6
Matrices, Tensors, Dyadics, and Rotation Operators

THIS chapter is intended to familiarize students with the mathematical symbols used in technical journals so that they may better understand newly published papers and to provide background for studying motions of rigid bodies in Chapter 7. These mathematical symbols allow many equations to be written in concise forms. Some topics, which are usually covered in a course of applied mathematics, will be introduced with a minimal amount of new physical concepts. To understand the subjects in this chapter better, students should have two courses in calculus and one course in differential equations and, specifically, some basic knowledge of matrix operations (see Appendix D).

Section 6.1 will show the relationship between two orthogonal coordinate systems under rotational motion relative to each other. Matrix notation and operations are introduced. Applications of matrix operations are given in Section 6.2 and in later sections dealing with the study of rotation of a symmetrical top. Section 6.3 introduces Cartesian tensors and dyadics including some basic operations. Applications of these are given in Sections 6.4, 6.5, and 6.6. Rotation operators are described in Section 6.7. The use of the rotation operator can simplify descriptions of complicated rotational motions. Some examples are given to illustrate this point. In general, this chapter provides background for studying the motions of rigid bodies.

6.1 Linear Transformation Matrices

From analytical geometry, we know that when x', y' axes are rotated with respect to the z axis by an angle of θ relative to the x, y axes as shown in Fig. 6.1; the relation of x', y' to x and y can be written as

$$x' = (\cos\theta)x + (\sin\theta)y \tag{6.1}$$

$$y' = (-\sin\theta)x + (\cos\theta)y \tag{6.2}$$

From Eqs. (6.1) and (6.2), we can solve easily for x, y in terms of x' and y' as

$$x = (\cos\theta)x' - (\sin\theta)y' \tag{6.3}$$

$$y = (\sin\theta)x' + (\cos\theta)y' \tag{6.4}$$

Equations (6.1) and (6.2) or (6.3) and (6.4) are examples of a linear transformation from one set of quantities to another. These quantities can be obtained in a different way. Considering a position vector $\boldsymbol{r}$ extending from the origin to the point P, we write

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = x'\boldsymbol{i}' + y'\boldsymbol{j}'$$

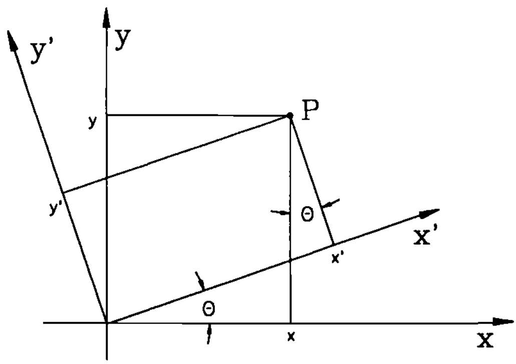

Fig. 6.1 Relation between prime and unprimed systems.

Then $\boldsymbol{r} \cdot \boldsymbol{i}'$ and $\boldsymbol{r} \cdot \boldsymbol{j}'$ will lead to Eqs. (6.1) and (6.2). Similarly, $\boldsymbol{r} \cdot \boldsymbol{i}$ and $\boldsymbol{r} \cdot \boldsymbol{j}$ will give Eqs. (6.3) and (6.4). Extending this technique to a three-dimensional vector, we have

$$x' = \cos(\boldsymbol{i}', \boldsymbol{i})x + \cos(\boldsymbol{i}', \boldsymbol{j})y + \cos(\boldsymbol{i}', \boldsymbol{k})z$$
$$y' = \cos(\boldsymbol{j}', \boldsymbol{i})x + \cos(\boldsymbol{j}', \boldsymbol{j})y + \cos(\boldsymbol{j}', \boldsymbol{k})z$$
$$z' = \cos(\boldsymbol{k}', \boldsymbol{i})x + \cos(\boldsymbol{k}', \boldsymbol{j})y + \cos(\boldsymbol{k}', \boldsymbol{k})z$$

where $\cos(\boldsymbol{i}', \boldsymbol{i})$ is the cosine function of the angle between $\boldsymbol{i}'$ and $\boldsymbol{i}$. To simplify the notation, we let

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z$$
$$a_{11} = \cos(\boldsymbol{i}', \boldsymbol{i}), \qquad a_{12} = \cos(\boldsymbol{i}', \boldsymbol{j}), \qquad a_{13} = \cos(\boldsymbol{i}', \boldsymbol{k})$$
$$a_{21} = \cos(\boldsymbol{j}', \boldsymbol{i}), \qquad a_{22} = \cos(\boldsymbol{j}', \boldsymbol{j}), \qquad a_{23} = \cos(\boldsymbol{j}', \boldsymbol{k})$$
$$a_{31} = \cos(\boldsymbol{k}', \boldsymbol{i}), \qquad a_{32} = \cos(\boldsymbol{k}', \boldsymbol{j}), \qquad a_{33} = \cos(\boldsymbol{k}', \boldsymbol{k})$$

Then we have

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

or

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \quad i = 1, 2, 3 \tag{6.5}$$

where a_{ij} are the direction cosines for all i and j. Because the magnitude of $\boldsymbol{r}$ does not change from one system to another, clearly

$$\sum_j x_j^2 = \sum_i x_i'^2 \tag{6.6}$$

With the use of Eq. (6.5) in Eq. (6.6), we have

$$\sum_j x_j^2 = \sum_j \left(\sum_k a_{jk}x_k\right)\left(\sum_\ell a_{j\ell}x_\ell\right) = \sum_{k,\ell} x_k x_\ell \left(\sum_j a_{jk}a_{j\ell}\right)$$

The commutative property of addition has been used in the preceding manipulation. For the two sides to agree, we let

$$\sum_j a_{jk}a_{j\ell} = \delta_{k,\ell} \tag{6.7}$$

where $\delta_{k,\ell}$ is the Kronecker delta function with the property

$$\delta_{k,\ell} \equiv \begin{cases} 1 & \text{as } k = \ell \\ 0 & \text{as } k \neq \ell \end{cases}$$

Equation (6.7) is known as the orthogonality condition on the direction cosines, and the transformation Eq. (6.5) consequently is called an orthogonal transformation, which transforms one set of orthogonal coordinates into another set.

The orthogonal transformation can be written in matrix notation as

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} \tag{6.8}$$

where

$$\boldsymbol{X}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let us review some basic operations from matrix algebra. Note that $\boldsymbol{A}$ is a square matrix. If $\boldsymbol{A}_{ij}$ is the cofactor of $\boldsymbol{a}_{ij}$ in the determinant of $\boldsymbol{A}$, then the matrix

$$(\boldsymbol{A}_{ji}) \equiv \text{transpose of } (\boldsymbol{A}_{ij})$$

is called the adjoint of $\boldsymbol{A}$. The reciprocal or inverse of a nonsingular matrix $\boldsymbol{A}$ is the adjoint of $\boldsymbol{A}$ divided by the determinant of $\boldsymbol{A}$. The reciprocal of $\boldsymbol{A}$ is denoted by the symbol $\boldsymbol{A}^{-1}$. Therefore,

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\text{Adj}(\boldsymbol{A}) \tag{6.9}$$

Multiplying Eq. (6.8) by $\boldsymbol{A}^{-1}$ leads to

$$\boldsymbol{A}^{-1}\boldsymbol{X}' = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{X}$$

or

$$X = A^{-1}X' \tag{6.10}$$

This is known as Cramer's rule. However, because A is formed by direction cosines of an orthogonal transformation, Eq. (6.10) can be simplified further to

$$X = A^T X' \tag{6.11}$$

where A^T = transpose of A, that is,

$$A^{-1} = A^T \tag{6.12}$$

The proof of Eq. (6.12) is given as follows. Based on

$$AA^{-1} = I$$

let $D = A^{-1}$, then

$$(AD)_{ij} = \sum_k a_{ik}\mathrm{d}_{kj} = (I)_{ij} = \delta_{ij}$$

Multiplying the preceding expression of $a_{i\ell}$ and taking summation over i gives

$$\sum_{i,k} a_{i\ell}(a_{ik}\mathrm{d}_{kj}) = \sum_i a_{i\ell}\delta_{i,j} = a_{j\ell}$$

The left-hand side of the preceding equation is

$$\sum_{i,k} a_{i\ell}(a_{ik}\mathrm{d}_{kj}) = \sum_{i,k}(a_{i\ell}a_{ik})\mathrm{d}_{kj} = \sum_k \delta_{\ell,k}\mathrm{d}_{kj} = \mathrm{d}_{\ell j}$$

Therefore,

$$a_{j\ell} = (A)_{j\ell} = (A^T)_{\ell j} = \mathrm{d}_{\ell j} = (A^{-1})_{\ell j}$$

or

$$A^{-1} = A^T$$

When a matrix satisfies the preceding equation, it is called an orthogonal matrix. To illustrate the use of Eq. (6.12), let us consider

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^T = A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We obtain

$$AA^T = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Example 6.1

Consider an airplane flying in a horizontal plane and measuring the wind velocity of a hurricane. A coordinate system (x', y', z'), which is attached to the airplane, is the moving system. Another system (x, y, z), which is fixed to earth with x–y plane parallel to the surface of the earth, is the fixed system. To simplify the problem, assume that $x'y'z'$ coordinates coincide with xyz coordinates at the beginning of operation; however, at the instant of measurement the airplane has yawed with respect to the z axis by an angle of θ. The wind velocity is successfully measured by the airplane in the $x'y'z'$ system. What is the velocity in xyz system?

Solution. It is known that $\boldsymbol{X}' = \boldsymbol{R}\boldsymbol{X}$ and $\boldsymbol{X} = \boldsymbol{R}^T\boldsymbol{X}'$. Applying this relationship for the transformation of velocity vector, we have

$$\boldsymbol{V} = \boldsymbol{R}^T\boldsymbol{V}'$$

where

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{R}^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{pmatrix} v'_x\cos\theta - v'_y\sin\theta \\ v'_x\sin\theta + v'_y\cos\theta \\ v'_z \end{pmatrix}$$

6.2 Application of Linear Transformation to Rotation Matrix

From the first course in dynamics, we know that six degrees of freedom are necessary to specify a solid body in motion: x, y, and z for a specific point on the body and θ, ϕ, and ψ for angular displacements of the body relative to a set of fixed axes. To illustrate the application of linear transformation, let us consider a solid body that is rotating without translational motion. Suppose that x, y, z

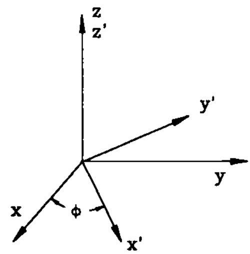

Fig. 6.2 Relative position between x' and x systems.

are fixed coordinates and that the prime system is attached to the rotating body. Consider the rotation in three steps as follows.

1) Let x', y', z' coincide with x, y, z first, and then rotate x', y', z' counterclockwise by angle ϕ about z as shown in Fig. 6.2. The relationship between the prime system and the fixed system is

$$X' = R_1 X$$

where

$$R_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

2) Let x'', y'', z'' coincide with x', y', z' first, and then rotate x'', y'', z'' counterclockwise by angle θ about x' as shown in Fig. 6.3. The relation between x'', y'', z'' and x, y, z is

$$X'' = R_2 X' = R_2(R_1 X) = (R_2 R_1) X$$

where

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{6.14}$$

The intersection of the x–y and x''–y'' planes is called the line of nodes.

