$

or

$$\ddot{X} = -\lambda X$$

where $\lambda = \omega^2$. Then Eq. (9.49) becomes

$$(A - \lambda I)X = 0$$

Because X is not zero, the determinant of the coefficients must be zero, i.e.,

$$|A - \lambda I| = 0 \tag{9.50}$$

This is the characteristic equation of the system. From this equation we can find n roots of λ_i, which are called eigenvalues. By substituting λ_i into Eq. (9.51), we can obtain the corresponding mode shape x_i, which is called an eigenvector. Note that, as λ_i is substituted into Eq. (9.49), there are most likely only $(n-1)$ independent equations, but there are various nx_i to be determined. One x_i can be chosen arbitrarily. It is convenient to add one condition as

$$\sum_i x_i^2 = 1 \tag{9.51}$$

In this way, x_i may be considered direction cosine throughout for two- and three-degree-of-freedom systems. For $n > 3$, the additional condition (9.51) is still valid to replace the condition that one x_i is arbitrarily chosen. Details will be shown in the examples.

Eigenvalue and eigenvector properties: different eigenvalues $\lambda_i \neq \lambda_j$. For the ith mode, we have

$$AX_i = \lambda_i X_i \tag{9.52}$$

If the transposed equation (9.52) is postmultiplied by X_j, then it becomes

$$(AX_i)^T X_j = \lambda_i X_i^T X_j$$

or

$$X_i^T A X_j = \lambda_i X_i^T X_j \tag{9.53}$$

On the other hand, for the jth mode, the equation is

$$AX_j = \lambda_j X_j$$

Premultiplying the preceding equation by X_i^T gives

$$X_i^T A X_j = \lambda_j X_i^T X_j \tag{9.54}$$

When Eq. (9.53) is subtracted by Eq. (9.54), we find

$$(\lambda_i - \lambda_j) X_i^T X_j = 0$$

Therefore, X_i and X_j are orthogonal.

In addition, consider the equation for the ith mode

$$KX_i = \lambda_i M X_i$$

Premultiplying the equation by X_j^T gives

$$X_j^T K X_i = \lambda_i X_j^T M X_i \tag{9.55}$$

Next, starting with the equation for the jth mode and premultiplying by X_i^T, we obtain

$$X_i^T K X_j = \lambda_j X_i^T M X_j$$

Taking the transpose of the preceding equation leads to

$$X_j^T K X_i = \lambda_j X_j^T M X_i \tag{9.56}$$

because M and K are symmetric matrices. Thus, subtracting Eq. (9.56) from Eq. (9.55) gives

$$0 = (\lambda_i - \lambda_j) X_j^T M X_i$$

For $\lambda_i \neq \lambda_j$, the preceding equation requires

$$X_j^T M X_i = 0 \tag{9.57}$$

It is also evident from Eq. (9.55) that

$$X_j^T K X_i = 0 \qquad \text{for } i \neq j \tag{9.58}$$

On the other hand, as $i = j$, we write

$$X_i^T M X_i = M_i$$

and

$$X_i^T K X_i = K_i$$

M_i and K_i are called generalized mass and generalized stiffness, respectively.

Eigenvalue and eigenvector properties: repeated eigenvalues $\lambda_i = \lambda_j$. Suppose that there are three roots from the characteristic equation, with $\lambda_1 = \lambda_2 = \lambda_0$ and $\lambda_3 \neq \lambda_0$. Then we have

$$\begin{aligned} Ax_1 &= \lambda_0 x_1 \\ Ax_2 &= \lambda_0 x_2 \\ Ax_3 &= \lambda_3 x_3 \end{aligned} \tag{9.59}$$

Multiplying the second equation by any constant b and adding it to the first gives

$$A(x_1 + bx_2) = \lambda_0 (x_1 + bx_2)$$

Thus a new eigenvector $x_{12} = x_1 + bx_2$ also satisfies the equation; hence, no unique eigenvector exists for λ_0. However, based on orthogonal properties of eigenvectors, we can choose x_1 to be perpendicular to x_3 and x_2 perpendicular to x_1 and x_3. Details will be shown in the example.

Principal or normal coordinates. With the properties of eigenvalues and eigenvectors already discussed, we can transform the equation of motion from Eq. (9.48)

$$M\ddot{X} + KX = 0 \tag{9.48}$$

to

$$\ddot{Y}_i + \omega_i^2 Y_i = 0$$

by the transformation of

$$X = PY \tag{9.60}$$

where P is called the modal matrix and is formed by eigenvectors. For a three-degree-of-freedom system

$$P = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_3 \right) = (X_1 \quad X_2 \quad X_3)$$

where X_1, X_2, X_3 are eigenvectors. With the transformation by Eq. (9.57), Eq. (9.56) becomes

$$MP\ddot{Y} + KPY = 0$$

Premultiplying the preceding equation by P^T gives

$$P^T MP\ddot{Y} + P^T KPY = 0 \tag{9.61}$$

Looking into details, we find

$$\begin{aligned} P^T MP &= (X_1 \quad X_2 \quad X_3)^T (M)(X_1 \quad X_2 \quad X_3) \\ &= \begin{pmatrix} X_1^T MX_1 & X_1^T MX_2 & X_1^T MX_3 \\ X_2^T MX_1 & X_2^T MX_2 & X_2^T MX_3 \\ X_3^T MX_1 & X_3^T MX_2 & X_3^T MX_3 \end{pmatrix} \\ &= \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \end{aligned}$$

where $M_i = X_i^T MX_i$ and Eq. (9.57) has been used for the zero terms. Similarly

$$P^T KP = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

Therefore, Eq. (9.61) becomes

$$M_i \ddot{Y}_i + K_i Y_i = 0 \qquad i = 1, 2, 3$$

which can be solved in a manner similar to that of the single-degree-of-freedom system. Once Y_i is found, the solution of the original equation can be obtained simply by applying the transformation equation

$$X(t) = PY$$

Example 9.6

Consider the system shown in Fig. 9.1 with $m_1 = m_2 = m$. Find the steady-state solution with the use of principal coordinates.

Solution. The equation of motion in matrix form is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{9.62}$$

and the eigenvalues and eigenvectors are found

$$\lambda_1 = \omega_1^2 = k/m, \qquad \lambda_2 = \omega_2^2 = 3(k/m)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\lambda_2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Hence the modal matrix P is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The transformation is

$$X = PY$$

Equation (9.62) then becomes

$$MP\ddot{Y} + KPY = 0$$

Premultiplying the preceding equation by P^T, we find

$$P^T MP\ddot{Y} + P^T KPY = 0$$

or

$$m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

This equation can be further simplified to

$$\ddot{y}_i + \omega_i^2 y_i = 0 \quad i = 1, 2 \tag{9.63}$$

The general solution of Eq. (9.63) is

$$y_i(t) = y_i(0) \cos \omega_i t + (1/\omega_i) \dot{y}_i(0) \sin \omega_i t$$

The initial conditions for the principal coordinates can be found from the transformation equation as follows:

$$Y = P^T X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$y_1(0) = \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)]$$

$$y_2(0) = \frac{1}{\sqrt{2}}[x_2(0) - x_1(0)]$$

Similar relationships can be found for $\dot{y}_i(0)$. Therefore

$$y_1(t) = \frac{1}{\sqrt{2}}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{\sqrt{2}\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t$$

$$y_2(t) = \frac{1}{\sqrt{2}}[x_2(0) - x_1(0)]\cos\omega_2 t + \frac{1}{\sqrt{2}\omega_2}[\dot{x}_2(0) + \dot{x}_1(0)]\sin\omega_2 t$$

To find the solution for x_1, x_2, we substitute $y_i(t)$ into the transformation equation

$$X = PY$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_2 \\ y_1 + y_2 \end{pmatrix}$$

Therefore

$$x_1 = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t$$
$$+ \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t + \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t$$

$$x_2 = \frac{1}{2}[x_1(0) + x_2(0)]\cos\omega_1 t + \frac{1}{2\omega_1}[\dot{x}_1(0) + \dot{x}_2(0)]\sin\omega_1 t$$
$$- \frac{1}{2}[x_1(0) - x_2(0)]\cos\omega_2 t - \frac{1}{2\omega_2}[\dot{x}_1(0) - \dot{x}_2(0)]\sin\omega_2 t$$

The preceding results agree completely with Eqs. (9.16) and (9.17).

Example 9.7

To illustrate a case of repeated roots in a characteristic equation, let us consider a particular system with the equation of motion as

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} 0 & -k & k \\ -k & 0 & k \\ k & k & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

When this equation is premultiplied by M^{-1}, the equation becomes

$$\ddot{X} + M^{-1}KX = 0$$

or

$$\ddot{X} + AX = 0 \tag{9.64}$$

where

$$A = M^{-1}K = \frac{k}{m}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{9.65}$$

1) Find the eigenvalues and eigenvectors, and 2) find the modal matrix P and carry out the product $P^T AP$.

Solution. 1) The steady-state solution of Eq. (9.64) may be assumed as

$$X = Ce^{i\omega t}$$

$$\ddot{X} = -C\omega^2 e^{i\omega t} = -C\lambda e^{i\omega t}$$

Substituting the preceding expression into Eq. (9.64) leads to

$$(A - \lambda I)X = 0 \tag{9.66}$$

Hence the characteristic equation is

$$|A - \lambda I| = 0$$

or

$$\lambda^3 - 3\left(\frac{k}{m}\right)^2 \lambda + 2\left(\frac{k}{m}\right)^3 = 0$$

$$\left(\lambda - \frac{k}{m}\right)^2 \left(\lambda + 2\frac{k}{m}\right) = 0$$

which gives the eigenvalues

$$\lambda_1 = \lambda_2 = k/m \qquad \text{and} \qquad \lambda_3 = -2k/m$$

To find the eigenvector corresponding to $\lambda_3 = -2k/m$, we substitute λ_3 into Eq. (9.66) and obtain

$$\frac{k}{m}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

In explicit form, we have

$$2x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

Note that there are only two independent equations in the preceding three equations. For example, the second equation can be obtained by the subtraction of the first equation from the third equation. Fortunately we can impose

$$x_1^2 + x_2^2 + x_3^2 = 1$$

Then we find the eigenvector

$$X_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{9.67}$$

For $\lambda_1 = \lambda_2 = k/m$, the equations become

$$-x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 - x_3 = 0$$

There is only one independent equation in the preceding equations. Taking $x_1 = x_3 - x_2$ leads to

$$X_1 = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

To satisfy the orthogonality condition $X_1^T X_3 = 0$, we have $x_2 = x_3$ and also to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$. Hence

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{9.68}$$

The second eigenvector for $\lambda = k/m$ then can be constructed from

$$X_2 = X_1 \times X_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{9.69}$$

2) The modal matrix P is obtained by collecting the eigenvectors from Eqs. (9.69–9.71)

$$P = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{9.70}$$

The product of $P^T AP$ is found to be

$$P^T AP = \frac{k}{m} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note that the result is a diagonal matrix of the eigenvalues.

Forced Vibration of Undamped Systems

The vibration of systems with multiple degrees of freedom activated by harmonic forcing functions can be treated quite simply as an extension of our matrix methods. Consider a system with three degrees of freedom and with forces $F_1(t) = q_1 e^{i\omega t}$, $F_2(t) = q_2 e^{i\omega t}$, and $F_3(t) = q_3 e^{i\omega t}$ being applied in the directions of x_1, x_2, and x_3, respectively. The equation of motion can be written as

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} e^{i\omega t} \tag{9.71}$$

By assuming the steady-state solution as

$$x_i = X_i e^{i\omega t} \tag{9.72}$$

we have

$$-M\omega^2 X + KX = Q$$

where M is the mass matrix, K is the stiffness matrix, and X is the column matrix of X_i. Further simplifying the equation gives

$$AX = Q$$

where $A = K - M\omega^2$. By premultiplying the equation by A^{-1}, we find immediately,

$$X = A^{-1}Q \tag{9.73}$$

Forced Vibration of Viscously Damped Systems

The differential equations of motion for a damped system having n degrees of freedom can be written in matrix form as

$$M\ddot{X} + C\dot{X} + KX = F \tag{9.74}$$

where M, C, K are $n \times n$ symmetric matrices and X, F are $n \times 1$ column matrices. To find the homogeneous solution, we set $F = 0$ and assume solutions of the form

$$x_i(t) = X_i e^{st} \quad i = 1, 2, \ldots, n$$

Substitution of the assumed solutions yields the matrix equation

$$s^2 MX + sCX + KX = 0$$

or

$$(s^2 M + sC + K)X = 0$$

To find the eigenvalues, we set the determinant of the coefficients to zero:

$$|s^2 M + sC + K| = 0$$

This is the characteristic equation. From this equation, we expect to find n roots or n eigenvalues. Then we can find n corresponding eigenvectors. However, the n roots are usually n pairs of complex conjugates and n eigenvectors are also in complex form. Therefore, very often we specify that the solution is the real part of the assumed solution, i.e.,

$$x_i(t) = \mathrm{Re}\left[X_i e^{s_i t}\right]$$

To find the particular solution of Eq. (9.74), we consider the following two special cases of damping systems.

Light damping. From the homogeneous undamped equation

$$M\ddot{X} + KX = 0$$

we obtain the eigenvalues and eigenvectors. From this we can transform X to principal coordinates Y by

$$X = PY$$

Substituting this transformation into Eq. (9.74) and premultiplying the equation by P^T, we have

$$P^T MP\ddot{Y} + P^T CP\dot{Y} + P^T KPY = P^T F \tag{9.75}$$

It has been shown previously that $P^T MP$ and $P^T KP$ are diagonal matrices. In general, $P^T CP$ results in a nondiagonal matrix. A frequently used approach for approximating the response of a system with light damping is to ignore all off-diagonal terms of the transformed damping matrix, then Eq. (9.75) becomes n uncoupled equations. Each can be solved by the methods used for a single-degree-of-freedom system already discussed.

