# 5
# Rockets and Space Vehicles

IN this chapter we shall study the dynamics of rockets and space vehicles in detail. We begin the study with a single-stage rocket in Section 5.1. In this section, we discuss thrust, air drag, stability, equation of motion, and conditions at the time of burnout. Multistage rockets are studied in Section 5.2. Advantages of multistage design are explained. The method of Lagrangian multiplier is employed to achieve optimum design for a multistage rocket. A numerical example is given to demonstrate the advantages of multistage design.

The orbit of a space vehicle is studied in Section 5.3. The space vehicle is modeled as a particle in a central force field. Different orbits may be achieved with different amounts of total mechanical energies. Special emphasis is placed on elliptical orbits. Numerical examples are given to illustrate the relationship between the velocity and position of a space vehicle for getting into an elliptical orbit.

Continuous propulsion in a rocket is discussed in Section 5.4. Usually this type of propulsion is provided by an electrical system. Because the thrust from electrical propulsion is small compared to the weight of the rocket, small perturbation method is applied for solving the equations of motion. The advantage of analytical method is that parameters involved in the result are seen clearly.

Interplanetary orbits of a space vehicle are discussed in Section 5.5. The launching time is small compared to the period required for an interplanetary trip; therefore, the thrust and time for launching are considered as an impulse. The space vehicle in orbit is still modeled as a particle in central force field. Numerical results of different trajectories are collected in Table 5.2. A detailed calculation for an elliptical trajectory of a space probe traveling from Earth to Mars is given for this subject. Special attention is paid to the space probe when it reaches Mars. With a proper impulse to reduce the speed of the probe, it will get into a spiral orbit around Mars so that a long-time observation can be carried out.

## 5.1 Single-Stage Rockets

Rockets differ from air-breathing jet engines that burn fuel with surrounding air. Rockets are self-contained, carrying both fuel and oxidizer. To understand better, we must look into details about the forces acting on the rocket. In general, there are three forces: thrust, gravity, and air drag. In addition, during early development of the space program, many rocket launches failed at the launching pad. What were the reasons behind this? Finally, we want to know what are the conditions of the rocket when fuel and oxidizer are burned. All of these interesting subjects will be explored in this section.

### *Thrust*

The thrust of a rocket can be determined by examining the performance of a rocket under static tests. The rocket is arranged schematically as shown in Fig. 5.1.

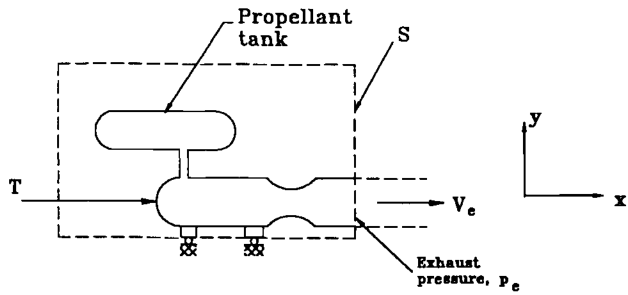


**Fig. 5.1 Rocket under static tests.**

Consider a stationary control surface that intersects the jet through the exit plane of the nozzle. Positive thrust acts in the direction opposite to $V_e$. The momentum equation for such a control volume is

$$\sum F = \frac{d}{dt}\int_v \rho V dv + \int_A \rho V(V_r \cdot dA) \tag{5.1}$$

$$\sum F = (T + A_e P_a - A_e P_e)i \tag{5.2}$$

where $V$ is the velocity of fluid, $V_r$ is the relative velocity between the fluid and the control volume, $P_a$ is the ambient pressure, $P_e$ is the exhaust pressure, and $A_e$ is the exit area of the nozzle. The first term on the right-hand side of Eq. (5.1) is

$$\frac{d}{dt}\int_v \rho V dv = 0$$

because $V = 0$. The second term is

$$\int_A \rho V(V_r \cdot dA) = \dot{m} V_e i$$

Therefore, we have the thrust,

$$T = \dot{m} V_e + A_e(P_e - P_a) \tag{5.3}$$

### *Gravity*

Because the gravitational force is inversely proportional to the distance squared between the center of the Earth and the mass center of the rocket, the gravity at different heights above the surface of the Earth can be expressed simply as

$$g = g_0\left(\frac{R_0}{R_0 + h}\right)^2 \tag{5.4}$$

where $g_0$ is the gravity at the surface of the Earth, $R_0$ is the average radius of the Earth, 6,371.23 km, and $h$ is the distance from the surface of the Earth.

### *Air Drag*

The air drag acting on the rocket can be estimated by

$$D = C_d \frac{1}{2} \rho v^2 A_f \tag{5.5}$$

where $C_d$ is the drag coefficient in the order of 0.1, $\rho$ is the air density, (0.075 lbm/ft$^3$ at sea level), $v$ is the rocket velocity, and $A_f$ is the frontal cross-sectional area of the rocket.

From Eq. (5.5), it is seen easily that the drag is a function of velocity and density of air. At the begining of the rocket journey, the velocity is very small; later on the density becomes very small. The atmospheric density is reduced to 1% of its sea-level value at an altitude of 100,000 ft. Therefore, the drag value is always much less than the thrust of a rocket. Because of that, in the estimate of conditions after burning of fuel and oxidizer, the drag term is often omitted.

### *Stability*

At the beginning of the launching process or shortly after the rocket leaves the launching pad, the forces acting on the rocket actually are thrust and gravity. It is easily seen that the thrust is produced by the exhaust gas at the exit of the nozzle. The sum of all the momentums of leaving particles $\sum_i \dot{m}_i V_{ei}$ is the major contribution to the thrust. The other part of the thrust is from pressure, which contributes a small fraction of the thrust. The vector sum of all $\dot{m}_i V_{ei}$ will locate the center of application of the thrust, C.T. as shown in Fig. 5.2. If C.T. is above the center of mass of the rocket, the situation is stable. Otherwise, the forces are not stable. The rocket most likely fails to be launched.

