8
Fundamentals of Small Oscillations

VIBRATION can be either destructive or beneficial to our daily life. The fatigue of a material, which may lead to the failure of a structure, is possibly caused by vibration. A machine is intentionally designed to be free of vibrations, but sometimes undesirable vibrations just cannot be avoided when it is in service. When a car is driven on the road, an unbalanced wheel or an out-of-round tire can cause it to vibrate. On the other hand, because of the oscillation of its pendulum, a mechanical clock can tell the time. Because of the vibration of its membrane, a loud speaker can produce music.

Because vibrations can be either useful or troublesome, it is desirable that we understand the causes and phenomena of vibrations and further how to control them according to our wishes. Developing the knowledge to accomplish this control is the purpose of Chapters 8 and 9.

As in previous chapters, the required mathematics for studying vibration is presented at the beginning of the chapter. The subjects of the mathematics needed are Fourier series, Fourier integral, and Fourier and Laplace transforms. They are presented in Sections 8.1 and 8.2. Because more functions can satisfy the conditions for the Laplace transform than for the Fourier transform, the Laplace transform method can be applied to many more cases. Section 8.3 presents some important properties of the Laplace transform. Tables of Laplace and Fourier transforms are included in Appendix F. However, we will not deal with the inverse Laplace transform in this chapter because the derivation of formulas involves some lengthy details from the theory of complex variables. A brief description of the inverse transform for some functions is given in Appendix G. The applications of Fourier and Laplace transforms are presented in this chapter. In Chapter 9, we will present applications of Fourier series and more applications of the Laplace transform.

In Section 8.4, we shall study forced vibration systems with single degrees of freedom.These systems are either with damping or without damping and are either harmonically or arbitrarily excited. Because a periodic force can be expanded into a Fourier series, an analysis for one harmonic excitation will suffice to demonstrate that for any other harmonic excitations. Applications of these vibration systems are presented as examples that include accelerometer, seismometer, and packaging. The meaning of the Richter scale, which is a measure of the magnitude of an earthquake, is explained in Example 8.6.

Transient vibration is studied in Section 8.5. This type of vibration is caused by a nonperiodic force. Depending on the type of forcing function applied, response of a general excitation may not be obtained by analytical integration; it can be always integrated numerically through the formulation of arbitrary excitations. The responses of the cases studied in this section are obtained by analytical integration.

Response and velocity spectra of transient vibration are studied in Section 8.6. Because design of a vibration system is often restricted by the maximum response, response spectra may be used for modifying the design so that the maximum response is within the acceptable range.

Section 8.7 is specifically devoted to the application of the Fourier transform for analyzing the response of a vibration system. As we shall study later, the amplitude of vibration as a function of time will be converted into that as a function of frequency by using Fourier transform. Because of limited time in practice, the method used in the vibration analyzer is modified and is called the discrete Fourier transform. Through this, many random vibrations can be analyzed and the amplitude of vibration can be displayed as a function of frequency. From there, we can detect the source of vibration.

8.1 Fourier Series and Fourier Integral

Fourier Series

A Fourier series is a useful tool for solving differential equations and for treating various problems involving periodic functions. It is an infinite series of trigonometric functions and, in general, is expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \tag{8.1}$$

where n is an integer, x can be any value from $-\infty$ to ∞, and a_n and b_n are coefficients.

A function that can be expanded into a Fourier series must satisfy the following conditions: 1) The function is periodic or 2) The function is piecewise continuous between x and $x + 2L$.

A function $f(x)$ is said to be periodic if it is defined for all x with a period of $2L$ such that

$$f(x + 2L) = f(x)$$

The function $f(x)$ shown in Fig. 8.1 is a piecewise continuous function. Note that it is impossible to expand a discrete function as shown in Fig. 8.2 into a Fourier series.

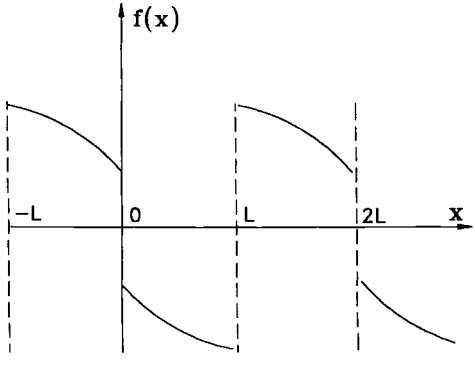

Fig. 8.1 Periodic and piecewise continuous function.

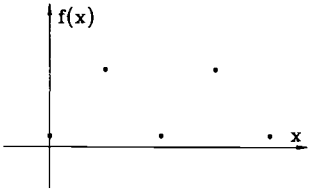

Fig. 8.2 Periodic and discrete function.

To facilitate the determination of the coefficients in a Fourier series, we introduce the following formulas for integrating trigonometric functions:

$$\begin{aligned}
&\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&\quad= \int_{-L}^{L}\left[\frac{1}{2}\cos(m+n)\frac{\pi x}{L}+\frac{1}{2}\cos(m-n)\frac{\pi x}{L}\right]\mathrm{d}x \\
&\quad= \frac{1}{2}\frac{L}{(m+n)\pi}\sin(m+n)\frac{\pi x}{L}\bigg|_{-L}^{L}+\frac{1}{2}\frac{L}{(m-n)\pi}\sin(m-n)\frac{\pi x}{L}\bigg|_{-L}^{L} \\
&\quad= 0 \qquad \text{if } m \not\equiv n \qquad (m, n \text{ are integers})
\end{aligned} \tag{8.2}$$

If $m = n$, then

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x &= \int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\frac{1}{2}\left(1+\cos 2m\frac{\pi x}{L}\right)\mathrm{d}x = L
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
&\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x \\
&\quad= \int_{-L}^{L}\left[\frac{1}{2}\cos(m-n)\frac{\pi x}{L}-\frac{1}{2}\cos(m+n)\frac{\pi x}{L}\right]\mathrm{d}x \\
&\quad= \frac{L}{2(m+n)\pi}\sin(m-n)\frac{\pi x}{L}\bigg|_{-L}^{L}-\frac{L}{2(m-n)\pi}\sin(m+n)\frac{\pi x}{L}\bigg|_{-L}^{L} \\
&\quad= 0 \qquad \text{if } m \neq n
\end{aligned} \tag{8.4}$$

If $m = n$, then

$$\begin{aligned}
\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x &= \int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \int_{-L}^{L}\frac{1}{2}\left(1-\cos\frac{2m\pi x}{L}\right)\mathrm{d}x = L
\end{aligned} \tag{8.5}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x$$
$$= \int_{-L}^{L}\left[\frac{1}{2}\sin(m+n)\frac{\pi x}{L} + \frac{1}{2}\sin(m-n)\frac{\pi x}{L}\right]\mathrm{d}x$$
$$= -\frac{1}{2}\left[\frac{L}{(m+n)\pi}\cos(m+n)\frac{\pi x}{L} + \frac{L}{(m-n)\pi}\cos(m-n)\frac{\pi x}{L}\right]_{-L}^{L}$$
$$= 0 \qquad \text{if } m \neq n \tag{8.6}$$

If $m = n$, then

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x = \int_{-L}^{L}\frac{1}{2}\sin\frac{2m\pi x}{L}\,\mathrm{d}x$$
$$= -\frac{L}{4m\pi}\left[\cos\frac{2m\pi x}{L}\right]_{-L}^{L}$$
$$= 0 \tag{8.7}$$

If $m = n = 0$,

$$\int_{-L}^{L} \cos o.x \cos o.x\,\mathrm{d}x = \int_{-L}^{L}\mathrm{d}x = 2L \tag{8.8}$$

$$\int_{-L}^{L} \sin Ox \sin Ox\,\mathrm{d}x = 0 \tag{8.9}$$

$$\int_{-L}^{L} \sin Ox \cos Ox\,\mathrm{d}x = 0 \tag{8.10}$$

Now we can conclude:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x = L\delta_{m,n} \tag{8.11}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x = L\delta_{m,n} \tag{8.12}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x = 0 \tag{8.13}$$

Equations (8.11–8.13) are known as orthogonality conditions.

Calculation of the coefficients in a Fourier series. If a function $f(x)$ satisfies the conditions for the Fourier series, it can be expanded into the form of Eq. (8.1). To determine the coefficient a_o, we multiply both sides of Eq. (8.1) by

$\cos 0x = 1$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = \int_{-L}^{L} \frac{1}{2}a_o\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos 0x\,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\cos 0x\,\mathrm{d}x$$

By using the orthogonality conditions, we get

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = a_o L$$

or

$$a_o = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x \tag{8.14}$$

To determine the coefficient a_n, we multiply both sides of Eq. (8.1) by $\cos(m\pi x/L)$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\mathrm{d}x = \frac{1}{2}a_o\int_{-L}^{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x$$
$$+\sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x$$
$$+\sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x$$

Because

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = 0$$
$$\sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,\mathrm{d}x = La_m$$
$$\sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

we obtain

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,\mathrm{d}x = La_m$$

The index m is a dummy index that can be replaced by any symbol. It is convenient

to use n, so that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 0, 1, 2, \ldots \tag{8.15}$$

Note that Eq. (8.15) includes the expression of Eq. (8.14). Similarly, to determine the coefficient b_n, we multiply both sides of Eq. (8.1) by $\sin(m\pi x/L)$ and integrate from $-L$ to L. We find

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 1, 2, 3, \ldots \tag{8.16}$$

Therefore, if $f(x)$ is a periodic function and is piecewise continuous, then it can be expanded into a Fourier series as

$$f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \quad n = 1, 2, 3, \ldots$$

It is worthwhile to mention that the integral limits in the preceding equations are not necessarily $-L$ and L. Because $f(x)$ is a periodic function of period of $2L$, the integral limits can be replaced by α and $\alpha + 2L$ where α is any real constant.

Example 8.1

Consider a function $f(x)$ that is known as a square wave and is defined as

$$f(x) = -h \qquad -L < x < o$$

$$f(x) = h \qquad o < x < L$$

where h is a constant. The function is periodic and is shown in Fig. 8.3. The

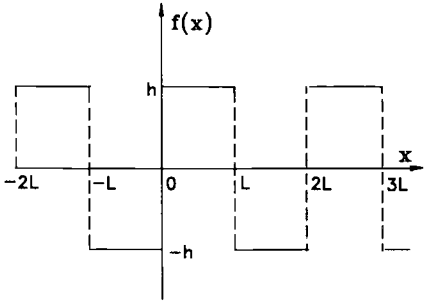

Fig. 8.3 Plot of the function.

interval $-L < x < L$ is so called the Fourier interval. Expand the function into a Fourier series.