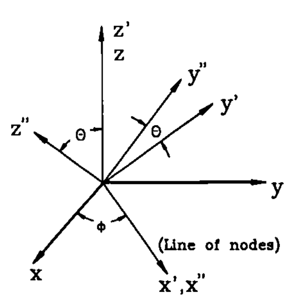

Fig. 6.3 Relative position between X'' and X' systems.

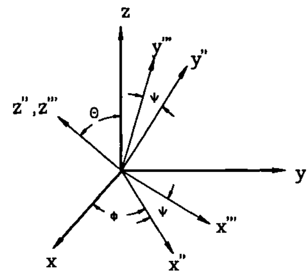

Fig. 6.4 Relative position between X''' and X''.

3) Let x''', y''', z''' coincide with x'', y'', z'' first, and then rotate x''', y''', z''' counterclockwise by angle ψ about z'' as shown in Fig. 6.4. Then the relation between x''', y''', z''' and x, y, z is

$$X''' = R_3 X'' = (R_3 R_2 R_1) X = RX$$

where

$$R_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.15}$$

$$R = R_3 R_2 R_1$$

$$R = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix} \tag{6.16}$$

The angles ϕ, θ, and ψ are known as Euler angles and are used to study the motion of a rotating top in Chapter 7.

6.3 Cartesian Tensors and Dyadics

A tensor is a quantity similar to a vector but with a much broader sense. Tensors can be scalars such as temperature or energy; tensors can be vectors; some tensors can represent stress, strain, or moment of inertia of a solid body. Furthermore, some high rank tensors can be used to express quantities in n-dimensional space with $n > 3$. The knowledge of tensors is essential for the study of general relativity theory. In this section, however, we are going to study only the Cartesian tensor. That means that the axes of the coordinate system, primed or unprimed, are perpendicular to each other.

Cartesian Tensor

A Cartesian tensor T in three-dimensional space is defined as a quantity that transforms according to the rule

$$T'_{lmn,\ldots} = \sum_{i,j,k=1}^{3} a_{li}\, a_{mj}\, a_{nk} \cdots T_{ijk,\ldots} \tag{6.17}$$

in a rotation from unprimed to primed coordinates where a_{li}s are direction cosines between the axes in unprimed and primed systems. The $T_{ijk,...}$ are called the components of the tensor and are functions of the unprimed coordinates; the $T'_{lmn,...}$ are the corresponding components in the primed system.

The rank of the tensor is defined by the total number of indices. Therefore, T is a zero-rank tensor that is a scalar such as temperature or energy; T_i is a first-rank tensor that is a vector such as velocity, force, and torque, etc.; T_{ij} is a second-rank tensor that represents nine-dimensional quantities in three-dimensional space such as stress and strain. In this chapter most of our attention will be devoted to the second-rank tensor.

A first-rank tensor is simply a vector. The transformation of a vector from unprimed system to primed system is

$$T'_\ell = \sum_i a_{\ell i} T_i$$

Consider that A_i and B_j are two first-rank tensors. Their transformations are

$$A'_\ell = \sum_i a_{\ell i} A_i \qquad \text{and} \qquad B'_\ell = \sum_j a_{\ell j} B_j$$

The dot product of $\boldsymbol{A}'$ and $\boldsymbol{B}'$ is

$$\sum_\ell A'_\ell B'_\ell = \sum_{\ell,i,j} (a_{\ell i} A_i)(a_{\ell j} B_j) = \sum_{i,j} \left(\sum_\ell a_{\ell i} a_{\ell j} \right) A_i B_j$$
$$= \sum_{i,j} (\delta_{i,j}) A_i B_j = \sum_i A_i B_i = \boldsymbol{A} \cdot \boldsymbol{B}$$

In other words, the dot product of any two vectors is invariant under the rotation of the coordinate system or is of zero rank. Therefore, it is also called isotropic tensor.

Second-Rank Tensor

To understand the second-rank tensor, let us consider

$$T_{ij} = x_i x_j$$

The complete expression of all the components of the tensor can be written in the form of matrix as

$$(T_{ij}) = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2^2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3^2 \end{pmatrix}$$

The transformation of T_{ij} from unprimed coordinate system to primed system is as follows:

$$\sum_{i,j} a_{\ell i} a_{mj} T_{ij} = \left(\sum_i a_{\ell i} x_i \right) \left(\sum_j a_{mj} x_j \right) = x'_\ell x'_m = T'_{\ell m}$$

Therefore $T_{ij} = x_i x_j$ is a second-rank tensor.

It is important to learn the process of contraction. Looking at

$$T'_{\ell m} = \sum_{i,j} a_{\ell i} a_{mj} T_{ij} \tag{6.18}$$

there are six indices in the right-hand side of the equation. Summing over i and j reduces the rank from six to two. This is called contraction. Note also the rank of a tensor must be the same on both sides of the equation. In many books, the summation sign is omitted in the equations. Automatic summation is to be done over a repeated index. This is known as the Einstein summation convention. For the sake of clarity, however, the summation sign is kept throughout this book.

Dyadic

Dyadic is closely related with vectors and second-rank tensors. A pair of vectors written in a definite order, such as $\boldsymbol{ij}$, is called a dyad, and a linear combination of dyads is known as a dyadic. For example, a second-rank tensor can be written into dyadic form as

$$\overleftrightarrow{T} = T_{11}\boldsymbol{ii} + T_{12}\boldsymbol{ij} + T_{13}\boldsymbol{ik} + T_{21}\boldsymbol{ji} + \cdots \tag{6.19}$$

Similarly,

$$\begin{aligned}\boldsymbol{AB} = {} & A_x B_x \boldsymbol{ii} + A_x B_y \boldsymbol{ij} + A_x B_z \boldsymbol{ik} + A_y B_x \boldsymbol{ji} + A_y B_y \boldsymbol{jj} + A_y B_z \boldsymbol{jk} \\ & + A_z B_x \boldsymbol{ki} + A_z B_y \boldsymbol{kj} + A_z B_z \boldsymbol{kk}\end{aligned}$$

is a dyadic.

Because vectors are used explicitly in the dyadic, many vector operations can be applied to dyadic operations. Let us study some fundamental operations as follows:

$$\begin{aligned}\boldsymbol{C} \cdot (\boldsymbol{AB}) = {} & C_x(A_x B_x \boldsymbol{i} + A_x B_y \boldsymbol{j} + A_x B_z \boldsymbol{k}) + C_y(A_y B_x \boldsymbol{i} + A_y B_y \boldsymbol{j} + A_y B_z \boldsymbol{k}) \\ & + C_z(A_z B_x \boldsymbol{i} + A_z B_y \boldsymbol{j} + A_z B_z \boldsymbol{k}) = (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B}\end{aligned}$$

The result shows that it is a vector in the direction of $\boldsymbol{B}$. On the other hand, the dot product of $(\boldsymbol{AB})$ with $\boldsymbol{C}$ from the right-hand side is

$$(\boldsymbol{AB}) \cdot \boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$$

The vector in the result is in the direction of $\boldsymbol{A}$. Therefore

$$\boldsymbol{C} \cdot (\boldsymbol{AB}) \neq (\boldsymbol{AB}) \cdot \boldsymbol{C} \tag{6.20}$$

A unit dyadic is defined as

$$\overleftrightarrow{1} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}$$

which possesses the property of

$$\overleftrightarrow{1} \cdot \omega = \omega = \omega \cdot \overleftrightarrow{1} \tag{6.21}$$

where ω represents any vector.

Now let us consider the transformation of a dyadic from an unprimed coordinate system to a primed coordinate system. Suppose that there is a relationship between vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ in the unprimed system as

$$\boldsymbol{U} = \overleftrightarrow{T} \cdot \boldsymbol{V} \tag{6.22}$$

where $\overleftrightarrow{T}$ is a dyad. Note that Eq. (6.22) can be written in matrix form as

$$\boldsymbol{U} = \boldsymbol{TV} \tag{6.23}$$

Transform $\boldsymbol{U}$ and $\boldsymbol{V}$ into $\boldsymbol{U}'$ and $\boldsymbol{V}'$ by premultiplying $\overleftrightarrow{A}$ to Eq. (6.22):

$$\boldsymbol{U}' = \overleftrightarrow{A} \cdot \boldsymbol{U} = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \boldsymbol{V} \tag{6.24}$$

The equivalent operation in the matrix form is

$$\boldsymbol{U}' = \boldsymbol{AU} = \boldsymbol{ATV}$$

However it is known in the matrix operation that

$$\boldsymbol{U}' = \boldsymbol{AT}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{V} = (\boldsymbol{ATA}^{-1})\boldsymbol{AV} = (\boldsymbol{ATA}^{-1})\boldsymbol{V}' = \boldsymbol{T}'\boldsymbol{V}'$$

which means that

$$\boldsymbol{T}' = \boldsymbol{ATA}^{-1} = \boldsymbol{ATA}^T \tag{6.25}$$

Applying this matrix manipulation to Eq. (6.24), we find that

$$\boldsymbol{U}' = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot (\overleftrightarrow{A}^T \overleftrightarrow{A}) \cdot \boldsymbol{V} = (\overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{A}^T) \cdot (\overleftrightarrow{A} \cdot \boldsymbol{V}) = \overleftrightarrow{T}' \cdot \boldsymbol{V}' \tag{6.26}$$

Therefore,

$$\overleftrightarrow{T}' = \overleftrightarrow{A} \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{A}^T \tag{6.27}$$

where $\overleftrightarrow{A}$ is a dyadic with direction cosines as the elements. Equation (6.27) can be written in tensor notation as

$$T'_{\ell m} = \sum_{i,j} (\boldsymbol{A})_{\ell i} (\boldsymbol{T})_{ij} (\boldsymbol{A}^{-1})_{jm} = \sum_{i,j} a_{\ell i} a_{mj} T_{ij}$$

which agrees with Eq. (6.18).

Example 6.2

Consider that a solid body is under rotational motion. It is rotating about the axes of symmetry. The axes of the coordinate system are chosen so they coincide with the axes of symmetry of the body. Express the relationship between angular momentum and the product of the moment of inertia and angular velocity of the chosen system in dyadic form. Find also the new relationship as the coordinate system is rotated about the z axis by an angle of ϕ.

