7
Dynamics of a Rigid Body

A RIGID body is a body with finite volume, mass, and shape that remains unchanged during the observation. Deformation of the body is not considered in this chapter. When a force and torque are applied to a rigid body, translational and rotational motions of the body will take place and are studied in this chapter. Because most objects can be modeled as a rigid body, the analysis of rigid-body dynamics is very useful and is the major subject of this book. Many general principles for the dynamics of particles studied in the preceding chapters provide necessary background for this chapter. Matrices and rotational operators from Chapter 6 are used extensively and should be reviewed before studying this chapter.

Fundamental principles are given in the first three sections, followed by three sections of specific examples. Section 7.1 introduces the general concept of a solid body in motion and explains how any motion always can be treated as a combination of translational and rotational motions. Section 7.2 derives the equation of motion for a mass in a moving frame of reference, which is, in general, motion relative to an inertial frame of reference. The foundation of the relations is known as Galilean transformation. Section 7.3 describes how to obtain the Euler's angular velocity using two different approaches: one uses matrix operation and the other the rotation operator. Both of them reach the same result. The difference between them is that, while the rotation operator rotates the position vector (as in Chapter 6), matrix operation uses the rotation of coordinates, not the vector. The use of these two approaches demonstrates how divergent methods can achieve the same result and also shows the usefulness of the rotation operator. Because the rotation operator was only recently rediscovered, its many applications have yet to be developed. A simple example for Euler's equations of motion is included in this section.

The second half of this chapter uses the physical concepts presented in the preceding chapters to solve both classical and contemporary problems. In Section 7.4 we deal with gyroscopic motion and use three examples for studying its fundamental principles. The first example demonstrates that a rotating propeller (or other rotating mechanisms such as turbines and compressors) can produce gyroscopic force, which tends to cause an airplane to dive or climb during yawing. The second example studies a single-degree-of-freedom gyro. The last example in this section explains the oscillation of the spinning axis in a gyro-compass caused by the Earth's rotation. The oscillation frequency of the axis about the meridian is determined. Section 7.5 is devoted to studying the motion of a heavy symmetrical top. The nutation and precession of the spinning axis are analyzed in detail. The nutation angle vs precession angle for three possible cases is integrated, and the results are presented. Section 7.6 studies a satellite in a circular orbit using the equation derived in Section 7.2 for a solid body in motion. This is the first example involving a solid body in general motion. The results of this study show that the yawing and rolling motions of a satellite always will generate torques about all three axes. We have only recently entered the space era and still must solve many dynamics problems related to the motion of space vehicles. This section opens that door.

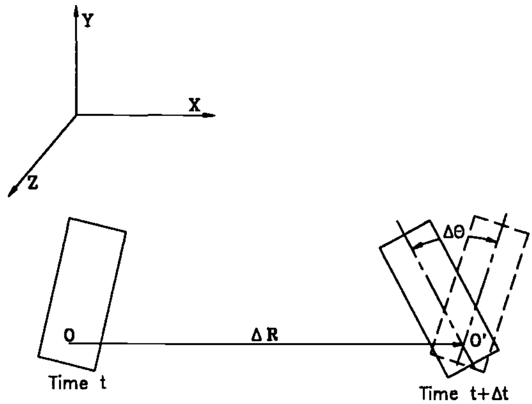

Fig. 7.1 General motion of a rigid body.

The examples included here are simple in comparison with many problems facing engineers today. However, I hope that the discussion and examples of this book will stimulate interest in and provide a firm foundation for further study and research work in this area.

7.1 Displacements of a Rigid Body

In three-dimensional space, six degrees of freedom are needed to specify the position of a solid body. Consider a coordinate system in translational motion with the body. Three degrees of freedom describe the origin of the coordinates and three degrees of freedom describe the rotational displacements of the body with respect to the three axes. As seen in earlier chapters, the origin of the moving coordinates is usually fixed at the center of mass of the body in order to simplify the equations. In certain cases, however, it is more convenient to place the origin of the moving coordinates elsewhere.

All rigid body motion can be reduced to translation combined with rotational motion as shown in Fig. 7.1. This is known as Chasles's theorem. If one point of the body is fixed, then the motion must be rotational only. The rotational displacements, no matter how complicated, always can be expressed by one rotation of the body with respect to an axis through the fixed point. This is known as Euler's theorem. In Section 6.7, we proved that two successive rotations about the axes through zero can be combined to a single rotation about an axis through zero. By a repeated application of that result, any number of successive rotations about the same point can be reduced to one rotation. This is another statement of Euler's theorem.

7.2 Relationship Between Derivatives of a Vector for Different Reference Frames

Vector in Moving Reference Frame Rotating Relative to Fixed or Inertial Reference System

Consider that xyz is a moving reference that rotates relative to a fixed reference denoted by XYZ. A vector $\boldsymbol{G}$ in the moving system can be expressed as

$$\boldsymbol{G} = \sum_i G_i \boldsymbol{e}_i$$

where e_i is a unit vector in the rotating system. G also can be expressed in the fixed system. Thus the time derivative of G, as seen from the fixed system, is obtained as

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\text{fixed}} = \sum_i \dot{G}_i e_i + \sum_i G_i \dot{e}_i$$

However,

$$\sum_i \dot{G}_i e_i = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\text{rotating}} = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{xyz}$$

where $(\mathrm{d}G/\mathrm{d}t)_{xyz}$ means the rate change of G as observed in the rotating system. On the other hand, in the fixed system the velocity of a point fixed in the rotating system is

$$v = \omega \times r \qquad \text{as } r = e_i$$

$$\dot{e}_i = \omega \times e_i$$

Hence

$$\sum_i G_i \dot{e}_i = \omega \times G$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\text{rotating}} + \omega \times G = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{xyz} + \omega \times G \quad (7.1)$$

Any vector G differentiated in the fixed coordinates equals the change of G in the rotating system plus $\omega \times G$ in the rotating system.

Velocities and Accelerations of a Particle in Different References

Suppose that XYZ is a fixed or inertial reference; xyz is a moving reference that is in both translational and rotational motion, as shown in Fig. 7.2. R is the position vector of the origin of xyz system and r and r' are position vectors of point P in XYZ and xyz systems, respectively.

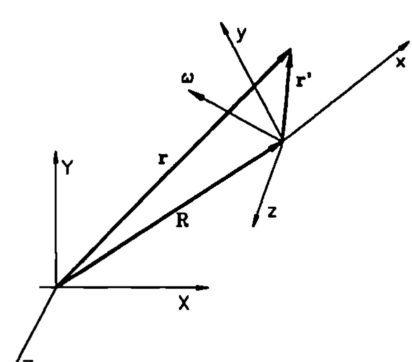

Fig. 7.2 Moving reference system relative to the inertial frame of reference.

Hence

$$r = R + r'$$

Differentiating with respect to time for the XYZ reference, we have

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{XYZ} = V_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}\right)_{XYZ} + \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{XYZ}$$

$$= \dot{\boldsymbol{R}} + \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{xyz} + \omega \times \boldsymbol{r}' = \dot{\boldsymbol{R}} + \boldsymbol{V}_{xyz} + \omega \times \boldsymbol{r}' \tag{7.2}$$

in which Eq. (7.1) has been used in the last step of manipulations. This means that the velocity of point P observed in the fixed reference system equals the vector sum of the velocity of the origin of the moving system, the velocity of point P in the moving system and the velocity of P due to rotation of the xyz system.

Differentiating Eq. (7.2) with respect to time for the XYZ reference system, we get

$$\left(\frac{\mathrm{d}\boldsymbol{V}_{XYZ}}{\mathrm{d}t}\right)_{XYZ} = \ddot{\boldsymbol{R}} + \left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{XYZ} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\omega \times \boldsymbol{r}')\right]_{XYZ}$$

$$\boldsymbol{a}_{XYZ} = \ddot{\boldsymbol{R}} + \left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{XYZ} + \omega \times \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{XYZ} + \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} \times \boldsymbol{r}' \tag{7.3}$$

Because

$$\left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{xyz} + \omega \times \boldsymbol{V}_{xyz}$$

$$\left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{xyz} + \omega \times \boldsymbol{r}'$$

Substituting into Eq. (7.3), we find

$$\boldsymbol{a}_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{xyz} + \ddot{\boldsymbol{R}} + \omega \times \boldsymbol{V}_{xyz} + \omega \times \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{xyz}$$

$$+ \omega \times (\omega \times \boldsymbol{r}') + \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} \times \boldsymbol{r}'$$

Note that

$$\left(\frac{\mathrm{d}\boldsymbol{V}_{xyz}}{\mathrm{d}t}\right)_{xyz} = \boldsymbol{a}_{xyz}, \qquad \left(\frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t}\right)_{xyz} = \boldsymbol{V}_{xyz}, \qquad \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)_{XYZ} = \dot{\omega}$$

Hence, we obtain

$$\boldsymbol{a}_{XYZ} = \boldsymbol{a}_{xyz} + \ddot{\boldsymbol{R}} + 2\omega \times \boldsymbol{V}_{xyz} + \omega \times (\omega \times \boldsymbol{r}') + \dot{\omega} \times \boldsymbol{r}' \tag{7.4}$$

where ω and $\dot{\omega}$ are the angular velocity and acceleration, respectively, of the xyz reference relative to the XYZ reference.

By Newton's law, the force on a particle is

$$F = ma_{XYZ}$$

In the moving reference, we find

$$ma_{xyz} = F - m\ddot{R} - m\dot{\omega} \times r' - m\omega \times (\omega \times r') - 2m\omega \times V_{xyz} \tag{7.5}$$

This expression gives the effective force acting on the mass as observed in the moving frame of reference. The meaning of each term is explained as follows:

1) The term $-m\ddot{R}$ is the inertial force caused by translational acceleration of the moving frame. For example, during the sudden acceleration of a car, passengers sense the force in the opposite direction to the direction of the acceleration.

2) The term $-m\dot{\omega} \times r'$ is the inertia force produced by angular acceleration of the rotating frame. A mass placed on a rotating disk will experience this inertia force in the direction opposite to the tangential acceleration.