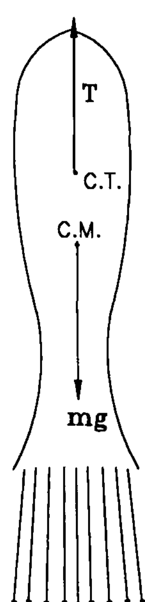


**Fig. 5.2 Stability.**

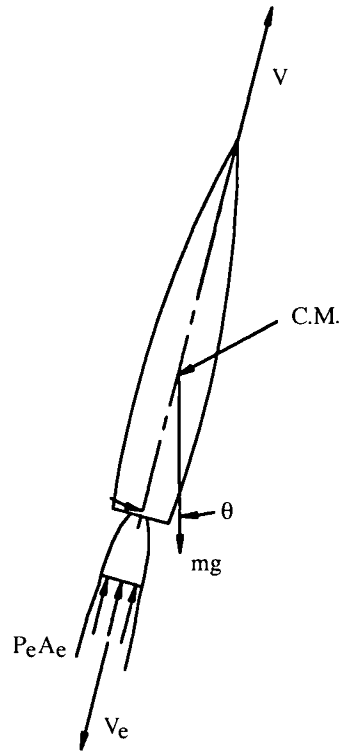


**Fig. 5.3 Motion of a rocket in gravitational field.**

One remark ought to be added here: the exit velocity $V_e$ is obviously very important to the location of C.T. However, the exit velocity is not determined completely by the contour of the nozzle. The expansion wave of the flow usually occuring at the corner of the exit will change the direction and magnitude of the exit velocity. Details of these topics are beyond the scope of this book.

### *Conditions of the Rocket at the Time of Burnout*

Consider that a rocket is launched at an angle of $\theta$ with the gravitational force as shown in Fig. 5.3. The equation of motion for the rocket along the axis of the rocket can be written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = T - D - mg\cos\theta \tag{5.6}$$

Note that as $T - D =$ net thrust denoted by $F$, the preceding equation agrees well with Eq. (2.19) $\times \sin\theta +$ Eq. (2.20) $\times \cos\theta$. Considering $T$, $D$, and $g$ in precise form, Eq. (5.6) becomes

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = (P_e - P_a)A_e + \dot{m}V_e - c_D\frac{1}{2}\rho v^2 A_f - mg_0\frac{R_0^2}{(R_0 + h)^2}\cos\theta \tag{5.7}$$

This equation can be integrated numerically, as shown previously in the integration of Eqs. (2.19) and (2.20). However, if only the major terms are kept in the equation, we can have the equation simplified to

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m}V_e - mg_0\cos\theta \tag{5.8}$$

Integrating the equation, we find

$$V_b = V_e \ln\frac{m_0}{m_b} - g_0(\cos\theta)_{\mathrm{av}}t_b \tag{5.9}$$

where $()_b$ is the quantity at the time of burnout, $m_0$ is the initial mass of the rocket, and $(\cos\theta)_{\mathrm{av}}$ is the integrated average value of $\cos\theta$. For a vertical flight the velocity is

$$V = V_e \ln\frac{m_0}{m} - g_0 t \tag{5.10}$$

where $m = m_0 - \dot{m}t$.

The altitude attained by the rocket at burnout is

$$h_b = \int_0^{t_b} v\,\mathrm{d}t = -V_e t_b \frac{\ln(m_0/m_b)}{(m_0/m_b) - 1} + V_e t_b - \frac{1}{2}g_0 t_b^2 \tag{5.11}$$

To see clearly the advantage of multistage design for rocket and save some writing, let us introduce mass ratio $R$ as

$$R = \frac{m_0}{m_b} \tag{5.12}$$

payload ratio

$$\lambda = \frac{\text{payload mass}}{\text{mass of propellant and structure}} = \frac{m_L}{m_p + m_s} \tag{5.13}$$

and the structure coefficient $\epsilon$ as

$$\epsilon = \frac{\text{structure mass}}{\text{mass of propellant and structure}} = \frac{m_s}{m_p + m_s} = \frac{m_b - m_L}{m_0 - m_L} \tag{5.14}$$

From the preceding equations it is clearly implied that

$$m_0 = m_L + m_p + m_s \tag{5.15}$$

and

$$m_b = m_L + m_s \tag{5.16}$$

Combining the expressions already introduced, the mass ratio can be written as

$$R = \frac{1+\lambda}{\epsilon+\lambda} \tag{5.17}$$

and the terminal velocity of the rocket at the burnout is

$$V_f = V_e \ln R - g_0 t_b = V_e \ln \frac{1+\lambda}{\epsilon+\lambda} - g_0 t_b \tag{5.18}$$

## 5.2 Multistage Rockets

From past observations, many rockets are designed in two or three stages. Theoretically speaking more stages always will make the terminal velocity higher. However, the practical design problem also must be considered carefully. The optimization of multistage rockets with respect to the distribution of mass has been treated in a number of interesting papers.* To simplify the problem, let us only consider the first term on the right-hand side of Eq. (5.18) and write $\Delta V_i$ for the increment of velocity of the $i$th stage of the rocket so that

$$\Delta V_i = V_e \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i} \tag{5.19}$$

The final velocity of $n$th stage is then

$$V_n = \sum_i \Delta V_i = V_e \sum_{i=1}^{n} \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i}$$

or

$$\frac{V_n}{V_e} = \sum_{l}^{n} \ln \frac{1+\lambda_i}{\epsilon_i+\lambda_i} = \sum_{i=1}^{n} F(\lambda_i) \tag{5.20}$$

Here we can maximize $V_n/V_e$ by adjusting the value of $\lambda_i$. On the other hand, for each stage, we have

$$\lambda_i = \frac{m_{0(i+1)}}{m_{0i} - m_{0(i+1)}}$$

$$\frac{m_{0i}}{m_{0(i+1)}} = \frac{1+\lambda_i}{\lambda_i}$$

where $m_{0i}$ is the initial mass of the $i$th stage of the rocket. That means

$$\frac{m_{01}}{m_L} = \frac{m_{01}}{m_{02}} \cdot \frac{m_{02}}{m_{03}} \cdots \frac{m_{0n}}{m_L} = \prod_{i=1}^{n} \left(\frac{1+\lambda_i}{\lambda_i}\right)$$