Solution. Because this function is periodic and piecewise continuous, we can expand it into a Fourier series in the form of Eq. (8.1). The coefficients are determined as follows:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{L}\int_{-L}^{0}(-h)\cos\frac{n\pi x}{L}\,\mathrm{d}x + \frac{1}{L}\int_{0}^{L} h\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= -\frac{h}{n\pi}\left(\sin\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} + \frac{h}{n\pi}\left(\sin\frac{n\pi x}{L}\right)\Bigg|_{0}^{L} = 0 \end{aligned}$$

that is true for all the values of n from 1 on up. For $n = 0$, we have zero divided by zero, hence we determine a_0 separately and we find

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x = \frac{1}{L}\int_{-L}^{0}(-h)\mathrm{d}x + \frac{1}{L}\int_{0}^{L} h\,\mathrm{d}x \\ &= \frac{1}{L}(-hl + hL) = 0 \end{aligned}$$

To determine the coefficients b_n, we use Eq. (8.16)

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{L}\left[\int_{-L}^{0}(-h)\sin\frac{n\pi x}{L}\,\mathrm{d}x + \int_{0}^{L} h\sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\ &= \frac{h}{n\pi}[1-\cos(n\pi)] - \frac{h}{n\pi}[\cos(n\pi)-1] \\ &= \frac{2h}{n\pi}[1-\cos(n\pi)] \\ &= \frac{2h}{n\pi}[1-(-1)^n] \end{aligned}$$

Hence

$$b_n = 4h/n\pi \qquad \text{as } n = 1, 3, 5\ldots$$
$$b_n = 0 \qquad \text{as } n = 2, 4, 6, \ldots$$

Therefore, the Fourier series for the square wave can be expressed as

$$f(x) = \frac{4h}{\pi}\left[\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right]$$

or

$$f(x) = \frac{4h}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \frac{(2m-1)\pi x}{L}$$

Fourier sine series and cosine series. If a function $f(x)$ satisfies the conditions for the Fourier series and is an odd function, i.e.,

$$f(-x) = -f(x)$$

then it can be expanded into a Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \tag{8.17}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \quad n = 1, 2, 3, \ldots \tag{8.18}$$

On the other hand, if a function $f(x)$ satisfies the condition for Fourier series and is an even function, $f(-x) = f(x)$, then it can be expanded into a Fourier cosine series as

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \tag{8.19}$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx \tag{8.20}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx \quad n = 1, 2, 3, \ldots \tag{8.21}$$

Fourier Integral

We have studied that a periodic and piecewise continuous function in a finite interval can be represented by a Fourier series. Now we shall generalize the method of Fourier series to include a piecewise continuous function as defined in an infinite interval. If a function $f(x)$ is piecewise continuous defined in the interval $0 < x < \infty$, and is an odd function, then we can write

$$f_0(x) = \sum_{n=0}^{\infty} B_n \frac{L}{\pi} \left(\sin \frac{n\pi x}{L} \right) \frac{\pi}{L} \tag{8.22}$$

For the ease of mathematical operation, we make changes in symbols in the preceding equation and let

$$B_n \frac{L}{\pi} = B(u_n) \qquad \frac{n\pi}{L} = u_n$$

and

$$\Delta u_n = u_{n+1} - u_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} = \Delta u$$

Then we have

$$f_0(x) = \sum_{n=0}^{\infty} B(u_n)(\sin u_n x)\Delta u \tag{8.23}$$

Now, considering the case as L approaches ∞, we have

$$\Delta u = \mathrm{d}u$$

and write u_n as u. Equation (8.23) becomes

$$f_0(x) = \int_0^{\infty} B(u) \sin ux \, \mathrm{d}u \tag{8.24}$$

where

$$B(u) = \lim_{L\to\infty} \frac{L}{\pi} B_n = \lim_{L\to\infty} \frac{L}{\pi}\left[\frac{2}{L}\int_0^L f_0(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x\right]$$

$$= \lim_{L\to\infty} \frac{2}{\pi}\int_0^L f_0(x) \sin ux \, \mathrm{d}x = \frac{2}{\pi}\int_0^{\infty} f_0(x) \sin ux \, \mathrm{d}x \tag{8.25}$$

In the preceding equation, x is a dummy variable that can be replaced by any symbol. By changing x to t, and combining Eqs. (8.24) and (8.25), we obtain

$$f_0(x) = \frac{2}{\pi}\int_0^{\infty} \sin ux \left[\int_0^{\infty} f_0(t) \sin ut \, \mathrm{d}t\right] \mathrm{d}u \tag{8.26}$$

This is known as the Fourier sine integral representation of $f_0(x)$.

Similarly, if $f(x)$ is piecewise continuous defined in the interval $0 < x < \infty$ and is an even function, then it can be represented by

$$f_e(x) = \int_0^{\infty} A(u) \cos ux \, \mathrm{d}u \tag{8.27}$$

where

$$A(u) = \frac{2}{\pi}\int_0^{\infty} f_e(t) \cos ut \, \mathrm{d}t \tag{8.28}$$

or

$$f_e(x) = \frac{2}{\pi}\int_0^{\infty} \cos ux \int_0^{\infty} f_e(t) \cos ut \, \mathrm{d}t \, \mathrm{d}u \tag{8.29}$$

In general, a function always can be expressed as a combination of even and odd

functions, i.e.,

$$f(x) = f_e(x) + f_0(x)$$

Using Eqs. (8.26) and (8.29) for even and odd functions, we have

$$\begin{aligned}
f(x) &= \frac{2}{\pi}\left[\int_0^\infty \cos ux \int_0^\infty f_e(t)\cos ut\,\mathrm{d}t\,\mathrm{d}u\right.\\
&\quad \left. + \int_0^\infty \sin ux \int_0^\infty f_0(t)\sin ut\ \mathrm{d}t\,\mathrm{d}u\right]\\
&= \frac{2}{\pi}\left\{\int_0^\infty \cos ux \frac{1}{2}\int_{-\infty}^\infty [f(t) - f_0(t)]\cos ut\,\mathrm{d}t\,\mathrm{d}u\right.\\
&\quad \left. + \int_0^\infty \sin ux \frac{1}{2}\int_{-\infty}^\infty [f(t) - f_e(t)]\sin ut\,\mathrm{d}t\,\mathrm{d}u\right\}\\
&= \frac{1}{\pi}\left\{\int_0^\infty \cos ux \int_{-\infty}^\infty f(t)\cos ut\,\mathrm{d}t\,\mathrm{d}u\right.\\
&\quad \left. + \int_0^\infty \sin ux \int_{\infty}^\infty f(t)\sin ut\,\mathrm{d}t\,\mathrm{d}u\right\}\\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos ux\cos ut + \sin ux \sin ut]\,\mathrm{d}t\,\mathrm{d}u\\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u \qquad (8.30)
\end{aligned}$$

Because the integrand is an even function of u, we can rewrite Eq. (8.30) in the form of

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u \qquad -\infty < x < \infty \qquad (8.31)$$

This expression is known as the complete Fourier integral representation of $f(x)$ for all values of x. The conditions for a function to be expressed in Eq. (8.31) are 1) the integral $\int_{-\infty}^\infty |f(t)|\mathrm{d}t$ must exist, and 2) $f(t)$ must be a single-valued function of the real variable t throughout the range $-\infty < t < \infty$. It may have several finite discontinuities.

Example 8.2

1) Find the Fourier sine integral representation of the function $f(x)$ which is given as

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & a < x < \infty \end{cases}$$

$$f(-x) = -f(x)$$

and is shown in Fig. 8.4.

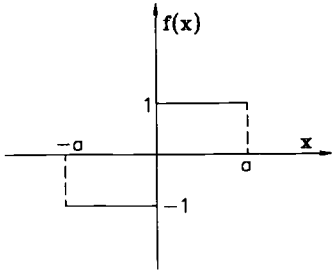

Fig. 8.4 Plot of the function.

2) Evaluate each term in the Fourier sine integral representation and prove that the result of the expression truly represents the original function as shown in Fig. 8.4.

Solution. 1) From Eq. (8.26), we have

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty \sin ux\left[\int_0^\infty f(t)\sin ut\,\mathrm{d}t\right]\mathrm{d}u \\
&= \frac{2}{\pi}\int_0^\infty \sin ux\int_0^a \sin ut\,\mathrm{d}t\,\mathrm{d}u \\
&= \frac{2}{\pi}\int_0^\infty \sin ux\left[-\frac{\cos ut}{u}\right]_0^a \mathrm{d}u \\
&= \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos ua}{u}\right)\sin ux\,\mathrm{d}u
\end{aligned}
$$

The result in the preceding equation is the one that we are looking for.

2) Rewrite the preceding equation as

$$
f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin ux}{u} - \frac{\cos ua\sin ux}{u}\right)\mathrm{d}u = \frac{2}{\pi}(I_1 - I_2)
$$

where

$$
I_1 = \int_0^\infty \frac{\sin ux}{u}\,\mathrm{d}u
$$

$$
I_2 = \int_0^\infty \frac{\cos ua\sin ux}{u}\,\mathrm{d}u
$$

Evaluate I_1, we find

$$
I_1 = \int_0^\infty \frac{\sin ux}{ux}\mathrm{d}(ux) = \int_0^\infty \frac{\sin z}{z}\mathrm{d}z = \frac{\pi}{2} \qquad \text{as } x > 0
$$

As $x < 0$,

$$
\begin{aligned}
I_1 &= \int_0^{-\infty} \frac{\sin z}{z}\,\mathrm{d}z \\
&= -\int_0^{\infty} \frac{\sin z^*}{z^*}\,\mathrm{d}z \qquad (\text{when } z = -z^*) \\
&= -\frac{\pi}{2}
\end{aligned}
$$

On the other hand, we rewrite I_2 in two parts as

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u + \sin(x-a)u}{u}\,\mathrm{d}u \\
&= I_{21} + I_{22}
\end{aligned}
$$

Evaluate I_{21}, we obtain

$$
\begin{aligned}
I_{21} &= \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u}{u}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^{\infty} \frac{\sin(x+a)u}{(x+a)u}\,\mathrm{d}[(x+a)u] = \frac{\pi}{4} \qquad \text{as } x > -a \\
I_{21} &= -\frac{\pi}{4} \qquad \text{as } x < -a
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_{22} &= \frac{1}{2}\int_0^{\infty} \frac{\sin(x-a)u}{u}\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^{\infty} \frac{\sin(x-a)u}{(x-a)u}\,\mathrm{d}[(x-a)u] = \frac{\pi}{4} \qquad \text{as } x > a \\
I_{22} &= -\frac{\pi}{4} \qquad \text{as } x < a
\end{aligned}
$$

Now the function $f(x)$ can be expressed as

$$
f(x) = \frac{2}{\pi}[I_1 - (I_{21} + I_{22})]
$$

We find that

$$
f(x) = \frac{2}{\pi}\left[-\frac{\pi}{2} - \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = 0
$$

as $x < -a$,

$$
f(x) = \frac{2}{\pi}\left[-\frac{\pi}{2} - \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = -1
$$

as $-a < x < 0$,

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\pi}{4}\right)\right] = 1$$

as $0 < x < a$, and

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right] = 0$$

as $x > a$. The preceding result proves that the Fourier sine integral representation is truly the original function.

Example 8.3

1) Find the Fourier cosine integral representation of the function $f(x)$ that is defined by

$$f(x) = \begin{cases} 1 & \text{as } 0 < x < a \\ 0 & \text{as } a < x < \infty \end{cases}$$

and

$$f(-x) = f(x)$$

2) Evaluate the Fourier cosine integral representation obtained in part 1 and prove that the result is the original function.

Solution. 1) From Eq. (8.29) we have

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty \cos ux \left[\int_0^\infty f(t)\cos ut\,\mathrm{d}t\right]\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \cos ux \int_0^a \cos ut\,\mathrm{d}t\,\mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \cos ux \left.\frac{\sin ut}{u}\right|_0^a \mathrm{d}u \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sin ua \cos ux}{u}\,\mathrm{d}u \end{aligned}$$

This is the Fourier cosine integral representation of the given functions.

2) Evaluate the integral, and we find

$$\begin{aligned} I &= \int_0^\infty \frac{\sin ua \cos ux}{u}\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^\infty \frac{\sin(x+a)u - \sin(x-a)u}{u}\,\mathrm{d}u = I_1 - I_2 \end{aligned}$$

where

$$I_1 = \frac{1}{2}\int_0^\infty \frac{\sin(x+a)u}{u}\,\mathrm{d}u$$

$$I_2 = \frac{1}{2}\int_0^\infty \frac{\sin(x-a)u}{u}\,\mathrm{d}u$$

Note that the integrals have been evaluated in the previous example. The results are collected as follows:

$$I_1 = \begin{cases} \pi/4 & \text{as } x > -a \\ -\pi/4 & \text{as } x < -a \end{cases}$$

$$I_2 = \begin{cases} \pi/4 & \text{as } x > a \\ -\pi/4 & \text{as } x < a \end{cases}$$

Now, the function $f(x)$ can be evaluated by

$$f(x) = \frac{2}{\pi}(I_1 - I_2)$$

We find that

$$f(x) = \frac{2}{\pi}\left[-\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 0$$

as $x < -a$,

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 1$$

as $-a < x < a$, and

$$f(x) = \frac{2}{\pi}\left[\frac{\pi}{4} - \left(\frac{\pi}{4}\right)\right] = 0$$

as $x > a$. The result reached agrees exactly with the original function.