3) The term $-m\omega \times (\omega \times r')$ is the centrifugal force term. When a satellite moves in a circular orbit, this force points outward from the center of Earth and is balanced completely by the gravitational force, causing astronauts to experience weightlessness in the orbit.

4) The term $-2m\omega \times V_{xyz}$ is the Coriolis force, which is the major cause of the counterclockwise rotation of hurricanes in the northern hemisphere. The Earth's rotation causes a component of rotational velocity to point outward from the surface of the Earth. To simplify the problem, let us consider only this component of the rotational velocity. The $-\omega \times V_{xyz}$ will cause the air to move in counterclockwise direction if air moves toward a low pressure center as observed from the top of the low pressure center. The rotational momentum of the air is nearly conserved. Hence, the tangential velocity increases as the air moves closer to the eye of the hurricane.

Apply Eqs. (7.2) and (7.4) to the velocities and accelerations of two points a and b of a rigid body. We imagine the xyz reference embedded in the rigid body with the origin at a as shown in Fig. 7.3. Clearly any point b of the body will

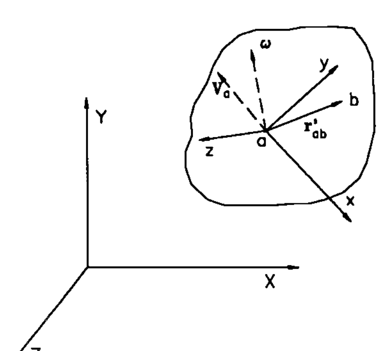

Fig. 7.3 Relative motion between two points in a rigid body in rotation.

not move relative to a and must have $V_{xyz} = 0$ and $a_{xyz} = 0$. Because the origin of xyz corresponds to point a, $\dot{R} = V_a$, $\ddot{R} = a_a$, velocity and acceleration for point b are

$$V_b = V_a + \omega \times r'_{ab} \tag{7.6}$$

$$a_b = a_a + \omega \times (\omega \times r'_{ab}) + \dot{\omega} \times r'_{ab} \tag{7.7}$$

The preceding two equations are often used in the dynamics of machinery.

7.3 Euler's Angular Velocity and Equations of Motion

Euler's angles have been mentioned in Sections 6.2 and 6.7. They are convenient for describing the motion of a rotating top. Before using them, however, we first need to find the angular velocities for the corresponding angles in three orthogonal coordinates. Many different ways are available to express angular velocity. The most elementary approach is to find the components of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ in the primed system directly; however, it is easy to make a mistake in this approach because $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ are not perpendicular. To express these in terms of perpendicular coordinates, the components of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$ must be found in the directions of those coordinates, which is not an easy task. The two methods described next are more systematic. By using the rotation matrix, the angular velocities are obtained through simple matrix operations, or the same result can be reached by using the rotation operator that rotates the vector itself (in this case, the unit vectors). These additional applications of the rotation matrix and rotation operator are described in greater detail below.

Euler's Angular Velocity Obtained Through Matrix Operation

Consider a position vector r' that is fixed in the rotating body of $x'''y'''z'''$ and is constant; the corresponding r in the fixed frame of reference is

$$r = R^{-1}r' \tag{7.8}$$

The time derivative of the equation is

$$\dot{r} = \dot{R}^{-1}r' = \dot{R}^T r' = R^T R r$$

Applying Eq. (7.1) here for $\dot{r}$ leads to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{XYZ} = \omega \times r = [\text{Matrix of}(\omega \times \overset{\leftrightarrow}{1})]r$$

Equating the preceding two equations gives a matrix of

$$(\omega \times \overset{\leftrightarrow}{1}) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \dot{R}^T R \tag{7.9}$$

To find ω in terms of $\dot{\phi}, \dot{\theta}$, and $\dot{\psi}$, we proceed as follows:

$$\begin{aligned}
\dot{\boldsymbol{R}}^T \boldsymbol{R} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1)^T\right)(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{R}_1^T\boldsymbol{R}_2^T\boldsymbol{R}_3^T)\right)(\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1) \\
&= \dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1 + \boldsymbol{R}_1^T\dot{\boldsymbol{R}}_2^T\boldsymbol{R}_2\boldsymbol{R}_1 + \boldsymbol{R}_1^T\boldsymbol{R}_2^T\dot{\boldsymbol{R}}_3^T\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1
\end{aligned}$$

These matrix products can be worked out rather simply, for example,

$$\begin{aligned}
\dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1 &= \dot{\phi}\begin{pmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \dot{\phi}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{R}}_2^T\boldsymbol{R}_2 &= \dot{\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\
&= \dot{\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

The final result of the matrix algebra is

$$\begin{aligned}
&\dot{\boldsymbol{R}}^T\boldsymbol{R} \\
&= \begin{pmatrix} 0 & -(\dot{\phi} + \dot{\psi}\cos\theta) & (\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi) \\ (\dot{\phi} + \dot{\psi}\cos\theta) & 0 & -(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) \\ -(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi) & (\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) & 0 \end{pmatrix}
\end{aligned}$$

Therefore, we find from Eq. (7.9)

$$\omega = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix} \tag{7.10}$$

for the components of angular velocity in the XYZ frame of reference. The velocity component can be expressed in any other primed frame of reference according to the transformation

$$\omega' = \boldsymbol{R}\omega$$

where $\boldsymbol{R}$ can be $\boldsymbol{R}_1$ and $\boldsymbol{R}_2\boldsymbol{R}_1$ or $\boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1$.

Euler's Angular Velocity Obtained Through Rotation Operator

Consider a position vector $\boldsymbol{r}$ with initial position $\boldsymbol{r}(0)$. After some time t, it is rotated to $\boldsymbol{r}(t)$, so that

$$\boldsymbol{r}(t) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{r}(0) \tag{7.11}$$

where

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta \overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1})$$

is the dyadic rotation operator defined in Section 6.7. Taking the time derivative of Eq. (7.11) gives

$$\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \boldsymbol{r}(0) = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \overleftrightarrow{R}^T \cdot \boldsymbol{r}(t) = \omega \times \boldsymbol{r}(t) = (\omega \times \overleftrightarrow{1}) \cdot \boldsymbol{r}(t)$$

which means

$$\omega \times \overleftrightarrow{1} = \frac{\mathrm{d}\overleftrightarrow{R}}{\mathrm{d}t} \cdot \overleftrightarrow{R}^T \tag{7.12}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overleftrightarrow{R}(\boldsymbol{n}, \beta) = \sin\beta[\dot{\beta}\boldsymbol{nn} - \dot{\beta}\overleftrightarrow{1} + (\dot{\boldsymbol{n}} \times \overleftrightarrow{1})] + \cos\beta\dot{\beta}(\boldsymbol{n} \times \overleftrightarrow{1})$$
$$+ (1 - \cos\beta)(\dot{\boldsymbol{n}}\boldsymbol{n} + \boldsymbol{n}\dot{\boldsymbol{n}}) \tag{7.13}$$

and

$$\overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta \overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) \tag{7.14}$$

The product of $(\mathrm{d}/\mathrm{d}t)\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T$ finally reaches the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T = \dot{\beta}\boldsymbol{n} \times \overleftrightarrow{1} + \sin\beta(\dot{\boldsymbol{n}} \times \overleftrightarrow{1}) + (1 - \cos\beta)(\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1} \tag{7.15}$$

In the derivation, the vector $\dot{\boldsymbol{n}}$ is assumed to be perpendicular to $\boldsymbol{n}$. The following identities are used for simplification:

$$\boldsymbol{A} \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{A} \times \boldsymbol{n}, \qquad (\boldsymbol{n} \times \overleftrightarrow{1}) \cdot \boldsymbol{A} = \boldsymbol{n} \times \boldsymbol{A} \tag{7.16a}$$

$$(\boldsymbol{A} \times \overleftrightarrow{1}) \cdot (\boldsymbol{B} \times \overleftrightarrow{1}) = \boldsymbol{BA} - \overleftrightarrow{1}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{7.16b}$$

$$(\dot{\boldsymbol{n}}\boldsymbol{n} - \boldsymbol{n}\dot{\boldsymbol{n}}) = (\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1}. \tag{7.16c}$$

Through the use of Eq. (7.12), we obtain

$$\omega = \dot{\beta}\boldsymbol{n} + \sin\beta\dot{\boldsymbol{n}} + (1 - \cos\beta)\boldsymbol{n} \times \dot{\boldsymbol{n}} \tag{7.17}$$

which is the result of this derivation. This is not quite meaningful, however, because $\dot{\boldsymbol{n}}$ is unknown. During the derivation, $\dot{\boldsymbol{n}}$ is assumed to be perpendicular to $\boldsymbol{n}$, but there are many $\dot{\boldsymbol{n}}$ that can satisfy the assumption. Therefore, ω cannot be obtained directly from Eq. (7.17). However, when the rotation is about a fixed axis, Eq. (7.17) reduces to

$$\omega = \dot{\beta}\boldsymbol{n}$$

as expected. On the other hand, from Eq. (7.17) $\dot{\boldsymbol{n}}$ can be expressed in terms of $\boldsymbol{n}$ and ω. The details of derivation are an assigned exercise in the problem section; $\dot{\boldsymbol{n}}$ is obtained as

$$\dot{\boldsymbol{n}} = -\frac{1}{2}\{(\boldsymbol{n} \times \omega) + \cot\frac{\beta}{2}[\boldsymbol{n} \times (\boldsymbol{n} \times \omega)]\} \tag{7.18}$$

Note that from this equation we can see that $\dot{\boldsymbol{n}}$ is perpendicular to $\boldsymbol{n}$ because $\dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 0$, with the components opposite to $(\boldsymbol{n} \times \omega)$ and $\boldsymbol{n} \times (\boldsymbol{n} \times \omega)$.