*Hill, P. G., and Peterson, C. R., *Mechanics and Thermodynamics of Propulsion,* McGraw-Hill, New York, 1983.

or

$$\frac{m_L}{m_{01}} = \prod_{i=1}^{n} \left( \frac{\lambda_i}{1+\lambda_i} \right) \tag{5.21}$$

Taking logarithmic form of the preceding equation, we obtain

$$\ell n \frac{m_L}{m_{01}} = \sum_{i=1}^{n} \ell n \left( \frac{\lambda_i}{1+\lambda_i} \right) = \sum_{i=1}^{n} G(\lambda_i) \tag{5.22}$$

which actually serves as a constraint equation for adjusting $\lambda_i$, because for a given design, the payload and the initial mass must be specified. Therefore, we reach the point that $(V_n/V_e)$ is to be maximized but subjected to the constraint equation of (5.22). This is a typical problem for the use of the Lagrange multiplier. Consider

$$L(\lambda_i) = F(\lambda_i) + \alpha G(\lambda_i) \tag{5.23}$$

where $\alpha$ is the Lagrange multiplier. Taking the derivative of Eq. (5.23) with respect to $\lambda_i$ and setting it to zero, we find

$$\frac{\partial L}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_i} + \alpha \frac{\partial G}{\partial \lambda_i} = 0$$

$$\frac{1}{1+\lambda_i} - \frac{1}{\epsilon + \lambda_i} + \frac{\alpha}{\lambda_i} - \frac{\alpha}{1+\lambda_i} = 0$$

which can be simplified to

$$\lambda_i = \frac{\alpha \epsilon_i}{1 - \alpha - \epsilon_i} \tag{5.24}$$

Then from Eq. (5.21), the Lagrange multiplier $\alpha$ can be determined by

$$\frac{m_L}{m_{01}} = \prod_{i=1}^{n} \left( \frac{\epsilon_i}{1-\epsilon_i} \right) \left( \frac{\alpha}{1-\alpha} \right) \tag{5.25}$$

or

$$\alpha = 1 \Bigg/ \left\{ 1 + \left[ \frac{m_{01}}{m_L} \prod_{i=1}^{n} \left( \frac{\epsilon_i}{1-\epsilon_i} \right) \right]^{\frac{1}{n}} \right\} \tag{5.26}$$

Then the value of $\lambda_i$ is determined by the value of $\alpha$ in Eq. (5.24).

## Example 5.1

To illustrate the advantage of multistage design, let us compare the terminal velocity of a single-stage rocket to that of a three-stage rocket. Suppose that the payload is 500 kg, the initial mass is 7500 kg, and the exhaust velocity is 3000 mps. The structure mass is 1000 kg.

*Solution.* For the single-stage rocket

$$\epsilon = \frac{m_s}{m_{01} - m_L} = \frac{1000}{7500 - 500} = 0.143$$

$$\lambda = \frac{m_L}{m_{01} - m_L} = \frac{500}{7500 - 500} = 0.0714$$

$$v_f = V_e \ln \frac{1 + \lambda}{\epsilon + \lambda} = 3000 \ln \frac{1 + 0.0714}{0.143 + 0.0714} = 4827 \text{ m/s}$$

For the three-stage rocket, by assuming

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.143$$

and from Eq. (5.26), we obtain

$$\alpha = \frac{1}{1 + (7500/500)^{\frac{1}{3}}(0.143/0.857)} = 0.70846$$

Using Eq. (5.24), we find

$$\lambda = \frac{\alpha\epsilon}{1 - \alpha - \epsilon} = \frac{0.70846 \times 0.143}{1 - 0.70846 - 0.143} = 0.68203$$

The terminal velocity at burnout is then obtained from Eq. (5.20):

$$\begin{aligned} V_f &= 3V_e \ln\left(\frac{1 + \lambda}{\epsilon + \lambda}\right) \\ &= 9000 \ln\left(\frac{1.68203}{0.143 + 0.68203}\right) \\ &= 6411 \text{ m/s} \end{aligned}$$

Certainly, this velocity is much higher than the velocity of the single-stage rocket.

## 5.3 Motion of a Particle in Central Force Field

Consider a system of two particles with mass $m_1$ and $m_2$. Let the center of mass $m_2$ be at the origin of $x$–$y$ plane. This plane contains the trajectory of $m_1$. Furthermore, let us consider the case $m_2 \gg m_1$ and write $M$ for $m_2, m$ for $m_1$. With the use of polar coordinates $(r, \theta)$, Lagrange's function for $m$ is

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Then the equations of motion for $m$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{5.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \tag{5.28}$$

From Eq. (5.28), we obtain the momentum in $\theta$ direction as

$$mr^2\dot{\theta} = \mathcal{L} \tag{5.29}$$

where $\mathcal{L}$ is a constant. This means that, as the particle moves in a central force field, its angular momentum is constant. With the information of Eq. (5.29), Eq. (5.27) becomes

$$m\ddot{r} - \frac{\mathcal{L}^2}{mr^3} = -\frac{\partial V}{\partial r} = F(r) \tag{5.30}$$

$F(r)$ is the force in the $r$ direction. Because the potential energy of the particle is a function of $r$ only, the force is a function of $r$. Equation (5.30) actually defines $r(t)$.