Solution. According to the given conditions, the components of angular momentum can be written as

$$L_i = I_i \omega_i \quad i = 1, 2, 3$$

In dyadic form

$$\boldsymbol{L} = (I_1 \boldsymbol{ii} + I_2 \boldsymbol{jj} + I_3 \boldsymbol{kk}) \cdot (\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}) = \overleftrightarrow{I} \cdot \omega \tag{6.28}$$

The angular momentum in the coordinate system rotated about the z axis by angle ϕ is

$$\begin{aligned} L' &= \overleftrightarrow{R}_1 \cdot \boldsymbol{L} = \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \omega \\ &= \left(\overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T\right) \cdot (\overleftrightarrow{R}_1 \cdot \omega) = \overleftrightarrow{I}' \cdot \omega' \end{aligned} \tag{6.29}$$

where

$$\overleftrightarrow{I}' = \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T \tag{6.30}$$

and

$$\omega' = \overleftrightarrow{R}_1 \cdot \omega \tag{6.31}$$

To clarify the preceding operations, let us express the quantities explicitly. We have

$$\begin{aligned} \overleftrightarrow{R}_1 &= \cos\phi \boldsymbol{i'i} + \sin\phi \boldsymbol{i'j} - \sin\phi \boldsymbol{j'i} + \cos\phi \boldsymbol{j'j} + \boldsymbol{k'k} \\ \boldsymbol{L}' &= \boldsymbol{i'} L_1' + \boldsymbol{j'} L_2' + \boldsymbol{k'} L_3' \end{aligned} \tag{6.32}$$

$$\begin{aligned} \boldsymbol{L}' = \overleftrightarrow{R}_1 \cdot \boldsymbol{L} &= \boldsymbol{i'}(L_1 \cos\phi + L_2 \sin\phi) \\ &+ \boldsymbol{j'}(-L_1 \sin\phi + L_2 \cos\phi) + \boldsymbol{k'} L_3 \end{aligned} \tag{6.33}$$

Therefore,

$$\begin{aligned} L_1' &= L_1 \cos\phi + L_2 \sin\phi \\ L_2' &= -L_1 \sin\phi + L_2 \cos\phi \\ L_3' &= L_3 \end{aligned}$$

To find $\overleftrightarrow{I}'$, we first write $\overleftrightarrow{R}_1^T$ explicitly as

$$\overleftrightarrow{R}_1^T = \cos\phi \boldsymbol{ii'} - \sin\phi \boldsymbol{ij'} + \sin\phi \boldsymbol{ji'} + \cos\phi \boldsymbol{jj'} + \boldsymbol{kk'} \tag{6.34}$$

then we obtain

$$
\begin{aligned}
\overleftrightarrow{I}' &= \overleftrightarrow{R}_1 \cdot \overleftrightarrow{I} \cdot \overleftrightarrow{R}_1^T \\
&= (\cos\phi \boldsymbol{i}'\boldsymbol{i} + \sin\phi \boldsymbol{i}'\boldsymbol{j} - \sin\phi \boldsymbol{j}'\boldsymbol{i} + \cos\phi \boldsymbol{j}'\boldsymbol{j} + \boldsymbol{k}'\boldsymbol{k}) \\
&\cdot (I_1 \boldsymbol{ii} + I_2 \boldsymbol{jj} + I_3 \boldsymbol{kk}) \cdot (\cos\phi \boldsymbol{ii}' - \sin\phi \boldsymbol{ij}' + \sin\phi \boldsymbol{ji}' + \cos\phi \boldsymbol{jj}' + \boldsymbol{kk}') \\
&= \boldsymbol{i}'\boldsymbol{i}'(I_1 \cos^2\phi + I_2 \sin^2\phi) + \boldsymbol{i}'\boldsymbol{j}'(-I_1 + I_2)\cos\phi \sin\phi \\
&+ \boldsymbol{j}'\boldsymbol{i}'(-I_1 + I_2)\cos\phi \sin\phi + \boldsymbol{j}'\boldsymbol{j}'(I_1 \sin^2\phi + I_2 \cos^2\phi) + \boldsymbol{k}'\boldsymbol{k}' I_3 \qquad (6.35)
\end{aligned}
$$

Similar to $\boldsymbol{L}'$, we find

$$
\boldsymbol{\omega}' = \overleftrightarrow{R}_1 \cdot \boldsymbol{\omega} = \boldsymbol{i}'(\omega_1 \cos\phi + \omega_2 \sin\phi) + \boldsymbol{j}'(-\omega_1 \sin\phi + \omega_2 \cos\phi) + \boldsymbol{k}'\omega_3
$$

Through $\boldsymbol{L}' = \overleftrightarrow{I}' \cdot \boldsymbol{\omega}'$ we finally obtain

$$
\begin{aligned}
L_1' &= L_1 \cos\phi + L_2 \sin\phi \\
&= (I_1 \cos^2\phi + I_2 \sin^2\phi)(\omega_1 \cos\phi + \omega_2 \sin\phi) \\
&+ (-I_1 + I_2)\cos\phi \sin\phi(-\omega_1 \sin\phi + \omega_2 \cos\phi) \\
&= I_1\omega_1 \cos\phi + I_2\omega_2 \sin\phi \qquad (6.36a)
\end{aligned}
$$

$$
\begin{aligned}
L_2' &= -L_1 \sin\phi + L_2 \cos\phi \\
&= (-I_1 + I_2)\cos\phi \sin\phi(\omega_1 \cos\phi + \omega_2 \sin\phi) \\
&+ (I_1 \sin^2\phi + I_2 \cos^2\phi)(-\omega_1 \sin\phi + \omega_2 \cos\phi) \\
&= -I_1\omega_1 \sin\phi + I_2\omega_2 \cos\phi \qquad (6.36b)
\end{aligned}
$$

$$
L_3' = L_3 \qquad (6.36c)
$$

The result shows that $\boldsymbol{L}'$ obtained from $\overleftrightarrow{I}' \cdot \boldsymbol{\omega}'$ is the same as that from $\overleftrightarrow{R}_1 \cdot \boldsymbol{L}$ and serves to illustrate the dyadic operation.

6.4 Tensor of Inertia

After having studied the fundamentals of dyadics, we are ready to learn some applications. Consider a rigid body in rotational motion. A set of rectangular coordinates is attached to the body and is rotating relative to a set of fixed space coordinates. The origins of the two systems coincide and are not in relative motion. Therefore, the body position can be specified by three angular coordinates such as Euler angles (Section 6.2). Without losing generality, let us consider that the solid body consists of many point masses and that the position vector of m_i is $\boldsymbol{r}_i$. Therefore, the velocity of point mass m_i is

$$
\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i
$$

and the angular momentum of the body is

$$\begin{aligned} \boldsymbol{L} &= \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \sum_i m_i \boldsymbol{r}_i \times (\omega \times \boldsymbol{r}_i) \\ &= \sum_i m_i \left[r_i^2 \omega - \boldsymbol{r}_i (\boldsymbol{r}_i \cdot \omega) \right] \\ &= \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \cdot \omega = \overleftrightarrow{I}_m \cdot \omega \end{aligned} \tag{6.37}$$

where $\overleftrightarrow{I}_m$ is the tensor of inertia expressed in dyadic form and

$$\overleftrightarrow{I}_m = \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \tag{6.38}$$

Expanding Eq. (6.37), we have

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \tag{6.39}$$

where

$$\begin{aligned} I_{xx} &= \sum_i m_i \left(r_i^2 - x_i^2 \right), & I_{xy} &= -\sum_i m_i x_i y_i, & I_{xz} &= -\sum_i m_i x_i z_i \\ I_{yx} &= -\sum_i m_i y_i x_i, & I_{yy} &= \sum_i m_i \left(r_i^2 - y_i^2 \right), & I_{yz} &= -\sum_i m_i y_i z_i \\ I_{zx} &= -\sum_i m_i z_i x_i, & I_{yz} &= -\sum_i m_i z_i y_i, & I_{zz} &= \sum_i m_i \left(r_i^2 - z_i^2 \right) \end{aligned}$$

In the preceding expressions, I_{ii} elements are called the moment of inertia and $I_{ij} (i \neq j)$ are the products of inertia.

Now let us relate the inertia tensor to the moment of inertia with respect to the rotational axis of the body. Let $\boldsymbol{n}$ be a unit vector along ω or

$$\omega = \omega \boldsymbol{n}$$

The moment of inertia with respect to the rotational axis is simply

$$\begin{aligned} I_m &= \boldsymbol{n} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i \right) \cdot \boldsymbol{n} \\ &= \sum_i m_i \left[r_i^2 - (\boldsymbol{r}_i \cdot \boldsymbol{n})^2 \right] \end{aligned} \tag{6.40}$$

Through the use of Eq. (6.40), the kinetic energy of the solid body rotating with velocity ω can be expressed in a familiar form as

$$
\begin{aligned}
T &= \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) \\
&= \frac{1}{2}\sum_i m_i \boldsymbol{r}_i \cdot (\boldsymbol{v}_i \times \boldsymbol{\omega}) = \frac{1}{2}\sum_i m_i \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times \boldsymbol{v}_i) \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \left[\sum_i m_i (\boldsymbol{r}_i \times \boldsymbol{v}_i)\right] = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L} \\
&= \frac{1}{2}\boldsymbol{\omega} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{\omega} = \frac{1}{2}\omega^2 \boldsymbol{n} \cdot \overleftrightarrow{I}_m \cdot \boldsymbol{n} = \frac{1}{2} I_m \omega^2
\end{aligned}
$$

With the definition of inertia tensor given in Eq. (6.38), the generalized parallel axis theorem can be derived easily as follows. Consider

$$\boldsymbol{r}_i = \boldsymbol{r}_i' + \boldsymbol{R}$$

as shown in Fig. 6.5.

The inertia tensor with respect to XYZ coordinates $\overleftrightarrow{I}_0$ then can be expressed as

$$
\begin{aligned}
\overleftrightarrow{I}_0 &= \sum_i m_i \left(r_i^2 \overleftrightarrow{1} - \boldsymbol{r}_i \boldsymbol{r}_i\right) \\
&= \sum_i m_i [(\boldsymbol{r}_i' + \boldsymbol{R}) \cdot (\boldsymbol{r}_i' + \boldsymbol{R})\overleftrightarrow{1} - (\boldsymbol{r}_i' + \boldsymbol{R})(\boldsymbol{r}_i' + \boldsymbol{R})] \\
&= \sum_i m_i [\boldsymbol{r}_i' \cdot \boldsymbol{r}_i' \overleftrightarrow{1} - \boldsymbol{r}_i' \boldsymbol{r}_i'] + \sum_i m_i [\boldsymbol{R} \cdot \boldsymbol{R} \overleftrightarrow{1} - \boldsymbol{R}\boldsymbol{R}] \\
&\quad + 2\left[\boldsymbol{R} \cdot \left(\sum_i m_i \boldsymbol{r}_i'\right)\right]\overleftrightarrow{1} - \boldsymbol{R}\left(\sum_i m_i \boldsymbol{r}_i'\right) - \left(\sum_i m_i \boldsymbol{r}_i'\right)\boldsymbol{R}
\end{aligned}
$$

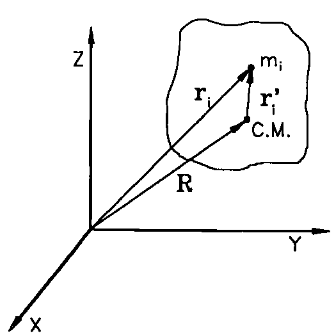

Fig. 6.5 General position of a body in *XYZ* coordinates.

Because the origin of the primed system is chosen at the center of mass,

$$\sum_i m_i \boldsymbol{r}'_i = 0$$

$$\overleftrightarrow{I}_0 = \sum_i m_i \left[r_i'^2 \overleftrightarrow{1} - \boldsymbol{r}'_i \boldsymbol{r}'_i\right] + M[R^2 \overleftrightarrow{1} - \boldsymbol{RR}]$$

$$= \overleftrightarrow{I}_c + M(R^2 \overleftrightarrow{1} - \boldsymbol{RR}) \tag{6.41}$$

where $\overleftrightarrow{I}_c$ is the inertia tensor of the solid body with respect to the primed axes with the origin at the center of mass. Equation (6.41) is known as the generalized parallel axis theorem.