Although ω cannot be obtained directly from Eq. (7.17), Eq. (7.12) can still lead us to find ω through the rotation operators. In Section 6.7 we have derived the rotation operator with respect to fixed frame of reference as given in Eq. (6.86) for the rotation through Euler angles:

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)$$

$$= \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi)$$

For the operator through the fixed axes, we have

$$\omega \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R} \cdot \overleftrightarrow{R}^T = \frac{d}{dt}[\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3] \cdot [\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3]^T$$

$$= \frac{d}{dt}[\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_3] \cdot \left[\overleftrightarrow{R}_3^T \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T\right]$$

$$= \frac{d}{dt}\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_1^T + \overleftrightarrow{R}_1 \cdot \frac{d}{dt}\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T$$

$$+ \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \frac{d}{dt}\overleftrightarrow{R}_3 \cdot \overleftrightarrow{R}_3^T \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T$$

$$= \omega_\phi \times \overleftrightarrow{1} + \overleftrightarrow{R}_1 \cdot \omega_\theta \times \overleftrightarrow{1} \cdot \overleftrightarrow{R}_1^T$$

$$+ \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \omega_\psi \times \overleftrightarrow{1} \cdot \overleftrightarrow{R}_2^T \cdot \overleftrightarrow{R}_1^T$$

where

$$\omega_\phi \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_1^T, \qquad \omega_\theta \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_2^T, \qquad \omega_\psi \times \overleftrightarrow{1} = \frac{d}{dt}\overleftrightarrow{R}_3 \cdot \overleftrightarrow{R}_3^T$$

have been used. Making use of the identity Eq. (6.76), which is rewritten as follows,

$$\overleftrightarrow{R} \cdot (\boldsymbol{v} \times \overleftrightarrow{1}) \cdot \overleftrightarrow{R}^T = (\overleftrightarrow{R} \cdot \boldsymbol{v}) \times \overleftrightarrow{1}$$

we find

$$\omega = \omega_\phi + \overleftrightarrow{R}_1 \cdot \omega_\theta + \overleftrightarrow{R}_1 \cdot \overleftrightarrow{R}_2 \cdot \omega_\psi$$

Note that in this equation

$$\omega_\phi = \dot{\phi}\boldsymbol{k}, \qquad \omega_\theta = \dot{\theta}\boldsymbol{i}, \qquad \omega_\psi = \dot{\psi}\boldsymbol{k}$$

so that

$$\begin{aligned}\omega &= \dot{\phi}\boldsymbol{k} + \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \dot{\theta}\boldsymbol{i} + \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \dot{\psi}\boldsymbol{k} \\ &= \dot{\phi}\boldsymbol{k} + \dot{\theta}\boldsymbol{i}' + \dot{\psi}\boldsymbol{k}'' \end{aligned} \tag{7.19}$$

which is certainly true. The ω can be expressed in any frame of reference. Rewriting the operators in Eq. (7.19) in detail, we have, in the fixed frame of XYZ,

$$\begin{aligned}\omega &= \dot{\phi}\boldsymbol{k} + [\boldsymbol{kk} + \cos\phi(\boldsymbol{ii} + \boldsymbol{jj}) + \sin\phi(\boldsymbol{k} \times \overleftrightarrow{1})] \cdot \dot{\theta}\boldsymbol{i} \\ &\quad + \overleftrightarrow{R}_1(\boldsymbol{k}\phi) \cdot [(1 - \cos\theta)\boldsymbol{ii} + \cos\theta\overleftrightarrow{1} + \sin\theta(\boldsymbol{i} \times \overleftrightarrow{1})] \cdot \dot{\psi}\boldsymbol{k} \\ &= \dot{\phi}\boldsymbol{k} + \dot{\theta}(\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}) + \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot [\cos\theta\boldsymbol{k} - \sin\theta\boldsymbol{j}]\dot{\psi} \\ &= \boldsymbol{i}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) + \boldsymbol{j}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi) \\ &\quad + \boldsymbol{k}(\dot{\phi} + \dot{\psi}\cos\theta) \end{aligned} \tag{7.20}$$

This result agrees well with Eq. (7.10), which was derived through matrix operations. In the rotation of axes for the Euler angles, because $\boldsymbol{i}' = \boldsymbol{i}''$, $\boldsymbol{k} = \boldsymbol{k}'$, the expression in Eq. (7.19) can be converted easily into the double-primed frame. Through the use of $\boldsymbol{k}' = \overleftrightarrow{R}_2^T(\boldsymbol{i}'', \theta) \cdot \boldsymbol{k}''$, we obtain

$$\omega = \boldsymbol{i}''\dot{\theta} + \boldsymbol{j}''(\dot{\phi}\sin\theta) + \boldsymbol{k}''(\dot{\psi} + \dot{\phi}\cos\theta) \tag{7.21}$$

This can be converted into the triple-primed frame with the use of

$$\boldsymbol{i}'' = \overleftrightarrow{R}_3^T(\boldsymbol{k}''', \psi) \cdot \boldsymbol{i}''', \qquad \boldsymbol{j}'' = \overleftrightarrow{R}_3^T(\boldsymbol{k}''', \psi) \cdot \boldsymbol{j}''', \qquad \boldsymbol{k}'' = \boldsymbol{k}'''$$

In the triple-primed frame of reference, we find

$$\begin{aligned}\omega &= \boldsymbol{i}'''(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi) + \boldsymbol{j}'''(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi) \\ &\quad + \boldsymbol{k}'''(\dot{\psi} + \dot{\phi}\cos\theta) \end{aligned} \tag{7.22}$$

Euler Equations of Motion

In an inertial frame of reference, such as the set of axes XYZ, for $\boldsymbol{G} = \boldsymbol{L} =$ angular momentum of a body, Eq. (7.1) gives

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{xyz} + \omega \times \boldsymbol{L} = \boldsymbol{N}$$

where the xyz frame is in rotational motion with velocity ω relative to the XYZ frame, and $\boldsymbol{N}$ is the torque applied to the body. In general, as given in Eq. (6.37) and Eq. (6.39)

$$L = \overleftrightarrow{I}_m \cdot \omega$$

We can simplify this expression by rotating the axes of xyz to coincide with the principal axes of the body and label them $x'y'z'$. Then we have

$$L_{x'} = I_1\omega_{x'}, \qquad L_{y'} = I_2\omega_{y'}, \qquad L_{z'} = I_3\omega_{z'}$$

The full set of Euler equations are reduced to

$$N_{x'} = I_1\dot{\omega}_{x'} - \omega_{y'}\omega_{z'}(I_2 - I_3) \tag{7.23a}$$

$$N_{y'} = I_2\dot{\omega}_{y'} - \omega_{z'}\omega_{x'}(I_3 - I_1) \tag{7.23b}$$

$$N_{z'} = I_3\dot{\omega}_{z'} - \omega_{x'}\omega_{y'}(I_1 - I_2) \tag{7.23c}$$

Let us first consider Euler's angular velocity for the preceding equations. Note that the triple-primed axes are actually the axes fixed in the rotating body, so that ω given in Eq. (7.22) is to be used for this set of equations:

$$\omega_{x'} = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$

$$\omega_{y'} = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\omega_{z'} = \dot{\psi} + \dot{\phi}\cos\theta$$

Differentiating with respect to time and substituting into Eqs. (7.23a–7.23c) gives

$$\begin{aligned} N_{x'} &= I_1[\ddot{\theta}\cos\psi + \ddot{\phi}\sin\theta\sin\psi] + (I_1 - I_2)[-\dot{\theta}\dot{\psi}\sin\psi + \dot{\phi}\dot{\psi}\sin\theta\cos\psi] \\ &\quad + (I_1 + I_2)\dot{\phi}\dot{\theta}\cos\theta\sin\psi + I_3[-\dot{\theta}\dot{\psi}\sin\psi + \dot{\phi}\dot{\psi}\sin\theta\cos\psi \\ &\quad - \dot{\theta}\dot{\phi}\cos\theta\sin\psi + \dot{\phi}^2\sin\theta\cos\theta\cos\psi] \end{aligned} \tag{7.24a}$$

$$\begin{aligned} N_{y'} &= I_2(-\ddot{\theta}\sin\psi + \ddot{\phi}\sin\theta\cos\psi) + (I_1 - I_2)[\dot{\theta}\dot{\psi}\cos\psi + \dot{\phi}\dot{\psi}\sin\theta\sin\psi] \\ &\quad + (I_1 + I_2)\dot{\theta}\dot{\phi}\cos\theta\cos\psi - I_3[\dot{\theta}\dot{\psi}\cos\psi + \dot{\theta}\dot{\phi}\cos\theta\cos\psi \\ &\quad + \dot{\phi}\dot{\psi}\sin\theta\sin\psi + \dot{\phi}^2\cos\theta\sin\theta\sin\psi] \end{aligned} \tag{7.24b}$$

$$\begin{aligned} N_{z'} &= I_3[\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta] - (I_2 - I_1)[-\dot{\theta}^2\cos\psi\sin\psi \\ &\quad + \dot{\theta}\dot{\phi}\sin\theta\cos 2\psi + \dot{\phi}^2\sin^2\theta\cos\psi\sin\psi] \end{aligned} \tag{7.24c}$$

The application of the preceding equations is demonstrated in the following example.

Example 7.1

A toy gyroscopic top is shown in Fig. 7.4. The gravitational force on the disk is W. If the disk is given a high angular velocity ω_s about its shaft oz' and one end of the shaft is placed on a pedestal, it is observed that the shaft and disk will not fall but will precess around the axis oZ because of torque $W\ell$ acting on the system. Find the angular velocity for precession.

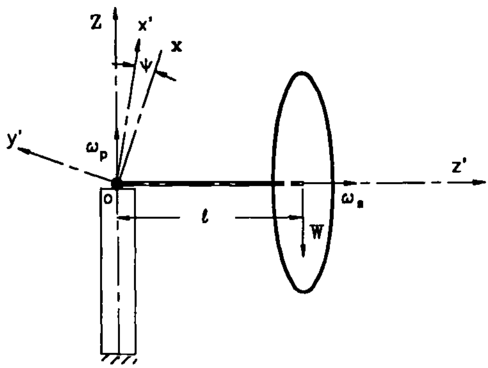

Fig. 7.4 Toy gyroscopic top.