To solve Eq. (5.30), we use the inverse square law for the force, i.e.,

$$F(r) = -\frac{GMm}{r^2} \tag{5.31}$$

where $G$ is the universal gravitational constant $= 6.670 \times 10^{-11}$ N·m$^2$/kg$^2$. Because $M$ and $m$ are known quantities, the force may be written simply as

$$F(r) = -\frac{k}{r^2}$$

where $k = GMm$. Now the equation becomes

$$m\ddot{r} - \frac{\mathcal{L}^2}{mr^3} = -\frac{k}{r^2} \tag{5.32}$$

To solve this equation analytically, we rearrange the equation. Because

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta} = \frac{\mathcal{L}}{mr^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \left(\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\left(\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)$$

Eq. (5.32) now becomes

$$\frac{\mathcal{L}}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\left(\frac{\mathcal{L}}{mr^2}\right)\frac{\mathrm{d}r}{\mathrm{d}\theta}\right]-\frac{\mathcal{L}^2}{mr^3}=-\frac{k}{r^2} \tag{5.33}$$

The preceding equation can be simplified further by changing the variable. Let $\mu = 1/r$, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta}=-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}$$

And we can write Eq. (5.33) as

$$-\mathcal{L}\mu^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathcal{L}}{m}\frac{\mathrm{d}\mu}{\mathrm{d}\theta}\right)-\frac{\mathcal{L}^2}{m}\mu^3=-k\mu^2$$

Simplifying leads to

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\theta^2}+\mu=\frac{mk}{\mathcal{L}^2} \tag{5.34}$$

Without losing generalization, we can write the solution of Eq. (5.34) as

$$\mu=\frac{mk}{\mathcal{L}^2}[1+\varepsilon\cos(\theta-\theta')] \tag{5.35}$$

where $\varepsilon$ and $\theta'$ are arbitrary constants of integration. To determine these constants, we put back the symbol $r$ for $1/\mu$.

$$r=\frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos(\theta-\theta')} \tag{5.36}$$

Differentiating Eq. (5.36) with respect to $\theta$, we find

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{\varepsilon\mathcal{L}^2}{mk}\frac{\sin(\theta-\theta')}{[1+\varepsilon\cos(\theta-\theta')]^2} \tag{5.37}$$

On the trajectory of $m$, there is a point called an apsidal point. At such a point, the $r$ is not changed as $\theta$ changes. Let us choose the $(r,\theta)$ coordinates in such a way that $\theta-\theta'=0$ at one apsidal point. On the other hand, using Eqs. (5.36) and (5.37), we have

$$\dot{r}=\frac{\mathcal{L}}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{\mathcal{L}}{mr^2}\left(\frac{\varepsilon\mathcal{L}^2}{mk}\right)\frac{\sin(\theta-\theta')}{[1+\varepsilon\cos(\theta-\theta')]^2}$$

$$=\frac{\mathcal{L}}{m}\left(\frac{mk}{\mathcal{L}^2}\right)^2\left(\frac{\varepsilon\mathcal{L}^2}{mk}\right)\sin(\theta-\theta')$$

$$\dot{r}=\frac{\varepsilon k}{\mathcal{L}}\sin(\theta-\theta') \tag{5.38}$$

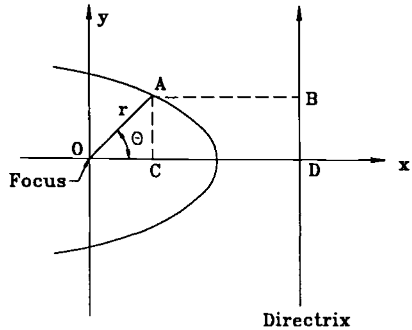


**Fig. 5.4 Geometry of a conic curve.**

The total energy of $m$ can be written as

$$E = T + V = \frac{1}{2}m\dot{r}^2 + \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r} = (\varepsilon^2 - 1)\frac{mk^2}{2\mathcal{L}^2}$$

Equations (5.36) and (5.38) are used in the process deriving the preceding equation. Hence

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} \tag{5.39}$$

Now the trajectory equation is

$$r = \frac{(\mathcal{L}^2/mk)}{1 + \varepsilon \cos\theta} \tag{5.40}$$

To understand the meaning of Eq. (5.40), let us review a part of analytical geometry for conic curves. A conic curve is defined as the locus of a point moving such that the ratio of its distance from a fixed point, the focus, to its distance from a fixed line, the directrix, is a constant $\varepsilon$. From Fig. 5.4, we have

$$\varepsilon = \frac{r}{AB}$$

or

$$r = \varepsilon(AB) = \varepsilon(CD) = \varepsilon(OD - r\cos\theta)$$

Rearranging leads to

$$r(1 + \varepsilon\cos\theta) = \varepsilon \cdot OD = \text{const} = C$$

Therefore

$$r = \frac{C}{1 + \varepsilon\cos\theta} \tag{5.41}$$

**Table 5.1 Different values of $\epsilon$ and $E$ for different orbits**

| Eccentricity, $\varepsilon$ | Energy, $E$ | Type of orbit |
|---|---|---|
| $>1$ | $>0$ | Hyperbola |
| $=1$ | $=0$ | Parabola |
| $<1$ but $>0$ | $<0$ | Ellipse |
| $=0$ | $-mk^2/(2\mathcal{L}^2)$ | Circle |

Compare this equation with Eq. (5.40); we find

$$C = \mathcal{L}^2/(mk)$$

That means the orbit of the particle in a central force field can be one of the conic curves. The focus is the center of central force field. Different conic curves result from different values of $\varepsilon$, which is called eccentricity. Because $E$ is directly related to $\varepsilon$, different orbits for different $\varepsilon$ and $E$ are given in Table 5.1.

Because the total energy of the particle $m$ dictates the type of orbit, let us look into the meaning of $E < 0$, i.e.,

$$E = T + V < 0$$

$$T < -V(r) = \frac{k}{r} = \frac{GMm}{r}$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) < \frac{GMm}{r}$$

or

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) < \frac{GM}{r}$$

The preceding equation says for an elliptical orbit, the velocity of the particle must be less than $\sqrt{2GM/r}$. As the velocity reaches the limiting value of $\sqrt{2GM/r}$, the particle will get on the parabolic orbit and will not come back. Hence this velocity is termed the escape velocity:

$$V_{\text{esc}} = \sqrt{\frac{2GM}{r}} \tag{5.42}$$

On the other hand, for a circular orbit, $\varepsilon = 0$. Let us examine the meaning of $\varepsilon = 0$. As

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} = 0$$

that means

$$1 + \frac{2\mathcal{L}^2 E}{mk^2} = 0$$

$$E = -\frac{mk^2}{2\mathcal{L}^2}$$

$$T + V = -\frac{mk^2}{2\mathcal{L}^2}$$

$$\frac{1}{2}m(\dot{r} + r^2\dot{\theta}^2) - \frac{GMm}{r} = -\frac{1}{2}m\frac{(GM)^2}{(r^2\dot{\theta})^2}$$