Proportional damping. If C is proportional to M and K

$$C = \alpha M + \beta K \tag{9.76}$$

where α and β are constants, then

$$P^T CP = \alpha P^T MP + \beta P^T KP$$

Thus Eq. (9.75) becomes uncoupled. Each principal coordinate will have the equation of motion of the form

$$\ddot{Y}_1 + \left(\alpha + \beta\omega_i^2\right)\dot{Y}_i + \omega_i^2 Y_i = f_i(t) \tag{9.77}$$

which can be solved by the methods discussed in Chapter 8.

9.3 Lumped Parameter Systems with Transfer Matrices

Many vibrational systems can be modeled as systems with lumped parameters. The method of transfer matrices is introduced here. This is a powerful tool for solving lumped parameter systems. To establish the method, we first apply the method for mass-spring systems. Then we will apply it to torsional systems and flexural beam systems. The method requires the knowledge of matrix operation, which has been reviewed in previous sections.

State Vectors and Transfer Matrices

To apply the method of lumped parameters to a vibration system, we divide the system into a number of appropriate sections. For each section, physical quantities are classified into two kinds of variables. One kind is known as the force, which includes force, torque, shear, and bending moment, and the other as the displacement, which includes linear displacement and angular displacement.

Now, we define two terms, state vector and transfer matrix, that are used in the method of lumped parameters. A state vector is a column matrix that has all of the components of the forces and displacement at a point i. The transfer matrix relates the state vectors from one location to another along the system.

Let us consider a mass and a spring as shown in Fig. 9.6a. We can formulate two equations: one is for the force, and the other for the displacement. For the force, we have

$$\sum f_i = m\ddot{x}_i$$

$$f_i - f_{i-1} = m\ddot{x}_i$$

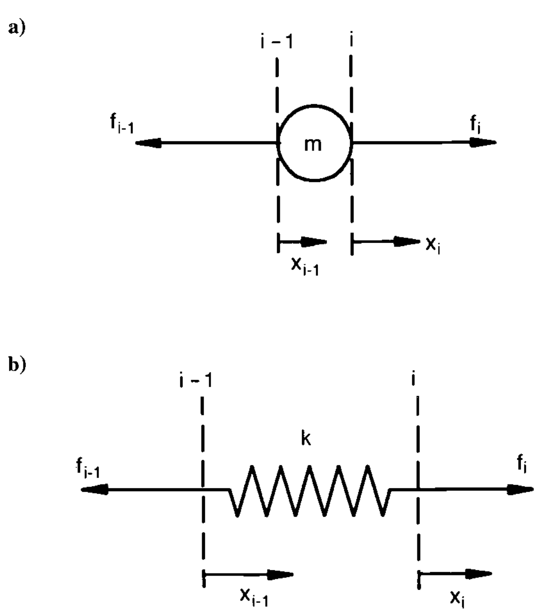

Fig. 9.6 State variables for a mass-spring system.

For a harmonic motion, the solution is assumed as $x_i = X_i e^{i\omega t}$. The forces applied must be in the same form $f = Fe^{i\omega t}$. Hence we have

$$F_i = F_{i-1} - m\omega^2 X_i$$

For the displacement, we have

$$x_i = x_{i-1}$$

or

$$X_i = X_{i-1}$$

In matrix form

$$\begin{pmatrix} X_i \\ F_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ F_{i-1} \end{pmatrix} \tag{9.78}$$

Next, let us consider the state variables around the spring as shown in Fig. 9.6b; we have

$$\sum F = 0$$

$$f_i = f_{i-1} = k(x_i - x_{i-1})$$

or

$$F_i = F_{i-1} = k(X_i - X_{i-1})$$

Again, in matrix form

$$\begin{pmatrix} X_i \\ F_i \end{pmatrix} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ F_{i-1} \end{pmatrix} \tag{9.79}$$

In Eqs. (9.78) and (9.79), the state vector is

$$Z_i = \begin{pmatrix} X_i \\ F_i \end{pmatrix}$$

The transfer matrices in Eq. (9.78) and (9.79) are, respectively,

$$A = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & -1 \end{pmatrix} \tag{9.80}$$

$$B = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \tag{9.81}$$

Therefore, for a system consisting of a spring and a mass as shown in Fig. 9.7, we can write the equations as

$$Z_i = BZ_{i-1}$$

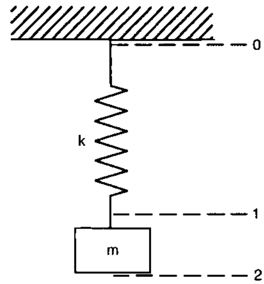

Fig. 9.7 Mass-spring system.

for the relationship across the spring and

$$Z_{i+1} = AZ_i$$

for that across the mass. The combined equation then is

$$Z_{i+1} = ABZ_{i-1} \tag{9.82}$$

Example 9.8

For a mass-spring system as shown in Fig. 9.7, determine the natural frequency of the system using state vectors and transfer matrices.

Solution. Applying the formulation given in the section, we have the transfer matrices as

$$B_{1-0} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix}$$

and

$$A_{2-1} = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix}$$

The equation of motion is

$$Z_2 = A_{2-1}B_{1-0}Z_0$$

In detail, we have

$$\begin{pmatrix} X_2 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ F_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1/k \\ -m\omega^2 & (1 - m\omega^2/k) \end{pmatrix} \begin{pmatrix} X_0 \\ F_0 \end{pmatrix} \tag{9.83}$$

Two different conditions for the vibrations may be illustrated as follows: 1) free vibration and 2) forced vibration.

1) The conditions for free vibration are $F_2 = 0$ and $X_0 = 0$. Then Eq. (9.83) leads to two equations

$$X_2 = F_0/k$$

and

$$0 = (1 - m\omega^2/k)F_0$$

which give the displacement of the mass in terms of the force in the spring and the natural frequency

$$\omega = \sqrt{k/m}$$

2) For a harmonically forced vibration $f_2 = Fe^{i\omega t}$, the magnitude of the force is F. Hence the conditions can be written as $F_2 = F$ and $X_0 = 0$. Again Eq. (9.83) gives

$$X_2 = F_0/k$$

and

$$F = (1 - m\omega^2/k)F_0$$

Rearranging leads to

$$X_2 = \frac{F}{k(1 - m\omega^2/k)} \tag{9.84}$$

which is the familiar result.

Transfer Matrices for Torsional Systems

Consider a disk and a bar as shown in Fig. 9.8a. As we have done in the last section, we formulate one equation for force and one for displacement. From the balance of torque, we have

$$t_i - t_{i-1} = J\ddot{\theta}_i \qquad \theta_i = \theta_{i-1}$$

Under the harmonic vibration, we express the state vector as

$$z_i = Z_i e^{i\omega t}$$

where the capital letter represents the magnitude. Therefore, the equations written in matrix form become

$$\begin{pmatrix} \Theta_i \\ T_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{i-1} \\ T_{i-1} \end{pmatrix} \tag{9.85}$$

Next consider Fig. 9.8b, and assume that the bar is an elastic torsional spring. Then we find

$$t_i = t_{i-1} = k(\theta_i - \theta_{i-1})$$

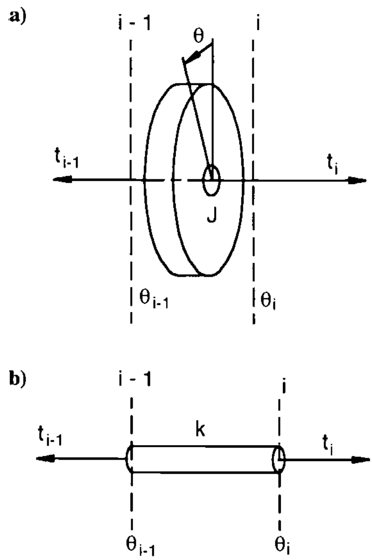

Fig. 9.8 State variables for a torsional system.

which can be written in matrix form as

$$\begin{pmatrix} \Theta_i \\ T_i \end{pmatrix} = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{i-1} \\ T_{i-1} \end{pmatrix} \tag{9.86}$$

Similar to the case of mass-spring system, if from $i - 1$ to i is a torsional spring we can have the equation

$$Z_i = BZ_{i-1}$$

where

$$B = \begin{pmatrix} 1 & 1/k \\ 0 & 1 \end{pmatrix} \tag{9.87}$$

For the case where from i to $i + 1$ is a disk, the equation is

$$Z_{i+1} = AZ_i \tag{9.88}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{pmatrix}$$

Then the combined equation from $i - 1$ to $i + 1$ is

$$Z_{i+1} = ABZ_{i-1} \tag{9.89}$$

Note that this formulation can be applied to many successive stations of the torsional system. Then the final equation is in the form of

$$Z_n = (AB)_n(AB)_{n-1}, \ldots, (AB)_1 Z_0 \tag{9.90}$$

Example 9.9

For the torsional system shown in Fig. 9.9, employ transfer matrix to find the relationship from station 0 to 3. Determine the natural frequencies for the principal modes.

Solution. To simplify the consideration while not losing generality, let us consider that the boundary conditions are $\theta_0 = 0$, $T_0 = 1$, $\theta_3 = \theta$, and $T_3 = 0$ where θ is arbitrary. The equation relating station 0 to 1 can be written as

$$\begin{pmatrix}\theta_1\\ T_1\end{pmatrix} = \begin{pmatrix}1 & 0\\ -3J\omega^2 & 1\end{pmatrix}\begin{pmatrix}1 & 1/(2k)\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 1/(2k)\\ -3J\omega^2 & 1-3J\omega^2/(2k)\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

In a similar manner, to relate stations 1 to 2 and 2 to 3, we have

$$\begin{pmatrix}\theta_3\\ T_3\end{pmatrix} = \begin{pmatrix}1 & 1/k\\ -J\omega^2 & (1-3J\omega^2/k)\end{pmatrix}\begin{pmatrix}1 & 1/(1.5k)\\ -2J\omega^2 & (1-2J\omega^2/1.5k)\end{pmatrix}$$

$$\times\begin{pmatrix}1 & 1/(2k)\\ -3J\omega^2 & (1-3J\omega^2/2k)\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

Carrying out the product of the matrices, we find

$$T_3 = \frac{1}{2k}\left[-J\omega^2 - 2J\omega^2\left(1-\frac{J\omega^2}{k}\right)\right]$$

$$+\left[-\frac{J\omega^2}{1.5k} + \left(1-\frac{J\omega^2}{k}\right)\left(1-\frac{2J\omega^2}{1.5k}\right)\right]\left(1-\frac{3J\omega^2}{2k}\right) \tag{9.91}$$

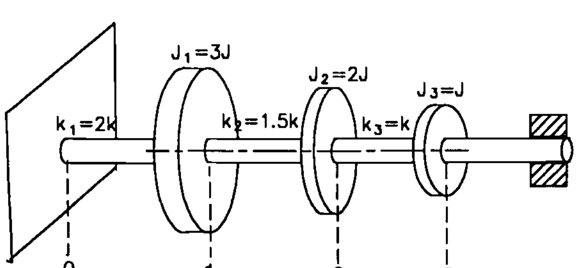

Fig. 9.9 States for the torsional system.

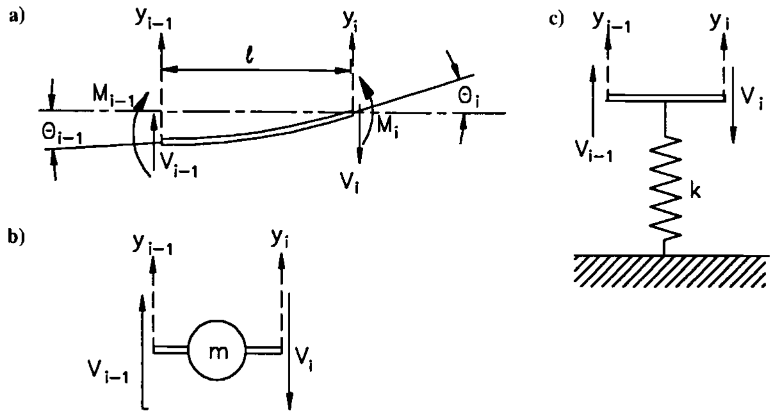

Fig. 9.10 Elements of lumped system for a vibrating beam.

To simplify the equation, let $\omega^2 = \lambda(k/J)$ and set $T_3 = 0$, we obtain

$$2\lambda^3 - \frac{41}{6}\lambda^2 + 6\lambda - 1 = 0$$

and find the three roots as

$$\lambda_1 = 0.2168, \qquad \lambda_1 = 1.0964, \qquad \lambda_3 = 2.1034$$

Therefore the natural frequencies of the principal modes are

$$\omega_1 = 0.4656\sqrt{\frac{k}{J}}, \qquad \omega_2 = 1.0471\sqrt{\frac{k}{J}}, \qquad \omega_3 = 1.4503\sqrt{\frac{k}{J}}$$

Transfer Matrices for Vibrating Beams

A beam is a continuous solid, but it also can be modeled as lumped masses connected by massless beam sections. The lateral vibration of the beam can be solved successfully by using state vectors and transfer matrices. The method originally developed by N. O. Myklestad and adapted by many textbooks* is discussed in this section.