To illustrate the generalized parallel axis theorem, let us consider a case where the center of mass of the body is at distance x on the x axis:

$$\boldsymbol{R} = x\boldsymbol{i}, \qquad R^2 = x^2, \qquad \boldsymbol{RR} = \boldsymbol{ii}x^2$$

$$M(R^2 \overleftrightarrow{1} - \boldsymbol{RR}) = M(x^2 \boldsymbol{jj} + x^2 \boldsymbol{kk})$$

$$\overleftrightarrow{I}_0 = \overleftrightarrow{I}_c + Mx^2(\boldsymbol{jj} + \boldsymbol{kk})$$

Writing the components of the moment of inertia in detail, we have

$$(\overleftrightarrow{I}_0)_{22} = (\overleftrightarrow{I}_c)_{22} + Mx^2 \tag{6.42a}$$

$$(\overleftrightarrow{I}_0)_{33} = (\overleftrightarrow{I}_c)_{33} + Mx^2 \tag{6.42b}$$

$$(\overleftrightarrow{I}_0)_{11} = (\overleftrightarrow{I}_c)_{11} \tag{6.42c}$$

$$(\overleftrightarrow{I}_0)_{ij} = (\overleftrightarrow{I}_c)_{ij} \qquad i \neq j \tag{6.42d}$$

In Eqs. (6.42a) and (6.42b), the difference between $\overleftrightarrow{I}_0$ and $\overleftrightarrow{I}_c$ is Mx^2 in $\boldsymbol{jj}$ and $\boldsymbol{kk}$ because the y' and z' axes are moved by x; however, because x, x' coincide, $(\overleftrightarrow{I}_0)_{11} = (\overleftrightarrow{I}_c)_{11}$. The results given in Eqs. (6.42a–6.42d) agree with the parallel axis theorem written with nine separate equations as given in the first course of dynamics.

6.5 Principal Stresses and Axes in a Three-Dimensional Solid

We have studied the fundamentals of matrices and tensors in Sections 6.1 and 6.3. Now let us apply them to determine the principal stresses in a solid. When forces and torques are applied to a three-dimensional homogeneous solid, three-dimensional stresses are set in the solid. As shown in Fig. 6.6, these stresses have nine components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy}. Because these stresses are in equilibrium, the summation of moments with respect to each axis must be equal to zero, and the results show that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$. The state of the stresses can be written in matrix form as

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{6.43}$$

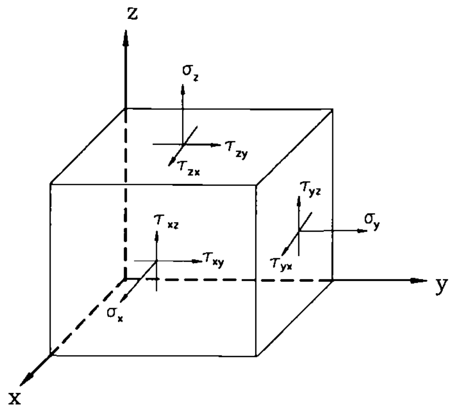

Fig. 6.6 Three-dimensional stress on a solid cube.

in which σ_i is the component of normal stresses and τ_{ij} is the component of shear stresses. The σ is a symmetric matrix. Figure 6.7 shows a tetrahedron formed by drawing three planes normal to the coordinate axes and a fourth plane with a directed normal $\boldsymbol{n}$ at a distance h from the point P that is at the origin. In the limit, as $h \to 0$, the tetrahedron will become of infinitesimal order with sides $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, and the inclined plane approaches P.

To find the principal stresses in a solid, we assume that the stress acting on the inclined plane is only a normal stress σ_n. The components of σ_n are σ_{nx}, σ_{ny}, and σ_{nz}. In the limit, as $h \to 0$, the equilibrium of all forces in the x direction requires

$$-\tfrac{1}{2}\tau_{yx}\mathrm{d}x\,\mathrm{d}z - \tfrac{1}{2}\tau_{zx}\mathrm{d}x\,\mathrm{d}y - \tfrac{1}{2}\sigma_x\mathrm{d}y\,\mathrm{d}z + \sigma_{nx}\mathrm{d}A = 0$$

where $\mathrm{d}A$ is the area of the inclined plane. Note that

$$\tfrac{1}{2}\mathrm{d}y\,\mathrm{d}z = \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{i} = \mathrm{d}A a_{nx}$$

$$\tfrac{1}{2}\mathrm{d}x\,\mathrm{d}z = \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{j} = \mathrm{d}A a_{ny}$$

$$\tfrac{1}{2}\mathrm{d}x\,\mathrm{d}y = \mathrm{d}A\,\boldsymbol{n}\cdot\boldsymbol{k} = \mathrm{d}A a_{nz}$$

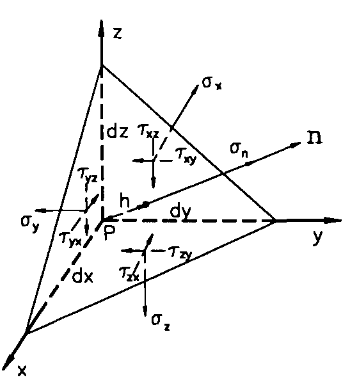

Fig. 6.7 Three-dimensional stress on a tetrahedron.

and

$$\boldsymbol{n} = a_{nx}\boldsymbol{i} + a_{ny}\boldsymbol{j} + a_{nz}\boldsymbol{k}$$

Hence,

$$\sigma_{nx} = \sigma_x a_{nx} + \tau_{yx} a_{ny} + \tau_{zx} a_{nz} \tag{6.44}$$

Similarly, we can find

$$\sigma_{ny} = \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{zy} a_{nz} \tag{6.45}$$

$$\sigma_{nz} = \tau_{zx} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz} \tag{6.46}$$

On the other hand,

$$\begin{aligned} \sigma_{nx} &= \sigma_n(\boldsymbol{n}\cdot\boldsymbol{i}) = \sigma_n a_{nx} \\ \sigma_{ny} &= \sigma_n a_{ny} \\ \sigma_{nz} &= \sigma_n a_{nz} \end{aligned} \tag{6.47}$$

Therefore, we find that the equations for the balance of forces are

$$\begin{aligned} \sigma_x a_{nx} + \tau_{yx} a_{ny} + \tau_{zx} a_{nz} &= \sigma_n a_{nx} \\ \tau_{xy} a_{nx} + \sigma_y a_{ny} + \tau_{zy} a_{nz} &= \sigma_n a_{ny} \\ \tau_{zx} a_{nx} + \tau_{yz} a_{ny} + \sigma_z a_{nz} &= \sigma_n a_{nz} \end{aligned} \tag{6.48}$$

Rewriting Eq. (6.48) in matrix form, we have

$$\boldsymbol{\sigma x} = \sigma_n \boldsymbol{x} \tag{6.49}$$

where

$$\boldsymbol{x} = \begin{pmatrix} a_{nx} \\ a_{ny} \\ a_{nz} \end{pmatrix} \tag{6.50}$$

From the formulation given, we will determine a_{nx}, a_{ny}, a_{nz}, and σ_n. In addition to the three equations in (6.48), we have

$$\boldsymbol{n}\cdot\boldsymbol{n} = 1 = a_{nx}^2 + a_{ny}^2 + a_{nz}^2 \tag{6.51}$$

Therefore, we have four equations to determine four unknowns.

Rearrange Eq. (6.49) as

$$\boldsymbol{\sigma x} = \sigma_n \mathbf{1}\boldsymbol{x}$$

or

$$(\sigma - \sigma_n \mathbf{1})\boldsymbol{x} = 0$$

Because $\boldsymbol{x}$ cannot be zero, the determinant of the coefficients must vanish, i.e.,

$$|\sigma - \sigma_n \mathbf{1}| = 0$$

Expanding the determinant, we find the functional relationship, called the characteristic equation,

$$\phi(\sigma_n) = \sigma_n^3 - I_1\sigma_n^2 + I_2\sigma_n - I_3 = 0 \tag{6.52}$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{xz}\tau_{yz}$$

The three roots of Eq. (6.52), say σ_1, σ_2, and σ_3, are called the principal stresses. Once the principal stresses have been obtained, the direction cosines of the normals of the planes can be found from Eqs. (6.48) and (6.51). The normals are known as principal axes. To illustrate the procedure in detail, let us study the following example.

Example 6.3

Given a stress matrix,

$$\sigma = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} \qquad \text{(MPa)}$$

find the principal stresses and the direction cosines of the principal axes.

Solution. The characteristic equation is

$$\begin{vmatrix} 7-\sigma_n & -2 & 0 \\ -2 & 6-\sigma_n & -2 \\ 0 & -2 & 5-\sigma_n \end{vmatrix} = -\sigma_n^3 + 18\sigma_n^2 - 99\sigma_n + 162 = 0$$

The three roots are

$$\sigma_1 = 3\ \text{MPa}, \qquad \sigma_2 = 6\ \text{MPa}, \qquad \sigma_3 = 9\ \text{MPa}$$

To find the direction cosines, let us use Eq. (6.48) in explicit form

$$(7-\sigma_n)a_{nx} - 2a_{ny} = 0$$

$$-2a_{nx} + (6-\sigma_n)a_{ny} - 2a_{nz} = 0$$

$$-2a_{ny} + (5-\sigma_n)a_{nz} = 0$$

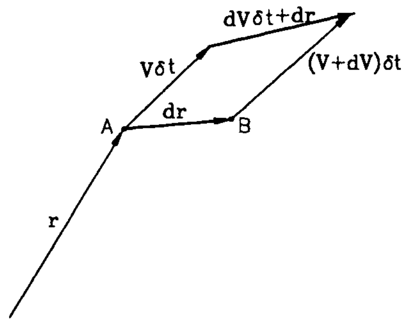

Fig. 6.8 Relative position between *A* and *B*.

For the first root $\sigma_1 = 3$ MPa,

$$4a_{nx} - 2a_{ny} = 0$$

$$-2a_{nx} + 3a_{ny} - 2a_{nz} = 0$$

$$-2a_{ny} + 2a_{nz} = 0$$

In the preceding three equations, only two equations are independent because the determinant of the coefficients is zero. However, we have

$$a_{nx}^2 + a_{ny}^2 + a_{nz}^2 = 1$$

and find

$$a_{nx} = \tfrac{1}{3}, \qquad a_{ny} = a_{nz} = \tfrac{2}{3}$$

Hence

$$\boldsymbol{n}_1 = \tfrac{1}{3}\boldsymbol{i} + \tfrac{2}{3}\boldsymbol{j} + \tfrac{2}{3}\boldsymbol{k} \qquad \text{for } \sigma_1 = 3 \text{ MPa}$$

Similarly, we find

$$\boldsymbol{n}_2 = \tfrac{2}{3}\boldsymbol{i} + \tfrac{1}{3}\boldsymbol{j} - \tfrac{2}{3}\boldsymbol{k} \qquad \text{for } \sigma_2 = 6 \text{ MPa}$$

$$\boldsymbol{n}_3 = -\tfrac{2}{3}\boldsymbol{i} + \tfrac{2}{3}\boldsymbol{j} - \tfrac{1}{3}\boldsymbol{k} \qquad \text{for } \sigma_3 = 9 \text{ MPa}$$

Note that the three axes are perpendicular to each other. It must be pointed out here that the technique illustrated here for principal stresses can be used also for finding principal strains in homogeneous materials and principal moments of inertia for solid bodies.

6.6 Viscous Stress in Newtonian Fluid

Suppose that a point A is located in a Newtonian fluid and is specified by the position vector $\boldsymbol{r}$ as shown in Fig. 6.8. The term *Newtonian fluid* implies the following postulates.

1) The fluid is continuous, and its stress tensor τ_{ij} is a linear function of the rates of strains.