For a circular orbit, $\dot{r} = 0$, and simplifying the preceding equation, we find

$$\frac{1}{2}r^2\dot{\theta}^2 = \frac{GM}{r} - \frac{(GM)^2}{2(r^2\dot{\theta})^2}$$

Using $v = r\dot{\theta}$, we have

$$\frac{1}{2}v^2 = \frac{GM}{r} - \frac{(GM)^2}{2r^2v^2}$$

$$(v^2 - GM/r)^2 = 0$$

$$v_{\text{cir}} = \sqrt{GM/r} \tag{5.43}$$

That means

$$v_{\text{esc}} = \sqrt{2}v_{\text{cir}}$$

and for an elliptical orbit, the velocity must satisfy the condition

$$v_{\text{cir}} < v_{\text{ell}} < v_{\text{esc}}$$

or

$$\sqrt{GM/r} < v_{\text{ell}} < \sqrt{2GM/r} \tag{5.44}$$

Just to have some feeling of the velocity of a planet on a circular orbit, let us calculate the velocity of the Earth around the sun. We have

$$v_{\text{Earth}} = \sqrt{\frac{GM_{\text{sun}}}{R_{\text{Earth}}}}$$

$$M_{\text{sun}} = 1.9866158 \times 10^{30}\ \text{kg}$$

$$R_{\text{Earth}} = 1.495 \times 10^{11}\ \text{m}$$

$$v_{\text{Earth}} = \sqrt{\frac{6.67 \times 10^{-11} \times 1.9866158 \times 10^{30}}{1.495 \times 10^{11}}} = 29771.4\ \text{m/s}$$

$$\simeq 30\ \text{km/s}$$

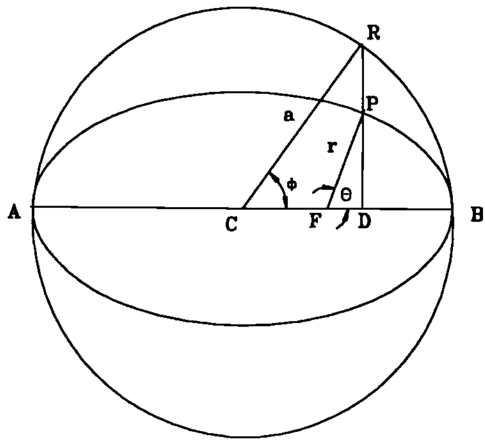


Fig. 5.5 Ellipse and auxiliary circle.

### *Elliptical Orbits*

In studying the orbits of satellites around the Earth, we are interested more in elliptical orbits. Let us look further into details about them. From Fig. 5.5, we have

$$FC = CD - FD = a\cos\phi - r\cos\theta$$

Also, we have

$$FC = a - BF = a - r_{\min} = a - \frac{\mathcal{L}^2}{mk(1+\varepsilon)}$$

Because

$$r_{\max} = \frac{\mathcal{L}^2}{mk(1-\varepsilon)}$$

$$r_{\min} + r_{\max} = \frac{\mathcal{L}^2}{mk}\left[\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right]$$

$$= \frac{2\mathcal{L}^2}{mk(1-\varepsilon^2)} = 2a$$

we find

$$a = \frac{\mathcal{L}^2}{mk(1-\varepsilon^2)} \tag{5.45}$$

$$r_{\min} = a(1-\varepsilon) \tag{5.46}$$

and

$$
\begin{aligned}
FC &= a - a(1-\varepsilon) = a\varepsilon \\
&= a\cos\phi - r\cos\theta = a\cos\phi - \frac{\mathcal{L}^2/mk - r}{\varepsilon} \\
&= a\cos\phi - \frac{a(1-\varepsilon^2)}{\varepsilon} + \frac{r}{\varepsilon}
\end{aligned}
$$

$$
r = a(1 - \varepsilon\cos\phi) \tag{5.47}
$$

So far, we have found the orbital equation $r = r(\theta)$ or $r = r(\phi)$, but there is no equation to have time $t$ explicitly involved. To relate $\phi$ to the time, let us start from the total energy $E$:

$$
E = \frac{m}{2}\dot{r}^2 + \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r}
$$

so that

$$
\dot{r} = \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{\mathcal{L}^2}{2mr^2}\right)}, \qquad \mathrm{d}r \Bigg/ \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{\mathcal{L}^2}{2mr^2}\right)} = \mathrm{d}t \tag{5.48}
$$

Because $E = \text{const}$, it remains the same at any value of $r$. Let us consider $E$ at $r = r_{\min}$.

$$
\begin{aligned}
E = T + V &= \frac{1}{2}mr_{\min}^2\dot{\theta}^2 - \frac{k}{r_{\min}} \\
&= \frac{1}{2}\frac{\mathcal{L}^2}{mr_{\min}^2} - \frac{k}{r_{\min}} \\
&= \frac{1}{2m}\left\{\mathcal{L}^2 \Big/ \left[\frac{\mathcal{L}^2}{mk(1+\varepsilon)}\right]^2\right\} - \left[k \Big/ \frac{\mathcal{L}^2}{mk(1+\varepsilon)}\right] \\
&= \frac{mk^2}{2\mathcal{L}^2}(\varepsilon^2 - 1) = -\frac{k}{2a}
\end{aligned} \tag{5.49}
$$

Substituting Eq. (5.49) into Eq. (5.48), we have

$$
\begin{aligned}
\mathrm{d}t &= r\,\mathrm{d}r \Bigg/ \sqrt{\frac{2}{m}\left(-\frac{k}{2a}r^2 + kr - \frac{\mathcal{L}^2}{2m}\right)} = \frac{\sqrt{ma/k}\,r\,\mathrm{d}r}{\sqrt{(a\varepsilon)^2 - (r-a)^2}} \\
&= \frac{\sqrt{ma/k}\,a(1-\varepsilon\cos\phi)a\varepsilon\sin\phi\,\mathrm{d}\phi}{\sqrt{(a\varepsilon)^2 - (-a\varepsilon\cos\phi)^2}} = \frac{\sqrt{ma}}{k}a(1-\varepsilon\cos\phi)\,\mathrm{d}\phi
\end{aligned} \tag{5.50}
$$