To formulate the governing equations, a portion of the beam is broken into three elements as shown in Fig. 9.10. They are the massless beam section, the mass section, and the load section.

The massless beam section as shown in Fig. 9.10a. There are four equations to relate state vector at station $i-1$ to that at station i. From $\sum F = 0$, we have

$$V_i = V_{i-1} \tag{9.92}$$

*Thomson, W. T., *Theory of Vibration with Applications*, 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, 1988.

From $\sum M = 0$, we obtain

$$M_i = M_{i-1} + V_{i-1}\ell \tag{9.93}$$

From the relationship between the change of slope and the moment applied in the beam, we have

$$\begin{aligned}\theta_i - \theta_{i-1} &= \frac{1}{EI}\int_0^\ell M(x)\,\mathrm{d}x = \frac{1}{EI}\left(M_{i-1} + \frac{1}{2}V_{i-1}\ell\right)\ell \\ \theta_i &= \theta_{i-1} + \frac{M_{i-1}\ell}{EI} + \frac{V_{i-1}\ell^2}{2EI}\end{aligned} \tag{9.94}$$

The deflection is found similarly

$$Y_i - Y_{i-1} = \int_0^\ell \theta(x)\,\mathrm{d}x = \theta_{i-1}\ell + \frac{M_{i-1}\ell^2}{2EI} + \frac{V_{i-1}\ell^3}{6EI}$$

$$Y_i = Y_{i-1} + \theta_{i-1}\ell + \frac{M_{i-1}\ell^2}{2EI} + \frac{V_{i-1}\ell^3}{6EI} \tag{9.95}$$

Combining Eqs. (9.92–9.95) leads to

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & \ell & \ell^2/2EI & \ell^3/6EI \\ 0 & 1 & \ell/EI & \ell^2/2EI \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.96}$$

Note that there are four terms in the state vector. Once the transfer matrix is determined for one section, it can be used for all sections of the same length and same flexural rigidity.

The mass section as shown in Fig. 9.10b. From the equation of motion

$$\sum F = m\ddot{y}$$

we have

$$V_{i-1} - V_i = m\ddot{y}$$

By assuming the harmonic vibration and keeping V_i for the magnitudes of harmonic shear forces

$$V_{i-1} - V_i = -mY_i\omega^2$$

For a rigid mass

$$Y_i = Y_{i-1}$$

$$V_i = V_{i-1} + m\omega^2 Y_{i-1}$$

$$\theta_i = \theta_{i-1}$$

$$M_i = M_{i-1}$$

Combining the preceding equations into one transfer matrix equation, we have

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.97}$$

The load section as shown in Fig. 9.10c. The balance of forces gives

$$V_{i-1} = V_i + kY_i$$

Because Y, θ, and M are not changed from station $i - 1$ to i, the transfer across a spring is simply

$$\begin{pmatrix} Y_i \\ \theta_i \\ M_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1} \\ \theta_{i-1} \\ M_{i-1} \\ V_{i-1} \end{pmatrix} \tag{9.98}$$

Now we have used three elements to model a vibrating beam and obtained three sets of matrix equations. Note that the dimensions of each term in the transfer matrix are different. It will be more convenient if all the terms are written in dimensionless form, especially if we use a computer to carry out the matrix operations. Let us define dimensionless variables as follows:

$$Y_i^* = \frac{Y_i}{\ell}, \qquad M_i^* = \frac{M_i \ell}{EI}, \qquad V_i^* = \frac{V_i \ell^2}{EI}$$

$$m^* = \frac{m\omega^2 \ell^3}{EI}, \qquad k^* = \frac{k\ell^3}{EI}$$

Then Eqs. (9.96–9.98) become respectively,

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.99}$$

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.100}$$

$$\begin{pmatrix} Y_i^* \\ \theta_i \\ M_i^* \\ V_i^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{i-1}^* \\ \theta_{i-1} \\ M_{i-1}^* \\ V_{i-1}^* \end{pmatrix} \tag{9.101}$$

Example 9.10

For the uniform cantilever beam shown in Fig. 9.11, find the transfer matrices and determine the natural frequencies and corresponding principal modes of vibration.

Solution. The boundary conditions are $Y_0^* = 0, \theta_0 = 0$, and $M_3^* = 0$, $V_3^* = 0$. The equation relating station 0 to 1 is

$$\begin{pmatrix} Y_1^* \\ \theta_1 \\ M_1^* \\ V_1^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m^* & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ M_0^* \\ V_0^* \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ m^* & m^* & \frac{1}{2}m^* & \left(1 + \frac{1}{6}m^*\right) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ m_0^* \\ V_0^* \end{pmatrix} \tag{9.102}$$

Similarly, for the relationship from station 0 to 2, the equation is

$$\begin{pmatrix} Y_2^* \\ \theta_2 \\ M_2^* \\ V_2^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ m^* & m^* & \frac{1}{2}m^* & \left(1 + \frac{1}{6}m^*\right) \end{pmatrix} \begin{pmatrix} Y_1^* \\ \theta_1 \\ M_1^* \\ V_1^* \end{pmatrix} \tag{9.103}$$
$$= A_1 Z_1 = A_1 A_1 Z_0 = A_2 Z_0$$

where Z_i is the state vector at station i and A_i is the transfer matrix.

$A_2 = A_1 A_1 =$

$$\begin{pmatrix} (1 + m^*/6) & (2 + m^*/6) & (2 + m^*/12) & (4/3 + m^*/36) \\ m^*/2 & (1 + m^*/2) & (2 + m^*/4) & (2 + m^*/12) \\ m^* & m^* & (1 + m^*/2) & (2 + m^*/6) \\ m^*(2 + m^*/6) & m^*(3 + m^*/6) & m^*(5/2 + m^*/12) & (1 + 3m^*/2 + m^{*2}/36) \end{pmatrix}$$

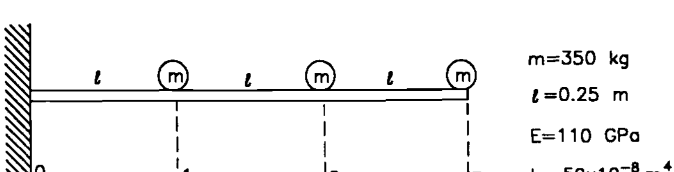

Fig. 9.11 Cantilever beam modeled as a lumped system.

Finally, the equation relating the state vector at 0 to that at 3 is

$$\begin{aligned} Z_3 &= A_3 Z_0 = A_1 A_1 A_1 Z_0 \\ A_3 &= A_1 A_1 A_1 \end{aligned} \tag{9.104}$$

The elements in A_3 are lengthy and not all are needed in the computations. The ones that are needed are worked out and given as follows:

$$a_{13} = 9/2 + m^* + m^{*2}/72$$

$$a_{14} = 9/2 + 16m^*/36 + m^{*2}/216$$

$$a_{33} = 1 + 3m^* + m^{*2}/12$$

$$a_{34} = 3 + 5m^*/3 + m^{*2}/36$$

$$a_{43} = 7m^* + 13m^{*2}/12 + m^{*3}/72$$

$$a_{44} = 1 + 6m^* + 17m^{*2}/36 + m^{*3}/216$$

The relationship between M_3^*, V_3^*, and M_0^*, V_0^* is

$$\begin{pmatrix} M_3^* \\ V_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} M_0^* \\ V_0^* \end{pmatrix} \tag{9.105}$$

Because M_0^* and V_0^* are not zero, the determinant of the coefficients must be zero, i.e.,

$$a_{33}a_{44} - a_{34}a_{43} = 0$$

When the preceding equation is expanded in detail, we find the expression for determining the natural frequencies as

$$26m^{*3} - 786m^{*2} + 2592m^* - 216 = 0$$

Three roots are obtained, and they are

$$m_1^* = 0.0855462, \qquad m_2^* = 3.66778, \qquad m_3^* = 26.4774$$

Newton's iteration method has been used in finding the preceding roots. The natural frequencies for principal modes are computed as follows. For m_1^*, we have

$$\omega_1^2 = \frac{EI}{m\ell^3} m_1^* = \frac{110 \times 10^9 \times 52 \times 10^{-8}}{350 \times (0.25)^3}(0.0855462) = 894.764$$

$$\omega_1 = 29.9126 \qquad (\mathrm{s}^{-1})$$

With the value of m^* determined, we find the numerical values of a_{33} and a_{34}:

$$a_{33} = 1.257248, \qquad a_{34} = 3.427934$$

Then from Eq. (9.105), we have

$$M_0^* = -2.726538\ V_0^* \tag{9.106}$$

To determine the shape of the fundamental mode, we use Eqs. (9.102–9.104) and find

$$Y_1^* = \tfrac{1}{2}M_0^* + \tfrac{1}{6}V_0^* = -1.196602\ V_0^*$$

$$Y_2^* = (2 + m^*/12)M_0^* + (4/3 + m^*/36)V_0^* = -4.136803\ V_0^*$$

$$\begin{aligned} Y_3^* &= (4.5 + m^* + m^{*2}/72)M_0^* \\ &\quad + (4.5 + 16m^*/36 + m^{*2}/216)V_0^* = -7.964889\ V_0^* \end{aligned}$$

In common practice, the mode shape is expressed as a ratio of magnitudes. Let us compute the ratios and find

$$Y_1^*/Y_3^* = 0.150235$$

$$Y_2^*/Y_3^* = 0.519380$$

That means as $Y_3^* = 1$,

$$Y_1^* = 0.150235, \qquad Y_2^* = 0.519380 \tag{9.107}$$

For $m_2^* = 3.66778$, we get

$$\omega_2^2 = 38362.88$$

$$\omega_2 = 195.864 \qquad (\text{s}^{-1})$$

$$Y_1^*/Y_3^* = -1.268889, \qquad Y_2^*/Y_3^* = -1.507484$$

Similarly, for $m_3^* = 26.4774$, we have

$$\omega_3^2 = 276938.47$$

$$\omega_3 = 526.249 \qquad (\text{s}^{-1})$$

$$Y_1^*/Y_3^* = 4.647174, \qquad Y_2^*/Y_3^* = -3.248295$$

9.4 Vibrations of Continuous Systems

Many practical systems that we deal with every day are continuous in nature. Therefore, without studying the vibrations of continuous systems, the knowledge of vibration analysis will not be complete. In this section, we will study some simple cases such as vibrating string, beam, and membrane. The materials involved are assumed to be homogeneous, isotropic, and obeying Hooke's law in stress and strain relations. In addition, because sound waves are a vibration of continuous medium, they also will be studied in this section. From this section, the reader will learn fundamentals in setting up a partial differential equation and methods for solving them. A Fourier series will be used in the solutions of the problems presented in the examples.

Vibrating String

Before deriving a partial differential equation for a vibrating string, we make the following assumptions:

1) The string is perfectly flexible, that is, it cannot resist any bending moments.
2) The vertical deflection y of the string is small compared with the length L.
3) The slope at any point of the deflected string is small compared with unity.
4) The tension T is constant at all times and at all points of the deflected string, and is large compared with the weight of the string.
5) The horizontal displacement of the string is negligible compared to the vertical displacement, that is, we have pure transverse vibrations.
6) The motion takes place only in the x–y plane.

Consider that the string is fixed at the ends and subjected to a constant tension of T. Let us take a small segment ds of the string as shown in Fig. 9.12, and let w be the weight per unit length of the string. From $\sum F = ma$, and

$$\sum F_y = -T \sin\alpha + T \sin\beta - w\,\mathrm{d}s$$

we can set up the equation of motion. With the preceding assumptions, we have

$$\mathrm{d}y \ll \mathrm{d}x, \qquad \mathrm{d}s \simeq \mathrm{d}x$$

$$\sin\alpha \simeq \tan\alpha, \qquad \sin\beta \simeq \tan\beta$$

Because,

$$\tan\alpha = \frac{\partial y}{\partial x}, \qquad \tan\beta = \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\mathrm{d}x$$

$$\sum F_y = T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x - w\,\mathrm{d}x \tag{9.108}$$

On the other hand,

$$a_y = \frac{\partial^2 y}{\partial t^2}$$

$$ma_y = \frac{\omega}{g}\mathrm{d}x\frac{\partial^2 y}{\partial t^2} \tag{9.109}$$

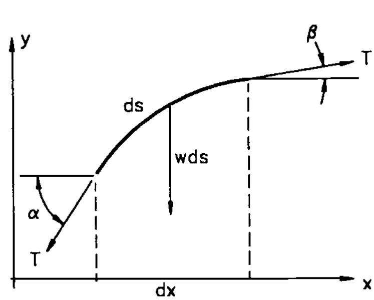

Fig. 9.12 Forces on a small segment of the string.

Combining Eqs. (9.108) and (9.109), we obtain

$$T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x - w\,\mathrm{d}x = \frac{w}{g}\mathrm{d}x\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{gT}{w}\frac{\partial^2 y}{\partial x^2} - g = \frac{\partial^2 y}{\partial t^2}$$

Let $a^2 = gT/w$ and because of fourth assumption, we drop the term g on the left-hand side and find

$$a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{9.110}$$

This is the partial differential equation for the transverse vibration of a string. It is also called the one-dimensional wave equation.

The boundary conditions for the case of vibrating string can be written as 1) $y(o,t) = 0$; 2) $y(L,t) = 0$; 3) $\partial y/\partial t(x,o) = g(x)$; and 4) $y(x,o) = f(x)$. Because the ends of the string are fixed, we have $y = 0$ at $x = 0$, and $x = L$ for all time t. The third and fourth conditions are the initial velocity and initial displacement of the string.