2) The fluid is isotropic, i.e., its properties are independent of direction, and therefore the deformation law is independent of the coordinate axes in which it is expressed.

3) When the fluid is at rest, the deformation law must reduce to the hydrostatic pressure condition, $\tau_{ij} = -p\delta_{i,j}$.

Consider that A is moving with velocity $\boldsymbol{V}$. In the vicinity of A, there is point B that is moving with velocity $\boldsymbol{V} + \mathrm{d}\boldsymbol{V}$. The velocity $\boldsymbol{V}$ and the change of velocity are written as

$$\boldsymbol{V} = V_1\boldsymbol{i} + V_2\boldsymbol{j} + V_3\boldsymbol{k}$$

$$\mathrm{d}\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial x}\mathrm{d}x + \frac{\partial \boldsymbol{V}}{\partial y}\mathrm{d}y + \frac{\partial \boldsymbol{V}}{\partial z}\mathrm{d}z = \mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{V}$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}r} = \nabla \boldsymbol{V} \cdot \boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit vector in the direction of $\mathrm{d}\boldsymbol{r}$:

$$\nabla \boldsymbol{V} = \frac{\partial V_1}{\partial x}\boldsymbol{ii} + \frac{\partial V_2}{\partial x}\boldsymbol{ij} + \frac{\partial V_3}{\partial x}\boldsymbol{ik} + \frac{\partial V_1}{\partial y}\boldsymbol{ji} + \frac{\partial V_2}{\partial y}\boldsymbol{jj} + \frac{\partial V_3}{\partial y}\boldsymbol{jk}$$
$$+ \frac{\partial V_1}{\partial z}\boldsymbol{ki} + \frac{\partial V_2}{\partial z}\boldsymbol{kj} + \frac{\partial V_3}{\partial z}\boldsymbol{kk}$$

$\nabla \boldsymbol{V}$ is called the strain rate dyadic. Note that $\nabla \boldsymbol{V}$ denotes strain as a function of time. Now let us define

$$\epsilon_{11} \equiv \frac{\partial V_1}{\partial x}, \qquad \epsilon_{22} \equiv \frac{\partial V_2}{\partial y}, \qquad \epsilon_{33} \equiv \frac{\partial V_3}{\partial z}$$

$$\epsilon_{12} \equiv \frac{1}{2}\left(\frac{\partial V_1}{\partial y} + \frac{\partial V_2}{\partial x}\right), \qquad \epsilon_{13} \equiv \frac{1}{2}\left(\frac{\partial V_3}{\partial x} + \frac{\partial V_1}{\partial z}\right), \qquad \epsilon_{23} \equiv \frac{1}{2}\left(\frac{\partial V_2}{\partial z} + \frac{\partial V_3}{\partial y}\right)$$

$$h_1 \equiv \frac{1}{2}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right), \qquad h_2 \equiv \frac{1}{2}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right), \qquad h_3 \equiv \frac{1}{2}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

The strain rate dyadic then becomes

$$\nabla \boldsymbol{V} = \overleftrightarrow{\epsilon} + \overleftrightarrow{\Omega} \tag{6.53}$$

with

$$\overleftrightarrow{\epsilon} = \epsilon_{11}\boldsymbol{ii} + \epsilon_{12}\boldsymbol{ij} + \epsilon_{13}\boldsymbol{ik} + \epsilon_{12}\boldsymbol{ji} + \epsilon_{22}\boldsymbol{jj} + \epsilon_{23}\boldsymbol{jk}$$
$$+ \epsilon_{13}\boldsymbol{ki} + \epsilon_{23}\boldsymbol{kj} + \epsilon_{33}\boldsymbol{kk} \tag{6.54}$$

and

$$\overleftrightarrow{\Omega} = h_3\boldsymbol{ij} - h_2\boldsymbol{ik} - h_3\boldsymbol{ji} + h_1\boldsymbol{jk} + h_2\boldsymbol{ki} - h_1\boldsymbol{kj} \tag{6.55}$$

Note that $\overleftrightarrow{\epsilon}$ is a symmetric dyadic and is called the pure strain rate dyadic, and $\overleftrightarrow{\Omega}$ is an antisymmetric dyadic and is the rotation dyadic.

To find the expression for viscous stress, let us consider a general stress dyadic:

$$\overleftrightarrow{\tau} = \tau_{11}\boldsymbol{ii} + \tau_{12}\boldsymbol{ij} + \tau_{13}\boldsymbol{ik} + \tau_{12}\boldsymbol{ji} + \tau_{22}\boldsymbol{jj} + \tau_{23}\boldsymbol{jk} + \tau_{13}\boldsymbol{ki} + \tau_{23}\boldsymbol{kj} + \tau_{33}\boldsymbol{kk} \tag{6.56}$$

Through the rotation of coordinate axes, we can find the principal stresses and also the principal axes. Along these principal axes, considered as the primed system, we have

$$\overleftrightarrow{\tau}' = \tau'_{11}\boldsymbol{i'i'} + \tau'_{22}\boldsymbol{j'j'} + \tau'_{33}\boldsymbol{k'k'} \tag{6.57}$$

$$(\nabla \boldsymbol{V})' = \epsilon'_{11}\boldsymbol{i'i'} + \epsilon'_{22}\boldsymbol{j'j'} + \epsilon'_{33}\boldsymbol{k'k'} \tag{6.58}$$

The relationship between the viscous stress and the rate of strain along the x' axis may be expressed as

$$\begin{aligned}
\tau'_{11} &= -p + c_1\epsilon'_{11} + c_2\epsilon'_{22} + c_2\epsilon'_{33} \\
&= -p + (c_1 - c_2)\epsilon'_{11} + c_2(\epsilon'_{11} + \epsilon'_{22} + \epsilon'_{33}) \\
&= -p + (c_1 - c_2)\frac{\partial V'_1}{\partial x'} + c_2(\nabla \cdot \boldsymbol{V}) \\
&= -p + (c_1 - c_2)\epsilon'_{11} + c_2(\nabla \cdot \boldsymbol{V})
\end{aligned} \tag{6.59}$$

Without losing generality, let $c_2 \equiv k - \frac{2}{3}\mu$ and $c_1 \equiv c_2 + 2\mu$ in which k and μ are to be determined. Hence

$$\tau'_{11} = -p + \left(k - \tfrac{2}{3}\mu\right)(\nabla \cdot \boldsymbol{V}) + 2\mu\epsilon'_{11} \tag{6.60}$$

To identify the constants, let us consider first

$$\tau'_{11} = \tau'_{22} = \tau'_{33}$$

$$\epsilon'_{11} = \epsilon'_{22} = \epsilon'_{33} = (\nabla \cdot \boldsymbol{V})/3$$

From Eqs. (6.57) and (6.60) we find

$$\begin{aligned}
\overleftrightarrow{\tau}' &= \left(k - \tfrac{2}{3}\mu\right)(\nabla \cdot \boldsymbol{V})\overleftrightarrow{1} + 2\mu\left[\tfrac{1}{3}(\nabla \cdot \boldsymbol{V})\right]\overleftrightarrow{1} - p\overleftrightarrow{1} \\
&= k(\nabla \cdot \boldsymbol{V})\overleftrightarrow{1} - p\overleftrightarrow{1} = \tau'_{11}\overleftrightarrow{1}
\end{aligned}$$

Hence

$$k = \frac{\tau'_{11} + P}{(\nabla \cdot \boldsymbol{V})} \tag{6.61}$$

Because $(\text{div}\, V)$ means the change of volume per unit volume, k is known as the coefficient of bulk viscosity. Now let us rotate the axes back to the unprimed coordinate system and consider the viscous stress in a general form as

$$\overleftrightarrow{\tau} = \left[-p + \left(k - \tfrac{2}{3}\mu\right)\nabla \cdot V\right]\overleftrightarrow{1} + 2\mu\overleftrightarrow{\epsilon} \tag{6.62}$$

Note that $\overleftrightarrow{\epsilon}$ is the only term affected by the rotation of coordinate axes. Because Eq. (6.62) is always true for all possible conditions, let us apply the equation to a case such that

$$V_1 = V_1(y), \qquad V_2 = V_3 = 0$$

Then

$$\begin{aligned} \epsilon_{12} &= \frac{1}{2}\frac{\partial V_1}{\partial y} = \frac{1}{2}\frac{\mathrm{d}V_1}{\mathrm{d}y} \\ \tau_{12} &= 2\mu\epsilon_{12} = \mu\frac{\mathrm{d}V_1}{\mathrm{d}y} \end{aligned} \tag{6.63}$$

where μ is known as the coefficient of viscosity. Therefore, Eq. (6.62) is the expression for the viscous stress in Newtonian fluid with k the coefficient of bulk viscosity and μ the coefficient of viscosity.

6.7 Rotation Operators

Earlier, in Chapter 3, we studied the collision of missiles in midair. In Example 3.1, the delay time after the first missile launch is given as 60 s. This interval includes the time to rotate the launching equipment to a proper angle. Certainly this operation could be done by using Euler angles, but that approach takes too much time. With the operation given in this section, we will find that the operation is simplified and saves time.

Consider that a position vector $\boldsymbol{r}$ is rotated with respect to vector $\boldsymbol{n}$ by angle β to $\boldsymbol{r}'$. The angle β is measured in a plane perpendicular to $\boldsymbol{n}$, containing the ends of vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ in that plane as shown in Fig. 6.9. Let $\boldsymbol{a}$ be a vector with the

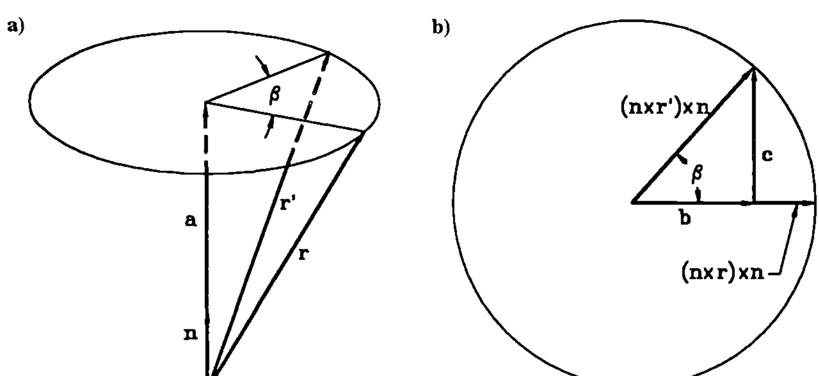

Fig. 6.9 Rotation of *r* about *n*.

direction of $\boldsymbol{n}$ and the magnitude of the component of $\boldsymbol{r}$ along $\boldsymbol{n}$, so that

$$\boldsymbol{a} = \boldsymbol{n}(\boldsymbol{r} \cdot \boldsymbol{n})$$

Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors in the circular plane, which is the top view of Fig. 6.9a looking down directly along $-\boldsymbol{n}$. Hence

$$\boldsymbol{r}' = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$$

The radius of the circle is $r \sin\theta$ or

$$|\boldsymbol{n} \times \boldsymbol{r}| = |(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n}| = |(\boldsymbol{n} \times \boldsymbol{r}') \times \boldsymbol{n}|$$

The vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are

$$\boldsymbol{b} = [(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n}] \cos\beta$$

$$\boldsymbol{c} = (\boldsymbol{n} \times \boldsymbol{r}) \sin\beta$$

Finally we have

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \cos\beta(\boldsymbol{n} \times \boldsymbol{r}) \times \boldsymbol{n} + \sin\beta(\boldsymbol{n} \times \boldsymbol{r}) \\ &= \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + [-\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \boldsymbol{r}(\boldsymbol{n} \cdot \boldsymbol{n})] \cos\beta + (\boldsymbol{n} \times \boldsymbol{r}) \sin\beta \\ &= (1 - \cos\beta)\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{r}) + \cos\beta\boldsymbol{r} + \sin\beta(\boldsymbol{n} \times \boldsymbol{r}) \end{aligned} \tag{6.64}$$

By defining a rotation operator as

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) \tag{6.65}$$

we obtain

$$\boldsymbol{r}' = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{r} \tag{6.66}$$

Note that $\boldsymbol{r}'$ is the vector $\boldsymbol{r}$ rotated about $\boldsymbol{n}$ by angle of β. The operator $\overleftrightarrow{R}$ is a function of $\boldsymbol{n}$ and β and is independent of coordinates. Note also that the operator $\overleftrightarrow{R}$ was first introduced by J. W. Gibbs in 1901* and has been further developed by C. Leubner and E. N. Moore.