Solution. Applying Eqs. (7.24a–7.24c) to this problem, we make some necessary assumptions and have

$$I_1 = I_2, \qquad I_3 = I$$

$$\theta = 90\,\text{deg}, \qquad \dot{\theta} = 0, \qquad \ddot{\theta} = \ddot{\phi} = \ddot{\psi} = 0, \qquad \dot{\psi} = \omega_s, \qquad \dot{\phi} = \omega_p$$

$$N_{x'} = W\ell\cos\psi, \qquad N_{y'} = -W\ell\sin\psi, \qquad N_{z'} = 0$$

Note that the torque produced by the weight is in the direction of x, and axes x, x', and y' are in the same plane. The x', y', and z' axes are embedded in the rotating top. Either from Eq. (7.24a) or (7.24b), we find

$$W\ell = I\dot{\phi}\dot{\psi} = I\omega_s\omega_p, \qquad \omega_p = W\ell/I\omega_s$$

Therefore, the angular velocity for precession is directly proportional to the torque produced by its own weight and inversely proportional to the angular momentum along the spinning axis.

7.4 Gyroscopic Motion

To study the motion of a gyroscope, it is convenient to consider the rotating body and the rotating coordinate system separately. Let the coordinate axes lie along the principal axes of the body but allow the body to spin in the rotating coordinate system with a rotating velocity of $\dot{\psi}$ along the z'' axis, as shown in Fig. 7.5. Hence,

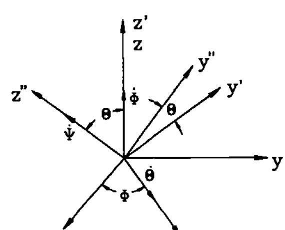

Fig. 7.5 Euler's angular velocities.

the angular velocity of the rotating frame of reference is

$$\boldsymbol{\Omega} = \dot{\theta}\boldsymbol{i}'' + \dot{\phi}\sin\theta\boldsymbol{j}'' + \dot{\phi}\cos\theta\boldsymbol{k}'' \tag{7.25}$$

and the angular velocity of the body is

$$\omega = \dot{\theta}\boldsymbol{i}'' + \dot{\phi}\sin\theta\boldsymbol{j}'' + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{k}'' \tag{7.26}$$

in which $\dot{\psi}$ is the angular velocity of the spin, $\dot{\phi}$ is the angular velocity of precession, and $\dot{\theta}$ is the angular velocity of nutation.

It is also assumed that there is always one point in the system that is fixed either in a fixed system or in an inertial frame of reference. This point may be the center of mass or one of the supports. Applying the Euler equations, we find

$$\left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{XYZ} = \left(\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\right)_{xyz} + \boldsymbol{\Omega}\times\boldsymbol{L} = \boldsymbol{N} \tag{7.27}$$

where $L_1 = I'\omega_1, L_2 = I'\omega_2, L_3 = I\omega_3$; I' is the mass moment of inertia with respect to the x or y axis; and I is the mass moment of inertia with respect to the z axis.

Substituting the expressions for Ω and ω in Eqs. (7.25) and (7.26) into Eq. (7.27) leads to

$$N_1 = I'\ddot{\theta} + (I - I')(\dot{\phi}^2\sin\theta\cos\theta) + I\dot{\phi}\dot{\psi}\sin\theta \tag{7.28a}$$

$$N_2 = I'\ddot{\phi}\sin\theta + 2I'\dot{\theta}\dot{\phi}\cos\theta - I(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\theta} \tag{7.28b}$$

$$N_3 = I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) \tag{7.28c}$$

Example 7.2

In Fig. 7.6, the propeller shaft of an airplane is shown. The propeller rotates at 2000 rpm clockwise (cw) when viewed from the rear and is driven by the engine through reduction gears. Suppose the airplane flies horizontally and makes a turn to the right at 0.2 rad/s as viewed from above. The propeller has a mass of 30 kg and moment of inertia of 25 kg-m^2. Find the gyroscopic forces that the propeller shaft exerts against bearings A and B, which are 150 mm apart.

Solution. To simplify the problem, it is assumed that, before the airplane begins making a turn, the whole system is in an inertial frame of reference for Eqs. (7.28a–7.28c) to apply. The angular momentum of the propeller is in the direction of z'' and is

$$\dot{\psi} = \omega_3 = \frac{2000(2\pi)}{60} = 209 \text{ rad/s}$$

$$L_3 = I\dot{\psi} = 25 \times 209 = 5225 \text{ kg-m}^2/\text{s}$$

$$\dot{\phi} = -0.2 \text{ rad/s}$$

$$\theta = 90 \text{ deg}, \qquad \dot{\theta} = \ddot{\theta} = \ddot{\phi} = \ddot{\psi} = 0$$

$$N_1 = I\dot{\psi}\dot{\phi} = -25(209)(0.2) = -1045 \text{ N-m}$$

$$F = \frac{|N_1|}{\ell} = \frac{+1045}{0.150} = 6967 \text{ N}$$

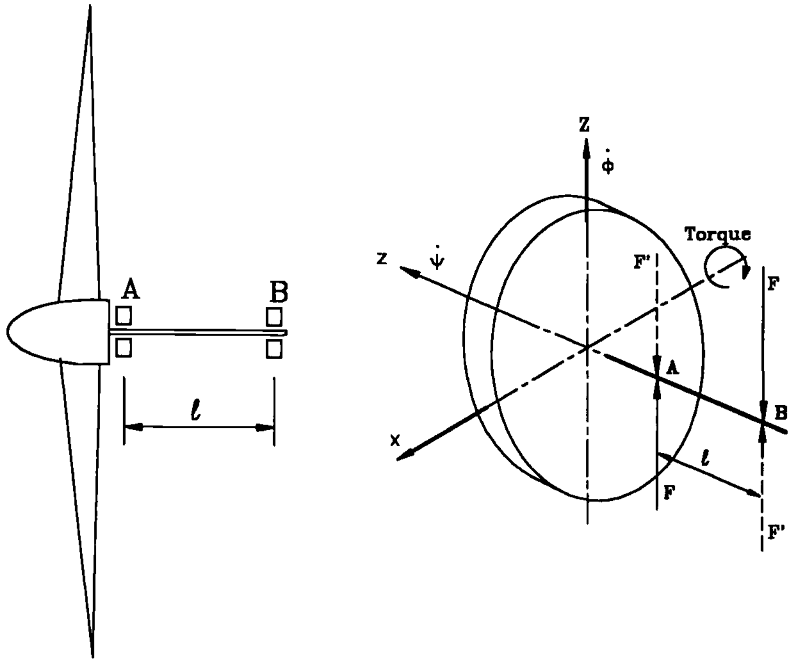

Fig. 7.6 Gyroscopic effect on propeller shaft.

Note that the force acting on the bearing F' is in the opposite direction of F. From the couple formed by F', we can see that the moment produced by F' causes the airplane to dive. On the other hand, if the airplane is turning to the left, then the moment from the bearings pitches the airplane upward.

Example 7.3

Shown in Fig. 7.7 is a single-degree-of-freedom gyro. The spin axis of disc E is held by a gimbal A that can rotate about bearings C and D. These bearings are supported by the gyro case which, in turn, is clamped to the vehicle to be guided. If the gyro case rotates about a vertical axis while the rotor is spinning about the horizontal axis, then the gimbal A will tend to rotate about CD in an attempt to align with the vertical. When gimbal A is restrained by a set of springs S with a combined torsional spring constant given as k_t, then the gyro is called a rate gyro. The neutral position of the springs is set at $\theta = \pi/2$. If the rotation of the gyro case is constant and the gimbal A assumes a fixed orientation relative to the vertical as a result of the restraining springs, we have a case of regular precession. The rotation of the gyro case gives the precession speed $\dot{\phi}$ about the precession axis, which is clearly the vertical axis. The nutation angle θ is then the orientation of gimbal A (i.e., the z axis) with respect to the Z axis.

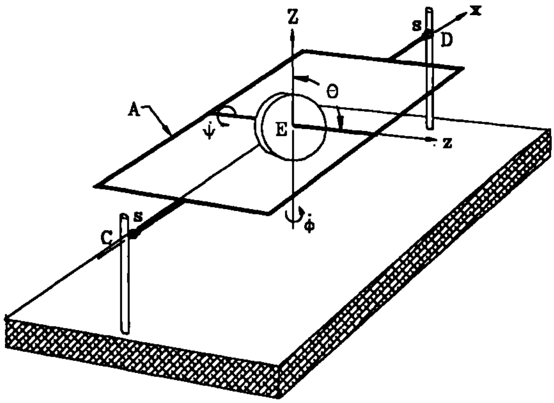

Fig. 7.7 Single-degree-of-freedom gyro.

Given the following data, what is θ for the condition of steady precession?

$I = 4 \times 10^{-4}$ kg-m^2
$I' = 2 \times 10^{-4}$ kg-m^2
$\dot{\psi} = 20{,}000$ rad/s
$k_t = 1.0$ N-m/rad
$\dot{\phi} = 0.1$ rad/s

Solution. Taking the x axis along CD, the z axis along the spinning axis of the rotor, and the Z axis for the precesion axis, we have, from Eq. (7.28a),

$$N_1 = k_t(\pi/2 - \theta) = (I - I')(\dot{\phi}^2 \sin\theta\cos\theta) + I\dot{\phi}\dot{\psi}\sin\theta$$

$$(\pi/2 - \theta) = 2 \times 10^{-4}[(0.1)^2 \sin\theta\cos\theta] + 4 \times 10^{-4}(0.1)(20{,}000)\sin\theta = (2 \times 10^{-6}\cos\theta + 0.8)\sin\theta$$

Neglecting $2 \times 10^{-6}\cos\theta$, which is much smaller than 0.8, the equation becomes

$$\pi/2 - \theta = 0.8\sin\theta$$

$$\theta = 53 \text{ deg}$$

In practice, the torque N_1 is measured. Because N_1 and the rotating velocity of the vehicle $\dot{\phi}$ are directly related, the required value of $\dot{\phi}$ can be calculated from the measured value of N_1.