Example 8.4

Find the complete Fourier integral representation of the functions which is given as

$$f(x) = \begin{cases} 0 & \text{as } -\infty < x < 0 \\ 1 & \text{as } 0 < x < a \\ 0 & \text{as } a < x < \infty \end{cases}$$

Solution. From Eq. (8.31), we have the complete Fourier integral representation of $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{a}[\cos ux\sin ut + \sin ux\sin ut]\,\mathrm{d}t\,\mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\cos ux\frac{\sin ut}{u} - \sin ux\frac{\cos ut}{u}\right]_0^a \mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\cos ux\sin ua}{u} + \sin ux\frac{1-\cos ua}{u}\right]\mathrm{d}u \end{aligned}$$

Note that the integrand is an even function of u. Hence we can write

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\cos ux\sin ua}{u} + \frac{\sin ux(1-\cos ua)}{u}\right]\mathrm{d}u$$

This result is the combination of the Fourier cosine and sine integral representations found in Examples 8.3 and 8.2 except the coefficient is reduced to one half, because the given function in this example is one half of the sum of the functions in the previous examples. In addition, it is worthwhile to point out that the expression that was reached also represents the inverse Fourier transform of the transformed function. Details of the Fourier transform are discussed in Section 8.2.

8.2 Fourier and Laplace Transforms

The Fourier transform is a powerful tool for solving differential equations. When it is applied to linear ordinary differential equations with constant coefficients, the differential equation is converted into an algebraic equation. Then the solution of the original differential equation is then reduced to the calculation of inverse transforms of the transformed functions obtained from the algebraic equation.

The Fourier transform also can be applied to linear partial differential equations with constant coefficients. If it is applied to one particular independent variable, the number of independent variables in the partial differential equations is reduced by one. The coefficients in the differential equations are not restricted to constants. Constant coefficients are simpler to apply to the method. The Laplace transform may be considered as a special case of the Fourier transform. Both transforms are closely related to the Fourier series and integral discussed in the previous section. To see how they are related, we shall begin the discussion with the complete Fourier integral representation of a function.

By Eq. (8.31), the complete Fourier integral representation of $f_1(x)$ is

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)\cos u(x-t)\,\mathrm{d}t\,\mathrm{d}u$$

Because

$$\cos u(x-t) = \tfrac{1}{2}\left[e^{iu(x-t)} + e^{-iu(x-t)}\right]$$

the equation becomes

$$f_1(x) = \frac{1}{2\pi}\left\{\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{iu(x-t)}\mathrm{d}t\,\mathrm{d}u + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{-iu(x-t)}\,\mathrm{d}t\,\mathrm{d}u\right\}$$

In the second part of the preceding equation, we make a change in one of the variables. First we change u to $-\nu$, and then change ν back to u, because ν is a dummy variable. Finally we have

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)e^{iu(x-t)}\mathrm{d}t\,\mathrm{d}u$$

or

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\int_{-\infty}^{\infty} e^{-iut} f_1(t)\,\mathrm{d}t\,\mathrm{d}u$$

Now let the Fourier transform of $f_1(t)$ be defined as

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-iut} f_1(t)\,\mathrm{d}t \tag{8.32}$$

Then the inverse transform of $\mathcal{F}(u)$ is

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \tag{8.33}$$

The conditions for the existence of the Fourier transform of $f_1(t)$ are the same as for the Fourier integral representation stated in Section 8.1.

To show that the Laplace transform is a special case of the Fourier transform, we consider the following function

$$f_1(t) = \begin{cases} 0 & \text{as } -\infty < t < 0 \\ e^{-\gamma t} f(t) & \text{as } 0 < t < \infty \end{cases}$$

where γ is a real number. The Fourier transform of $f_1(t)$ then becomes

$$\mathcal{F}(u) = \int_{0}^{\infty} e^{-(\gamma+iu)t} f(t)\,\mathrm{d}t \tag{8.34}$$

and the inverse transform is

$$\begin{aligned} e^{-\gamma t} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(u)e^{iut}\,\mathrm{d}u \\ f(t) &= \frac{e^{\gamma t}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+iu)t}\mathcal{F}(u)\,\mathrm{d}u \end{aligned} \tag{8.35}$$

In this equation, we make changes in variables and let $s = \gamma + iu$ and $\mathcal{F}(u) = F(s)$ then $\mathrm{d}s = i\,\mathrm{d}u$. Thus, we write

$$\mathcal{F}(u) = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

This is defined as the Laplace transform of $f(t)$. The symbol for the transform is

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \tag{8.36}$$

From Eq. (8.35), the inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s)\,\mathrm{d}s \tag{8.37}$$

The conditions for the existence of the Laplace transform of a function $f(t)$ are as follows:

1) The term $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \le t < T$, where $t_1 > 0$ and $T > t_1$.

2) The term $t^n|f(t)|$ is bounded near $t = 0$ for some number n when $n < 1$.

3) The term $e^{-s_0 t}|f(t)|$ is bounded for large values of t for some positive real number s_0. Therefore, $f(t)$ may be infinite as $t \to 0$ or finite as $t \to \infty$. The Laplace transform of such a function is still possible.

8.3 Properties of Laplace Transforms

In this section, we shall establish, by detailed calculations, the properties of the Laplace transforms of functions that are very important in solving differential equations. With the sample calculations shown in this section, it would be easy for the reader to establish other formulas of the Laplace transforms given in Appendix F.

The differentiation of $f(t)$ is

$$\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = s\mathcal{L}[f(t)] - f(0) \tag{8.38}$$

Proof: By using the method of integration by parts, we find that

$$\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = \int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{d}t = e^{-st} f(t)|_0^\infty$$
$$+ s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = sF(s) - f(0+)$$

where 0+ means on the positive side of zero.

For the second derivative,

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = s^2 F(s) - sf(0+) - \frac{\mathrm{d}f}{\mathrm{d}t}(0+) \tag{8.39}$$

Proof:

$$\mathcal{L}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right] = \int_0^\infty e^{-st}\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\,\mathrm{d}t = e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_0^\infty$$

$$+ s\int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = s[sF(s) - f(0+)] - \frac{\mathrm{d}f}{\mathrm{d}t}(0+)$$

$$= s^2F(s) - sf(0+) - \frac{\mathrm{d}f}{\mathrm{d}t}(0+)$$

The integration of $f(t)$ is

$$\mathcal{L}\left[\int_0^t f(u)\,\mathrm{d}u\right] = \frac{1}{s}F(s)$$

Proof:

$$\mathcal{L}\left[\int_0^t f(u)\,\mathrm{d}u\right] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,\mathrm{d}u\right]\mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(u)\,\mathrm{d}u\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \frac{1}{s}F(s) \tag{8.40}$$

If the lower limit in the integral is not zero, then

$$\mathcal{L}\left[\int_a^t f(u)\,\mathrm{d}u\right] = \mathcal{L}\left[\left(\int_0^t - \int_0^a\right) f(u)\,\mathrm{d}u\right]$$

$$= \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(u)\,\mathrm{d}u \tag{8.41}$$

The translation property is

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

Proof:

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)\,\mathrm{d}t$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t = F(s-a) \tag{8.42}$$

If

$$f(t) = \begin{cases} 0 & \text{as } t < a (a \geq 0) \\ g(t-a) & \text{as } t \geq a \end{cases}$$

then

$$F(s) = e^{-as}\,G(s) \tag{8.43}$$

where $G(s)$ is the transformed function of $g(t)$.

Proof:

$$\mathcal{L}[f(t)] = \int_a^\infty e^{-st} g(t-a)\,\mathrm{d}t$$

Let $x = t - a$, then

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-s(x+a)} g(x)\,\mathrm{d}x$$

$$= e^{-sa} \int_0^\infty e^{-sx} g(x)\,\mathrm{d}x = e^{-sa} G(s)$$

Let

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \tag{8.44}$$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s) = \int_0^\infty (-t) e^{-st} f(t)\,\mathrm{d}t = (-1)\mathcal{L}[t f(t)]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) = \int_0^\infty (-t)^2 e^{-st} f(t)\,\mathrm{d}t = (-1)^2 \mathcal{L}[t^2 f(t)] \cdots$$

Therefore

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$$

If $\mathcal{L}[f(t)] = F(s)$ and if $[f(t)/t] \leq (M/t^n)$ as $t \to 0+$ with $n < 1$, $M =$ finite, then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,\mathrm{d}s \tag{8.45}$$

Proof:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\,\mathrm{d}t$$

$$\int_s^\infty F(s)\,\mathrm{d}s = \int_s^\infty \left[\int_0^\infty f(t) e^{-st}\,\mathrm{d}t\right] \mathrm{d}s$$

Because s and t are independent, the integration order can be changed:

$$\begin{aligned}\text{LHS} &= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\mathrm{d}s\right]\mathrm{d}t \\ &= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty \mathrm{d}t \\ &= \int_0^\infty \frac{f(t)}{t}e^{-st}\mathrm{d}t = \mathcal{L}\left[\frac{f(t)}{t}\right]\end{aligned}$$

The convolution

$$\mathcal{L}\left[\int_0^t f(t-u)g(u)\mathrm{d}u\right] = F(s)G(s) \tag{8.46}$$

or

$$\begin{aligned}\mathcal{L}[f * g] &= F(s)G(s) \\ \mathcal{L}[g * f] &= F(s)G(s)\end{aligned}$$

Proof:

$$\begin{aligned}F(s)G(s) &= \left[\int_0^\infty e^{-sv}f(v)\mathrm{d}v\right]\left[\int_0^\infty e^{-su}g(u)\mathrm{d}u\right] \\ &= \int_0^\infty\int_0^\infty e^{-s(v+u)}f(v)g(u)\mathrm{d}v\mathrm{d}u \\ &= \int_0^\infty g(u)\left[\int_0^\infty e^{-s(v+u)}f(v)\mathrm{d}v\right]\mathrm{d}u\end{aligned}$$

Let $v = t - u$, $\mathrm{d}v = \mathrm{d}t$, then

$$\int_0^\infty e^{-s(v+u)}f(v)\mathrm{d}v = \int_u^\infty e^{-st}f(t-u)\mathrm{d}t$$

Therefore

$$F(s)G(s) = \int_0^\infty\left[\int_u^\infty e^{-st}f(t-u)g(u)\mathrm{d}t\right]\mathrm{d}u$$

The integration shown in Fig. 8.5 can be represented by the triangular area bounded by $t = u$ and $u = 0$. By interchanging the order of integration and changing the

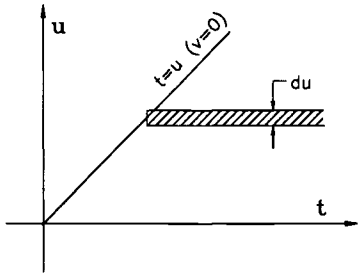

Fig. 8.5 Integration by *t* then *u*.

limits as shown in Fig. 8.6, we can obtain the equivalent result:

$$F(s)G(s) = \int_0^\infty \left[\int_0^t e^{-st} f(t-u)g(u)\,\mathrm{d}u\right]\mathrm{d}t$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(t-u)g(u)\,\mathrm{d}u\right]\mathrm{d}t$$

$$= \mathcal{L}\left[\int_0^t f(t-u)g(u)\,\mathrm{d}u\right]$$

By changing variables in Eq. (8.46) $t - u = v$, we have

$$\int_0^t f(t-u)g(u)\,\mathrm{d}u = \int_t^0 f(v)g(t-v)\,\mathrm{d}(-v)$$

$$= \int_0^t f(v)g(t-v)\,\mathrm{d}v$$

Therefore, we can easily establish

$$f * g = g * f$$

Singularity functions (Dirac delta function) are

$$\mathcal{L}[\delta(t-t_1)] = e^{-t_1 s} \tag{8.47}$$

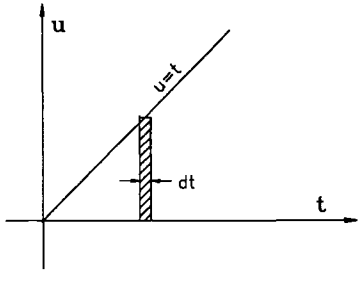

Fig. 8.6 Integration by *u* then *t*.