Furthermore, because $mr^2\dot{\theta} = \mathcal{L} = \text{const}$ or $r^2\dot{\theta} = \text{const}$,

$$\tfrac{1}{2}r^2 \mathrm{d}\theta = \mathrm{d}A$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \text{const} = \frac{\mathcal{L}}{2m} = \frac{\pi ab}{T}$$

This is known as Kepler's law of areas. Where $T$ is the period of the motion, $\pi ab$ is the area of the ellipse. Making use of the relations from analytical geometry,

$$b = a\sqrt{1-\varepsilon^2}$$

$$\varepsilon = (c/a)$$

we find

$$T = \frac{2\pi abm}{\mathcal{L}} = 2\pi a^2\sqrt{1-\varepsilon^2}\frac{m}{\mathcal{L}}$$

$$= 2\pi a^2 \frac{\sqrt{1-\varepsilon^2}m}{\sqrt{mka(1-\varepsilon^2)}} = 2\pi\sqrt{\frac{ma^3}{k}} \tag{5.51}$$

Using Eq. (5.51) in Eq. (5.50), we obtain

$$\mathrm{d}t = \frac{T}{2\pi}(1-\varepsilon\cos\phi)\mathrm{d}\phi$$

Therefore

$$\frac{2\pi t}{T} = \phi - \varepsilon\sin\phi \tag{5.52}$$

Collecting all the results together, now we have

$$r = \frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos\theta}, \qquad \varepsilon = \sqrt{1+\frac{2\mathcal{L}^2E}{mk^2}}$$

$$r = a(1-\varepsilon\cos\phi), \qquad a = \frac{\mathcal{L}^2}{mk}\frac{1}{(1-\varepsilon^2)}$$

$$\frac{2\pi t}{T} = \phi - \varepsilon\sin\phi, \qquad T = 2\pi\sqrt{\frac{ma^3}{k}}$$

### Example 5.2

Consider a particle that is moving in an elliptical orbit about a fixed focus because of an inverse-square law of attraction. 1) Find the points in the orbit at which the magnitude of the radial velocity $\dot{r}$ is maximum, and 2) prove that the possible values of corresponding $\dot{\theta}$ are

$$\sqrt{\frac{k}{mr^3}\left(1-\frac{b}{a}\right)} < \dot{\theta} < \sqrt{\frac{k}{mr^3}\left(1+\frac{b}{a}\right)}$$

*Solution.* 1) Rewrite the equations of motion for a particle in an elliptical orbit:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2} \tag{5.53}$$

$$mr^2\dot{\theta} = \mathcal{L} \tag{5.54}$$

As $\dot{r} \to \dot{r}_{\max}$, $\ddot{r} = 0$, then from Eq. (5.53), we have

$$mr\dot{\theta}^2 = \frac{k}{r^2}$$

$$mr\left(\frac{\mathcal{L}}{mr^2}\right)^2 = \frac{k}{r^2}$$

or

$$r = \frac{\mathcal{L}^2}{mk} \tag{5.55}$$

On the other hand,

$$r = \frac{\mathcal{L}^2/(mk)}{1+\varepsilon\cos\theta}$$

$$\dot{r} = \frac{k\varepsilon\sin\theta}{\mathcal{L}}$$

$$\ddot{r} = \frac{k\varepsilon}{\mathcal{L}}\cos\theta\dot{\theta} = 0$$

That means $\theta = \pi/2$ or $3\pi/2$:

$$\dot{r}_{\max} = \frac{k\varepsilon}{\mathcal{L}} \quad \text{at} \quad \theta = \frac{\pi}{2}, \qquad r = \frac{\mathcal{L}^2}{mk} \tag{5.56}$$

2) For a particle in an elliptical orbit,

$$E = \frac{1}{2}m(\dot{r} + r^2\dot{\theta}^2) - \frac{k}{r} < 0$$

$$\dot{\theta}^2 < \frac{1}{r^2}\left(\frac{2k}{mr} - \dot{r}^2\right) = \frac{2k}{mr^3} - \left(\frac{\varepsilon k}{\mathcal{L}}\right)^2\frac{1}{r^2} = \frac{2k}{mr^3} - \frac{\varepsilon^2k^2}{m^2r^6\dot{\theta}^2}$$

or

$$\dot{\theta}^4 - \frac{2k}{mr^3}\dot{\theta}^2 + \frac{\varepsilon^2 k^2}{m^2 r^6} < 0 \tag{5.57}$$

Consider

$$f(\dot{\theta}^2) = \dot{\theta}^4 - \frac{2k}{mr^3}\dot{\theta}^2 + \frac{\varepsilon^2 k^2}{m^2 r^6} = 0$$

$$\dot{\theta}^2 = \frac{1}{2}\left[\frac{2k}{mr^3} \pm \sqrt{\frac{4k^2}{m^2 r^6} - \frac{4\varepsilon^2 k^2}{m^2 r^6}}\right]$$

$$= \frac{k}{mr^3}[1 \pm \sqrt{1-\varepsilon^2}] = \frac{k}{mr^3}\left(1 \pm \frac{b}{a}\right) = \begin{cases}\dot{\theta}_1^2 \\ \dot{\theta}_2^2\end{cases} \tag{5.58}$$

Therefore, choosing the values of $\dot{\theta}$ between the two roots from Eq. (5.58), Eq. (5.57) is satisfied, i.e.,

$$f(\dot{\theta}^2) = (\dot{\theta}^2 - \dot{\theta}_1^2)(\dot{\theta}^2 - \dot{\theta}_2^2) < 0$$

### Example 5.3

A weather satellite is to be launched. The requirement of such a satellite is that it must stay above the same point on the surface of the Earth all the time. Determine the radius of the circular orbit above a point located along the line from the center of Earth perpendicular to the Earth's rotating axis with mass of Earth $= 5.975 \times 10^{24}$ kg.