Example 7.4

We shall now explain the effect of the Earth's rotation on the operation of the gyro-compass. The gyro-compass is a two-degree-of-freedom gyroscope as shown in Fig. 7.8a with torsional springs restricting the x axis. This device gives the direction to the geometric north pole (not the magnetic north pole) if it is set to that direction at the beginning of observation. In this example we will see that the Earth's rotation causes some oscillation of the spinning axis about the meridian.

For simplicity, we consider a gyro-compass at a fixed position on the Earth's surface. The body axis z of the gyro-compass can rotate in plane T tangent to the Earth's surface as shown in Fig. 7.8b, where the z axis is at an angle α with the tangent to the meridian line. Because the angle α may vary with time, there is a possible angular velocity $\dot{\alpha}$ normal to the plane T. The y axis is a radial line from the center of Earth at o, and, therefore, is always collinear with $\dot{\alpha}$. The x axis then is chosen to form a right-hand triad and is in plane T. An inertial reference XYZ is chosen at the center of the Earth so that the Z axis is along the north–south axis. The gyroscope rotates with spin velocity $\dot{\psi}$ along z and swinging velocity $\dot{\alpha}$ along y and precession velocity $\dot{\phi}$ along Z, where $\dot{\phi}$ is the angular velocity of the Earth, a constant vector of small magnitude. For convenience, another Z axis has been set up at the gyroscope. The angle between the Z axis and the tangent to the meridian designated as λ is the latitude of the position of the gyro-compass. Note that the nutation velocity $\dot{\theta}$ is not used here. It is a function of α, λ, and $\dot{\alpha}$.

Because the axes xyz are not fixed to the body, we must use Eq. (7.27) for the equation of motion. We have

$$L_1 = I'\omega_1, \qquad L_2 = I'\omega_2, \qquad L_3 = I\omega_3$$

and

$$\omega_1 = -\dot{\phi}\cos\lambda\sin\alpha$$

$$\omega_2 = \dot{\alpha} + \dot{\phi}\sin\lambda$$

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha$$

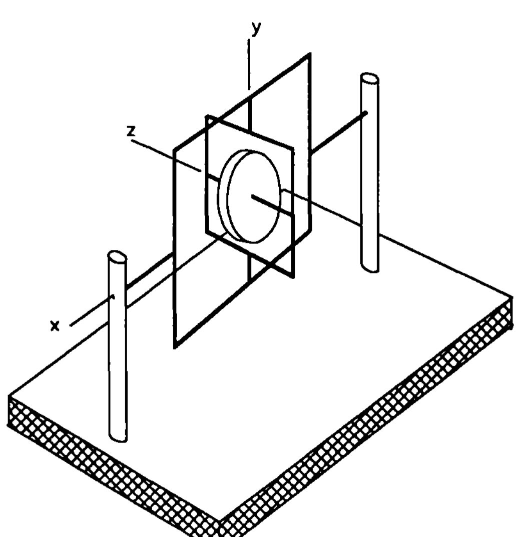

Fig. 7.8a Two-degree-of-freedom gyro (rotation about x axis restricted).

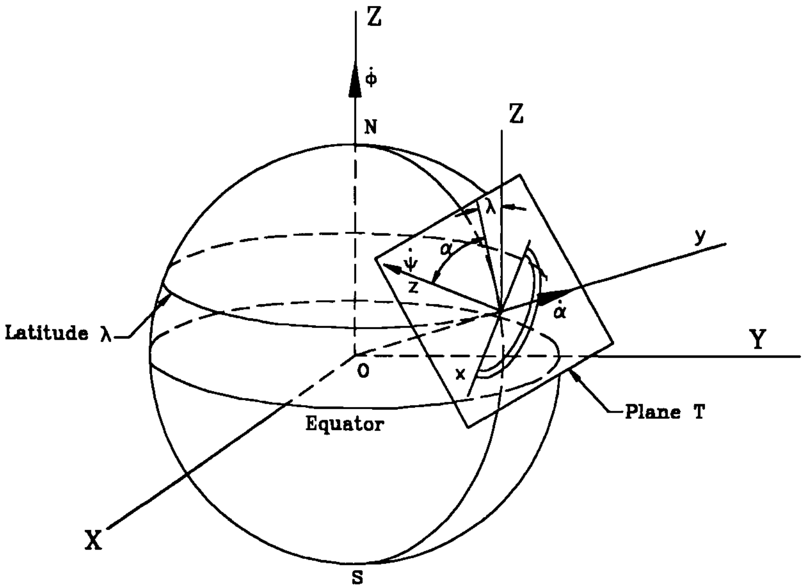

Fig. 7.8b Gyro-compass in oscillation.

The angular velocity components of xyz are

$$\Omega_1 = -\dot{\phi}\cos\lambda\sin\alpha$$

$$\Omega_2 = \dot{\alpha} + \dot{\phi}\sin\lambda$$

$$\Omega_3 = \dot{\phi}\cos\lambda\cos\alpha$$

Now we can write Eq. (7.27) as

$$\begin{aligned} N_1\boldsymbol{i} + N_2\boldsymbol{j} + N_3\boldsymbol{k} = {} & I'\frac{\mathrm{d}}{\mathrm{d}t}(-\dot{\phi}\cos\alpha\sin\alpha)\boldsymbol{i} \\ & + I'\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{j} + I\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)\boldsymbol{k} \\ & + I'(\dot{\alpha} + \dot{\phi}\sin\lambda)[(-\dot{\phi}\cos\lambda\sin\alpha)\boldsymbol{k} - \dot{\phi}\cos\lambda\cos\alpha\boldsymbol{i}] \\ & + I'(-\dot{\phi}\cos\lambda\sin\alpha)[\dot{\phi}\cos\lambda\cos\alpha\boldsymbol{j} - (\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{k}] \\ & + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)[(\dot{\alpha} + \dot{\phi}\sin\lambda)\boldsymbol{i} + (\dot{\phi}\cos\lambda\sin\alpha)\boldsymbol{j}] \end{aligned}$$

Dividing the preceding equation into three components, we find

$$\begin{aligned} N_1 = {} & I'[(-\dot{\phi}\dot{\alpha}\cos\lambda\cos\alpha) + (\dot{\alpha} + \dot{\phi}\sin\lambda)(-\dot{\phi}\cos\lambda\cos\alpha)] \\ & + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)(\dot{\alpha} + \dot{\phi}\sin\lambda) \end{aligned} \tag{7.29a}$$

$$N_2 = I'(\ddot{\alpha} - \dot{\phi}^2\cos^2\lambda\sin\alpha\cos\alpha) + I(\dot{\psi} + \dot{\phi}\cos\lambda\cos\alpha)\dot{\phi}\cos\lambda\sin\alpha \tag{7.29b}$$

$$N_3 = I(\ddot{\psi} - \dot{\phi}\dot{\alpha}\cos\lambda\sin\alpha) \tag{7.29c}$$

Now let us consider the external torques acting on the gyro-compass system. Because the spin axis is kept in the plane T, a proper amount of N_1 must be applied along the x axis. There are no torques along y and z axes, i.e., $N_2 = N_3 = 0$, and since $\dot{\phi}$ is expected to be much smaller than $\dot{\psi}$, and $\dot{\phi}^2 \ll \ddot{\alpha}$, Eqs. (7.29b) and (7.29c) are reduced to

$$I'\ddot{\alpha} + I\dot{\psi}\dot{\phi}\cos\lambda\sin\alpha = 0 \tag{7.30a}$$

$$I(\ddot{\psi} - \dot{\phi}\dot{\alpha}\cos\lambda\sin\alpha) = 0 \tag{7.30b}$$

Note that $\dot{\psi}$ is the spin velocity of the rotor, $\dot{\psi} \gg \dot{\phi}$, and $\dot{\psi} \gg \dot{\alpha}$. Equation (7.30b) may be approximated as $\ddot{\psi} = 0$, i.e., as $\dot{\psi}$ is a constant. Then Eq. (7.30a) can be written in the form of

$$\ddot{\alpha} + c\alpha = 0 \tag{7.31}$$

where

$$c = \frac{I\dot{\psi}\dot{\phi}\cos\lambda}{I'}$$

We also assume that α is much less than one. Equation (7.31) means that the Earth's rotation can cause the spin axis to oscillate about the meridian. The frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{I\dot{\psi}\dot{\phi}\cos\lambda}{I'}} \tag{7.32}$$

Plugging realistic values into this equation, let $\dot{\psi} = 20{,}000$ rad/s, $\dot{\phi} = 7.2722 \times 10^{-5}$ rad/s, $\lambda = 20$ deg, and $I = 2I'$, then we find

$$f = 0.263 \text{ cycle/s}$$

or the period of oscillation is 3.8 s.

7.5 Motion of a Heavy Symmetrical Top

The motion of a rotating top is well known and is a good example to learn how powerful mathematical techniques are used to extract a great deal of physical information using minimal calculation. Nutation and precession will be studied in detail.