Proof: The Dirac delta function is defined as

$$\delta(t) = \begin{cases} \lim\limits_{t_0 \to 0} \dfrac{1}{t_0} & \text{as } 0 < t < t_0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathcal{L}[\delta(t)] = \int_0^\infty e^{-st}\delta(t)\mathrm{d}t = \lim_{t_0 \to 0} \int_0^{t_0} \frac{1}{t_0} e^{-st}\mathrm{d}t$$

$$= \lim_{t_0 \to 0} \frac{1}{st_0}(1 - e^{-st_0}) = \lim_{t_0 \to 0} \frac{se^{-st_0}}{s} = 1$$

The Laplace transform of $\delta(t - t_1)$ can be obtained by using Eq. (8.43) and the preceding result. Thus we write

$$\mathcal{L}[\delta(t - t_1)] = e^{-t_1 s}\mathcal{L}[\delta(t)] = e^{-t_1 s}$$

Sometimes, it is useful to know the Laplace transform of the first derivative of Dirac delta function that can be written as

$$\mathcal{L}[\delta'(t - t_1)] = se^{-t_1 s} \tag{8.48}$$

Proof: The first derivative of the Dirac delta function is defined as

$$\delta'(t) = \lim_{t_0 \to 0} \frac{\delta(t) - \delta(t - t_0)}{t_0}$$

which is shown in Fig. 8.7. In other words

$$f(t) = \frac{1}{t_0^2} \qquad \text{as } 0 < t < t_0$$

$$f(t) = -\frac{1}{t_0^2} \qquad \text{as } t_0 < t < 2t_0$$

$$f(t) = 0 \qquad \text{elsewhere}$$

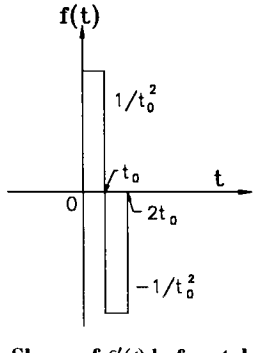

Fig. 8.7 Shape of $\delta'(t)$ before taking limit.

and the derivative of the Dirac delta function is defined as

$$\delta'(t) \equiv \lim_{t_0 \to 0} f(t)$$

Hence, the Laplace transform of the function is

$$\begin{aligned}
\mathcal{L}[\delta'(t)] &= \lim_{t_0 \to 0} \frac{1}{t_0^2} \left\{ \int_0^{t_0} e^{-st} \mathrm{d}t - \int_{t_0}^{2t_0} e^{-st} \mathrm{d}t \right\} \\
&= \lim_{t_0 \to 0} \frac{1}{st_0^2} [1 - e^{-st_0} + e^{-2st_0} - e^{-st_0}] \\
&= \lim_{t_0 \to 0} \frac{1}{st_0^2} (1 - e^{-st_0})^2 = s
\end{aligned}$$

Again, with the use of Eq. (8.43), we have

$$\mathcal{L}[\delta'(t - t_1)] = e^{-t_1 s} \mathcal{L}[\delta'(t)] = s e^{-t_1 s}$$

8.4 Forced Harmonic Vibration Systems with Single Degree of Freedom

Any system consisting of a mass and a spring or the equivalent is capable of vibration. Because of friction, practical problems are usually modeled as a system consisting of a mass, a spring, and a damper as shown in Fig. 8.8. The damper is modeled as a viscous device with damping force proportional to the velocity $\dot{x}$. The weight of the mass is balanced by the spring force $k\Delta$, where k is the spring constant and Δ is the initial deformation of the spring. $F(t)$ is an external force applied to the mass. For the system, the balance of forces gives

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{8.49}$$

To understand the behavior of the system, first we examine the solution of the homogeneous equation:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{8.50}$$

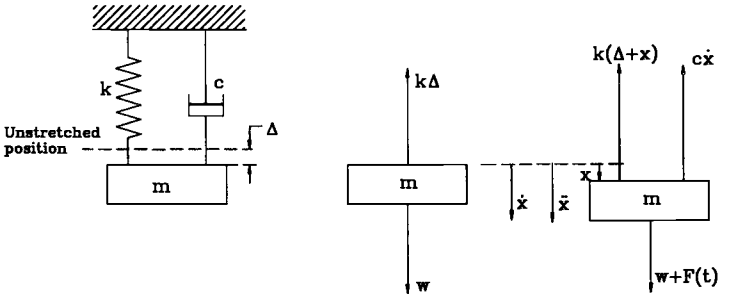

Fig. 8.8 Single-degree-of-freedom system.

Assuming that the solution of the equation is an exponential function of time,

$$x = Xe^{st}$$

where X and s are arbitrary, we find

$$(ms^2 + cs + k)Xe^{st} = 0$$

Because X and e^{st} cannot be zero for all values of t, we must have

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

or

$$S_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

That means

$$\begin{aligned} x &= Ae^{s_1 t} + Be^{s_2 t} \\ &= \exp\left[-\frac{c}{2m}t\right]\left\{A\exp\left[t\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]\right. \\ &\quad \left. + B\exp\left[-t\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right]\right\} \end{aligned} \tag{8.51}$$

The preceding equation can have three possible results, which are discussed in detail as follows.

Case 1 Overdamped motion

As $(c/2m)^2 > k/m$, the two roots of s are real-negative. The vibration will be damped out rapidly.

Case 2 Critically damped motion

As $(c/2m)^2 = k/m$, the two roots of s are the same. The solution of Eq. (8.50) can be written as

$$x = (C + Dt)e^{-\omega_n t}$$

where ω_n = natural frequency = $\sqrt{k/m}$.

By applying initial conditions we find

$$C = x(0)$$

$$D = \dot{x}(0) + \omega_n x(0)$$

Therefore

$$x = \{x(0) + [\dot{x}(0) + \omega_n x(0)]t\}e^{-\omega_n t} \tag{8.52}$$

The oscillation also will be damped because of $e^{-\omega_n t}$.

Case 3 Underdamped motion

Finally, let us consider the case as $(c/2m)^2 < k/m$. We define the frequency of damped oscillation as

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{8.53}$$

so that

$$\begin{aligned} x &= e^{-\frac{c}{2m}t}[Ae^{i\omega_d t} + Be^{-i\omega_d t}] \\ &= e^{-\frac{c}{2m}t}[c_1 \sin \omega_d t + c_2 \cos \omega_d t] \end{aligned}$$

where

$$c_1 = i(A - B), \qquad c_2 = A + B$$

By applying the initial conditions, the constants are found as

$$c_1 = \frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]$$

and $c_2 = x(0)$. Hence

$$x = \exp\left[-\frac{c}{2m}t\right]\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]\sin \omega_d t + x(0)\cos \omega_d t\right\} \tag{8.54}$$

For this case, the amplitude of oscillation is still decreasing continually because of the damping effect, and the frequency of oscillation is reduced from the natural frequency as given in Eq. (8.53).

In all cases, the solution of the homogeneous equation indicates that the amplitude of the vibration will be vanishing as the time increases.

Forced Harmonic Vibration

Harmonic excitation is commonly produced by the unbalance in a rotating machinery. The force is often periodic and is represented by $F(t)$ in Eq. (8.49). Because the periodic function can be expressed in a Fourier series, we can take one term first for $F(t)$ when analyzing the problem. Because the differential equation is linear, the principle of superposition allows us to study a particular case first. The final result is the summation of solutions from all the cases considered.

Now let us consider $F(t) = F_0 \sin \omega t$. From Eq. (8.49), we have

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t = \mathrm{Im}(F_0 e^{i\omega t}) \tag{8.55}$$

where F_0 is the real constant. Let the solution be

$$x = \mathrm{Im}(Ae^{i\omega t})$$

Then we have

$$\dot{x} = \mathrm{Im}(+iA\omega e^{i\omega t})$$
$$\ddot{x} = \mathrm{Im}(-A\omega^2 e^{i\omega t})$$

Substituting the preceding expressions into Eq. (8.55) and dropping the symbol for imaginary, we find

$$(-m\omega^2 + ic\omega + k)Ae^{i\omega t} = F_0 e^{i\omega t}$$

$$A = \frac{F_0/m}{k/m - \omega^2 + (ic\omega/m)}$$
$$= \frac{F_0}{m}\left\{\frac{(k/m - \omega^2) - i(c\omega/m)}{[k/m - \omega^2]^2 + (c\omega/m)^2}\right\}$$

Let

$$D = (k/m - \omega^2)^2 + (c\omega/m)^2$$

$$\cos\phi = \frac{(k/m - \omega^2)}{\sqrt{D}}, \qquad \sin\phi = \frac{c\omega/m}{\sqrt{D}} \tag{8.56}$$

where ϕ is called the phase angle between the vibrating mass and the excitation, then,

$$A = \frac{F_0}{m\sqrt{D}}(\cos\phi - i\sin\phi) = \frac{F_0}{m\sqrt{D}}e^{-i\phi}$$
$$x = \mathrm{Im}\left[\frac{F_0}{m\sqrt{D}}e^{i(\omega t - \phi)}\right]$$
$$= \frac{F_0}{m\sqrt{D}}\sin(\omega t - \phi) \tag{8.57}$$

The general solution is the summation of the homogeneous solution and the particular solution. Therefore, we have

$$x = \frac{F_0}{m\sqrt{D}}\sin(\omega t - \phi)$$
$$+ \exp\left[-\frac{c}{2m}t\right]\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2m}x(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\} \tag{8.58}$$

If the excitation force is a general periodic function, then

$$f(t) = \sum_i c_i f_i(t) \tag{8.59}$$

where c_i is a known constant and $f_i(t)$ may be any sinusoidal function in the Fourier series, although it is not restricted to sinusoidal functions. Because the principle of superposition is applicable, then, for each force $f_i(t)$, we can write the corresponding displacement as $x_i(t)$ so that the differential equation becomes

$$m\ddot{x}_i + c\dot{x}_i + kx_i = f_i(t) \tag{8.60}$$

After $x_i(t)$ is obtained, the general particular solution can be written as

$$x(t) = \sum_i c_i x_i(t) \tag{8.61}$$

Therefore, the solution given in Eq. (8.57) represents the typical form of the particular solution for the study of a harmonically excited motion. In addition, it is worthwhile to mention that $x(t)$ is the displacement of mass m and is often called the response of the system.

Example 8.5

Consider a spring mass system. The mass M is constrained to move only in the vertical direction. The vibration is excited by a rotating machine with an unbalanced mass m as shown in Fig. 8.9. The unbalanced mass m is at an eccentricity of e and is rotating with angular velocity ω. 1) Formulate an equation for describing the dynamic behavior of the system. 2) Find the amplitude of the vibration of M as a function of the force of excitation $me\omega^2$. 3) Find the expression for the phase angle between the vibration of M and the excitation. 4) Determine also the complete solution of the equation obtained in part 1.

Solution. 1) Let x be the displacement of the mass $(M - m)$ from its static equilibrium position, and $x + e\sin\omega t$ for the displacement of the unbalanced mass m. The equation of motion is

$$(M - m)\ddot{x} + m\frac{d^2}{dt^2}(x + e\sin\omega t) = -kx - c\dot{x}$$

Rearranging the terms, we have

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t \tag{8.62}$$

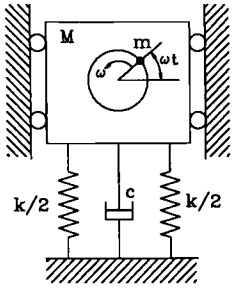

Fig. 8.9 Rotating machine with an unbalanced mass.