*Solution.* For a circular orbit, the velocity of the satellite is

$$v = \sqrt{GM/r}$$

Because the satellite must be moving with $v = r\omega$, where $\omega$ is the rotating speed of the Earth,

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = 7.2722 \times 10^{-5} \text{ rad/s}$$

$$\omega r = \sqrt{GM/r}$$

we find

$$r = \frac{(GM)^{1/3}}{\omega^{2/3}} = \frac{(6.67 \times 10^{-11} \times 5.975 \times 10^{24})^{1/3}}{(7.2722 \times 10^{-5})^{2/3}}$$

$$= 4.22387 \times 10^4 \qquad \text{(km)}$$

### Example 5.4

A satellite enters its orbit at a velocity of 8045 m/s at an altitude of 644 km. The velocity is parallel to the Earth's surface. Find the equation for the orbit and the maximum altitude from the Earth's surface the satellite will reach. The average radius of Earth is 6436 km and the mass of Earth is $5.975 \times 10^{24}$ kg.

*Solution.* From the given data, we have

$$r = 644{,}000 + 6{,}436{,}000 = 7{,}080{,}000 \qquad \text{(m)}$$

$$\mathcal{L} = mr^2\dot{\theta} = mrv = m(7{,}080{,}000)(8045)$$

$$= m(5.69586 \times 10^{10}) \qquad (\text{kg} \cdot \text{m}^2/\text{s})$$

$$E = \frac{\mathcal{L}^2}{2mr^2} - \frac{k}{r}$$

$$k = GMm = (6.67 \times 10^{-11})(5.975 \times 10^{24})m$$

$$= m(3.9853 \times 10^{14}) \qquad (\text{N} \cdot \text{m}^2)$$

$$E = \frac{(m5.69586 \times 10^{10})^2}{2m(7{,}080{,}000)^2} - \frac{m(3.9853 \times 10^{14})}{7{,}080{,}000}$$

$$= -m(23{,}928{,}536) \qquad (\text{N} \cdot \text{m})$$

$$\varepsilon = \sqrt{1 + \frac{2\mathcal{L}^2 E}{mk^2}} = \sqrt{1 - \frac{2(m5.69586 \times 10^{10})^2 m(23{,}928{,}536)}{m(m3.9853 \times 10^{14})^2}} = 0.1498$$

$$\mathcal{L}^2/(mk) = \frac{(m5.69586 \times 10^{10})^2}{m^2(3.9853 \times 10^{14})} = 8{,}140{,}622 \qquad \text{(m)}$$

Hence, the orbital equation is

$$r = \frac{\mathcal{L}^2/(mk)}{1 + \varepsilon \cos\theta} = \frac{8140622}{1 + 0.1498\cos\theta}$$

$$r_{\max} = 9{,}574{,}950 \qquad \text{(m)} \qquad \text{as } \theta = \pi$$

The maximum altitude is

$$h_{\max} = r_{\max} - r_{\text{Earth}} = 9{,}574{,}950 - 6{,}436{,}000$$

$$= 3{,}138{,}950 \qquad \text{(m)}$$

$$= 3139 \qquad \text{(km)}$$

## 5.4 Space Vehicle with Electrical Propulsion (equations solved by small perturbation method)

Electrical propulsion systems are known very low in thrust as compared to the gravity of the space vehicle at the Earth's surface. Because electrons are emitted from the sun constantly, space vehicles can collect electrons in orbit and the electrical propulsion system can function properly while the vehicle is traveling in space. Consider that a low thrust is oriented along the tangential direction of the orbit. The equations of motion are then

$$m(\ddot{r} - r\dot{\theta}^2) = -(k/r^2) \tag{5.59}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \Gamma_\theta \tag{5.60}$$

where $\Gamma_\theta$ is the electrical thrust in the tangential direction. To solve Eqs. (5.59) and (5.60), we introduce dimensionless variables as follows:

$$\rho = \frac{r}{r_0}, \qquad \tau = \sqrt{\frac{GM}{r_0^3}}t, \qquad \nu = \frac{\Gamma_\theta}{mg} = \frac{\Gamma_\theta r_0^2}{GMm} \tag{5.61}$$

where $r_0$ is the initial orbit radius and $g$ is the gravitational acceleration at the initial orbit radius. Now we can write

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{GM}{r_0}}\frac{\mathrm{d}\rho}{\mathrm{d}\tau} \tag{5.62}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{GM}{r_0^3}}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} \tag{5.63}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{GM}{r_0^2}\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} \tag{5.64}$$

With the use of these expressions, Eq. (5.60) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\rho^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) = \nu\rho \tag{5.65}$$

and Eq. (5.59) becomes

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} = \rho\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - \frac{1}{\rho^2} \tag{5.66}$$

Recall that the parameter $\nu$ is the ratio of the thrust from electrical rocket to the gravitational force at the beginning point while $r = r_0$ and is in the order of $10^{-3}$. Therefore, this is a typical case to be solved by small perturbation method. To solve Eqs. (5.65) and (5.66), the initial conditions are assumed to be $\ddot{r} = \dot{r} = 0$ and $r = r_0$; the thrust is initiated at $t = 0$, and

$$\dot{\theta} = \sqrt{GM/r_0^3}$$

In dimensionless variables, that means

$$\rho = 1, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} = 0 \tag{5.67}$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 1, \quad \theta = 0 \tag{5.68}$$

From Eq. (5.66) we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \left[\frac{1}{\rho}\left(\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} + \frac{1}{\rho^2}\right)\right]^{\frac{1}{2}} \tag{5.69}$$

Substituting Eq. (5.69) into Eq. (5.65), we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\rho^3\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2} + \rho\right]^{\frac{1}{2}} = \nu\rho \tag{5.70}$$

To solve this equation, let us assume the solution can be expressed as

$$\rho = \rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots \tag{5.71}$$

where $\rho$ is a function of $\tau$ and has a magnitude in order of unity. With the use of Eq. (5.71), Eq. (5.70) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg[(\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)^3\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)$$
$$+ (\rho_0 + \nu\rho_1 + \nu^2\rho_2 + \cdots)\Bigg]^{\frac{1}{2}}$$
$$= \nu\rho_0 + \nu^2\rho_1 + \cdots$$