When

$$\boldsymbol{n} = \boldsymbol{k}$$

$$\overleftrightarrow{R}(\boldsymbol{k}, \beta) = \boldsymbol{k}\boldsymbol{k} + \cos\beta(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j}\boldsymbol{j}) + \sin\beta(\boldsymbol{j}\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j})$$

Properties of the Operator $\overleftrightarrow{R}(n, \beta)$

1) As $\beta = 0$,

$$\overleftrightarrow{R}(\boldsymbol{n}, 0) = \overleftrightarrow{1} \tag{6.67}$$

*Gibbs, J. W., *Vector Analysis*, Scribner, New York, 1901, Chap. 6.

2) When $\boldsymbol{n}$ is rotated about $\boldsymbol{n}$ itself, $\boldsymbol{n}'$ is $\boldsymbol{n}$ or

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{n} = \boldsymbol{n} \tag{6.68}$$

3) Two consecutive rotations about the same axis $\boldsymbol{n}$ by angles of α and β will expect a result of

$$\overleftrightarrow{R}(\boldsymbol{n}, \alpha) \cdot \overleftrightarrow{R}(\boldsymbol{n}, \beta) = \overleftrightarrow{R}(\boldsymbol{n}, \alpha + \beta) \tag{6.69}$$

The preceding equation, however, requires a mathematical proof, which is given as follows.

It is easily verified that

$$\boldsymbol{A} \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{A} \times \boldsymbol{n} \tag{6.70}$$

or

$$(\boldsymbol{n} \times \overleftrightarrow{1}) \cdot \boldsymbol{A} = \boldsymbol{n} \times \boldsymbol{A}$$

also

$$(\boldsymbol{n} \times \overleftrightarrow{1}) \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{nn} - \overleftrightarrow{1} \tag{6.71}$$

and

$$\boldsymbol{nn} \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{nn} \times \boldsymbol{n} = 0 \tag{6.72}$$

Using Eqs. (6.70–6.72), we have

$$\begin{aligned}
\overleftrightarrow{R}(\boldsymbol{n}, \alpha) \cdot \overleftrightarrow{R}(\boldsymbol{n}, \beta) &= [(1 - \cos\alpha)\boldsymbol{nn} + \cos\alpha \overleftrightarrow{1} + \sin\alpha(\boldsymbol{n} \times \overleftrightarrow{1})] \\
&\quad \cdot [(1 - \cos\beta)\boldsymbol{nn} + \cos\beta \overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1})] \\
&= (1 - \cos\alpha)(1 - \cos\beta)\boldsymbol{nn} + (1 - \cos\beta)\cos\alpha \boldsymbol{nn} \\
&\quad + \sin\alpha(1 - \cos\beta)[(\boldsymbol{n} \times \overleftrightarrow{1}) \cdot \boldsymbol{nn}] + (1 - \cos\alpha)\cos\beta \boldsymbol{nn} \\
&\quad + \cos\alpha\cos\beta \overleftrightarrow{1} + \sin\alpha\cos\beta(\boldsymbol{n} \times \overleftrightarrow{1}) + (1 - \cos\alpha)\sin\beta[\boldsymbol{nn} \cdot (\boldsymbol{n} \times \overleftrightarrow{1})] \\
&\quad + \cos\alpha\sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) + \sin\alpha\sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) \cdot (\boldsymbol{n} \times \overleftrightarrow{1}) \\
&= \boldsymbol{nn}[1 - \cos\alpha - \cos\beta + \cos\alpha\cos\beta + \cos\alpha - \cos\alpha\cos\beta \\
&\quad + \cos\beta - \cos\alpha\cos\beta + \sin\alpha\sin\beta] + \overleftrightarrow{1}(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\
&\quad + \boldsymbol{n} \times \overleftrightarrow{1}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= [1 - \cos(\alpha + \beta)]\boldsymbol{nn} + \cos(\alpha + \beta)\overleftrightarrow{1} + \sin(\alpha + \beta)(\boldsymbol{n} \times \overleftrightarrow{1}) \\
&= \overleftrightarrow{R}[\boldsymbol{n}, (\alpha + \beta)]
\end{aligned}$$

4)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{1} \tag{6.73}$$

Here $\overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$ is the transpose of $\overleftrightarrow{R}$ and carries the similar sense as in matrix notation. In the operator $\overleftrightarrow{R}$, $\boldsymbol{nn}$ is symmetric and the transpose of $\boldsymbol{n} \times \overleftrightarrow{1}$ gives $-(\boldsymbol{n} \times \overleftrightarrow{1})$; hence,

$$\overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = (1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \overleftrightarrow{R}(\boldsymbol{n}, -\beta)$$

and

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{R}(\boldsymbol{n}, -\beta)$$
$$= \overleftrightarrow{R}(\boldsymbol{n}, \beta - \beta) = \overleftrightarrow{R}(\boldsymbol{n}, 0) = \overleftrightarrow{1}$$

5)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{V} = \boldsymbol{V} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) \tag{6.74}$$

Proof:

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{V} = [(1 - \cos\beta)\boldsymbol{nn} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1})] \cdot \boldsymbol{V}$$
$$= (1 - \cos\beta)\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{V}) + \cos\beta\boldsymbol{V} + \sin\beta(\boldsymbol{n} \times \boldsymbol{V})$$
$$= (1 - \cos\beta)(\boldsymbol{V} \cdot \boldsymbol{n})\boldsymbol{n} + \cos\beta\boldsymbol{V} \cdot \overleftrightarrow{1} - \sin\beta\boldsymbol{V} \cdot (\boldsymbol{n} \times \overleftrightarrow{1})$$
$$= \boldsymbol{V} \cdot \overleftrightarrow{R}(\boldsymbol{n}, -\beta) = \boldsymbol{V} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$$

6)

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{T}' \tag{6.75}$$

Proof: Because

$$\overleftrightarrow{T} = T_{11}\boldsymbol{ii} + T_{12}\boldsymbol{ij} + T_{13}\boldsymbol{ik} + T_{21}\boldsymbol{ji} + \cdots$$

each term in the preceding equation may be represented by $\boldsymbol{AB}$. Without losing generality, let us consider $\overleftrightarrow{T} = \boldsymbol{AB}$, then

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \overleftrightarrow{T} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta) = \overleftrightarrow{R}(\boldsymbol{n}, \beta) \cdot \boldsymbol{AB} \cdot \overleftrightarrow{R}^T(\boldsymbol{n}, \beta)$$
$$= (\overleftrightarrow{R} \cdot \boldsymbol{A})(\boldsymbol{B} \cdot \overleftrightarrow{R}^T)$$
$$= \boldsymbol{A}'(\overleftrightarrow{R} \cdot \boldsymbol{B}) \qquad \text{[Eq. (6.74) used]}$$
$$= \boldsymbol{A}'\boldsymbol{B}' = \overleftrightarrow{T}'$$

7)

$$[\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\boldsymbol{V}]\times\overleftrightarrow{1}=\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot(\boldsymbol{V}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{n},\beta) \tag{6.76}$$

Proof:

$$(\overleftrightarrow{R}\cdot\boldsymbol{V})\times\overleftrightarrow{1}=\boldsymbol{V}'\times\overleftrightarrow{1}=(V_1'\boldsymbol{i}'+V_2'\boldsymbol{j}'+V_3'\boldsymbol{k}')\times\overleftrightarrow{1}'$$

in which $\overleftrightarrow{1}'=\overleftrightarrow{R}\cdot\overleftrightarrow{1}\cdot\overleftrightarrow{R}^T=\overleftrightarrow{R}\cdot\overleftrightarrow{R}^T=\overleftrightarrow{1}$ has been used. Hence

$$(\overleftrightarrow{R}\cdot\boldsymbol{V})\times\overleftrightarrow{1}=-V_3'\boldsymbol{i}'\boldsymbol{j}'+V_2'\,\boldsymbol{i}'\boldsymbol{k}'+V_3'\boldsymbol{j}'\boldsymbol{i}'-V_1'\boldsymbol{j}'\boldsymbol{k}'-V_2'\boldsymbol{k}'\boldsymbol{i}'+V_1'\boldsymbol{k}'\boldsymbol{j}'$$

$$=(\boldsymbol{V}\times\overleftrightarrow{1})'=\overleftrightarrow{R}\cdot(\boldsymbol{V}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T$$

Eq. (6.75) is used in the last step.

8) If a unit vector $\boldsymbol{n}$ is rotated to $\boldsymbol{n}'$ by $\overleftrightarrow{R}(\boldsymbol{m},\alpha)$

$$\boldsymbol{n}'=\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}$$

then

$$\overleftrightarrow{R}'(\boldsymbol{n}',\beta)=\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha) \tag{6.77}$$

Proof: The relationship between $\overleftrightarrow{R}(\boldsymbol{n}',\beta)$ and $\overleftrightarrow{R}(\boldsymbol{m},\alpha)$ is

$$\overleftrightarrow{R}(\boldsymbol{n}',\beta)=\overleftrightarrow{R}[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n},\beta]$$

$$=(1-\cos\beta)[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}][\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]$$

$$+\ \cos\beta\overleftrightarrow{1}+\sin\beta[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]\times\overleftrightarrow{1}$$

Using Eqs. (6.74) and (6.76), we have

$$\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

and

$$[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}]\times\overleftrightarrow{1}=\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

Then

$$\overleftrightarrow{R}(\boldsymbol{n}',\beta)=(1-\cos\beta)[\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\boldsymbol{n}][\boldsymbol{n}\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)]$$

$$+\ \cos\beta\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)+\sin\beta\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot(\boldsymbol{n}\times\overleftrightarrow{1})\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

$$=\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot[(1-\cos\beta)\boldsymbol{n}\boldsymbol{n}+\cos\beta\overleftrightarrow{1}+\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1})]\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

$$=\overleftrightarrow{R}(\boldsymbol{m},\alpha)\cdot\overleftrightarrow{R}(\boldsymbol{n},\beta)\cdot\overleftrightarrow{R}^T(\boldsymbol{m},\alpha)$$

Applications of the Rotation Operator

Rotation of coordinate system through Euler angles ϕ, θ, and ψ. Suppose that we rotate the coordinate first with respect to $\boldsymbol{k}$ by an angle of ϕ. The position vector $\boldsymbol{r}$ is rotated with the rotation of coordinates. The new vector $\boldsymbol{r}'$ can be expressed as

$$\boldsymbol{r}' = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r} \tag{6.78}$$