2) Note that this equation is identical to Eq. (8.55) except that m is replaced by M and F_0 by $me\omega^2$. Using Eq. (8.57) with the change of symbols, we find

$$x(t) = \frac{me\omega^2}{M\sqrt{D}} \sin(\omega t - \phi)$$

Let X be the amplitude of x, then we write

$$X = \frac{me\omega^2}{M\sqrt{D}} \tag{8.63}$$

where $D = (k/M - \omega^2)^2 + (c\omega/M)^2$.

3) The phase angle ϕ can be determined by Eq. (8.56):

$$\tan\phi = \frac{c\omega/M}{(k/M - \omega^2)} \tag{8.64}$$

Note that the phase angle ϕ is less than 90 deg if $\omega^2 < k/M$ and is greater than 90 deg as $\omega^2 > k/M$.

4) The complete solution is similar to Eq. (8.58) and can be written as

$$x(t) = \frac{me\omega^2}{M\sqrt{D}} \sin(\omega t - \phi) + \exp\left(-\frac{ct}{2M}\right)\left\{\frac{1}{\omega_d}\left[\dot{x}(0) + \frac{c}{2M}x(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\} \tag{8.65}$$

Example 8.6

A seismometer is an instrument for measuring the intensity of an earthquake. The design of the instrument is shown in Fig. 8.10. 1) Formulate an equation for describing the dynamic behavior of the instrument due to the ground vibration. 2) Find the relationship between the amplitudes of ground vibration and the instrument. 3) Discuss the essential factors in the design of the seismometer. 4) Suppose that the measured result is 7 on the Richter scale. What is the actual magnitude of the earthquake?

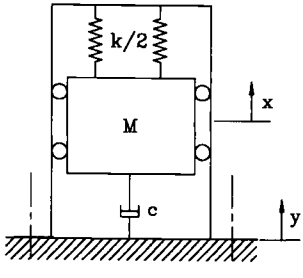

Fig. 8.10 Schematic diagram of a seismometer.

Solution. 1) Assume that the case is rigid and that the displacement of M is x and of the ground is y. The equation of motion for this system can be written as

$$M\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y) \tag{8.66}$$

Because the instrument actually only senses the relative motion between M and the case, i.e.,

$$z = x - y$$

Equation (8.66) becomes

$$M\ddot{z} + c\dot{z} + kz = -M\ddot{y} \tag{8.67}$$

Assuming that the vibration of ground is in sinusoidal motion, $y = Y \sin \omega t$, we find then the equation

$$M\ddot{z} + c\dot{z} + kz = MY\omega^2 \sin \omega t \tag{8.68}$$

2) Note that the preceding equation is in the same form as Eq. (8.55) except the symbols are changed. Using Eq. (8.57), we find

$$z(t) = \frac{Y\omega^2}{\sqrt{D}} \sin(\omega t - \phi)$$

where

$$\tan \phi = \frac{c\omega/m}{k/m - \omega^2}$$

$$D = (k/M - \omega^2)^2 + (c\omega/M)^2$$

Therefore, the amplitude of vibration detected by the instrument is

$$Z = \frac{Y\omega^2}{\sqrt{(k/m - \omega^2)^2 + (c\omega/M)^2}} \tag{8.69}$$

3) Equation (8.69) can be simplified if we define the following.

Natural frequency of undamped oscillation:

$$\omega_n = \sqrt{k/M} \tag{8.70}$$

Critical damping coefficient:

$$c_c = 2M\omega_n \tag{8.71}$$

Damping factor:

$$\zeta = c/c_c \tag{8.72}$$

Then

$$\frac{c\omega}{M} = \frac{c\omega}{k}\frac{k}{M} = \frac{c}{c_c}\frac{c_c\omega}{k}\frac{k}{M} = \zeta\frac{2M\omega_n\omega}{k}\frac{k}{M} = 2\zeta\omega\omega_n$$

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{8.73}$$

From this equation we can easily see that

$$Z \approx Y$$

if $\omega \gg \omega_n$ and $\zeta \ll 1$. Therefore we always use a large mass, weak spring, and small damper such that $\omega/\omega_n \to \infty$ and $\zeta \to 0$.

4) The Richter scale is defined as

$$R \equiv \log_{10}(Y/Y_s) \tag{8.74}$$

Y_s is the standard magnitude of an earthquake that is 1 μ, or 10^{-6} m. For a Richter scale value of 7

$$\log_{10}(Y/Y_s) = 7$$

$$Y = 10^7 \times 10^{-6}\ \text{m} = 10\ \text{m}$$

Therefore, the amplitude of the earthquake is 10 m.

Example 8.7

An accelerometer is an instrument that directly measures the vibration of a moving object. Then the amplitude of the vibration is converted into acceleration. Once the acceleration is measured, the velocity and displacement can be obtained by integration. The accelerometer is a very important instrument in a submarine. The general construction of the meter consists of a spring, a mass, and a damper similar to the seismometer shown in Fig. 8.10. For this instrument, discuss the essential factors in the design so that the measured quantity truly represents the acceleration.

Solution. Because the construction of the accelerometer is similar to that of the seismometer, Eq. (8.73) is applicable and we have

$$Z = \left(\frac{\omega}{\omega_n}\right)^2 \frac{Y}{f(\omega/\omega_n, \zeta)} \tag{8.75}$$

where

$$f(\omega/\omega_n, \zeta) = \sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2} \tag{8.76}$$

In this equation, $Y\omega^2$ is the acceleration, Z is measured, and ω_n^2 is fixed in the design. From this equation, we see that $f(\omega/\omega_n, \zeta) \simeq 1$ as $\omega/\omega_n \ll 1$ and $\varsigma = 0.65$ to 0.70. Therefore in the design of accelerometer, the natural frequency chosen should be very high and the damping factor should be about 0.70.

Example 8.8

Often in engineering practices, vibration of machinery cannot be totally eliminated. To reduce the effect of vibration to adjacent parts, vibration absorber is usually installed between the machine and the supporting ground. The vibration absorber can be modeled by a combination of springs and a damper as shown in Fig. 8.11a.

For the vibration absorber, 1) find the force transmitted from the vibrating mass M to the supporting ground, 2) find the expression for the ratio of the transmitted force and the excitation force, and 3) discuss how to reduce the ratio of forces in part 2.

Solution. 1) Because the present system is similar to the one shown in Fig. 8.8, Eq. (8.55) and its solution can be applied here. Rewrite Eq. (8.57) as

$$x = \mathrm{Im}\{X \exp[i(\omega t - \phi)]\} \tag{8.77}$$

where $X = F_0/(M\sqrt{D})$, then we have

$$\dot{x} = \mathrm{Im}\{X\omega i \exp[i(\omega t - \phi)]\} = \mathrm{Im}\{X\omega \exp[i(\omega t - \phi + \pi/2)]\} \tag{8.78}$$

and

$$\ddot{x} = \mathrm{Im}\{-X\omega^2 \exp[i(\omega t - \phi)]\} = \mathrm{Im}\{X\omega^2 \exp[i(\omega t - \phi + \pi)]\} \tag{8.79}$$

By drawing $F_0 \exp[i\omega t]$, $MX\omega^2 \exp[i(\omega t - \phi + \pi)]$, $c\omega X \exp[i(\omega t - \phi + \pi/2)]$, and $kX \exp[i(\omega t - \phi)]$ on a graph as shown in Fig. 8.11b, we obtain the total force transmitted to the support as

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (c\omega/k)^2} \tag{8.80}$$

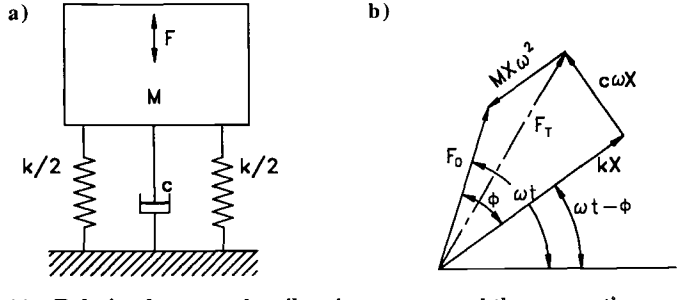

Fig. 8.11 Relation between the vibration system and the supporting ground.

2) Because

$$X = \frac{F_0}{M\sqrt{D}}$$

$$kX = \frac{F_0(k/M)}{\sqrt{(k/M - \omega^2)^2 + (c\omega/M)^2}}$$

$$= \frac{F_0}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (c\omega/k)^2}}$$

$$= \frac{F_0}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2}}$$

The ratio of the transmitted force to the exciting force is

$$\frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2}} \tag{8.81}$$

3) From Eq. (8.81), we easily can see that to reduce the value of the ratio, we set the damping factor ς to zero and choose $\omega/\omega_n \gg 1$. Then we find

$$\frac{F_T}{F_0} = \left(\frac{\omega_n}{\omega}\right)^2 \tag{8.82}$$

Therefore the ratio is reduced if we use weak springs and no damper.

Example 8.9

Figure 8.12 depicts a simplified model of a spring-supported vehicle traveling over a rough road. 1) Find the equation for the amplitude of vibrating mass m, and

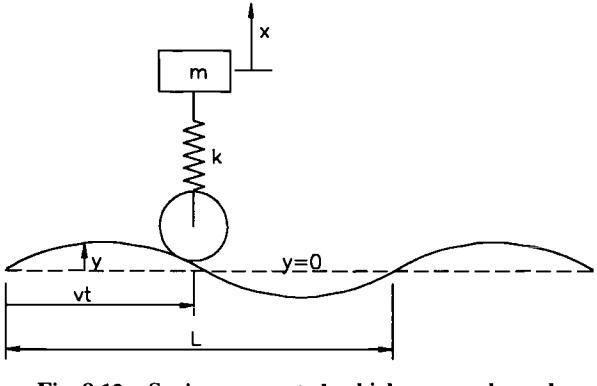

Fig. 8.12 Spring-supported vehicle on rough road.

determine the most unfavorable speed. 2) Suppose that the vehicle is now traveling on a flat road but its tire is cut by a hard braking. Note that this out-of-round wheel is equivalent to driving the vehicle on a road with repeated holes. Determine also the most unfavorable speed.

Solution. 1) For the balance of forces in the vertical direction, we have

$$m\ddot{x} = -k(x - y)$$

or

$$m\ddot{x} + kx = ky$$

The rough road can be modeled as

$$y = Y \sin \omega t$$

where $\omega = 2\pi v/L$. Therefore the equation of motion is

$$m\ddot{x} - kx = kY \sin \omega t \tag{8.83}$$

This equation is similar to Eq. (8.55) except $c = 0$. Hence the solution of Eq. (8.55) can be used. We have

$$X = \frac{kY}{m\sqrt{D}} = \frac{Y}{(\omega/\omega_n)^2 - 1}$$

or

$$\frac{X}{Y} = \frac{1}{(2\pi v/L\omega_n)^2 - 1}$$

The most unfavorable speed occurs when the system is in resonance, i.e.,

$$\begin{aligned} \omega &= \omega_n \\ 2\pi v/L\omega_n &= 1 \\ v &= (L/2\pi)\sqrt{k/m} \end{aligned} \tag{8.84}$$

2) When the tire is out-of-round, the velocity of the vehicle is directly related to ω as

$$v = R\omega$$

The governing equation for the motion still can be written as Eq. (8.83). The condition of resonance is now

$$\begin{aligned} \omega = v/R = \omega_n &= \sqrt{k/m} \\ v &= R\sqrt{k/m} \end{aligned} \tag{8.85}$$

where R is the radius of the tire.