We choose the symmetry axis of the top as the z axis and fix the supporting point of the top at the origin of coordinates. The center of mass is located on the z axis at distance ℓ from the origin as shown in Fig. 7.9. The Euler angles were originally designed for the treatment of a rotating top, and they will prove to be very convenient. To find the equations of motion, Lagrangian techniques are applied because they are simpler than the Euler equations. With this in mind, we write the Lagrangian function as

$$L = T - V = \tfrac{1}{2}I'(\omega_x^2 + \omega_y^2) + \tfrac{1}{2}I\omega_z^2 - Mg\ell\cos\theta \tag{7.33}$$

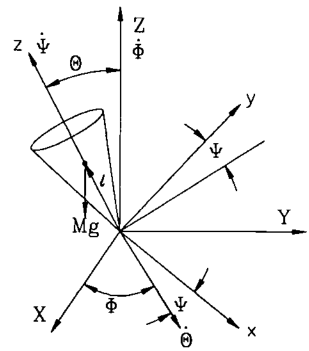

Fig. 7.9 Coordinates for the heavy symmetrical top.

in which $I_1 = I_2 = I'$ because of symmetry and $I_3 = I$ have been used. Clearly, the angular velocity of the top expressed in the body axes is most convenient. From Eq. (7.22) we have

$$\omega = (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\boldsymbol{i} + (-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\boldsymbol{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{k} \tag{7.34}$$

With the use of Eq. (7.34), the Lagrangian function, Eq. (7.33) becomes

$$L = \tfrac{1}{2}I'(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mg\ell\cos\theta \tag{7.35}$$

As discussed in Chapter 4, ψ and ϕ are ignorable coordinates because they do not appear in the Lagrangian function. Consequently, the two angular momenta P_ψ and P_ϕ are constant, i.e.,

$$P_\psi \equiv \frac{\partial L}{\partial\dot{\psi}} \equiv I(\dot{\psi} + \dot{\phi}\cos\theta) = I\omega_z = \text{const} \tag{7.36}$$

$$P_\phi \equiv \frac{\partial L}{\partial\dot{\phi}} = I'(\sin^2\theta)\dot{\phi} + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

$$= (I'\sin^2\theta + I\cos^2\theta)\dot{\phi} + I\dot{\psi}\cos\theta = \text{const} \tag{7.37}$$

Furthermore, because no frictional dissipation is assumed in this analysis, the total energy $E = T + V$ is constant. In view of Eq. (7.36), subtraction of E by $\frac{1}{2}I\omega_z^2$ is still constant.

$$E' = E - \tfrac{1}{2}I\omega_z^2 = \tfrac{1}{2}I'(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + Mg\ell\cos\theta = \text{const}$$

From Eq. (7.36) we have

$$\dot{\psi} = \omega_z - \dot{\phi}\cos\theta \tag{7.38}$$

Substitution of $\dot{\psi}$ into Eq. (7.37) gives

$$(I'\sin^2\theta + I\cos^2\theta)\dot{\phi} + I\cos\theta(\omega_z - \dot{\phi}\cos\theta)$$

$$= I'(\sin^2\theta)\dot{\phi} + I\omega_z\cos\theta = \text{const} = I'B$$

Rearranging, we find

$$\dot{\phi} = \frac{B - A\cos\theta}{\sin^2\theta} \tag{7.39}$$

where $A = (I/I')\omega_z$.

Then, from Eq. (7.38), we have

$$\dot{\psi} = \omega_z - \frac{\cos\theta}{\sin^2\theta}(B - A\cos\theta) \tag{7.40}$$

Substituting Eq. (7.39) into the expression for energy E' gives

$$E' = \frac{1}{2}I'\left[\dot{\theta}^2 + \frac{(B - A\cos\theta)^2}{\sin^2\theta}\right] + Mg\ell\cos\theta = \text{const}$$

or

$$(\sin^2\theta)\dot{\theta}^2 = (C - D\cos\theta)\sin^2\theta - (B - A\cos\theta)^2 \tag{7.41}$$

where

$$C = \frac{2E'}{I'}, \qquad D = 2\frac{Mg\ell}{I'}$$

Equation (7.41) is a first-order differential equation. The nutational motion of the rotating shaft can be predicted from this equation. Having found the function $\theta(t)$, the precession of the top can be obtained through Eq. (7.39), and the variation of spinning velocity can be found from Eq. (7.40). Equation. (7.41) is a nonlinear equation, however, which cannot be integrated analytically. Much information may be obtained without integration of these equations. Let us change the variable in the equation with

$$\mu = \cos\theta$$

Then we have

$$\dot{\mu}^2 = (C - D\mu)(1 - \mu^2) - (B - A\mu)^2 = f(\mu)$$

The result $\theta(t)$ of the preceding equation depends highly on the behavior of the function $f(\mu)$. By introducing proper numerical values for A, B, C, and D, the variations of $f(\mu)$ are obtained as shown in Fig. 7.10. It easily is seen that there are three roots. From the plot, two roots are between 0 and 1 and are reasonable roots because $0 \leq \cos\theta \leq 1$; the third root is impossible. Note that at those two roots $\dot{\mu} = 0$, i.e., $\dot{\theta} = 0$, so that θ reaches minimum or maximum at these roots; also note that $f(\mu)$ is positive between these two roots so that $\dot{\mu} = \pm\sqrt{f(\mu)}$ or $\mathrm{d}\mu$ can be positive and negative. With this understanding, Eqs. (7.41) and (7.40) are numerically integrated. Three different possible cases are given in Figs. 7.11. For

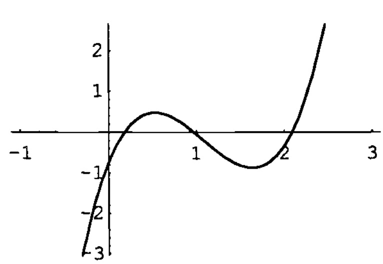

Fig. 7.10 Plot of $f(\mu)$ with A = 2, B = 1.8, C = 2.5, and D = 2.

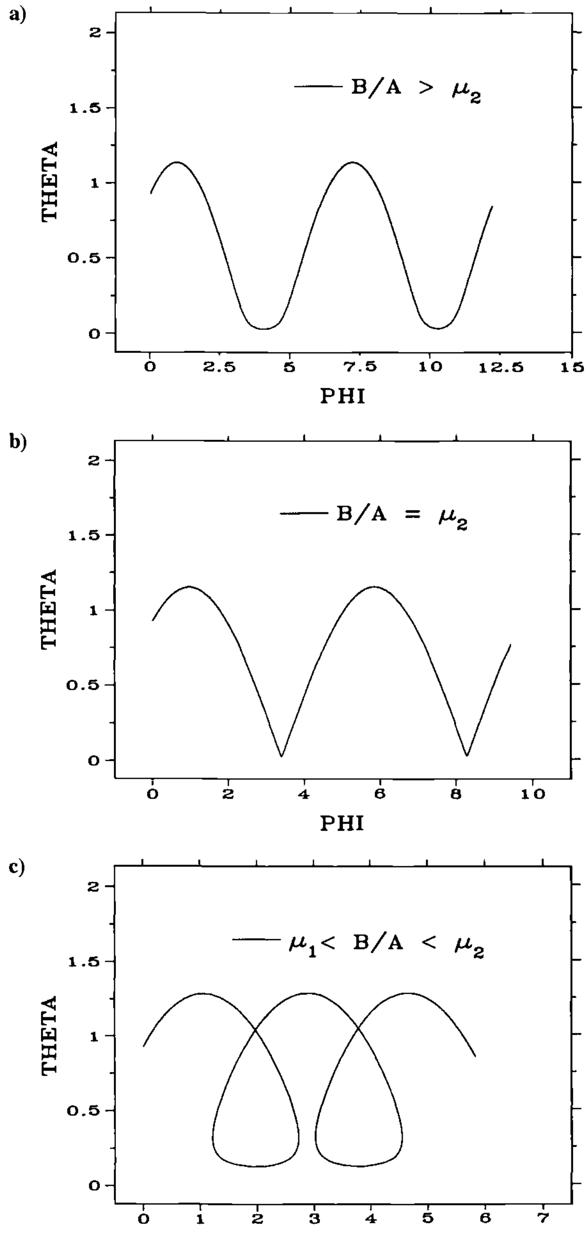

Fig. 7.11 Three different possible nutations.

Fig. 7.11a, where the value of B/A is greater than μ_2, the motion is called regular precession with nutation because precession occurs at nearly constant speed. For Fig. 7.11b, where the value of $B/A = \mu_2, \dot{\theta} = 0$, and $\dot{\phi} = 0$ as $\mu = \mu_2$, so that cusps are shown at θ_{min}. For Fig. 7.11c, the value of B/A is between the first two roots, so that $\dot{\phi}$ can be positive and negative. Consequently, loops are shown in this case. In the numerical integration, because the two integral limits are the roots in the denominator of the integrand, Simpson's one-third rule with $d\mu = 0.0001$ is employed for integration and with a further reduced interval near the integral limits.

7.6 Torque on a Satellite in Circular Orbit

During the last four decades, we have launched many objects into space and have encountered many engineering problems specific to motion in orbit. As the motion of airplanes was well studied in the beginning of the 20th century, the motion of the space station moving in orbit now requires diligent study so that some induced motions during flight operations can be anticipated and delicate space vehicles are designed to endure the additional stresses they may encounter. Certainly there are many possible ways to analyze the problem. The following approach was first given by E. Neal Moore.*

Consider a satellite moving in a circular orbit around Earth. The coordinate system xyz is so chosen that the z axis is from the center of Earth pointing outward through the center of mass of the satellite. A plane contains the z axis and the orbit curve is called the orbit plane. The y axis is in the orbit plane. The angular velocity ω of the satellite relative to the Earth is perpendicular to that plane. The x axis is antiparallel to ω. The origin of the x, y, z coordinates is chosen at the center of mass of the satellite. The body of the satellite is not fixed in the xyz system so that it can pitch, roll, and yaw relative to the axes of xyz system. With the coordinate system chosen, now let us consider that a small element dm as shown in Fig. 7.12 and consider that the frame of reference in the Earth is the inertial frame of reference.

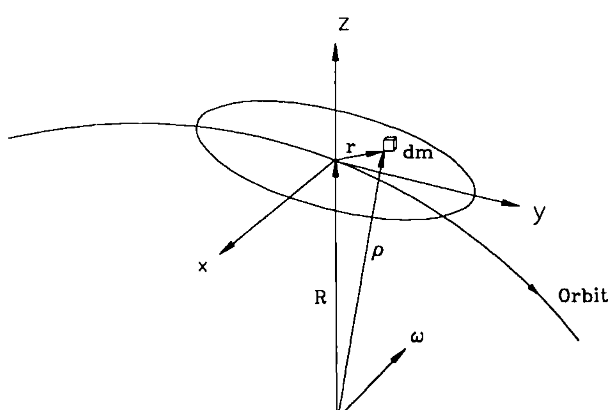

Fig. 7.12 Satellite in a circular orbit.

*Moore, E. N., *Theoretical Mechanics*, Wiley, New York, 1983, Chap. 6.