Note that this operation is not the same as in the operation of a rotation matrix

$$\boldsymbol{r}'_m = \boldsymbol{R}_1 \boldsymbol{r} \tag{6.79}$$

where $\boldsymbol{r}'_m$ is the vector $\boldsymbol{r}$ in the rotated coordinates; $\boldsymbol{r}$ itself is not rotated. To emphasize this difference, let us consider

$$\boldsymbol{r} = \boldsymbol{i}$$

$$\overleftrightarrow{R}_1\left(\boldsymbol{k}, \frac{\pi}{2}\right) = \boldsymbol{kk} + (\boldsymbol{ji} - \boldsymbol{ij})$$

$$\boldsymbol{r}' = \boldsymbol{i}' = (\boldsymbol{kk} + \boldsymbol{ji} - \boldsymbol{ij}) \cdot \boldsymbol{i} = \boldsymbol{j} \tag{6.80}$$

This means the vector $\boldsymbol{i}$ is rotated to $\boldsymbol{j}$ after the coordinate axis $\boldsymbol{i}$ is rotated about $\boldsymbol{k}$ by an angle of $\pi/2$. On the other hand, the operation of Eq. (6.79) by rotation matrix will have a totally different result. Let us see the following case:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{R}_1 = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{r}'_m = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{6.81}$$

This $x'_2 = -1$ means that the vector $\boldsymbol{i}$ is not moved and is now along $-\boldsymbol{j}'$ after the whole coordinate system is rotated about $\boldsymbol{k}$ by an angle of $\pi/2$. The operation of a rotation matrix also can be expressed as a dyadic operation:

$$\boldsymbol{r}'_m = \overleftrightarrow{R}_1 \cdot \boldsymbol{r} \tag{6.82}$$

in which $\overleftrightarrow{R}_1 = \boldsymbol{i}'\boldsymbol{j} - \boldsymbol{j}'\boldsymbol{i} + \boldsymbol{k}'\boldsymbol{k}$, and $\boldsymbol{r} = \boldsymbol{i}$. Hence, in the rotated coordinate system, the unit vector becomes

$$\boldsymbol{r}'_m = -\boldsymbol{j}' \tag{6.83}$$

Now, continuing to consider the rotation of position vector $\boldsymbol{r}'$ with the coordinates from Eq. (6.78), let us rotate $\boldsymbol{r}'$ about $\boldsymbol{i}'$ by an angle of θ. Then we have

$$\boldsymbol{r}'' = \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \boldsymbol{r}' = \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r}$$

Next we rotate $\boldsymbol{r}''$ about $\boldsymbol{k}''$ by an angle of ψ, and we find

$$\begin{aligned}\boldsymbol{r}''' &= \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \boldsymbol{r}'' \\ &= \overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \boldsymbol{r} \end{aligned} \tag{6.84}$$

where $\boldsymbol{r}'''$ is the final form of the position vector $\boldsymbol{r}$ after being rotated about $\boldsymbol{k}$ by angle of ϕ, rotated about $\boldsymbol{i}'$ by θ and rotated about $\boldsymbol{k}''$ by ψ . Note that $\boldsymbol{i}'$ and $\boldsymbol{k}''$ are unit vectors along rotated coordinates. It will be more convenient to rotate $\boldsymbol{r}$ with respect to fixed axes. With the use of Eq. (6.77), we can express

$$\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_1^T(\boldsymbol{k}, \phi)$$

Taking the dot product with $\overleftrightarrow{R}_1(\boldsymbol{k}, \phi)$ from the right leads to

$$\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \tag{6.85}$$

Similarly,

$$\begin{aligned}\overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) &= [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)] \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)]^T \\ &= [\overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi)] \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot \left[\overleftrightarrow{R}_1^T(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2^T(\boldsymbol{i}', \theta)\right]\end{aligned}$$

Multiplying from the right by $\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1$ gives

$$\begin{aligned}\overleftrightarrow{R}_3(\boldsymbol{k}'', \psi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) &= \overleftrightarrow{R}_2(\boldsymbol{i}', \theta) \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \\ &= \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi)\end{aligned} \tag{6.86}$$

Equation (6.85) has been used in the last step of the manipulation. The result reached in Eq. (6.86) shows that the Euler angles ϕ, θ, ψ can be replaced by rotating the position vector $\boldsymbol{r}$ with respect to unprimed axes in a reversed order of ψ, θ, ϕ.

Applying the preceding results to a vector $\boldsymbol{r}$ fixed in space but with the coordinate system rotated, the relation between primed system and unprimed system may be derived by

$$\boldsymbol{r} = x_1'''\boldsymbol{i}''' + x_2'''\boldsymbol{j}''' + x_3'''\boldsymbol{k}''' = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$$

where

$$\boldsymbol{i}''' = \overleftrightarrow{R} \cdot \boldsymbol{i}, \qquad \boldsymbol{j}''' = \overleftrightarrow{R} \cdot \boldsymbol{j}, \qquad \boldsymbol{k}''' = \overleftrightarrow{R} \cdot \boldsymbol{k} \tag{6.87}$$

or

$$r = \sum_k x_k''' \varepsilon_k''' = \sum_j x_j \varepsilon_j$$

Taking the dot product of the preceding equation with ε_i''' leads to

$$x_i''' = \sum_k x_k'''(\varepsilon_k''' \cdot \varepsilon_i''') = \sum_j x_j(\varepsilon_j \cdot \varepsilon_i''') = \sum_j x_j(\varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i)$$

Note that $\varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i = \varepsilon_j \cdot \varepsilon_i'''$ = direction cosine between ε_j and ε_i'''. Therefore

$$a_{ij} = \varepsilon_j \cdot \overleftrightarrow{R} \cdot \varepsilon_i$$

For $\overleftrightarrow{R} = \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi)$,

$$\begin{aligned} a_{ij} &= \varepsilon_j \cdot \overleftrightarrow{R}_1 \cdot \overleftrightarrow{1} \cdot \overleftrightarrow{R}_2 \cdot \overleftrightarrow{1} \cdot \overleftrightarrow{R}_3 \cdot \varepsilon_i \\ &= \sum_{k,\ell} [\varepsilon_j \cdot \overleftrightarrow{R}_1(\boldsymbol{k}, \phi) \cdot \varepsilon_k][\varepsilon_k \cdot \overleftrightarrow{R}_2(\boldsymbol{i}, \theta) \cdot \varepsilon_\ell][\varepsilon_\ell \cdot \overleftrightarrow{R}_3(\boldsymbol{k}, \psi) \cdot \varepsilon_i] \end{aligned} \quad (6.88)$$

The preceding equation can be easily used to verify that the result agrees well with $(a_{ij}) = \boldsymbol{R}_3\boldsymbol{R}_2\boldsymbol{R}_1$ given in Eq. (6.16).

Combination of two successive rotations about different axes by one rotation. Suppose a rigid body to be rotated by two steps. First it is rotated about the $\boldsymbol{k}'$ axis by an angle of ϕ and then it is rotated about the $\boldsymbol{k}$ axis by an angle of ψ. The directions of $\boldsymbol{k}$ and $\boldsymbol{k}'$ are known, and the plane containing them is

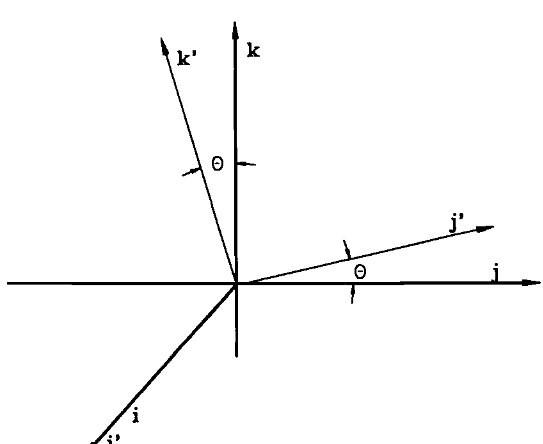

Fig. 6.10 True angle θ between axes k and k'.

determined. Choose the x axis perpendicular to the plane. Suppose the true angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ is θ, as shown in Fig. 6.10, then

$$\boldsymbol{k}' = \overleftrightarrow{R} \cdot \boldsymbol{k} = -(\sin\theta)\boldsymbol{j} + (\cos\theta)\boldsymbol{k} \tag{6.89}$$

And the two consecutive rotations may be expressed by

$$\overleftrightarrow{R}_1 = (1 - \cos\phi)\boldsymbol{k}\boldsymbol{k} + \cos\phi\overleftrightarrow{1} + \sin\phi(\boldsymbol{k} \times \overleftrightarrow{1})$$

and

$$\overleftrightarrow{R}_2 = (1 - \cos\psi)\boldsymbol{k}'\boldsymbol{k}' + \cos\psi\overleftrightarrow{1} + \sin\psi(\boldsymbol{k}' \times \overleftrightarrow{1})$$

According to Euler's theorem that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that point, these two rotations can be combined into one, i.e.,

$$\overleftrightarrow{R}(\boldsymbol{n}, \beta) = \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1 \tag{6.90}$$

The theorem is established if $\boldsymbol{n}$ and β are determined uniquely. To determine them, let us start from

$$(1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} + \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1$$

and taking the transpose of both sides,

$$(1 - \cos\beta)\boldsymbol{n}\boldsymbol{n} + \cos\beta\overleftrightarrow{1} - \sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = (\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1)^T = \overleftrightarrow{R}_1^T \cdot \overleftrightarrow{R}_2^T$$

The subtraction of the preceding two equations gives

$$\sin\beta(\boldsymbol{n} \times \overleftrightarrow{1}) = \tfrac{1}{2}\left[\overleftrightarrow{R}_2 \cdot \overleftrightarrow{R}_1 - \overleftrightarrow{R}_1^T \cdot \overleftrightarrow{R}_2^T\right] \tag{6.91}$$

After the right hand of the equation is expanded in detail, the following identities are used for simplification:

$$\boldsymbol{i} \times \overleftrightarrow{1} = \boldsymbol{k}\boldsymbol{j} - \boldsymbol{j}\boldsymbol{k}, \qquad \boldsymbol{j} \times \overleftrightarrow{1} = \boldsymbol{i}\boldsymbol{k} - \boldsymbol{k}\boldsymbol{i}, \qquad \boldsymbol{k} \times \overleftrightarrow{1} = \boldsymbol{j}\boldsymbol{i} - \boldsymbol{i}\boldsymbol{j},$$

$$(\boldsymbol{k}' \times \overleftrightarrow{1}) \cdot \boldsymbol{k} = \boldsymbol{k}' \times \boldsymbol{k} = -\sin\theta\boldsymbol{i}$$

$$(\boldsymbol{k} \times \overleftrightarrow{1}) \cdot (\boldsymbol{k}' \times \overleftrightarrow{1}) = (\boldsymbol{k} \times \overleftrightarrow{1}) \times \boldsymbol{k}' = -\boldsymbol{j}\boldsymbol{j}' - \boldsymbol{i}\boldsymbol{i}'\cos\theta$$

and

$$\boldsymbol{j}' = \overleftrightarrow{R}(\boldsymbol{i}, \theta) \cdot \boldsymbol{j} = \cos\theta\boldsymbol{j} + \sin\theta\boldsymbol{k}$$

Finally Eq. (6.91) is reduced to

$$\sin\beta(\boldsymbol{n}\times\overleftrightarrow{1}) = 2\left[\cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta\right]\cdot\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}\times\overleftrightarrow{1})\right.$$

$$\left. + \sin\frac{\psi}{2}\cos\frac{\phi}{2}(\boldsymbol{k}'\times\overleftrightarrow{1}) + \sin\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}'\times\boldsymbol{k})\times\overleftrightarrow{1}\right]$$

$$= 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}(\boldsymbol{n}\times\overleftrightarrow{1}) \tag{6.92}$$

To identify the β and $\boldsymbol{n}$ in the preceding equation, let us consider a special case of $\theta = 0$, then $\boldsymbol{k} = \boldsymbol{k}' = \boldsymbol{n}$:

$$\sin\frac{\beta}{2}\cos\frac{\beta}{2}(\boldsymbol{k}\times\overleftrightarrow{1}) = \cos\left(\frac{\psi+\phi}{2}\right)\cdot\sin\left(\frac{\psi+\phi}{2}\right)(\boldsymbol{k}\times\overleftrightarrow{1})$$

Hence

$$\cos\frac{\beta}{2} = \cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta \tag{6.93}$$

$$\boldsymbol{n} = \frac{1}{\sin(\beta/2)}\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}\boldsymbol{k} + \sin\frac{\psi}{2}\cos\frac{\phi}{2}\boldsymbol{k}' + \sin\frac{\psi}{2}\sin\frac{\phi}{2}(\boldsymbol{k}'\times\boldsymbol{k})\right] \tag{6.94}$$

Because $\boldsymbol{n}$ and β are properly determined, Euler's theorem, that two consecutive rotations with respect to two different axes can be combined into one rotational movement, is proved. Let us use an example to illustrate this concept as follows.