8.5 Transient Vibration

When a spring-mass system is suddenly subjected to a nonperiodic force, a transient vibration is produced, because a steady-state oscillation is usually not present. Let us consider that an impulse is applied to a spring-mass system. The equation for the dynamic behavior of the system can be written as

$$m\ddot{x} + kx = \hat{F}\delta(t - t_1) \tag{8.86}$$

where $\hat{F}$ is an impulse and $\delta(t - t_1)$ is a Dirac delta function with a dimension of t^{-1}. Taking the Laplace transform of the equation, we have

$$m[s^2\bar{x}(s) - sx(0) - \dot{x}(0)] + k\bar{x}(s) = \hat{F}e^{-t_1 s}$$

where $\bar{x}(s)$ is the transformed function of $x(t)$. For $x(0) = \dot{x}(0) = 0$,

$$\bar{x}(s) = \frac{\hat{F}e^{-t_1 s}}{ms^2 + k}$$

Taking the inverse transform by using Eq. (8.43) and the Laplace transform table, we find

$$x(t) = \begin{cases} 0 & \text{as } t < t_1 \\ \frac{\hat{F}}{m\omega_n}\sin\omega_n(t - t_1) & \text{as } t > t_1 \end{cases} \tag{8.87}$$

By introducing a unit step function

$$H(t - t_1) = \begin{cases} 0 & \text{as } t < t_1 \\ 1 & \text{as } t > t_1 \end{cases} \tag{8.88}$$

we can write the solution of Eq. (8.86) as

$$x(t) = \frac{\hat{F}}{m\omega_n}\sin\omega_n(t - t_1)H(t - t_1) \tag{8.89}$$

Example 8.10

A simple spring-mass system is subjected to a repeated impulse $\hat{F}$ of finite duration at intervals of τ as shown in Fig. 8.13. Find the transient response.

Solution. Following Eq. (8.86) we write the equation of motion as

$$m\ddot{x} + kx = \hat{F}\sum_{n=0}^{N}\delta(t - n\tau) \tag{8.90}$$

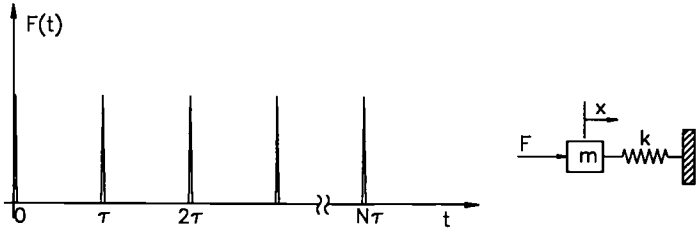

Fig. 8.13 Repeated impulse on a spring-mass system.

where N is a finite integer. Taking the Laplace transform and for $x(0) = \dot{x}(0) = 0$, we have

$$\bar{x}(s) = \frac{\hat{F}}{ms^2 + k} \sum_{n=0}^{N} e^{-n\tau s}$$

Therefore, from the inverse Laplace transform of the preceding equation, we find the transient response

$$x(t) = \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin \omega_n (t + n\tau) H(t - n\tau) \tag{8.91}$$

Note that, at time t_1, N is the integer of t_1/τ in the preceding equation because of the unit step functions; if $t_1 > N\tau$, then N is the total number of impulses used in Eq. (8.90). Two particular cases may be discussed here:

1) As $\omega_n \tau = 2\pi$, then

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t + n2\pi) H(t - n\tau) \\ &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t) H(t - n\tau) \\ &= \frac{N_1 \hat{F}}{m\omega_n} \sin(\omega_n t) \qquad N_1 = \text{integer}(t_1/\tau) \end{aligned} \tag{8.92}$$

This equation shows that the amplitude of $x(t)$ increases with time as indicated by N_1 in the expression. This means that the impulse is in resonance with the system.

2) As $\omega_n \tau = \pi$, then

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} \sin(\omega_n t - n\pi) H(t - n\tau) \\ &= \sum_{n=0}^{N} \frac{\hat{F}}{m\omega_n} (-1)^n \sin(\omega_n t) H(t - n\tau) \end{aligned} \tag{8.93}$$

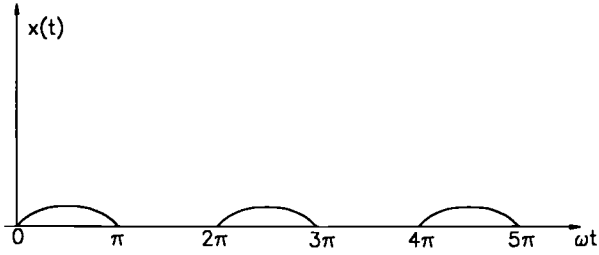

Fig. 8.14 Transient response at minimum amplitude.

The transient response of the preceding expression is shown in Fig. 8.14. If N is odd, the response is 0 as $N\tau < t < (N+1)\tau$; if N is even, the response is

$$\frac{\hat{F}}{m\omega_n}\sin\omega_n t$$

as $N\tau < t < (N+1)\tau$.

Next let us consider that an impulse is applied to a spring-mass system with a damper. Then the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = \hat{F}\delta(t) \tag{8.94}$$

with $x(0) = \dot{x}(0) = 0$. Taking the Laplace transform, we have

$$ms^2\bar{x}(s) + cs\bar{x}(s) + k\bar{x}(s) = \hat{F}$$

or

$$\bar{x}(s) = \frac{\hat{F}}{ms^2 + cs + k}$$

The transient response is the inverse Laplace transform of the equation. By using the table of Laplace Transforms in Appendix F, we find

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{\hat{F}}{ms^2 + cs + k}\right] \\ &= \frac{\hat{F}}{m}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \\ &= \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t) \end{aligned} \tag{8.95}$$

Furthermore, let us consider a general system with an arbitrary excitation. The equation of motion is in the form of

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{8.96}$$

with $x(0) = \dot{x}(0) = 0$. Taking the Laplace transform, we obtain

$$\bar{x}(s) = \frac{\bar{f}(s)}{ms^2 + cs + k} = \frac{\bar{f}(s)}{m} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let

$$\frac{1}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \bar{g}(s)$$

then

$$\bar{x}(s) = \bar{f}(s)\bar{g}(s)$$

Using the convolution, Eq. (8.46), and Eq. (8.95), we find

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[\bar{f}(s)\bar{g}(s)] = \int_0^t f(\eta)g(t-\eta)\mathrm{d}\eta \\ &= \frac{1}{m\omega_n\sqrt{1-\zeta^2}} \int_0^t f(\eta)e^{-\zeta\omega_n(t-\eta)} \sin[\omega_n\sqrt{1-\zeta^2}(t-\eta)]\mathrm{d}\eta \end{aligned} \tag{8.97}$$

For a special case, $c = 0$ or $\varsigma = 0$, then

$$x(t) = \frac{1}{m\omega_n} \int_0^t f(\eta) \sin[\omega_n(t-\eta)]\mathrm{d}\eta \tag{8.98}$$

Example 8.11

Determine the response of an undamped system with a single degree of freedom subjected to the following excitation:

$$f(t) = \begin{cases} F_0 \sin \omega t & \text{as } 0 < t < \pi/\omega \\ 0 & \text{as } t < 0 \quad \text{and} \quad t > \pi/\omega \end{cases}$$

Solution. Using Eq. (8.98) for $0 < t < \pi/\omega$, we have

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n} \int_0^t \sin \omega\eta \sin[\omega_n(t-\eta)]\mathrm{d}\eta \\ &= \frac{F_0}{2m\omega_n} \int_0^t \{\cos[(\omega+\omega_n)\eta - \omega_n t] - \cos[(\omega-\omega_n)\eta + \omega_n t]\}\mathrm{d}\eta \\ &= \frac{F_0}{k\left[1-(\omega/\omega_n)^2\right]} \left[\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t\right] \end{aligned} \tag{8.99}$$

For $t > \pi/\omega$, we find

$$x(t) = \frac{F_0}{m\omega_n}\int_0^{\frac{\pi}{\omega}} \sin\omega\eta \sin[\omega_n(t-\eta)]\mathrm{d}\eta$$

$$= \frac{F_0\omega_n/\omega}{k\left[1-(\omega_n/\omega)^2\right]}\left[\sin\omega_n t + \sin\omega_n\left(t-\frac{\pi}{\omega}\right)\right] \tag{8.100}$$

Example 8.12

Determine the response of a damped system with a single degree of freedom subjected to the following excitation

$$f(t) = F_0 H(t) = \begin{cases} F_0 & \text{as } t > 0 \\ 0 & \text{as } t < 0 \end{cases}$$

Solution. Using Eq. (8.97) for a damped system, we find

$$x(t) = \frac{-F_0}{m\omega_n\sqrt{1-\zeta^2}}\int_0^t \exp\left[\zeta\omega_n(\eta-t)\right]\sin[\omega_n\sqrt{1-\zeta^2}(\eta-t)]\mathrm{d}\eta$$

$$= \frac{F_0}{m\omega_n^2}\left\{1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\zeta\sin(\omega_n\sqrt{1-\zeta^2}t)+\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}t)]\right\}$$

Let $\zeta = \sin\psi$, then $\sqrt{1-\zeta^2} = \cos\psi$; the response becomes

$$x(t) = \frac{F_0}{k}\left\{1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_n\sqrt{1-\zeta^2}t-\psi)\right\} \tag{8.101}$$

where

$$\tan\psi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

Example 8.13

A mass m is packaged in a box as shown in Fig. 8.15. The box is dropped through height h. Determine the maximum force transmitted to mass m and the required rattle space at the instant of impact when the box reaches the ground. Assume that the impact can be represented by an impulse.

1) Assume the mass of the box is much greater than m, so that the free fall of the box is not influenced by the relative motion of the mass m.

2) On striking the floor, the impact depends greatly on the material properties of the box and the floor. It is reasonable to have the impact represented by an impulse.

3) Assume that the box is rigid. No deformation is considered.

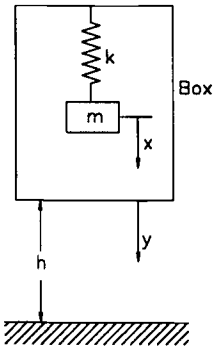

Fig. 8.15 Packaging analysis.

Based on the preceding assumptions, the equation for the system can be written as

$$m(\ddot{x} + \ddot{y}) = -kx$$

$$m\ddot{x} + kx = -m\ddot{y}$$

or

$$\ddot{x} + \omega_n^2 x = -\ddot{y} \tag{8.102}$$

where x is the displacement of mass m from its equilibrium position relative to the box and y is the displacement of the box from its initial position. Taking the Laplace transform of the preceding equation, we find

$$\bar{x}(s) = [x(0) + y(0)]\frac{s}{s^2 + \omega_n^2} + [\dot{x}(0) + \dot{y}(0)]\frac{1}{s^2 + \omega_n^2} - \frac{s^2\bar{y}(s)}{s^2 + \omega_n^2}$$

The response of $x(t)$ is obtained from

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[\bar{x}(s)] \\ &= [x(0) + y(0)]\cos\omega_n t + \frac{1}{\omega_n}[\dot{x}(0) + \dot{y}(0)]\sin\omega_n t - \mathcal{L}^{-1}\left[\frac{s^2\bar{y}(s)}{s^2 + \omega_n^2}\right] \end{aligned}$$

For the time of the free falling,

$$x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$$

$$y(t) = \tfrac{1}{2}gt^2, \qquad \bar{y}(s) = (g/s^3)$$

$$\begin{aligned} x(t) &= -\mathcal{L}^{-1}\left[\frac{g}{s(s^2 + \omega_n^2)}\right] \\ &= -\frac{g}{\omega_n^2}(1 - \cos\omega_n t) \end{aligned}$$

At the instant of impact,

$$t_0 = \sqrt{2h/g}$$

$$x(t_0) = -\left(g/\omega_n^2\right)(1 - \cos\omega_n t_0)$$

$$\dot{x}(t_0) = -(g/\omega_n)\sin\omega_n t_0$$

These quantities become the initial conditions for the second phase of the problem after the impact of the box with the floor.