Here a reader may question the number of boundary conditions necessary for solving partial differential equations. In solving the ordinary differential equations, we know, in general, the number of boundary conditions equals the order of differential equations. Because one integral constant will appear when the equation is integrated once, such a constant must be determined by one boundary condition. In solving the partial differential equations, we may state that the number of boundary conditions needed for solving the problem equals the number of necessary conditions needed for determining the arbitrary functions after integrating the partial differential equation.

Solution of the Vibrating String with Initial Displacement

First let us consider the problem of the vibrating string with the initial displacement. Note that the equation of motion is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{9.110}$$

where $a^2 = Tg/w$. The boundary conditions are

$$y(0,t) = 0, \qquad y(L,t) = 0$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad y(x,0) = f(x) \tag{9.111}$$

where $f(x)$ is known. To solve such a problem, we assume

$$y(x,t) = X(x)T(t) \tag{9.112}$$

This method is known as a separation of variables. Substituting this expression into Eq. (9.110) leads to

$$XT'' = a^2X''T$$

or

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T}$$

where $T'' = \mathrm{d}^2T/\mathrm{d}t^2$ and $X'' = \mathrm{d}^2X/\mathrm{d}x^2$.

Now the right-hand side of the preceding equation is independent of x and the left-hand side is independent of t. Because they are equal, their common value must be a constant, say λ. Hence

$$\frac{X''}{X} = \lambda, \qquad \frac{T''}{a^2T} = \lambda$$

or

$$X'' - \lambda X = 0, \qquad T'' - \lambda a^2 T = 0 \tag{9.113}$$

Thus, we have two ordinary differential equations. To satisfy the boundary conditions, the value of λ must be less than zero, i.e., $\lambda = -\beta^2$ where β is real. Hence Eqs. (9.113) become

$$X'' + \beta^2 X = 0, \qquad T'' + \beta^2 a^2 T = 0 \tag{9.114}$$

The solution then can be written as

$$y(x,t) = (A\cos\beta x + B\sin\beta x)(C\cos\beta at + D\sin\beta at)$$

where A, B, C, D, and β are to be determined. Applying the first boundary condition gives

$$0 = y(0,t) = AT(t)$$

Hence $A = 0$. Applying the second boundary condition leads to

$$0 = y(L,t) = (B\sin\beta L)T(t)$$

That means $\sin\beta L$ must be zero, so that

$$\beta L = \pm n\pi \quad n = 1, 2, 3, \ldots$$

or

$$\beta = \pm\frac{n\pi}{L}$$

The function $X(x)$ can be written in the form of

$$\begin{aligned} X(x) &= B_n^* \sin\frac{n\pi}{L}x + B_{-n}^* \sin\left(-\frac{n\pi}{L}\right) \\ &= (B_n^* - B_{-n}^*) \sin\frac{n\pi}{L}x \\ &= B_n \sin\frac{n\pi}{L}x \end{aligned}$$

so we just consider

$$\beta = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots$$

and

$$y_n(x,t) = B_n \sin\frac{n\pi}{L}x\left[c_n \cos\frac{n\pi}{L}at + D_n \sin\frac{n\pi}{L}at\right]$$

On the other hand, to determine the constants in $T(t)$, we apply the third boundary condition

$$\frac{\partial y}{\partial t}(x,0) = 0$$

$$X(x)T'(0) = 0$$

$$T'(0) = -\beta aC \sin 0 + \beta aD \cos 0 = \beta aD = 0$$

Hence

$$D = 0$$

$$T(t) = C \cos\beta at$$

Because $\beta = n\pi/L$ and n is an integer, with $T_n(t)$ for a specific n, we have

$$T_n(t) = C_n \cos\frac{n\pi a}{L}t$$

Combining $X_n(x)$ and $T_n(t)$, we get

$$\begin{aligned} X_n(x)T_n(t) &= B_n \sin\frac{n\pi}{L}xC_n \cos\frac{n\pi a}{L}t \\ &= b_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a}{L}t \end{aligned}$$

where $b_n = B_nC_n$. Now this is a solution of the partial differential equation and satisfies three boundary conditions for all n, where $n = 1, 2, 3, \ldots$. Because the wave equation is a linear partial differential equation which has the property that any linear combination of solutions is its solution. Thus, the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \cos\frac{n\pi a}{L}t \tag{9.115}$$

To determine b_n, we apply the fourth boundary condition

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \tag{9.116}$$

Although $f(x)$ is defined in $0 \le x \le L$, because we are only interested in this interval, we can prolong the function in $(-L \le x \le 0)$ and consider it as a periodic odd function in the whole space, then with the use of Fourier sine series, b_n can be found as

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Therefore the complete solution becomes

$$y(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(z) \sin \frac{n\pi z}{L} dz \right] \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t \tag{9.117}$$

On the other hand with the use of the formula from trigonometry, we can write

$$\sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t = \frac{1}{2} \left[\sin \frac{n\pi}{L}(x - at) + \sin \frac{n\pi}{L}(x + at) \right]$$

then the solution becomes

$$y(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x - at) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x + at)$$

Because

$$f(z) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} z$$

we have

$$f(x - at) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x - at)$$

and

$$f(x + at) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x + at)$$

The solution is simply

$$y(x,t) = \tfrac{1}{2}[f(x - at) + f(x + at)] \tag{9.118}$$

Before looking into the physical meaning of the two functions $f(x - at)$ and $f(x + at)$, we first recall that they are the functions representing the vertical displacements of the string along the whole space of x from $-\infty$ to ∞ because they are derived from the Fourier series expansion of $f(x)$. Then we recognize that they are different from $f(x)$ given in Eq. (9.111), which is true only for x from 0 to L. The two functions $f(x-at)$ and $f(x+at)$ represent two waves traveling in opposite directions along the string, each with velocity a. To show this, we make the following observations.

Consider $f(x-at)$. At $t=0$, $y(x)=1/2f(x)$ is the half of the initial displacement. At any later time t_1, it defines the curve $1/2f(x-at_1)$. The two curves are identical except that the latter is translated to the right a distance at_1. Thus, the configuration moves along the string without distortion a distance at_1 in t_1 units of time. The velocity of this progression is therefore a.

Similarly the function $f(x+at)$ defines a configuration of $y(x)=1/2f(x)$ that moves to the left along the string with constant velocity a. Hence, the entire configuration is the sum of the two functions.

Example 9.11

A string of length of 10 units is fixed at both ends and given the initial displacement as

$$f(x)=\frac{x(10-x)}{1000} \qquad \text{for } 0<x<10 \tag{9.119}$$

It is released from rest. Assume that the string has $a^2=10,000$ units. Determine its subsequent motion.

Solution. According to the solution derived in the section, we assume that $f(x)$ is an odd periodic function as shown in Fig. 9.13, then we have

$$y(x,0)=f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x$$

$$y(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(z)\sin\frac{n\pi}{L}z\mathrm{d}z\right]\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t$$

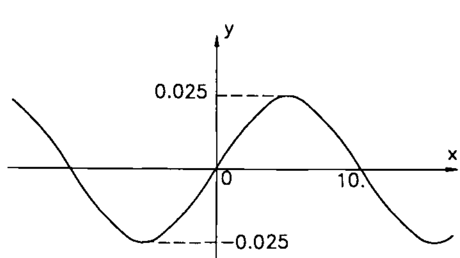

Fig. 9.13 Initial displacement of the string with an odd function assumed.

Evaluating b_n for the given $f(x)$, we find

$$b_n = \frac{2}{10}\int_0^{10} \frac{x(10-x)}{1000}\sin\frac{n\pi}{10}x\,\mathrm{d}x$$

$$= \frac{2}{5(n\pi)^3}[1-(-1)^n] = \begin{cases} 0 & n = \text{even} \\ \frac{4}{5(n\pi)^3} & n = \text{odd} \end{cases}$$

Hence

$$y(x,t) = \frac{4}{5\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{10}x\cos(10n\pi t)$$

Note that in the process of deriving the solution of the wave equation, the initial displacement function has been assumed to be a periodic odd function. The given initial displacement, Eq. (9.118), is true only between two ends. If the expression $f(x)$ is true for all the values of x, then Eq. (9.117) can be used directly for the solution, which is illustrated in the following example.

Example 9.12

A string stretching to infinity in both directions is given the initial displacement

$$f(x) = \frac{1}{1+8x^2}$$

and released from rest. (One remark must be made here. In many engineering problems, the term "infinity" means that the boundary is far away from space reached by the motion. For this particular example, infinity means before a reflected wave is observed.) Find the displacement during its subsequent motion.

Solution. Using Eq. (9.117), we have the solution as simply

$$y(x,t) = \frac{1}{2}[f(x-at)+f(x+at)]$$

$$= \frac{1}{2}\left[\frac{1}{1+8(x-at)^2}+\frac{1}{1+8(x+at)^2}\right]$$

Solution of the Vibrating String with Initial Velocity and Displacement

Now let us consider the problem of the vibrating string stretching from $-\infty$ to ∞. Rewrite the equation of motion, Eq. (9.110), as

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

with the boundary conditions given as

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x) \tag{9.120}$$

The general solution of Eq. (9.110) is

$$y(x,t) = y_1(x - at) + y_2(x + at)$$

where $y_1(x - at)$ and $y_2(x + at)$ are arbitrary. The task here is to relate these two functions with the given function $f(x)$ and $g(x)$. Applying Eqs. (9.120), we have

$$y(x,0) = f(x) = y_1(x) + y_2(x) \tag{A}$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) = -ay_1'(x) + ay_2'(x) \tag{B}$$

Dividing the second of the preceding equations by a and then integrating, we find

$$-y_1(x) + y_2(x) = \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x \tag{C}$$

Combining Eqs. (A) and (C), gives

$$y_1(x) = \frac{1}{2}\left[f(x) - \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x\right] \tag{D}$$

$$y_2(x) = \frac{1}{2}\left[f(x) + \frac{1}{a}\int_{x_0}^{x} g(x)\mathrm{d}x\right] \tag{E}$$

With the form of y_1 and y_2 known, we can now write

$$\begin{aligned} y(x,t) &= y_1(x - at) + y_2(x + at) \\ &= \frac{1}{2}\left[f(x - at) - \frac{1}{a}\int_{x_0}^{x-at} g(x)\mathrm{d}x\right] + \frac{1}{2}\left[f(x + at) + \frac{1}{a}\int_{x_0}^{X+at} g(x)\mathrm{d}x\right] \\ &= \frac{1}{2}\left[f(x - at) + f(x + at) + \frac{1}{a}\int_{x-at}^{x+at} g(x)\mathrm{d}x\right] \end{aligned}$$

Transverse Vibrations of a Beam

Consider a beam of length L loaded by a variable load $w(x, t)$. To simplify the problem, assumptions are made as follows:

1) The weight of the beam is included in the load w.
2) The vertical deflection y is small compared to the length L.
3) The slope of the deflection curve is much smaller than unity.
4) The horizontal displacement of the beam is negligible compared to the vertical displacement; that is, we have pure transverse motion.
5) The assumptions for beam theory hold: Every layer of material is free to expand and contract longitudinally and laterally under stress as if it is separated from other layers; the tensile and compressive moduli of elasticity are equal; and the cross section remains a plane surface.

Now let us consider a small segment ds of a bent beam as shown in Fig. 9.14. Let e be the amount of length changed from its original length ds on the fiber UV.

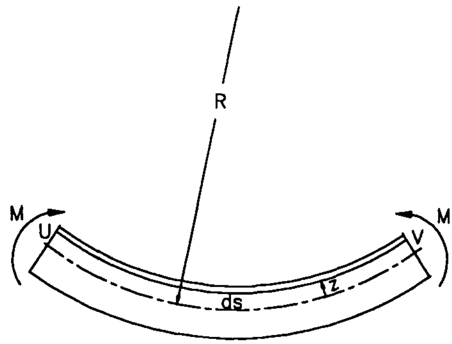

Fig. 9.14 Small segment of a beam.