Applying Eq. (7.5) with the gravitational force applied to the satellite as the only external force, we have

$$\mathrm{d}\boldsymbol{F} = dm\boldsymbol{a} = -GMdm\frac{\boldsymbol{\rho}}{\rho^3} - dm\ddot{\boldsymbol{R}} - dm\dot{\boldsymbol{\omega}} \times \boldsymbol{r} - dm\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2dm(\boldsymbol{\omega} \times \boldsymbol{v}) \tag{7.42}$$

where $\boldsymbol{v}$ is the velocity of dm as observed in the xyz system. As the body is rotating relative to the moving coordinate system with angular velocity $\boldsymbol{\omega}'$, then

$$\boldsymbol{v} = \boldsymbol{\omega}' \times \boldsymbol{r}$$

where $\boldsymbol{r}$ is the position vector of dm. The torque acting on the body of the satellite about the center of mass because of its own motion is obtained by integration of torque over the whole body

$$\boldsymbol{N} = \int_{\text{boby}} \boldsymbol{r} \times \mathrm{d}\boldsymbol{F} \tag{7.43}$$

Making use of the fact that $\boldsymbol{R} = R\boldsymbol{K}$

$$\dot{\boldsymbol{R}} = R\dot{\boldsymbol{K}} = R\boldsymbol{\omega} \times \boldsymbol{K} = \boldsymbol{\omega} \times \boldsymbol{R}$$

$$\ddot{\boldsymbol{R}} = \boldsymbol{\omega} \times \dot{\boldsymbol{R}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{R}), \qquad \boldsymbol{\omega} = \text{const}$$

In addition, $\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r}$. Substituting these expressions into Eq. (7.42) leads to

$$\mathrm{d}\boldsymbol{F} = -GMdm\frac{\boldsymbol{\rho}}{\rho^3} - dm\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - 2dm(\boldsymbol{\omega} \times \boldsymbol{v}) \tag{7.44}$$

The first term on the right-hand side of the preceding equation is called the gravity term; the second term is the centrifugal term, and the third term is the Coriolis term. They are to be examined separately as follows.

1) For the gravity term, because

$$\rho^2 = R^2 + r^2 + 2\boldsymbol{R} \cdot \boldsymbol{r}$$

$$\rho^3 = [R^2 + r^2 + 2\boldsymbol{R} \cdot \boldsymbol{r}]^{\frac{3}{2}} = R^3\left[1 + \left(\frac{r}{R}\right)^2 + 2\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right]^{\frac{3}{2}}$$

$$\frac{1}{\rho^3} = \frac{1}{R^3}\left[1 + \left(\frac{r}{R}\right)^2 + 2\frac{\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right]^{-\frac{3}{2}}$$

$$\cong \frac{1}{R^3}\left[1 - \frac{3\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right] \qquad \text{for } R \gg r$$

and because the satellite is in circular orbit

$$\frac{GMm}{R^2} = mR\omega^2$$

$$\frac{GM}{R^3} = \omega^2$$

The torque produced by the gravitational effect is found to be

$$\boldsymbol{N}_g = -\int GMdm\frac{\boldsymbol{r} \times \rho}{\rho^3}$$

$$= -\omega^2 \int dm(\boldsymbol{r} \times \rho)\left(1 - \frac{3\boldsymbol{R} \cdot \boldsymbol{r}}{R^2}\right)$$

With the use of $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{R} = R\boldsymbol{k}$, the equation is simplified to

$$\boldsymbol{N}_g = -\omega^2 \int dm(\boldsymbol{r} \times \boldsymbol{R})\left(1 - \frac{3z}{R}\right) = -\omega^2 \int dmR(-x\boldsymbol{j} + y\boldsymbol{i})\left(1 - \frac{3z}{R}\right)$$

$$= 3\omega^2 \int dmz(-x\boldsymbol{j} + y\boldsymbol{i}) = 3\omega^2(-I_{yz}\boldsymbol{i} + I_{xz}\boldsymbol{j}) \tag{7.45}$$

where

$$I_{yz} = -\int zydm \tag{7.46a}$$

$$I_{xz} = -\int xzdm \tag{7.46b}$$

2) For the centrifugal term, because

$$\omega = -\omega\boldsymbol{i}$$

$$\boldsymbol{N}_{\text{cen}} = -\int dm\boldsymbol{r} \times [\omega \times (\omega \times \rho)] = -\omega^2 \int dm\boldsymbol{r} \times [\boldsymbol{i} \times (\boldsymbol{i} \times \rho)]$$

Now, because

$$\boldsymbol{i} \times (\boldsymbol{i} \times \rho) = \boldsymbol{i} \times [\boldsymbol{i} \times (R\boldsymbol{k} + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})] = -y\boldsymbol{j} - (R + z)\boldsymbol{k}$$

we find

$$\boldsymbol{N}_{\text{cen}} = -\omega^2 \int dm\boldsymbol{r} \times [y\boldsymbol{j} + (R + z)\boldsymbol{k}] = \omega^2 \int dm(xy\boldsymbol{k} - xz\boldsymbol{j})$$

$$= \omega^2(I_{xz}\boldsymbol{j} - I_{xy}\boldsymbol{k}) \tag{7.47}$$

where

$$I_{xz} = -\int xzdm \tag{7.48a}$$

$$I_{xy} = -\int xydm \tag{7.48b}$$

3) The Coriolis term is

$$N_{\text{cor}} = -2\int dm\boldsymbol{r} \times (\omega \times \boldsymbol{v})$$

Because

$$\boldsymbol{v} = \omega' \times \boldsymbol{r} = (\omega'_x\boldsymbol{i} + \omega'_y\boldsymbol{j} + \omega'_z\boldsymbol{k}) \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$
$$= \omega'_x y\boldsymbol{k} - \omega'_x z\boldsymbol{j} - \omega'_y x\boldsymbol{k} + \omega'_y z\boldsymbol{i} + \omega'_z x\boldsymbol{j} - \omega'_z y\boldsymbol{i}$$

$$\omega \times \boldsymbol{v} = -\boldsymbol{i}\omega \times \boldsymbol{v} = \omega(\omega'_x y - \omega'_y x)\boldsymbol{j} + \omega(\omega'_x z - \omega'_z x)\boldsymbol{k}$$

$$\boldsymbol{r} \times (\omega \times \boldsymbol{v}) = \omega(\omega'_y xz - \omega'_z xy)\boldsymbol{i} + \omega(-\omega'_x xz + \omega'_z x^2)\boldsymbol{j}$$
$$+ \omega(\omega'_x xy - \omega'_y x^2)\boldsymbol{k}$$

we obtain

$$N_{\text{cor}} = -2\int dm\boldsymbol{r} \times (\omega \times \boldsymbol{v})$$
$$= 2\omega[(\omega'_y I_{xz} - \omega'_z I_{xy})\boldsymbol{i} + (\omega'_z I - \omega'_x I_{xz})\boldsymbol{j} + (\omega'_x I_{xy} - \omega'_y I)\boldsymbol{k}] \tag{7.49}$$

where $I = -\int x^2 dm$ and ω' is the angular velocity of the satellite relative to the xyz axes.

The addition of Eqs. (7.45), (7.47), and (7.49) will give the torque produced on the satellite because of its own motion. However, in these equations, the various I are computed in the moving coordinates. In other words, I changes with time. This is not convenient to apply. It is better to relate I to the principal moments of inertia. Let R be a rotational transformation matrix and I' the principal moment of inertia. Assume that at the beginning of observation, the xyz axes are coincided with the principal axes of the body. Note that

$$\boldsymbol{I}' = \boldsymbol{RIR}^{-1}$$
$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} \tag{7.50}$$

Now let us consider pitching of the satellite, which means the satellite rotates about the x axis by an angle of θ_p with a speed of $\dot{\theta}_p$. We have

$$\omega'_x = \dot{\theta}_p, \qquad \omega'_y = \omega'_z = 0$$

Because the body is rotated about the x axis counterclockwise by an angle of θ_p, the rotational transformation matrix is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_p & \sin\theta_p \\ 0 & -\sin\theta_p & \cos\theta_p \end{pmatrix}$$

We find

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2\cos^2\theta_p + I_3\sin^2\theta_p & (I_2 - I_3)\cos\theta_p\sin\theta_p \\ 0 & (I_2 - I_3)\cos\theta_p\sin\theta_p & I_2\sin^2\theta_p + I_3\cos^2\theta_p \end{pmatrix}$$

Hence

$$I_{xx} = I_1, \qquad I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0$$

$$I_{yz} = (I_2 - I_3)\cos\theta_p \sin\theta_p = I_{zy}$$

Thus the torque produced on the satellite because of pitching is simply

$$\begin{aligned} \boldsymbol{N}_p = \boldsymbol{N}_g &= 3\omega_2(-I_{yz}\boldsymbol{i}) \\ &= -3\omega^2(I_2 - I_3)\cos\theta_p \sin\theta_p \boldsymbol{i} \\ &= -\tfrac{3}{2}\omega^2(I_2 - I_3)\sin 2\theta_p \boldsymbol{i} \end{aligned} \tag{7.51}$$

Next, let us consider rolling of the satellite about the y axis by an angle of θ_R with a speed of $\dot{\theta}_R$, i.e.,

$$\omega_x' = 0, \qquad \omega_y' = \dot{\theta}_R, \qquad \omega_z' = 0$$

then we have

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta_R & 0 & -\sin\theta_R \\ 0 & 1 & 0 \\ \sin\theta_R & 0 & \cos\theta_R \end{pmatrix}$$