After the impact of the box with the floor, consider $t = 0$ at the instant of impact:

$$x(0) = -\frac{g}{\omega_n^2}(1 - \cos\omega_n t_0)$$

$$\dot{x}(0) = -\frac{g}{\omega_n}\sin\omega_n t_0$$

$$\ddot{y}(t) = -\frac{\hat{F}}{m_b}\delta(t)$$

Therefore $\mathcal{L}[\ddot{y}(t)] = -\hat{F}/m_b$;

$$\bar{x}(s) = x(0)\frac{s}{s^2 + \omega_n^2} + \dot{x}(0)\frac{1}{s^2 + \omega_n^2} + \frac{\hat{F}}{m_b}\frac{1}{s^2 + \omega_n^2}$$

where m_b is the mass of the box. We find the response

$$\begin{aligned} x(t) &= x(0)\cos\omega_n t + \frac{1}{\omega_n}\left[\dot{x}(0) + \frac{\hat{F}}{m_b}\right]\sin\omega_n t \\ &= -\frac{g}{\omega_n^2}(1 - \cos\omega_n t_0)\cos\omega_n t + \frac{1}{\omega_n}\left[\frac{\hat{F}}{m_b} - \frac{g}{\omega_n}\sin\omega_n t_0\right]\sin\omega_n t \\ &= \frac{g}{\omega_n^2}\sqrt{(1 - \cos\omega_n t_0)^2 + \left[\frac{\hat{F}\omega_n}{m_b g} - \sin\omega_n t_0\right]^2}\,\sin(\omega_n t - \phi) \end{aligned}$$

$$\tan\phi = \frac{1 - \cos\omega_n t_0}{(\hat{F}\omega_n/m_b g) - \sin\omega_n t_0}$$

For a special case, as $(\hat{F}\omega_n/m_b g) \gg 1$,

$$\begin{aligned} x(t) &= \frac{g}{\omega_n^2}\frac{\hat{F}\omega_n}{m_b g}\sin(\omega_n t - \phi) \\ &= \frac{\hat{F}}{m_b\omega_n}\sin(\omega_n t - \phi) \end{aligned}$$

Because $x(t)$ is the displacement of m relative to the box, the maximum space required for m to travel before reaching the wall of the box is

$$x_{\max} = \frac{g}{\omega_n^2}\sqrt{(1-\cos\omega_n t_0)^2 + \left(\frac{\hat{F}\omega_n}{m_b g} - \sin\omega_n t_0\right)^2} \tag{8.103}$$

and the maximum force applied is simply $kx_{\max}$.

8.6 Response Spectrum

A response spectrum is a plot of the maximum peak response as a function of the product of the natural frequency of the oscillator and the characteristic time of the applied force.

From the information revealed in the plot, we can modify the design so that the peak response will be within the expected range. To illustrate the use of response spectrum, let us see the following example.

Example 8.14

Determine the response spectrum for a mass-spring system subjected to a force as a function of time given as follows:

$$\begin{aligned} f(t) &= F_0(t/t_1) && \text{as } 0 < t < t_1 \\ &= F_0 && \text{as } t > t_1 \end{aligned}$$

Solution. For the interval of $0 < t < t_1$, the response is obtained from Eq. (8.98):

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\int_0^t \frac{F_0\eta}{t_1}\sin\omega_n(t-\eta)\mathrm{d}\eta \\ &= \frac{F_0}{k}\left[\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right] \end{aligned} \tag{8.104}$$

For the time $t_1 < t < \infty$, the response is

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n}\left[\int_0^{t_1}\frac{F_0\eta}{t_1}\sin\omega_n(t-\eta)\,\mathrm{d}\eta + \int_{t_1}^t F_0\sin\omega_n(t-\eta)\mathrm{d}\eta\right] \\ &= \frac{F_0}{k}\left[1 + \frac{1}{\omega_n t_1}\sin\omega_n(t-t_1) - \frac{1}{\omega_n t_1}\sin\omega_n t\right] \end{aligned} \tag{8.105}$$

Examining Eqs. (8.104) and (8.105), we can see easily that the response from Eq. (8.105) is higher than that from Eq. (8.104). Hence the response spectrum is determined from Eq. (8.105). To find maximum $x(t)$, we differentiate $x(t)$ with respect to time and set it to zero and obtain

$$\cos\omega_n(t_p - t_1) - \cos\omega_n t_p = 0 \tag{8.106}$$

where t_p is a particular time such that $\dot{x}(t) = 0$, as $t = t_p$. Recalling that

$$\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

we conclude that t_p must satisfy the equation

$$\sin\omega_n\left(t_p - \frac{t_1}{2}\right)\sin\omega_n\frac{t_1}{2} = 0$$

which yields the solution

$$t_p = \frac{n\pi}{\omega_n} + \frac{t_1}{2} \qquad n = 1, 2, 3, \ldots$$

or

$$\omega_n(t_p - t_1) = 2n\pi - \omega_n t_p \tag{8.107}$$

From Eq. (8.106), we also find

$$\tan\omega_n t_p = \frac{1-\cos\omega_n t_1}{\sin\omega_n t_1}$$

$$\sin\omega_n t_p = -\sqrt{\frac{1}{2}(1-\cos\omega_n t_1)} \tag{8.108}$$

Using Eq. (8.107), we obtain

$$\sin\omega_n(t_p - t_1) = \sin(-\omega_n t_p) = -\sin\omega_n t_p \tag{8.109}$$

Then, the peak amplitude is found as

$$x_{\max} = \frac{F_0}{k}\left[1 + \frac{1}{\omega_n t_1}\sqrt{2(1-\cos\omega_n t_1)}\right] \tag{8.110}$$

The response spectrum is plotted in Fig. 8.16, which shows that $kx_{\max}/F_0 \to 1$ as $\omega_n t_1$ approaches infinity. Therefore, if the desired response is small, the natural frequency should be chosen as high as possible; otherwise set at

$$\omega_n t_1 = 2n\pi \qquad n = 1, 2, 3, \ldots$$

A velocity spectrum is a plot of the maximum velocity of the mass m versus time for a single-degree-of-freedom oscillator. It is often used for analyses of earthquakes or other ground shock situations. With the formulation of the seismometer given in Example 8.4, we can write

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \tag{8.111}$$

where $z = x - y$ is the relative displacement between the mass m and the coil

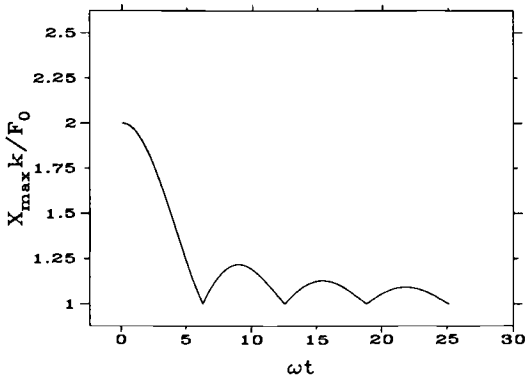

Fig. 8.16 Response spectrum.

attached to the ground. Assuming $z(0) = \dot{z}(0) = 0$ and using Eq. (8.97), we have

$$z(t) = \frac{-1}{\omega_n\sqrt{1-\zeta^2}} \int_0^t \ddot{y}(\eta) \exp[-\zeta\omega_n(t-\eta)] \sin\sqrt{1-\zeta^2}\omega_n(t-\eta)\,\mathrm{d}\eta$$

Recalling the differential formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{A(x)}^{B(x)} f(x,t)\,\mathrm{d}t \\ &= \int_A^B \frac{\partial f(x,t)}{\partial x}\mathrm{d}t + f(x,B)\frac{\mathrm{d}B}{\mathrm{d}x} - f(x,A)\frac{\mathrm{d}A}{\mathrm{d}x} \qquad (8.112)\end{aligned}$$

we find

$$\dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t,\eta)\,\mathrm{d}\eta = \int_0^t \frac{\partial f(t,\eta)}{\partial t}\mathrm{d}\eta + f(t,t)$$

or

$$\begin{aligned}\dot{z}(t) = \frac{-1}{\omega_n\sqrt{1-\zeta^2}} \int_0^t \ddot{y}(\eta) \exp[-\zeta\omega_n(t-\eta)][-\zeta\omega_n \sin\sqrt{1-\zeta^2}\omega_n(t-\eta) \\ + \omega_n\sqrt{1-\zeta^2}\cos\sqrt{1-\zeta^2}\omega_n(t-\eta)]\,\mathrm{d}\eta \qquad (8.113)\end{aligned}$$

Let

$$A = \int_0^t \ddot{y}(\eta)e^{\zeta\omega_n\eta}\cos\sqrt{1-\zeta^2}\omega_n\eta\,\mathrm{d}\eta$$

$$B = \int_0^t \ddot{y}(\eta)e^{\zeta\omega_n\eta}\sin\sqrt{1-\zeta^2}\omega_n\eta\,\mathrm{d}\eta$$

Equation (8.113) can be written as

$$\dot{z}(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[(A\zeta - B\sqrt{1-\zeta^2})\sin\sqrt{1-\zeta^2}\omega_n t$$
$$- (A\sqrt{1-\zeta^2} + B\zeta)\cos\sqrt{1-\zeta^2}\omega_n t]$$

or

$$\dot{z}(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sqrt{A^2+B^2}\sin(\sqrt{1-\zeta^2}\omega_n t - \phi) \tag{8.114}$$

$$\tan\phi = \frac{B\zeta + A\sqrt{1-\zeta^2}}{A\zeta - B\sqrt{1-\zeta^2}} \tag{8.115}$$

If $\dot{z}(t)$ is plotted against time, it would appear as an amplitude modulated wave. Because

$$|\dot{z}(t)|_{\max} = \left|\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sqrt{A^2+B^2}\right|_{\max} \tag{8.116}$$

the envelope of the amplitude modulated wave is the velocity spectrum. Two terms also may be mentioned here: pseudo-response spectrum is the plot of

$$|z|_{\max} = \frac{1}{\omega_n}|\dot{z}(t)|_{\max}$$

and pseudo-acceleration spectrum is the plot of

$$|\ddot{z}|_{\max} = \omega_n|\dot{z}(t)|_{\max}$$

8.7 Applications of Fourier Transforms

The Fourier transform was introduced in Section 8.2. In this section we shall study its applications. The Fourier transform of a function $f(t)$ can be written as

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-iut} f(t)\,\mathrm{d}t \tag{8.117}$$

and the inverse transform of $\mathcal{F}(u)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}\mathcal{F}(u)\,\mathrm{d}u \tag{8.118}$$

In these equations, the dimension of u is 1/time, which is the same as the frequency dimension. This means that $\mathcal{F}(u)$ is the function in the frequency domain. If the function $f(t)$ is the amplitude of vibration as a function of time, then, after taking the Fourier transform, we obtain $\mathcal{F}(u)$, which is the amplitude of vibration as

a function of frequency. When the result of $\mathcal{F}(u)$ is displayed, we can see the amplitudes of vibration at different frequencies. From this result, we may find the source of vibration and devise a way to suppress it. Therefore, the Fourier transform is useful for analyzing vibration. We will present more details of this application later in this section.

Another application of the Fourier transform is that it can be used to solve differential equations as shown in the following example.