We have

$$\frac{\mathrm{d}s - e}{\mathrm{d}s} = \frac{R - z}{R}$$

or

$$\frac{z}{R} = \frac{e}{\mathrm{d}s}$$

where R is the radius of curvature of the deflection curve. The strain is defined positive for tension and negative for compression; thus,

$$\epsilon = -\frac{e}{\mathrm{d}s} = -\frac{z}{R}$$

By using Hooke's law,

$$\sigma = E\epsilon = -\frac{zE}{R}$$

The force acting on the area $\mathrm{d}A$ is then

$$\mathrm{d}F = \sigma \,\mathrm{d}A = -\frac{Ez}{R}\mathrm{d}A$$

Because the tensile and compressive forces are equal over any cross section, the total force acting over the whole cross section is zero:

$$F = -\int_A \frac{Ez}{R}\,\mathrm{d}A = -\frac{E}{R}\int_A z\,\mathrm{d}A = 0$$

This result means that the neutral axis passed through the centroid of the cross-sectional area. On the other hand, under equilibrium, the internal bending moment created by the stress σ must be the same as the external moment M:

$$M = \int_A (-z)\sigma \,\mathrm{d}A = \frac{E}{R}\int_A z^2 \mathrm{d}A = \frac{EI}{R}$$

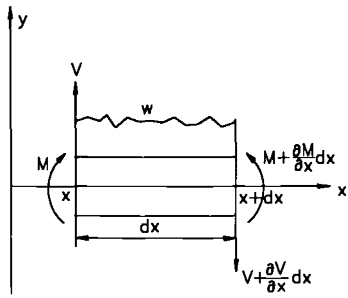

Fig. 9.15 Load on a segment of a beam.

where I is the moment of inertia of the area about the neutral axis. From studies in mathematics, we also learn that the curvature of a plane curve is given by the equation

$$\frac{1}{R} = \frac{\mathrm{d}^2y/\mathrm{d}x^2}{[1+(\mathrm{d}y/\mathrm{d}x)^2]^{3/2}} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

because $\mathrm{d}y/\mathrm{d}x \ll 1$. Therefore

$$M = EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \tag{A}$$

Referring to Fig. 9.15, we can compute the sum of the force in the y direction

$$\sum F_y = V - \left(V + \frac{\partial V}{\partial x}\,\mathrm{d}x\right) - w\,\mathrm{d}x = -\frac{\partial V}{\partial x}\mathrm{d}x - w\,\mathrm{d}x \tag{B}$$

and the inertial force is

$$ma_y = \frac{w}{g}\mathrm{d}x\frac{\partial^2 y}{\partial t^2} \tag{C}$$

Equating Eq. (B) to (C), we find

$$-\frac{\partial V}{\partial x} - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2} \tag{D}$$

On the other hand, taking moments about the point x, we have

$$\begin{aligned}\sum M &= M - \left(M + \frac{\partial M}{\partial x}\mathrm{d}x\right) + \left(V + \frac{\partial V}{\partial x}\mathrm{d}x\right)\mathrm{d}x + w\,\mathrm{d}x\frac{\mathrm{d}x}{2}\\ &= -\frac{\partial M}{\partial X}\mathrm{d}x + V\,\mathrm{d}x + \frac{\partial V}{\partial x}(\mathrm{d}x)^2 + \frac{w}{2}(\mathrm{d}x)^2\end{aligned}$$

Neglecting high order terms and setting $\sum M = 0$ leads to

$$V = \frac{\partial M}{\partial x} \tag{E}$$

Substituting Eq. (E) into Eq. (D), we get

$$-\frac{\partial^2 M}{\partial x^2} - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2}$$

Using Eq. (A), then we obtain

$$-\frac{\partial^2}{\partial X^2}\left(EI\frac{\partial^2 y}{\partial X^2}\right) - w = \frac{w}{g}\frac{\partial^2 y}{\partial t^2} \tag{9.121}$$

This is the partial differential equation for the transverse vibration of a beam. Note that upward y is positive, but downward w is positive in Eq. (9.121). If we are interested only in studying the free vibration of the beam, the load term is dropped, and Eq. (9.121) becomes

$$\frac{w}{g}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial X^2}\right) = 0$$

The equation can be further simplified for $EI = \text{const}$:

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0 \tag{9.122}$$

where $a^2 = EIg/w$.

Solving the partial differential equation, we must use some necessary boundary conditions. Boundary conditions for two popular beams follow.

1) Boundary conditions for simply supported beams:

$$y(0, t) = 0$$

$$y(L, t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(0, t) = 0 \qquad \text{for } M = 0 \qquad \text{at } x = 0$$

$$\frac{\partial^2 y}{\partial x^2}(L, t) = 0 \qquad \text{for } M = 0 \qquad \text{at } x = L$$

$$y(x, 0) = f(x)$$

$$\frac{\partial y}{\partial t}(x, 0) = g(x)$$

2) Boundary conditions for built-in beams:

$$y(0, t) = 0$$

$$y(L, 0) = 0$$

$$\frac{\partial y}{\partial x}(0, t) = 0 \qquad \text{for slope } = 0 \qquad \text{at } x = 0$$

$$\frac{\partial y}{\partial x}(L, t) = 0 \qquad \text{for slope } = 0 \qquad \text{at } x = L$$

$$y(x,0) = f(x)$$
$$\frac{\partial y}{\partial t}(x,0) = g(x)$$

Example 9.13

A simply supported beam is given the initial displacement $f(x)$ and released from rest. Determine its subsequent motion.

Solution. The conditions given establish Eq. (9.122) as the equation of motion; it is rewritten here for convenience:

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0 \tag{9.122}$$

We shall seek a separable solution of the form

$$y(x,t) = X(x)T(t)$$

and we have

$$X\frac{d^2T}{dt^2} + a^2 T\frac{d^4X}{dx^4} = 0$$

or

$$-\frac{a^2}{X}\frac{d^4X}{dx^4} = \frac{1}{T}\frac{d^2T}{dt^2} \tag{9.123}$$

Because the left side of Eq. (9.123) is a function of x alone, and the right side is a function of t only, the common value for the equation must be a constant, say λ. Thus,

$$-\frac{a^2}{X}\frac{d^4X}{dx^4} = \frac{1}{T}\frac{d^2T}{dt^2} = \lambda$$

To satisfy the boundary conditions, it is found that λ must be negative. Let $\lambda = -\omega^2$, then we have two ordinary differential equations:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{A}$$

$$\frac{d^4X}{dx^4} - \frac{\omega^2}{a^2}X = 0 \tag{B}$$

The solution of Eq. (A) is known as

$$T(t) = C_1 \sin\omega t + C_2 \cos\omega t \tag{C}$$

The solution for Eq. (B) is assumed as

$$X(x) = Ae^{sx} \tag{D}$$

where A and s are constant. Substituting the assumed solution into Eq. (B) gives

$$\left(s^4 - \frac{\omega^2}{a^2}\right)Ae^{sx} = 0$$

From the equation we obtain the four roots

$$S_1 = \sqrt{\frac{\omega}{a}} = \alpha \qquad S_2 = -\sqrt{\frac{\omega}{a}} = -\alpha$$

$$S_3 = i\sqrt{\frac{\omega}{a}} = i\alpha \qquad s_4 = -i\sqrt{\frac{w}{a}} = -i\alpha$$

The solution is then

$$X(x) = A_1 e^{\alpha x} + A_2 e^{-\alpha x} + A_3 e^{+i\alpha x} + A_4 e^{-i\alpha x} \tag{E}$$

where A_1, A_2, A_3, and A_4 are arbitrary. Without loss of generality, we can write the solution as

$$X(x) = C_3 \sinh \alpha x + C_4 \cosh \alpha x + C_5 \sin \alpha x + C_6 \cos \alpha x$$

The solution of Eq. (9.122) is then

$$y(x,t) = (C_1 \sin \omega t + C_2 \cos \omega t)(C_3 \sinh \alpha x + C_4 \cosh \alpha x + C_5 \sin \alpha x + C_6 \cos \alpha x) \tag{F}$$

with

$$\omega = \alpha^2 a$$

The constants appearing in the solution and the natural frequencies are determined by applying the boundary conditions. For a simply supported beam, the boundary conditions are

$$y(0,t) = y(L,t) = 0$$

$$\frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^2 y}{\partial x^2}(L,t) = 0$$

Applying the boundary conditions to Eq. (F) gives

$$C_4 + C_6 = 0$$

$$C_3 \sinh \alpha L + C_4 \cosh \alpha L + C_5 \sin \alpha L + C_6 \cos \alpha L = 0$$

$$C_4 - C_6 = 0$$

$$C_3 \sinh \alpha L + C_4 \cosh \alpha L - C_5 \sin \alpha L - C_6 \cos \alpha L = 0$$

From the four preceding equations, we find $C_3 = C_4 = C_6 = 0$, and

$$C_5 \sin \alpha L = 0$$

Therefore, the natural frequencies can be determined from

$$\alpha L = n\pi \quad n = 1, 2, 3, \ldots$$

and we obtain

$$\omega_n = \alpha_n^2 a = \left(\frac{n\pi}{L}\right)^2 a = (n\pi)^2 \sqrt{\frac{EIg}{wL^4}} \tag{9.124}$$

With the natural frequencies determined, the general solution Eq. (F) becomes

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi}{L} x$$

where $A_n = (C_1 C_5)_n$ and $B_n = (C_2 C_5)_n$. The constants A_n and B_n can be determined by initial conditions of the motion. For this example, $y(x, 0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

By assuming $f(x)$ as a periodic odd function, we obtain

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, \mathrm{d}x$$

Because initial velocity is zero, A_n must be zero. Therefore, the complete solution is

$$y(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(z) \sin \frac{n\pi z}{L} \mathrm{d}z \right] \sin \frac{n\pi x}{L} \cos \omega_n t \tag{9.125}$$

Example 9.14

A simply supported beam of length L is subjected to a concentrated harmonic force $F_0 \sin \omega_f t$ as shown in Fig. 9.16. Determine its subsequent motion.

Solution. The governing equation is

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = F_0 \sin \omega_f t \delta(x - a) \tag{A}$$

To find the response of the forced vibration, we consider the forcing function as

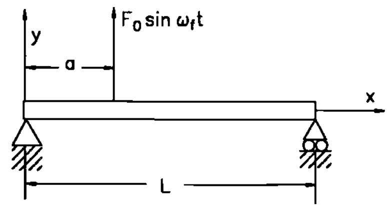

Fig. 9.16 Concentrated harmonic force acting on a simple beam.

an odd periodic function with period of $2L$ as shown in Fig. 9.17, and expand the function into a Fourier sine series as

$$\delta(x-a) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

where

$$b_n = \frac{2}{L} \int_0^L \delta(x-a) \sin \frac{n\pi}{L} x \, \mathrm{d}x = \frac{2}{L} \sin \frac{n\pi a}{L}$$

Therefore Eq. (A) becomes

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g} \frac{\partial^2 y}{\partial t^2} = \frac{2F_0}{L} \left(\sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \right) \sin \omega_f t \tag{B}$$

To find the forced response, we assume the solution as

$$y(x,t) = f(x) \sin \omega_f t \tag{C}$$

Substituting Eq. (C) into Eq. (B) gives

$$EI \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} - \frac{w}{g} \omega_f^2 f = \frac{2F_0}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \tag{D}$$

where the common factor $\sin \omega_f t$ on both sides of the equation has been dropped. The particular solution of Eq. (D) is assumed as

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

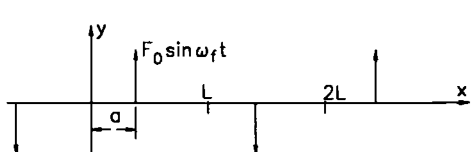

Fig. 9.17 Concentrated force assumed as a periodic odd function of *x*.

With this, Eq. (D) becomes

$$\frac{\pi^4 EI}{L^4}\sum_{n=1}^{\infty} n^4 A_n \sin\frac{n\pi x}{L} - \frac{w}{g}\omega_f^2 \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$= \frac{2F_0}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi a}{L}\sin\frac{n\pi x}{L}$$

Equating coefficients of $\sin(n\pi x/L)$ gives

$$A_n = \frac{2F_0 L^3}{n^4\pi^4 EI - \frac{w}{g}L^4\omega_f^2}\sin\frac{n\pi a}{L} \tag{9.126}$$

Therefore the forced response is obtained as

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\omega_f t \tag{9.127}$$

From the denominator of Eq. (9.126), we find that a resonant condition is

$$\omega_f = n^2\pi^2\sqrt{\frac{EIg}{wL^4}}$$

A few remarks must be made before we end the section. In this analysis, the mass of the beam is considered in the inertial force, but the weight of the beam is neglected in the load. This means that the initial deflection caused by weight is small compared to the dynamic deflection. Examples given are solved successfully. The beam involved is supported simply. For other end conditions the solutions may become complicated. To fully understand the subject, additional references will be needed.

Vibration of a Circular Membrane

Suppose that a piece of membrane is mounted on a drum. The tension in the membrane is shown in Fig. 9.18. Our first task is to find the equation of motion for the vibrating membrane. To simplify the considerations, assumptions are made as follows.

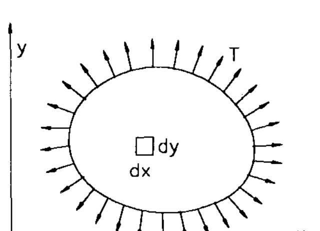

Fig. 9.18 Membrane tension.

1) Tension measured as force per unit length is normal to the boundary of the element and is constant throughout the membrane.

2) The total tension on the boundary is large compared to the weight of the membrane.

3) The membrane is so thin that it cannot resist any bending moment, i.e., there is no bending stress.

4) The vertical deflection w is small compared to the diameter of the membrane.

5) The slopes of the deflection surface are small compared to unity.

6) The lateral displacements are negligible compared with the vertical displacements.

Consider a differential element of the membrane with area $dxdy$. To analyze the force acting on this element, let us enlarge the element as shown in Fig. 9.19. Here P is applied pressure. The sum of forces in the z direction then can be computed as

$$\sum F_z = -T\,\mathrm{d}y \tan\alpha + T\,\mathrm{d}y \tan\beta - T\,\mathrm{d}x \tan\gamma + T\,\mathrm{d}x \tan\delta + P\,\mathrm{d}x\,\mathrm{d}y \quad \text{(A)}$$

Because slopes are small, the following relations have been used in the preceding equation:

$$\sin\alpha \simeq \tan\alpha, \qquad \sin\beta \simeq \tan\beta$$

$$\sin\gamma \simeq \tan\gamma, \qquad \sin\delta \simeq \tan\delta$$

Because w is the vertical displacement of the membrane, in the xz plane we have

$$\tan\alpha = \frac{\partial w}{\partial x}$$

$$\tan\beta = \frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\mathrm{d}x$$

Similarly in the yz plane, we have

$$\tan\gamma = \frac{\partial w}{\partial y}$$

$$\tan\delta = \frac{\partial w}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\mathrm{d}y$$

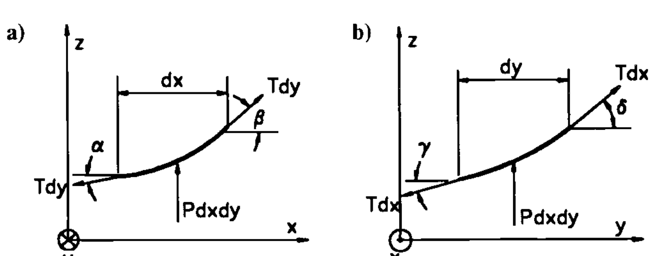

Fig. 9.19 Forces on a segment of membrane.