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{I}'\boldsymbol{R} = \begin{pmatrix} I_1\cos^2\theta_R + I_3\sin^2\theta_R & 0 & (-I_1 + I_3)\cos\theta_R \sin\theta_R \\ 0 & I_2 & 0 \\ (-I_1 + I_3)\cos\theta_R\sin\theta_R & 0 & I_1\sin^2\theta_R + I_3\cos^2\theta_R \end{pmatrix}$$

or

$$I_{xx} = I_1\cos^2\theta_R + I_3\sin^2\theta_R$$

$$I_{yy} = I_2$$

$$I_{zz} = I_1\sin^2\theta_R + I_3\cos^2\theta_R$$

$$I_{xz} = I_{zx} = (-I_1 + I_3)\cos\theta_R\sin\theta_R$$

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = 0$$

The I in Eq. (7.49) is

$$\begin{aligned} I &= -\int x^2 dm = -\frac{1}{2}\int (r^2 + x^2 - y^2 - z^2) dm \\ &= -\frac{1}{2}\int [(r^2 - y^2) - (r^2 - x^2) + (r^2 - z^2)] dm \\ &= -\frac{1}{2}[I_{yy} - I_{xx} + I_{zz}] \\ &= -\frac{1}{2}[I_2 - (I_1\cos^2\theta_R + I_3\sin^2\theta_R) + (I_1\sin^2\theta_R + I_3\cos^2\theta_R)] \\ &= -\frac{1}{2}[I_2 - (I_1 - I_3)\cos 2\theta_R] \end{aligned}$$

With the use of I as just obtained, we find the torque produced on the satellite because of rolling is

$$
\begin{aligned}
\boldsymbol{N}_{\text{rol}} &= \boldsymbol{N}_g + \boldsymbol{N}_{\text{cer}} + \boldsymbol{N}_{\text{cor}} \\
&= 3\omega^2 I_{xz}\boldsymbol{j} + \omega^2 I_{xz}\boldsymbol{j} + 2\omega(\dot\theta_R I_{xz}\boldsymbol{i} - \dot\theta_R I\boldsymbol{k}) \\
&= -\omega\dot\theta_R(I_1 - I_3)\sin 2\theta_R\,\boldsymbol{i} - 2\omega^2(I_1 - I_3)\sin 2\theta_R\boldsymbol{j} \\
&+ \omega\dot\theta_R[I_2 - (I_1 - I_3)\cos(2\theta_R)]\boldsymbol{k}
\end{aligned}
\tag{7.52}
$$

Similarly we can find that the torque acting on the satellite because of yawing about the z axis is

$$
\begin{aligned}
\boldsymbol{N}_{\text{yaw}} = &-\omega\dot\theta_y(I_1 - I_2)\sin(2\theta_y)\boldsymbol{i} - \omega\dot\theta_y[I_3 - (I_1 - I_2)\cos(2\theta_y)]j \\
&-\tfrac{1}{2}\omega^2(I_1 - I_2)\sin(2\theta_y)\boldsymbol{k}
\end{aligned}
\tag{7.53}
$$

Therefore, rolling and yawing can produce rotations about all three axes.

From here one can easily suggest a project that is to carry out the proper operational procedure so that the torques generated by the motions of the satellite are balanced. Furthermore, it is easy to recognize the need for a great deal more research for a satellite in an elliptical orbit.

Problems

7.1. Prove that

$$(\dot{\boldsymbol{n}}\boldsymbol{n} - \boldsymbol{n}\dot{\boldsymbol{n}}) = (\boldsymbol{n} \times \dot{\boldsymbol{n}}) \times \overleftrightarrow{1}$$

7.2. Verify Eq. (7.17) through direct evaluation in detail of $(\mathrm{d}\overleftrightarrow{R}/\mathrm{d}t) \cdot \overleftrightarrow{R}^T$.

7.3. Given

$$\omega = \dot\beta\boldsymbol{n} + (1 - \cos\beta)(\boldsymbol{n} \times \dot{\boldsymbol{n}}) + \sin\beta\dot{\boldsymbol{n}}$$

prove that by assuming $\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0$

$$\boldsymbol{n} \times \omega = -(1 - \cos\beta)\dot{\boldsymbol{n}} + \sin\beta(\boldsymbol{n} \times \dot{\boldsymbol{n}})$$

and

$$\boldsymbol{n} \times (\boldsymbol{n} \times \omega) = -\sin\beta\dot{\boldsymbol{n}} - 2\sin^2(\beta/2)(\boldsymbol{n} \times \dot{\boldsymbol{n}})$$

Consequently,

$$\dot{\boldsymbol{n}} = -\frac{1}{2}\left\{(\boldsymbol{n} \times \omega) + \cot\frac{\beta}{2}[(\boldsymbol{n} \cdot \omega)\boldsymbol{n} - \omega]\right\}$$

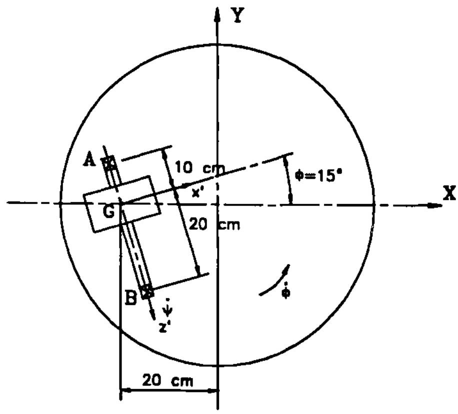

Fig. P7.4

7.4. A round plate rotates about the z axis perpendicular to the x–y plane with an angular velocity $\dot{\phi}$. Mounted on this revolving plate are two bearings A and B that retain a shaft and mass rotating at the angular velocity $\dot{\psi}$ as shown in Fig. P7.4. An $x'y'z'$ system is selected and fixed to the shaft and mass in such a way that the z' axis is along the shaft, x' is perpendicular to the z' axis, and y' is parallel to the Z axis. The mass center G defines the center of this system. The angular velocity $\dot{\psi}$ is observed from a position on the rotating plate. Let the mass be 10 kg, its radius of gyration be $r = 10$ cm, and its angular velocity $\dot{\psi} = 350$ rad/s. Using $\dot{\phi} = 5$ rad/s in the direction shown, find the bearing reactions.

7.5. The rotor of a jet airplane engine is supported by two bearings as shown in Fig. P7.5. The rotor assembly, consisting of the shaft, compressor, and turbine, has a mass of 820 kg and a moment of inertia with respect to its shaft of 45 kg-m^2; its center of mass is lying at point G. The rotor is rotating at 10,000 rpm cw when viewed from the rear. The speed of the airplane is 970 km/h, and it is pulling out of a dive along a path 1530 m in radius. Determine the magnitude and direction of

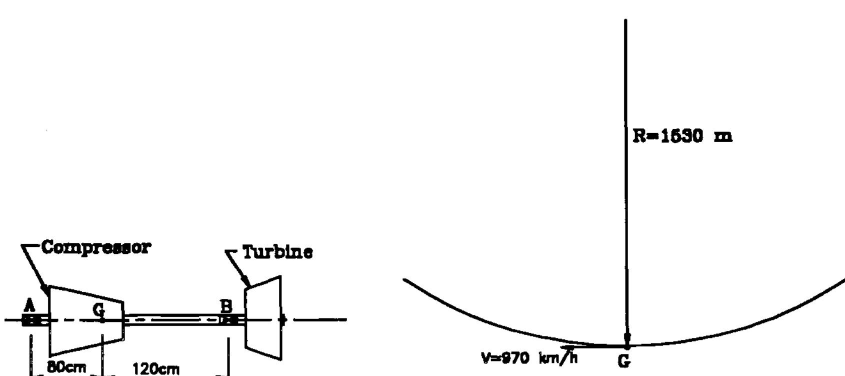

Fig. P7.5

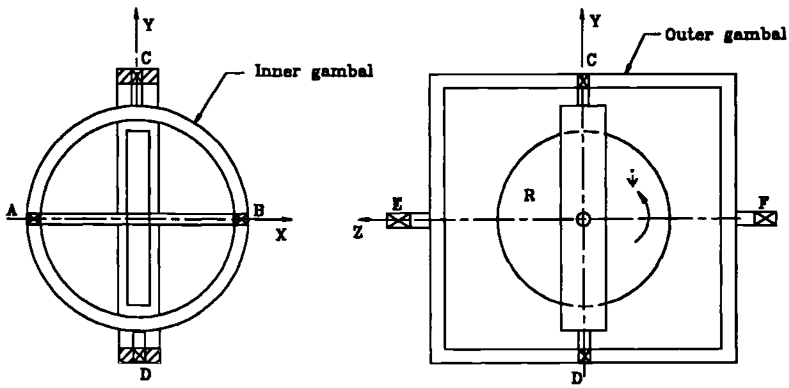

Fig. P7.7

the combined forces that the shaft exerts against the bearings due to the gyroscopic effect and the centrifugal effect.

7.6. The jet airplane in Problem 7.5 is traveling at 850 km/h in a horizontal plane and makes a clockwise turn of 2.0 km radius when viewed from above. The rotor is rotating at 9000 rpm cw when viewed from the rear. Determine the magnitude and direction of the gyroscopic forces that the shaft exerts against the bearings. Will the forces make the front of the plane tilt upward or downward?

7.7. In Fig. P7.7 a gyroscope used in instrument applications is illustrated. The rotor R is mounted in gimbals so that it is free to rotate about all three axes. In the figure A, B, C, D, E, and F are precision bearings. The rotor has a moment of inertia with respect to its axis $I = 0.0025$ kg-m^2 and is rotating at 12,000 rpm. Suppose that the instrument experiences a precession of 1 deg/h about the Z axis. Determine the magnitude and direction of the torque applied to cause the precession.

7.8. A heavy symmetric top is spun with its axis of symmetry in the vertical position initially. Find the conditions that will cause the top to remain vertical.

7.9. Derive Lagrange's equation for the coordinate θ of a heavy symmetrical top. Then solve this relation for the precession angular velocity $\dot{\phi}$ when there are no nutation velocity and acceleration present. From this result, show that there is a minimum valve of ω_z for which precession is possible. Finally, for ω_z higher than the minimum value, show that there are two permissible values of $\dot{\phi}$, corresponding to the cases of fast and slow precession.

7.10. Show that the total torque in yawing motion of a spacecraft in a circular orbit is given by Eq. (7.53).

7.11. Find the torques produced on a satellite in an elliptical orbit caused by its motions of pitching, rolling, and yawing.