Example 8.15

For an undamped vibration system with a single degree of freedom, apply the Fourier transform to determine the response $x(t)$ subjected to the excitation given below:

$$f(t) = \begin{cases} F_0 & \text{for } -T < t < T \\ 0 & \text{for } t < -T \quad \text{and} \quad t > T \end{cases}$$

Solution. Taking the Fourier transform of the equation

$$m\ddot{x} + kx = f(t)$$

or

$$\ddot{x} + \omega_n^2 x = (1/m) f(t)$$

with $\dot{x}(0) = x(0) = 0$, we obtain

$$(iu)^2 \bar{x}(u) + \omega_n^2 \bar{x}(u) = (1/m)\mathcal{F}(u) \tag{8.119}$$

where $\bar{x}(u)$ is the Fourier transform of $x(t)$, and

$$\mathcal{F}(u) = \int_{-T}^{T} F_0 e^{-iut} \mathrm{d}t = \frac{F_0}{iu}(e^{iuT} - e^{-iuT})$$

Hence, Eq. (8.119) becomes

$$\bar{x}(u) = \frac{F_0}{k} \frac{1}{iu\left[1 - (u/\omega_n)^2\right]}(e^{iuT} - e^{-iuT}) \tag{8.120}$$

The response $x(t)$ is the inverse transform of $\bar{x}(u)$, i.e.,

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{x}(u) e^{iut} \mathrm{d}u \\ &= \frac{F_0}{k} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{iuT} - e^{-iuT}}{u\left[1 - (u/\omega_n)^2\right]} e^{iut} \mathrm{d}u \end{aligned}$$

Using the expansion of partial fractions

$$\frac{1}{u\left[1-(u/\omega_n)^2\right]}=\frac{1}{u}-\frac{1}{2(u-\omega_n)}-\frac{1}{2(u+\omega_n)}$$

we have

$$x(t)=\frac{F_0}{k2\pi i}\int_{-\infty}^{\infty}\left[\frac{1}{u}-\frac{1}{2(u-\omega_n)}-\frac{1}{2(u+\omega_n)}\right]$$
$$\cdot\{\exp[iu(t+T)]-\exp[iu(t-T)]\}\,\mathrm{d}u \tag{8.121}$$

To evaluate the preceding integrals, we must perform contour integrations in the complex plane as given in Appendix G. We present the results here; namely,

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i & \text{as } \lambda>0\end{cases}$$

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u-\omega_n}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i e^{i\omega_n\lambda} & \text{as } \lambda>0\end{cases}$$

$$\int_{-\infty}^{\infty}\frac{e^{iu\lambda}}{u+\omega_n}\mathrm{d}u=\begin{cases}0 & \text{as } \lambda<0\\ 2\pi i e^{-i\omega_n\lambda} & \text{as } \lambda>0\end{cases}$$

Note that $\lambda=t+T$ or $t-T$, when the preceding results are used for Eq. (8.121), we find

$$x(t)=0 \qquad \text{as } t<-T \tag{8.122}$$

$$x(t)=\frac{F_0}{k}\frac{1}{2\pi i}\left\{2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t+T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t+T)\}\right]$$
$$=\frac{F_0}{k}[1-\cos\omega_n(t+T)] \qquad \text{as } -T<t<T \tag{8.123}$$

$$x(t)=\frac{F_0}{k}\frac{1}{2\pi i}\left\{\left[2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t+T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t+T)]\right]\right.$$
$$\left.-\left[2\pi i-\frac{1}{2}(2\pi i)\exp[i\omega_n(t-T)]-\frac{1}{2}(2\pi i)\exp[-i\omega_n(t-T)]\right]\right\}$$
$$=\frac{F_0}{k}[\cos\omega_n(t-T)-\cos\omega_n(t+T)] \qquad \text{as } t>T \tag{8.124}$$

In analyzing vibration in the field, because of limited time, samples of data are taken in a finite interval of T_0. The samples are assumed to be a periodic function with period of T_0 and with N points in the interval. Hence, the samples can be represented as

$$f(t)\delta(t-kT) \quad k=0,1,2,\ldots,N-1$$

Note that $\delta(t - kT)$ is assumed to be dimensionless. Because of the characteristics of the function $f(t)\delta(t - kT)$, its Fourier transform cannot be performed in the usual manner. To satisfy the conditions for Fourier transform as stated previously, the transform of this function is performed approximately in discrete manner. Thus the transform is known as the discrete Fourier transform. Consider the Fourier transform

$$\mathcal{F}(n) = \int_{-\infty}^{\infty} \exp\left[-\frac{i2\pi nt}{T_0}\right] f(t)\,\mathrm{d}t$$

in which the frequency u has been replaced by $2\pi n/T_0$. The interval between datum points is T, which is also assumed as the interval of the delta function. Thus $NT = T_0$. Therefore, the discrete Fourier transform of $f(t)\delta(t - kT)$ is performed as

$$\begin{aligned}
\mathcal{F}(n) &= \sum_{k=0}^{N-1} \int_{-\frac{T}{2}}^{T_0+\frac{T}{2}} f(t)\delta(t - kT) \exp\left[-i\frac{2\pi nt}{T_0}\right]\mathrm{d}t \\
&= T\sum_{k=0}^{n-1} f(kT)e^{-i2\pi nkT/T_0} \\
&= T\sum_{k=0}^{n-1} f(kT)e^{-i2\pi nk/N} \qquad n = 0, 1, \ldots, N-1 \qquad (8.125)
\end{aligned}$$

Similarly, the corresponding discrete inverse Fourier transform is manipulated as follows:

$$\begin{aligned}
f(kT) &= \int_{-\infty}^{\infty} \mathcal{F}(f)e^{i2\pi kTf}\mathrm{d}f \\
&= \sum_{n=0}^{N-1} \int_{-\frac{1}{NT}}^{\frac{N+1}{NT}} \mathcal{F}(f)\delta(f - n\Delta f)e^{i2\pi kTf}\mathrm{d}f \\
&= \frac{1}{NT}\sum_{n=0}^{N-1} \mathcal{F}(n\Delta f)e^{i2\pi kTn\Delta f} \\
&= \frac{1}{NT}\sum_{n=0}^{N-1} \mathcal{F}(n)e^{i2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \qquad (8.126)
\end{aligned}$$

Note that in the preceding derivation, the width of the Dirac delta function is Δf, (i.e., $1/NT = \Delta f$, or $T\Delta f = 1/N$). Further, the data of $\mathcal{F}(f)$ available are only N points in the range of f from 0 to $1/T$; outside of the range are assumed to be zero. Also notice that Δf is omitted in the last expression of the transformed function $\mathcal{F}(n)$.

From Eq. (8.125), the vibration amplitude in the frequency domain is determined. Because rotating speeds of moving parts are usually known, their frequencies can be calculated. Once the frequencies of vibration are known, the source

of vibration can be determined. Consequently any undesirable vibration may be eliminated.

Problems

8.1. Determine the Fourier series for the rectangular pulses given here:

$$f(x) = 1 \qquad \text{as } 0 < x < \pi$$
$$f(x) = 0 \qquad \text{as } \pi < x < 2\pi$$
$$f(x) = f(x + 2\pi)$$

8.2. Find the Fourier integral representations for the function:

$$f(x) = x \qquad \text{as } 0 < x < a$$
$$= 0 \qquad \text{as } x > a$$

with the two different conditions: $f(x) = f(-x)$ and $f(-x) = -f(x)$.

8.3. Expand the following function in a Fourier series of period 2π:

$$f(t) = \begin{cases} 0 & \text{as } -\pi < t < 0 \\ t(\pi - t) & \text{as } 0 < t < \pi \end{cases}$$

8.4. Find Fourier transforms of the derivatives of a function: $f'(x)$ and $f''(x)$.

8.5. Determine the Laplace transforms of the two following functions: $f(t) = \sin \omega t$ and $f(t) = a(1 - e^{-t})$, where a and ω are constant.

8.6. A cylinder of mass m and mass moment of inertia J is free to roll without slipping but is restrained by a spring k (Fig. P8.6).

(a) Find the equation of motion and determine the natural frequency of oscillation.

(b) Suppose that the cylinder is pulled away from its equilibrium position horizontally by distance a and is released at rest. Find the response of the system.

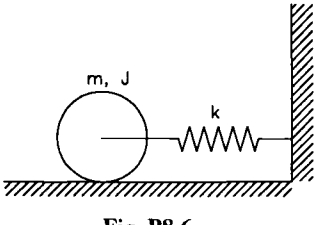

Fig. P8.6

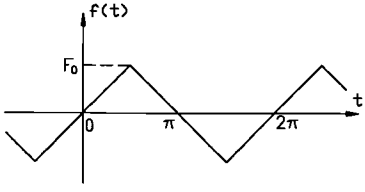

Fig. P8.8

8.7. A damped spring-mass system is started in oscillation under the initial conditions: $x = 0, \dot{x} = v_0$.

(a) Determine the equations of motion when 1) $\zeta = 0.5$, 2) $\zeta = 1.0$, and 3) $\zeta = 2.0$.

(b) Find the responses for the three cases.

8.8. If the periodic force shown in Fig. P8.8 is applied to an undamped spring-mass system, determine the responses of the system subjected to various harmonics.

8.9. The system shown in Fig. P8.9 models a vehicle with an out-of-round tire traveling on a flat ground. For a constant vehicle speed v, find the equation of motion and the steady-state solution, and obtain the speed under the resonance condition.

8.10. The differential equation of motion for a certain undamped spring-mass system is $4\ddot{x} + 14400x = p(t)$, where the forcing function $p(t)$ is defined by Fig. P8.10. The initial conditions are $x = \dot{x} = 0$. Determine analytically the displacement x for the range of time from 0 to 0.2.

8.11. An undamped spring-mass system is excited by a force$F = te^{-t}$. Assume that the initial conditions are $x = \dot{x} = 0$. Determine the response using Laplace transforms.

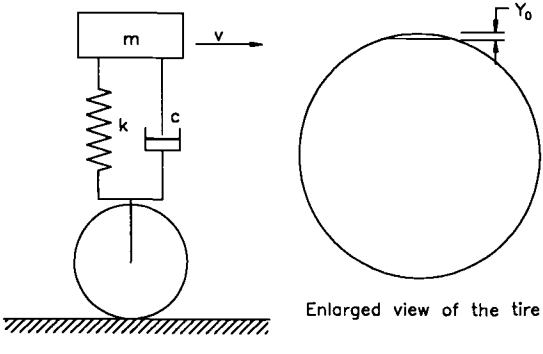

Fig. P8.9

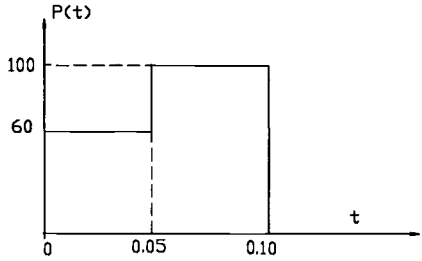

Fig. P8.10

8.12. Determine the response spectrum for an undamped spring-mass system subjected to a rectangular pulse given by

$$f(t) = \begin{cases} F_0 & \text{as } 0 < t < t_0 \\ 0 & \text{as } t > t_0 \end{cases}$$

8.13. The support of a simple pendulum is given a harmonic motion $y_0 = Y_0 \sin \omega t$ along a vertical line, as shown in Fig. P8.13. Find the equation of motion for the system under a small amplitude of oscillation. Determine the steady-state solution.

8.14. Consider the system shown in Fig. P8.14, where the displacements of masses m_1 and m_2 are x_1 and x_2 measured from fixed reference positions, and the amount of the streching of the spring is given by

$$x = x_1 - x_2$$

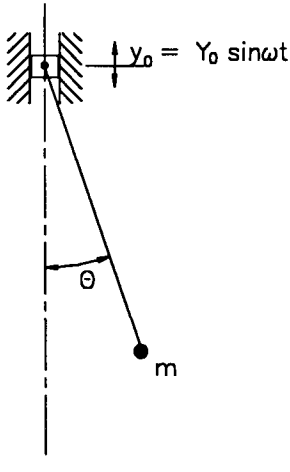

Fig. P8.13

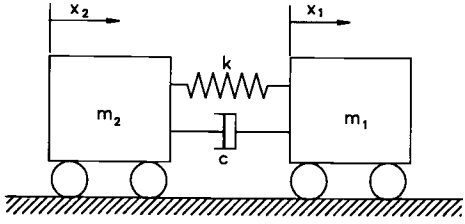

Fig. P8.14

Assuming no friction between the masses and the support and arbitrary initial conditions, find the equation of motion for the system and the response $x(t)$.