Substituting these expressions into Eq. (A) gives

$$\sum F_z = T\,\mathrm{d}x\,\mathrm{d}y\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + P\,\mathrm{d}x\,\mathrm{d}y \tag{B}$$

On the other hand, the mass of the element is

$$m = \rho\,\mathrm{d}x\,\mathrm{d}y \tag{C}$$

and the acceleration is

$$a_z = \frac{\partial^2 w}{\partial t^2} \tag{D}$$

where ρ is the mass per unit area of the membrane. The equation of motion then can be written as

$$\rho\,\mathrm{d}x\,\mathrm{d}y\frac{\partial^2 w}{\partial t^2} = T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\mathrm{d}x\,\mathrm{d}y + P\,\mathrm{d}x\,\mathrm{d}y$$

or

$$\frac{\partial^2 w}{\partial t^2} = a^2\nabla^2 w + \frac{1}{\rho}P(x, y) \tag{9.128}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad a^2 = \frac{T}{\rho}$$

Equation (9.128) can be applied to cylindrical coordinates that require the expression of ∇^2 as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

For the study of free vibration, the pressure term is dropped, and Eq. (9.128) becomes

$$\frac{\partial^2 w}{\partial t^2} = a^2\nabla^2 w \tag{9.129}$$

For simplicity, we consider a special case, that is, the membrane is initially deflected into a radially symmetrical form and is released from rest. The equation of motion is reduced to

$$\frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{2}\frac{\partial w}{\partial r}\right) \tag{9.130}$$

And the boundary conditions are

$$
\begin{aligned}
w(R, t) &= 0 \\
w(r, 0) &= f(r) \\
\frac{\partial w}{\partial t}(r, 0) &= 0
\end{aligned}
\tag{9.131}
$$

where $r = R$ is the boundary of the membrane. Assume the solution as

$$w(r, t) = R(r)T(t) \tag{A}$$

Substituting the expression into Eq. (9.130) gives

$$a^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = \frac{T''}{T} = -\omega^2$$

where $R'' = (\mathrm{d}^2R/\mathrm{d}r^2)$, $R' = (\mathrm{d}R/\mathrm{d}r)$, $T'' = (\mathrm{d}^2T/\mathrm{d}t^2)$, and ω^2 is the arbitrary real constant. From which we get two ordinary differential equations as

$$rR'' + R' + \left(\frac{\omega}{a}\right)^2 rR = 0 \tag{B}$$

and

$$T'' + \omega^2 T = 0 \tag{C}$$

Equation (B) is known as Bessel's equation of order 0 with a parameter ω/a. The solution of Bessel's equation is

$$R(r) = AJ_0\left(\frac{\omega r}{a}\right) + BY_0\left(\frac{\omega r}{a}\right) \tag{D}$$

Because Y_0 approaches infinity as $r \to 0$, B must be zero. The solution of Eq. (C) is

$$T(t) = C\cos\omega t + D\sin\omega t \tag{E}$$

Combining Eqs. (D) and (E), we find

$$w(r, t) = J_0\left(\frac{\omega r}{a}\right)(C\cos\omega t + D\sin\omega t)$$

Because the initial velocity is zero, we have

$$D = 0.$$

Applying the first boundary condition gives

$$w(R, 0) = 0 = J_0\left(\frac{\omega R}{a}\right)$$

From this, the natural frequencies are determined. For example, the smallest root of $J_0 = 0$ is

$$\frac{\omega_1 R}{a} = 2.405$$

Hence, in general, we can write

$$J_0\left(\frac{\omega_n R}{a}\right) = 0 \quad n = 1, 2, 3, \ldots$$

for all the natural frequencies. The general solution becomes

$$w(r,t) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\omega_n r}{a}\right) \cos \omega_n t \tag{9.132}$$

To determine C_n, we apply the boundary condition Eq. (9.131)

$$w(r,0) = f(r) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\omega_n r}{a}\right)$$

And with the use of the properties given in Appendix H, we find

$$C_n = \frac{2}{R^2[J_1(\omega_n R/a)]^2} \int_0^R r f(r) J_0\left(\frac{\omega_n r}{a}\right) \mathrm{d}r \tag{9.133}$$

Example 9.15

A circular membrane is fixed on its edge and given an initial displacement as

$$f(r) = 1 - r^2/100$$

It is released from rest. Assume that the diameter of the membrane is 20 units and the property of the membrane has $a^2 = 10{,}000$ units. Determine its subsequent motion.

Solution. From Eq. (9.133) we can compute the coefficients as

$$C_n = \frac{2}{R_1^2[J_1(\omega_n R/a)]^2} \int_0^R r \left[1 - \frac{r^2}{100}\right] J_0\left(\frac{\omega_n r}{a}\right) \mathrm{d}r$$

From the properties of Bessel functions given in Appendix F, we have

$$\int_0^L x J_0(x) \mathrm{d}x = L J_1(L)$$

$$\int_0^L x^3 J_0(x) \mathrm{d}x = L^3 J_1(L) - 2L^2 J_2(L)$$

Hence

$$C_n = \frac{2}{R^2[J_1(\omega_n R/a)]^2}\left\{\left(\frac{aR}{\omega_n}\right)J_1\left(\frac{\omega_n R}{a}\right)\right.$$

$$\left.-\frac{1}{100}\left[\left(\frac{aR^3}{\omega_n}\right)J_1\left(\frac{\omega_n R}{a}\right)-2\left(\frac{aR}{\omega_n}\right)^2 J_2\left(\frac{\omega_n R}{a}\right)\right]\right\}$$

$$= \frac{2}{[J_1(\omega_n R/a)]^2}\left\{\left[\left(\frac{a}{\omega_n R}\right)-\left(\frac{a}{\omega_n R}\right)\frac{R^2}{100}\right]\right.$$

$$\left.\times J_1\left(\frac{\omega_n R}{a}\right)+\frac{1}{50}\left(\frac{a}{\omega_n}\right)^2 J_2\left(\frac{\omega_n R}{a}\right)\right\}$$

$$= \frac{J_2(\omega_n R/a)}{25[J_1(\omega_n R/a)]^2}\left(\frac{a}{\omega_n}\right)^2$$

We determine ω_n from $J_0(\omega_n R/a) = 0$. Then with the use of the table of Bessel functions (Appendix H), we find

$$C_1 = 1.81152 \qquad \omega_1 = 24.05\ (\mathrm{s}^{-1})$$

$$C_2 = -0.139890 \qquad \omega_2 = 55.20\ (\mathrm{s}^{-1})$$

$$C_3 = 0.0455503 \qquad \omega_3 = 86.54\ (\mathrm{s}^{-1})$$

The solution then can be written as

$$w(r,t) = C_1 J_0\left(\frac{\omega_1 r}{a}\right)\cos\omega_1 t + C_2 J_0\left(\frac{\omega_2 r}{a}\right)\cos\omega_2 t$$

$$+C_3 J_0\left(\frac{\omega_3 r}{a}\right)\cos\omega_3 t + \cdots \tag{9.134}$$

Sound Waves in Fluid

Sound waves in air or water are longitudinal pressure waves propagating under an isentropic process. As the sound wave propagates, the change in pressure is small compared with the ambient pressure. Because of isentropic process, the change in density of the fluid is small compared with the original density. Because the viscous force plays no role in the sound wave, the equations involved in the phenomena are the continuity and the momentum equations only. These can be written as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0 \tag{9.135}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla \boldsymbol{V} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_s \nabla\rho = -a^2\left(\frac{\nabla p}{\rho}\right) \tag{9.136}$$

where $a = \sqrt{(\mathrm{d}P/\mathrm{d}\rho)_s}$ is the propagating speed of the sound wave. Based on the facts observed, we can express

$$\rho = \rho_0 + \epsilon\rho_1, \qquad V = \epsilon V_1 \tag{A}$$

where $\epsilon \ll 1$. From Eq. (9.135) we have to the ϵ order

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot V_1 = 0 \tag{B}$$

From Eq. (9.136) we obtain, also to the ϵ order,

$$\frac{\partial V_1}{\partial t} = -a^2 \frac{\nabla \rho_1}{\rho_0} \tag{C}$$

Differentiating Eq. (B) with respect to time t and substituting Eq. (C) into it gives

$$\frac{\partial^2 \rho_1}{\partial t^2} - a^2 \nabla^2 \rho_1 = 0 \tag{9.137}$$

This is known as a wave equation. We have studied it in rectangular and cylindrical coordinates. Now let us study the wave equation in spherical coordinates such that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{9.138}$$

However, to simplify the mathematics, we study a special case that is spherically symmetric, so that Eq. (9.137) becomes

$$\frac{\partial^2 \rho_1}{\partial t^2} - a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \rho_1}{\partial r}\right) = 0 \tag{9.139}$$

This equation can be rearranged to

$$\frac{\partial^2 r\rho_1}{\partial t^2} = a^2 \frac{\partial^2 r\rho_1}{\partial r^2} \tag{9.140}$$

The solution of the preceding equation, similar to Eq. (9.117), can be written as

$$r\rho = f(r - at) + F(r + at) \tag{9.141}$$

As in the case of one-dimensional rectangular coordinates, the first term represents a wave advancing in the direction of r increasing, that is to say, a divergent wave, and the second term represents a converging wave. The latter does not possess much interest. To illustrate the physical meaning of the solution, let us consider the following example.

Example 9.16

Suppose that fireworks explode in the air; the initial change in density is

$$\Delta\rho = b \qquad \text{as } r < r_0$$
$$\Delta\rho = 0 \qquad \text{as } r > r_0$$

Determine the density change in the air during the propagating of the wave.

Solution. This is a case of divergent wave. Hence, only the first term in Eq. (9.141) is to be considered. The change in density of air is simply

$$\Delta\rho = b/r \qquad \text{as } 0 < r - at < r_0 \tag{9.142}$$

$$\Delta\rho = 0 \qquad \text{as } r - at < 0 \qquad \text{and} \qquad r - at > r_0 \tag{9.143}$$

This means that the higher density occurs in the spherical shell with the origin of the sphere where the fireworks exploded and with the thickness of r_0. This change in density is inversely proportional to the radius of the sphere. In other words, it will vanish as r approaches the infinite. The sphere is bounded by the radius of $(r_0 + at)$. Outside the sphere, there is no change in density. Also, the change vanishes as $r < at$. That is why the sound of the explosion can be heard only for a brief moment.

9.5 Nonlinear Vibrations

So far, we have studied many vibrating systems with linear characteristics. In discussing these systems, it was assumed that the force in a spring is proportional to the deformation. It was assumed also that, in the case of damping, the frictional force is a linear function of the velocity of motion. As a result of these assumptions, we had vibration systems represented by linear differential equations. However, there are practical problems in which these assumptions are no longer satisfactory to describe the actual motions. Such systems are called systems with nonlinear characteristics and are represented by nonlinear differential equations. In this section, we will deal with nonlinear vibration systems.

We may recall that the difference between a linear and a nonlinear differential equation is quite simple. If a differential equation contains products of unknown variables or products of unknown variable with the derivatives of unknown variables, the equation is nonlinear. Otherwise, the equation is linear.

As we learned in Chapter 8 and previous sections of this chapter, there are many analytical methods for solving linear differential equations. Because the principle of superposition is applicable to a linear equation, its general solution is the combination of all possible solutions.

For nonlinear differential equations, however, there are no definite methods for solving them analytically. Small perturbation methods may be considered as the systematic approach for solving them. One of the small perturbation methods, which is commonly used, has been introduced in Chapter 5 and will not be repeated here. On the other hand, because of the advancement of computer technology, many nonlinear problems whose solutions are not possible many years ago can be solved now. The following example illustrates this point.

Example 9.17

While a shaft is rotating at a high speed, the centrifugal force produced by the unbalanced disk can pull the shaft to a bow shape. This motion is known as the whirling of a rotating shaft. The sketch of the system is shown in Fig. 9.20. To consider the major dynamic properties of the motion, make some necessary assumptions and determine the equations of motion for the shaft rotating with and without acceleration; also find the maximum deflections for the two different conditions.

Solution. The following assumptions are made for this analysis.

1) The disk is rigid and is always perpendicular to the shaft.

2) The mass of the shaft is neglected.

3) Inertial forces lie in the plane of symmetry perpendicular to the shaft.

4) Damping is present and is assumed to be directly proportional to the precession speed of the shaft.

5) The supports are rigid, and the bearing flexibilities are neglected.

6) A torsional deformation is present, but vibration due to torsion will not be considered.