Example 6.4

A slab of size $a \times b$ and a thickness of t is placed vertically in x–z plane at the beginning. Suppose that the slab experiences two different kinds of rotations while the point at the origin remains fixed. Consider two different cases: 1) the slab is rotated first about the x axis by 90 deg, then about the y' axis by 90 deg; 2) the slab is rotated first about the y axis by 90 deg and then about the x' axis by 90 deg.

1) Perform the rotations through two steps.

2) Combine the rotations into one step and show the same results reached as in part 1.

Solution. 1a) The slab is rotated about the x axis by 90 deg then about the y' axis by 90 deg. At the beginning the unit normal vector of the slab is denoted by $\boldsymbol{n}'$ which is parallel to the y axis, or $\boldsymbol{n} = \boldsymbol{j}$ (see Fig. 6.11).

After it is rotated about the x axis by 90 deg, the normal vector becomes

$$\boldsymbol{n}' = \overleftrightarrow{R}(\boldsymbol{i}, 90\,\text{deg})\cdot\boldsymbol{j}$$

$$= (\boldsymbol{i}\boldsymbol{i} + \boldsymbol{i}\times\overleftrightarrow{1})\cdot\boldsymbol{j} = \boldsymbol{k}, \qquad \boldsymbol{n}' = \boldsymbol{j}' = \boldsymbol{k}$$

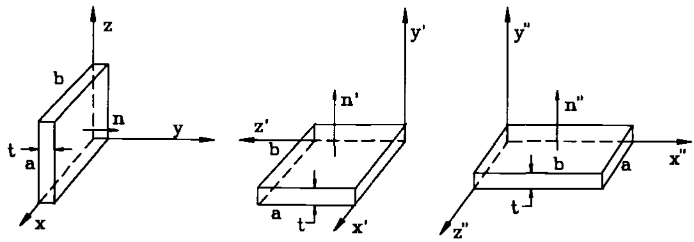

Fig. 6.11 Slab is rotated according to case 1a.

Finally, after the second rotation about the y' axis by 90 deg, the new normal vector is

$$n'' = \overleftrightarrow{R}(j', 90\text{ deg}) \cdot k = \overleftrightarrow{R}(k, 90\text{ deg}) \cdot k = (kk + k \times \overleftrightarrow{1}) \cdot k = k \qquad (6.95)$$

1b) The slab is rotated about the y axis by 90 deg then about the x' axis by 90 deg (see Fig. 6.12). Similarly, as in case 1a, the normal vectors of the slab are denoted by n, n', and n'' for three positions of the slab. We find

$$n = j$$

$$n' = \overleftrightarrow{R}(j, 90\text{ deg}) \cdot j = j, \qquad i' = -k$$

and

$$n'' = \overleftrightarrow{R}(i', 90\text{ deg}) \cdot j = \overleftrightarrow{R}(-k, 90\text{ deg}) \cdot j = (kk - k \times \overleftrightarrow{1}) \cdot j = i \qquad (6.96)$$

2a) The slab is rotated with respect to n_1 and by an angle of β_1 for the case of 1a:

$$\cos\frac{\beta_1}{2} = \cos^2 45\text{ deg} - \sin^2 45\text{ deg} \cos 90\text{ deg} = \frac{1}{2}$$

$$\frac{\beta_1}{2} = 60\text{ deg} \qquad \beta_1 = 120\text{ deg}$$

$$n_1 = \frac{1}{\sin(\beta/2)}[\cos 45\text{ deg} \sin 45\text{ deg}\, i + \sin 45\text{ deg} \cos 45\text{ deg}\, j'$$

$$+ \sin^2 45\text{ deg}(j' \times i)] = \frac{1}{\sqrt{3}/2}\left[\frac{1}{2}i + \frac{1}{2}k + \frac{1}{2}j\right] = \frac{1}{\sqrt{3}}(i + j + k)$$

$$n'' = \overleftrightarrow{R}(n_1, 120\text{ deg}) \cdot j = [(1 - \cos 120\text{ deg})n_1 n_1$$

$$+ \cos 120\text{ deg}\overleftrightarrow{1} + \sin 120\text{ deg}(n_1 \times \overleftrightarrow{1})] \cdot j$$

$$= \frac{3}{2}n_1\frac{1}{\sqrt{3}} - \frac{1}{2}j + \frac{1}{2}(k - i) = k$$

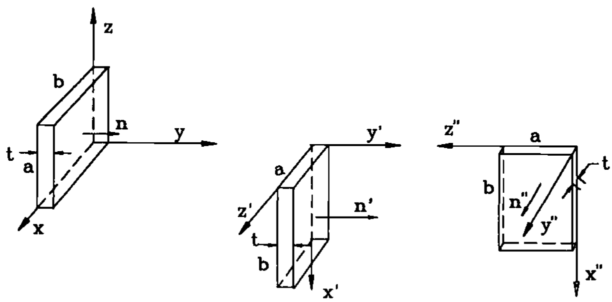

Fig. 6.12 Slab is rotated according to case 1b.

The same result is reached as given in Eq. (6.95), but here is done only by one rotation.

2b) The slab is rotated with respect to $\boldsymbol{n}_2$ by angle of β_2 for case 1b:

$$\cos\frac{\beta_2}{2} = \cos^2 45\,\text{deg} - \sin^2 45\,\text{deg}\cos 90\,\text{deg} = \frac{1}{2}, \qquad \beta_2 = 120\,\text{deg}$$

$$\boldsymbol{n}_2 = \frac{1}{\sin(\beta/2)}[\cos 45\,\text{deg}\sin 45\,\text{deg}\,\boldsymbol{j} + \sin 45\,\text{deg}\cos 45\,\text{deg}\,\boldsymbol{i}' + \sin^2 45\,\text{deg}(\boldsymbol{i}' \times \boldsymbol{j})] = \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$$

$$\boldsymbol{n}'' = \overleftrightarrow{R}(\boldsymbol{n}_2, 120\,\text{deg}) \cdot \boldsymbol{j} = [(1 - \cos 120\,\text{deg})\boldsymbol{n}_2\boldsymbol{n}_2 + \cos 120\,\text{deg}\overleftrightarrow{1} + \sin 120\,\text{deg}\,(\boldsymbol{n}_2 \times \overleftrightarrow{1}] \cdot \boldsymbol{j} = \frac{3}{2}\boldsymbol{n}_2\frac{1}{\sqrt{3}} - \frac{1}{2}\boldsymbol{j} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}(\boldsymbol{k} + \boldsymbol{i}) = \boldsymbol{i}$$

The same result is found as given in Eq. (6.96) for two steps of rotation. More details may be shown if the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are rotated as $\boldsymbol{n}$. That approach has been assigned as an exercise for readers to complete in the problems section.

Problems

6.1. Verify that the following transformations are orthogonal:

(a)

$$x' = (\cos\theta)x + (\sin\theta)y$$
$$y' = (-\sin\theta)x + (\cos\theta)y$$

(b)

$$x_1' = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3$$

$$x_2' = x_2$$

$$x_3' = -\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_3$$

6.2. Prove that the product of two orthogonal transformations is an orthogonal transformation.

6.3. Given a stress matrix

$$\begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix} \qquad \text{(ksi)}$$

find the principal stresses and the corresponding principal axes.

6.4. It is known that the moments and products of inertia of area A for the centroidal axes are

$$I_{xx} = 40 \text{ ft}^4, \qquad I_{yy} = 20 \text{ ft}^4, \qquad I_{xy} = -4 \text{ ft}^4$$

Find the principal moments of inertia and the corresponding principal axes in the x–y plane.

6.5. Similar to the derivation of viscous stress in Newtonian fluid as given in Section 6.6, derive the expression of stress tensor in homogeneous solid as a function of strains.

6.6. Prove that

$$\boldsymbol{A}\cdot(\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{A} \times \boldsymbol{n}$$

and

$$(\boldsymbol{n} \times \overleftrightarrow{1})\cdot\boldsymbol{A} = \boldsymbol{n} \times \boldsymbol{A}$$

6.7. Prove that

$$(\boldsymbol{n} \times \overleftrightarrow{1})\cdot(\boldsymbol{n} \times \overleftrightarrow{1}) = \boldsymbol{n}\boldsymbol{n} - \overleftrightarrow{1}$$

6.8. Suppose that the angle between two unit vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$ is θ as shown in Fig. 6.10. Prove that

$$(\boldsymbol{k} \times \overleftrightarrow{1}) \cdot (\boldsymbol{k}' \times \overleftrightarrow{1}) = (\boldsymbol{k} \times \overleftrightarrow{1}) \times \boldsymbol{k}' = -\boldsymbol{j}\boldsymbol{j}' - \boldsymbol{i}\boldsymbol{i}' \cos\theta$$

6.9. A slab of size $a \times b$ and thickness of t is placed vertically in x–z plane at the beginning as shown in Fig. 6.11. Suppose that the slab is rotated but with the point at the origin fixed. First, the slab is rotated about the x axis by 30 deg then rotated about the y' axis by 60 deg.

(a) Perform the rotations through two steps.

(b) Combine the rotations into one step and show the same results reached as in part (a).

6.10. Consider Example 6.4. Let $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ be the unit vectors of the initial coordinate system. The vectors are $\boldsymbol{i}'\boldsymbol{j}'$ and $\boldsymbol{k}'$ after the first rotation and the unit vectors are $\boldsymbol{i}''\boldsymbol{j}''$ and $\boldsymbol{k}''$ after the second rotation. Find the relationships between these unit vectors for the rotations considered in the example.

6.11. Suppose that $\boldsymbol{I}_{xx}, \boldsymbol{I}_{yy}$, and $\boldsymbol{I}_{zz}$ are given and the products of inertia are zero. Find the moment of inertia matrix when the coordinate system is rotated about the z axis by an angle of θ.