The geometry of the system is described as follows: The center of the mass of the disk is at point G at a distance e away from the center s of the shaft. The point o is the intersection of the straight line connecting two supports and the plane of symmetry. The center s is away from point o by a distance of r. The angle θ between line os and the reference line is the precession angle of the shaft, and $\dot{\theta}$

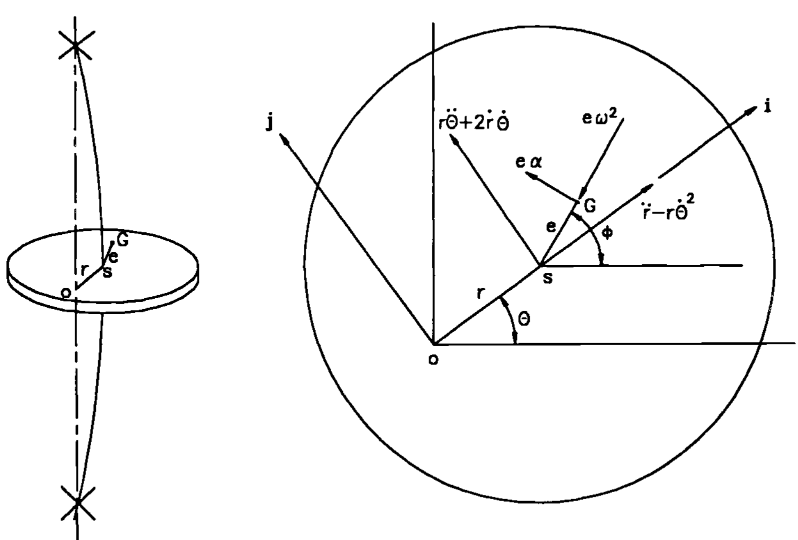

Fig. 9.20 Geometry of a whirling shaft.

is the precession speed that is considered to be different from the rotating speed ω of the shaft. The formulations for the two cases are considered separately.

Shaft rotating at a constant speed. It is known that the acceleration of point G relative to a fixed coordinate system can be expressed as

$$\boldsymbol{a}_G = \boldsymbol{a}_s + \boldsymbol{a}_{G/s} \tag{9.144}$$

where $\boldsymbol{a}_s$ is the acceleration of point s relative to point o and $\boldsymbol{a}_{G/s}$ is the relative acceleration between point G and point s. As the acceleration components are expressed along radial and tangential directions, they are found to be

$$\begin{aligned}\boldsymbol{a}_G = {} & [(\ddot{r} - r\dot{\theta}^2) - e\omega^2\cos(\omega t - \theta)]\boldsymbol{i} \\ & + [(r\ddot{\theta} + 2\dot{r}\dot{\theta}) - e\omega^2\sin(\omega t - \theta)]\boldsymbol{j}\end{aligned} \tag{9.145}$$

The equations of motion then can be written in the radial and tangential directions as

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta} - e\omega^2\cos(\omega t - \theta)]$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2\sin(\omega t - \theta)]$$

which can be rewritten into a familiar form of

$$\ddot{r} + c\dot{r}/m + (k/m - \dot{\theta}^2)r = e\omega^2\cos(\omega t - \theta) \tag{9.146}$$

$$r\ddot{\theta} + (cr/m + 2\dot{r})\dot{\theta} = e\omega^2\sin(\omega t - \theta) \tag{9.147}$$

Shaft rotating with acceleration. While the shaft is rotating with an angular acceleration α, additional acceleration in the tangential direction must be taken into account in considering $\boldsymbol{a}_{G/s}$. The acceleration of point G relative to a fixed system becomes

$$\begin{aligned}\boldsymbol{a}_G = {} & [\ddot{r} - r\dot{\theta}^2 - e\omega^2\cos(\phi - \theta) - e\alpha\sin(\phi - \theta)]\boldsymbol{i} \\ & + [r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2\sin(\phi - \theta) + e\alpha\cos(\phi - \theta)]\boldsymbol{j}\end{aligned}$$

where ϕ is the rotating angular displacement of the shaft. The equations of motion for describing the whirling of the shaft become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e\omega^2\cos(\phi - \theta) - e\alpha\sin(\phi - \theta)] \tag{9.148}$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e\omega^2\sin(\phi - \theta) + e\alpha\cos(\phi - \theta)] \tag{9.149}$$

Although the effect of the angular acceleration α to the motion of whirling is to be explored, it is reasonable to simplify the considerations by setting $\omega = \alpha t$ and $\phi = \alpha t^2/2$. By doing these, Eqs. (9.148) and (9.149) become

$$-kr - c\dot{r} = m[\ddot{r} - r\dot{\theta}^2 - e(\alpha t)^2\cos(\alpha t^2/2 - \theta) - e\alpha\sin(\alpha t^2/2 - \theta)] \tag{9.150}$$

$$-cr\dot{\theta} = m[r\ddot{\theta} + 2\dot{r}\dot{\theta} - e(\alpha t)^2\sin(\alpha t^2/2 - \theta) + e\alpha\cos(\alpha t^2/2 - \theta)] \tag{9.151}$$

Maximum deflection as shaft rotates at a constant speed. Equations (9.146) and (9.147) are nonlinear equations of $r(t)$ and $\theta(t)$. The exact solution can only be obtained numerically. Before solving them it is proper to convert the variables into dimensionless forms. Let

$$r^* = r/e \tag{9.152a}$$

$$t^* = \omega_n t \tag{9.152b}$$

and

$$c = c_c\zeta = 2m\omega_n\zeta \tag{9.152c}$$

where c_c is the critical damping coefficient, $\omega_n = \sqrt{k/m}$, and ζ is the damping ratio. By introducing the preceding dimensionless variables, Eqs. (9.146) and (9.147) become

$$\ddot{r}^* + 2\zeta\dot{r}^* + (1 - \dot{\theta}^{*2})r^* = (\omega/\omega_n)^2\cos(\omega t^*/\omega_n - \theta) \tag{9.153a}$$

$$\ddot{\theta}^* + 2(\zeta + \dot{r}^*/r^*)\dot{\theta}^* = (\omega/\omega_n)^2/r^*\sin(\omega t^*/\omega_n - \theta) \tag{9.153b}$$

These equations can be solved numerically with the use of the Runge–Kutta method. However, for a special case, as the whirling speed $\dot{\theta}$ is equal to the rotating speed ω of the shaft, it is called the synchronous whirl. Thus we have

$$\dot{\theta}^* = \omega/\omega_n \tag{9.154}$$

Under this condition,

$$\ddot{\theta}^* = \ddot{r}^* = \dot{r}^* = 0, \qquad \theta = (\omega/\omega_n)t^* + \beta \tag{9.155}$$

Equations (9.151a) and (9.151b) reduce to

$$[1 - (\omega/\omega_n)^2]r^* = (\omega/\omega_n)^2\cos\beta \tag{9.156}$$

$$2\zeta(\omega/\omega_n)r^* = (\omega/\omega_n)^2\sin\beta \tag{9.157}$$

where β is the phase angle between θ and ωt. Squaring Eqs. (9.156) and (9.157) and adding them together, we find

$$r^* = \frac{(\omega/\omega_n)^2}{\{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2\}^{1/2}} \tag{9.158}$$

Here we easily can see the maximum deflection increases as ω approaches ω_n. The numerical solution has been obtained by Ying.* It is lengthy. Details are revealed in the reference.

*Ying, S. J., "Transient Whirling of a Rotating Shaft with an Unbalanced Disk," *Rotating Machinery Dynamics,* ASME Pub. H0400B, Vol. 2, pp. 537–543, Sept. 1987.

Maximum deflection as shaft rotates with a constant acceleration. The equations of motion for describing the whirling of a shaft rotating with acceleration are given in Eqs. (9.150) and (9.151). By introducing the dimensionless quantities given in Eqs. (9.152a–9.152c), Eqs. (9.150) and (9.151) become

$$-r^* - 2\zeta \dot{r}^* = \ddot{r}^* - (\alpha^* t^*)^2 \cos(\alpha^* t^{*2}/2 - \theta) - \alpha^* \sin(\alpha^* t^{*2}/2 - \theta) \quad (9.159)$$

$$-2\zeta r^* \dot{\theta}^* = r^* \ddot{\theta}^* + 2\dot{r}^* \dot{\theta}^* - (\alpha^* t^*)^2 \sin(\alpha^* t^{*2}/2 - \theta) + \alpha^* \cos(\alpha^* t^{*2}/2 - \theta) \quad (9.160)$$

where $\alpha^* = \alpha/\omega_n^2$. Equations (9.159) and (9.160) are solved numerically by the Runge–Kutta method as given in Appendix A. The initial conditions are chosen as follows:

$$r^*(0) = 0.001, \qquad \theta(0) = 0, \qquad \dot{r}^*(0) = 0, \qquad \dot{\theta}^*(0) = 1.0$$

Because it is interesting to see the growth of r^* as the shaft rotates, the value of $r^*(0)$ should be as small as possible. However, a low $r^*(0)$ value could cause instability in the numerical computations. The term $r^*(0) = 0.001$ is a compromised quantity. The increment of time Δt^* used in the computation is 0.001, which satisfies the convergence criterion in all the cases calculated because further decrease in Δt^* does not change the results significantly. On the other hand, the range of time in the calculation is determined as follows. It is reasonable to assume that the maximum deflection will reach the peak in the range $0 < \omega/\omega_n < 3$. In all of the calculations, the number of maximum time steps is limited by $\omega/\omega_n = 3$. That is

$$\alpha t_{\max}/\omega_n = 3$$

or

$$t^*_{\max} = 3/\alpha^*$$

In this way $t^*_{\max}$ is determined for each value of α^* assigned. For example, as $\alpha^* = 0.01$, 300, 000 steps are calculated for the determination of maximum dimensionless deflection $R^*_{\max}$, and for $\alpha^* = 0.50$, 6000 steps are calculated. To find the effect of acceleration on the motion of rotating shaft, the range of α^* used for calculations is from 0.01 to 0.59 with an increment of 0.01. The results of the maximum dimensionless deflection vs dimensionless acceleration are plotted by a computer and are given in Fig. 9.21. From the curves shown in the figure, it easily is seen that for low damping factors $\zeta < 0.2$ the values of maximum deflection are higher at low acceleration. That means that while the shaft is rotating with low acceleration, the system has more time to stay in the neighborhood of resonance and $R^*_{\max}$ is occurring at low value of α^*. For systems with high damping factors $\zeta > 0.2$, the magnitude of whirling increases slightly with α^*. This is caused by the fact that the inertial force is not enough to overcome the damping force at low values of α^*.

Therefore, for slightly damped cases ($\zeta < 0.2$) the shaft should be operated with its highest possible acceleration to reach its operational speed; on the contrary, for

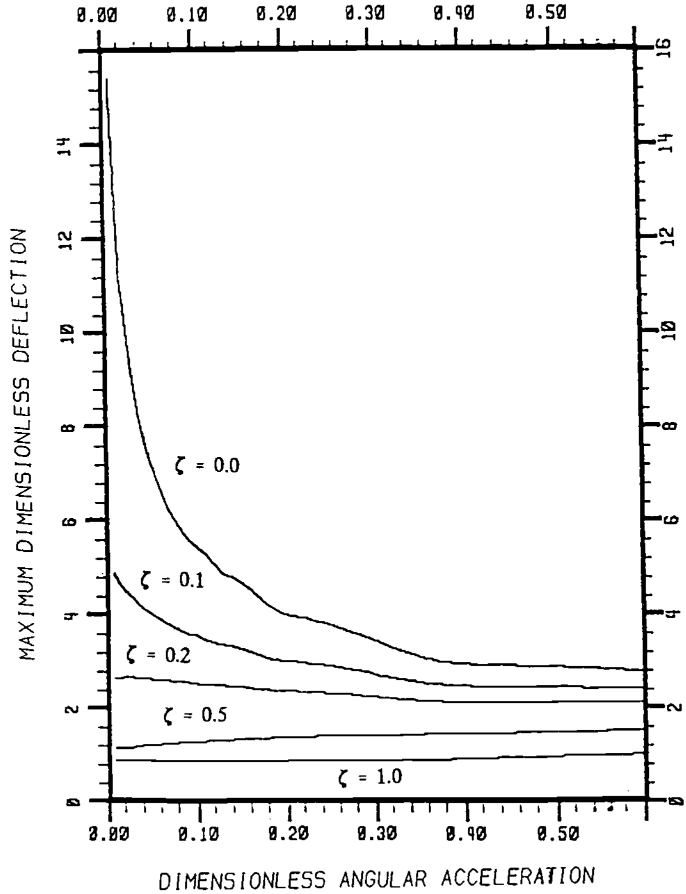

Fig. 9.21 Maximum deflection vs angular acceleration.

highly damped systems ($\zeta > 0.2$) the shaft should be operated with the lowest possible acceleration to reach its operational speed.

9.6 Stability of Vibrating Systems

Stability analysis is important in the study of vibrating systems. From the result of analysis, we can predict whether the amplitude of vibration will grow with time or not. For linear systems, we can determine the stability from the roots of characteristic equations. If the real parts of the roots are negative, the amplitudes of oscillations will decrease exponentially with the time; the system is stable. If the real parts are zero, then the harmonic motion will continue indefinitely, and the motion is still stable. However, for nonlinear systems, there is no characteristic equation so that we cannot predict the stability from the roots of characteristic equation. We must take a different approach for the analysis. Furthermore, we know that there is no analytical method to find the exact response of a nonlinear system. The following is the introduction of this new stability analysis. First we need to learn some new terminologies, and then we can discuss new concepts.

Phase Plane

The differential equation describing a nonlinear system may have the general form of

$$\ddot{x} + f(\dot{x}, x, t) = 0 \tag{9.161}$$

where the function f contains at least one term of the product of $x, \dot{x}$, or x and $\dot{x}$, such as $x^2, \dot{x}^2$, or $x\dot{x}$. If the function does not have the time t explicitly stated in the expression, then the system is known as an autonomous system that will be discussed in this section, and Eq. (9.159) becomes

$$\ddot{x} + f(\dot{x}, x) = 0 \tag{9.162}$$

In the study of stability, we define

$$\dot{x} = y \tag{9.163a}$$

$$\dot{y} = -f(x, y) \tag{9.163b}$$

Equation (9.163b) is actually the new form of Eq. (9.160). Consider x and y as the Cartesian coordinates. The x–y plane is called the phase plane.

Dividing Eq. (9.161b) by Eq. (9.161a), we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f(x, y)}{y} \tag{9.164}$$

Integrating the equation gives

$$y = g(x)$$

which can be plotted in the phase plane and is called the trajectory. If the trajectory is bounded by a circle with finite radius, then x and y are limited; the system is stable. If at some points, $y = 0$ and $f(x, y) = 0$, the slope is indeterminate. We define such a point as a singular point. Further discussion will be presented for the integration of Eq. (9.164) around the singular point to determine whether the system is stable or unstable.

Example 9.18

Consider a simple pendulum. The differential equation of motion can be written as

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \tag{9.165}$$

where $\omega^2 = g/L$, g is gravitational acceleration, and L is the length of the pendulum. Find the function for the trajectory. In the process of integration, an arbitrary constant will be present. Plot the trajectories in the phase plane for different arbitrary constants. Discuss whether the system is stable or unstable.

Solution. Let

$$\theta = x, \qquad \dot{\theta} = y$$

$$\omega^2 \sin\theta = \omega^2 \sin x = f(x, y)$$

then

$$\dot{y} = -\omega^2 \sin x$$

$$\frac{dy}{dx} = -\frac{f(x, y)}{y} = -\frac{\omega^2 \sin x}{y}$$

$$y\,dy = -\omega^2 \sin x\,dx$$

$$\tfrac{1}{2}y^2 + \omega^2(1 - \cos x) = E \tag{9.166}$$

where E is the arbitrary constant to be determined by the initial conditions. Note that E is proportional to the total energy of the system.

Equation (9.166) is the equation for trajectories. Three different trajectories are plotted as shown in Fig. 9.22 for $E = \omega^2, 2\omega^2, 3\omega^2$ with $\omega^2 = 1$. We notice that for $E < 2\omega^2$ we obtain closed trajectories, so that the motion repeats itself. This implies that the motion is stable. For $E > 2\omega^2$, the trajectories are open and the motion is unstable with the pendulum going over the top. The trajectories corresponding to $E = 2\omega^2$ separate the two types of motion, oscillatory and rotary, for which reason these trajectories are called separatrices. Note that at $x = \pm(2j + 1)\pi\,(j = 0, 1, 2, \ldots)$ and $y = 0$ the points are singular points. A more general discussion will be given in the next subsection.

Stability Around a Singular Point

Equation (9.164) can be expressed in the general form of

$$\frac{dy}{dx} = \frac{p(x, y)}{y} \tag{9.167}$$

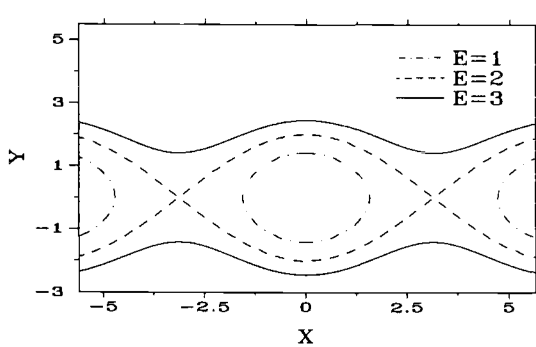

Fig. 9.22 Three trajectories.

The singular points of the equation are specified by

$$p(x, y) = y = 0 \tag{9.168}$$

Equation (9.167) is actually combined from the following two equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = p(x, y) \tag{9.169}$$

Let us construct a new set of coordinates u, v parallel to x, y with the origin at the singular point x_s and y_s, i.e.,

$$x = x_s + u, \qquad y = y_s + v$$

Because x_s and y_s are definite constants

$$\frac{dy}{dx} = \frac{dv}{du} \tag{9.170}$$

Expanding $p(x, y)$ into the Taylor series about the singular point (x_s, y_s), we obtain for $p(x, y)$

$$p(x, y) = p(x_s, y_s) + \left(\frac{\partial p}{\partial u}\right)_s u + \left(\frac{\partial p}{\partial v}\right)_s v$$

$$+ \frac{1}{2}\left(\frac{\partial^2 p}{\partial u^2}\right)_s u^2 + \cdots = cu + ev \tag{9.171}$$

Then Eq. (9.167) becomes

$$\frac{dv}{du} = \frac{cu + cv}{v}$$

or

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = cu + ev$$

which can be rewritten in the matrix form as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c & e \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{9.172}$$

With the use of modal matrix discussed in Section 9.2, the preceding equation can be transformed into the equation for principal mode:

$$\begin{pmatrix} u \\ v \end{pmatrix} = (P) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Then Eq. (9.170) becomes

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\xi = e^{\lambda_1 t}$$

$$\eta = e^{\lambda_2 t}$$

The solutions for u and v are

$$u = u_1 e^{\lambda_1 t} + u_2 e^{\lambda_2 t}$$

$$v = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t}$$

It is evident, then, that the stability of the system around the singular point depends on the eigenvalues λ_1 and λ_2 determined from the characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ c & (e - \lambda) \end{vmatrix} = 0$$

$$\lambda_{1,2} = \frac{e}{2} \mp \sqrt{\left(\frac{e}{2}\right)^2 + c}$$

Thus, if $(e/2)^2 + c < 0$, the motion is oscillatory; if $(e/2)^2 + c > 0$, the motion is aperiodic; if $e > 0$, the system is unstable; and if $e < 0$, the system is stable.

Example 9.19

Let us consider once again the pendulum of Example 9.18 governed by the differential equation

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 \sin x$$

Determine the stability around the singular points that have been found as

$$x = \pm j\pi \quad j = 0, 1, 2, \ldots$$

$$y = 0$$

Solution. Around $x = y = 0$,

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 \sin x$$

To use Eq. (9.167), we write

$$p(x, y) = -\omega^2 \sin x$$

When it is expanded around the singular point $x = y = 0$, we have

$$p(x, y) = -\omega^2 u$$

That means

$$\dot{u} = v, \qquad \dot{v} = -\omega^2 u$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The characteristic equation is simply

$$\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = -\omega^2$$

$$\lambda_{1,2} = \pm i\omega$$

Because the roots are pure, imaginary complex conjugate, we conclude that the motion in the neighborhood of the origin is stable.

Around the singular point $x = \pi$, $y = 0$,

$$p(x, y) = -\omega^2 \sin x$$

When it is expanded around the singular point $x = \pi$, $y = 0$, we have

$$p(x, y) = \omega^2 u$$

That means

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The characteristic equation for eigenvalues is

$$\begin{vmatrix} -\lambda & 1 \\ \omega^2 & -\lambda \end{vmatrix} = \lambda^2 - \omega^2 = 0$$

$$\lambda_{1,2} = \pm\omega$$

Because the roots are real but opposite in sign, the singular point is a saddle point. Clearly, the motion around $x = \pi$, $y = 0$ is unstable.

Problems

9.1. Two simple pendula of length s and bob mass m swing in a common vertical plane and are attached to two different support points. The masses are connected by a spring of constant k as shown in Fig. 4.4. The equations of motion are derived in Example 4.3 and are rewritten as follows:

$$ms^2\ddot{\theta}_1 + mgs\theta_1 + ks^2(\theta_1 - \theta_2) = 0$$

$$ms^2\ddot{\theta}_1 + mgs\theta_2 - ks^2(\theta_1 - \theta_2) = 0$$

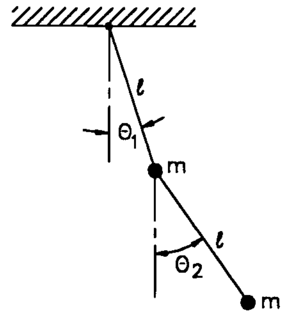

Fig. P9.2

Find the natural frequencies and the principal-mode solution for small oscillations of the system.

9.2. Determine the differential equations of motion for the double pendulum shown in Fig. P9.2. Find the natural frequencies and amplitude ratios for small oscillations of the system.

9.3. A two-degree-of-freedom system as shown in Fig. P9.3 is excited by a harmonic force $F_1 = F_0 \sin \omega_f t$. The physical constants for the system are $m_1 = 8$ kg, $m_2 = 4$ kg, $k_1 = 8.0$ kN/m, and $k_2 = 1.5$ kN/m. Using the Laplace transform method, determine the solution for the forced vibration with $F_0 = 2$N and $\omega_f = 2$ Hz. Assume that the initial displacements and velocities are zero.

9.4. Determine the solution of the vibrating system given in Example 9.7 with the use of the method of principal coordinates.

9.5. For a cantilevered beam with a uniform cross section, as shown in Fig. P9.5, find the transfer matrices from state 0 to 2. Determine the natural frequency of the system.

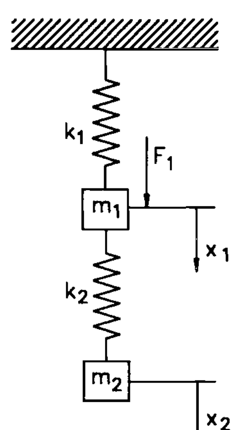

Fig. P9.3

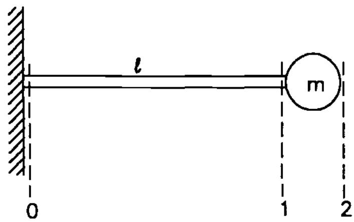

Fig. P9.5

9.6. A two-degree-of-freedom system consists of two equal springs and two equal masses as shown in Fig. P9.6. Using state vectors and transfer matrices, obtain the natural frequencies and mode shapes for the system.

9.7. Solve the problem of the vibrating string for the following boundary conditions: $y(0, t) = 0$; $y(L, t) = 0$; $\partial y/\partial t(x, 0) = 0$; and $y(x, 0) = f(x)$ as shown in Fig. P9.7.

9.8. A uniform string stretching from $-\infty$ to ∞ is originally displaced into the curve

$$y = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the displacement of the string as a function of x and t.

9.9. Derive the differential equation of motion for a longitudinal vibration along a uniform rod with length L.

9.10. Consider a simply supported beam of length L. The initial displacement of the beam is

$$\frac{M_0}{2EI}x(L - X)$$

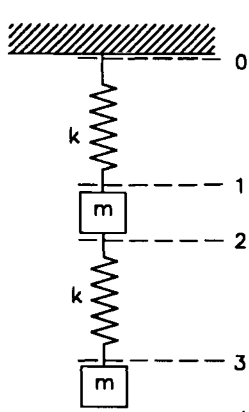

Fig. P9.6

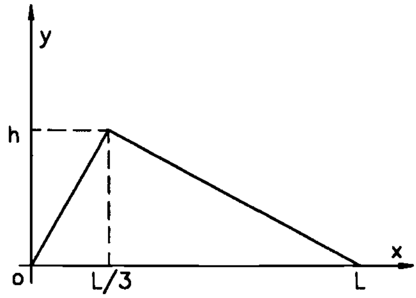

Fig. P9.7

and it is released from rest. Obtain the transverse motion $y(x, t)$ of the beam.

9.11. For a freely vibrating square membrane of length L, supported along the boundary $x = 0, x = L, y = 0, y = L$, suppose that the membrane is deflected in the form

$$w(x, y, 0) = f(x, y)$$

and is released from rest. Prove that the expression for the transverse vibration $w(x, y, t)$ of the membrane is

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L} \cos \omega_{mn} t$$

where

$$a_{mn} = \frac{4}{L^2} \int_0^L \int_0^L f(x, y) \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L} \mathrm{d}x \, \mathrm{d}y$$

9.12. The differential equation of motion of a damped pendulum can be written in the form

$$\ddot{\theta} + 2\zeta\omega\dot{\theta} + \omega^2 \sin\theta = 0$$

(a) Transform the equation into the equation for the phase plane and determine the singular points.

(b) Choose a value of ω, and plot curves in the phase plane for two cases: $\zeta = 0.1$, and $\zeta = 2$.

(c) Examine the motion in the neighborhood of the singular points.

9.13. Using the Runge–Kutta method, obtain the numerical solution $\theta(t)$ for

$$\ddot{\theta} + w^2 \sin\theta = 0$$

with $\omega^2 = 50$ (rad/s^2), $\theta(0) = \pi/2$, and $\dot{\theta}(0) = 0.1$(rad/s). Plot the numerical results for $0 < t < 2$.

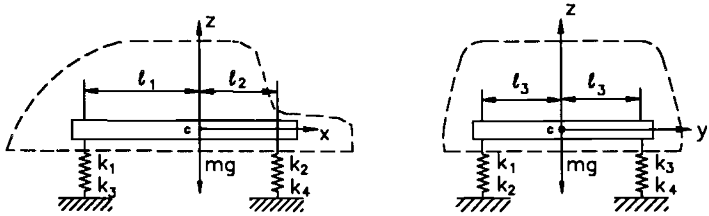

Fig. P9.14

9.14. Model the vibration of an automobile as a solid body supported by four springs as shown in Fig. P9.14. Obtain the differential equations of motion for the system under small oscillation.

9.15. Suppose that one of the four wheels is not balanced on the automobile modeled in Problem 9.14. Obtain the differential equations of motion for the system, and find the subsequent motion of the vibrating car during driving.

9.16. A circular membrane with radius of 10 cm is fixed on its edge. Suppose that the membrane is deflected initially in the form

$$w(r, 0) = \frac{10 - r}{10}$$

and is released from rest. Find the expression for the transverse vibration $w(r, t)$ of the membrane. Assume $a = 300$ m/s.