After carrying out the product in the preceding equation and breaking down the terms according to the orders of $\nu$, we find the following equations. To the zeroth order of $\nu$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\rho_0^3\frac{\mathrm{d}^2\rho_0}{\mathrm{d}\tau^2} + \rho_0\right]^{\frac{1}{2}} = 0 \tag{5.72}$$

or

$$\rho_0^3\frac{\mathrm{d}^2\rho_0}{\mathrm{d}\tau^2} + \rho_0 = c$$

The solution of this nonlinear differential equation is simply

$$\rho_0 = c$$

When the initial condition is applied, we find

$$\rho_0 = 1 \tag{5.73}$$

To the first order of $\nu$, we have

$$\frac{1}{2}\frac{d}{d\tau}\left(\frac{d^2\rho_1}{d\tau^2}+\rho_1\right)=1 \tag{5.74}$$

The solution of Eq. (5.74) is

$$\rho_1 = 2\tau - 2\sin\tau \tag{5.75}$$

Therefore, the solution of Eq. (5.70) up to the first order of $\nu$ is

$$\rho = 1 + \nu(2\tau - 2\sin\tau) + 0(\nu^2) \tag{5.76}$$

With the use of Eq. (5.76) in Eq. (5.69), the solution for $\theta(\tau)$ is obtained as

$$\theta = \tau - \nu(4\cos\tau + 1.5\tau^2 - 4) + 0(\nu^2) \tag{5.77}$$

From Eqs. (5.76) and (5.77) it is clear that the trajectory of the space vehicle is a spiral. The increment of the radius is proportional to the tangential thrust and the initial angular speed. The results are plotted in Fig. 5.6. Equations (5.65) and (5.66) can be solved numerically by the Runge–Kutta method. The disadvantage of numerical method is that the parameters involved cannot be seen immediately.

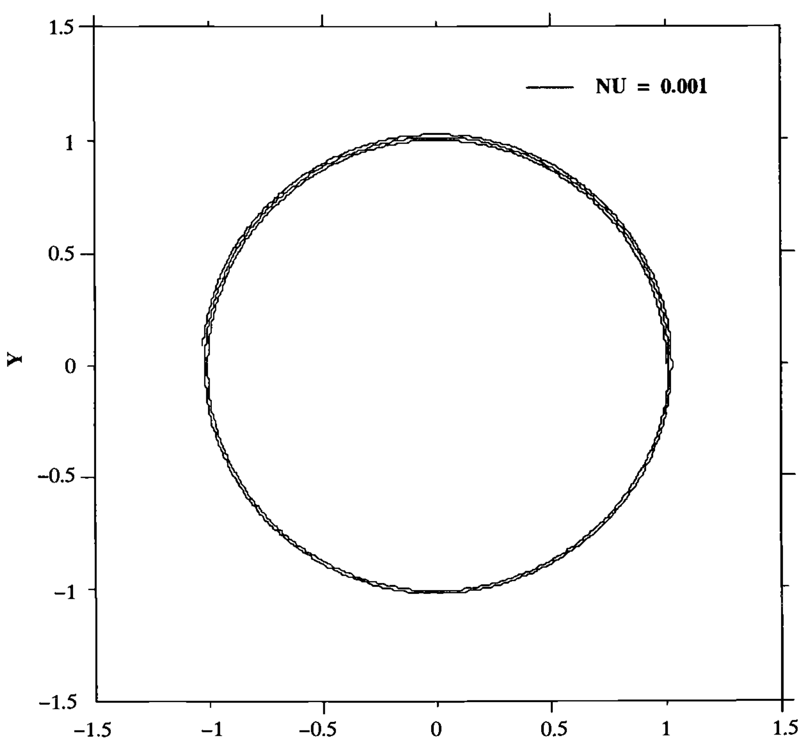


**Fig. 5.6 Spiral orbit of an electrical rocket.**

## 5.5 Interplanetary Trajectories

As a space vehicle moves in space, there is often more than one gravitational force acting on it. Therefore the equation of motion for the vehicle can be written as

$$m\ddot{\boldsymbol{r}} = \sum_i \boldsymbol{F}_i \tag{5.78}$$

where $\boldsymbol{F}_i = (GM_i m/|\boldsymbol{r} - \boldsymbol{r}_i|^3)(\boldsymbol{r}_i - \boldsymbol{r}) =$ the gravitational force from $M_i$. For example, when a spaceship travels from Earth to Mars, it is subjected to the gravitational forces from Earth, the sun, and Mars. However as the spaceship leaves Earth, the gravitational force will shift from Earth to the sun. To estimate the gravitational force from Earth, it is found that

$$F_{\text{Earth}} < \tfrac{1}{10} F_{\text{sun}}$$

as the spaceship moves away from the Earth by 1/1000 of the circumference of the Earth orbit around the sun. Hence it is not a bad approximation that the whole journey is divided into three segments; in each segment the ship is subjected to one gravitational force, so that the equations and solutions developed in Section 5.3 can be applied. The first segment is for the Earth's gravitational field, the second is for the sun's gravitational field, and the third for Mars's field. As the space vehicle reaches the escape velocity from the surface of Earth, it will stay in the Earth's circular orbit around the sun. In the second part of the journey, the trajectory is elliptical and is called transfer orbit. The last part of the journey is in Mars's circular orbit with the radius larger than that of Earth's circular orbit. Hohmann* studied the interplanetary trajectory first and used three impulses for the Earth to Mars journey. The total velocity increment for the vehicle to reach the Mars orbit is

$$\Delta U_{\text{Hohmann}} = U_{\text{esc}} + \Delta U_T + \Delta U_{\text{Mars}} \tag{5.79}$$

where $U_{\text{esc}}$ is the velocity required for the vehicle to escape the gravitational field of Earth, $\Delta U_T$ is the increment of velocity as the vehicle moves from the Earth's circular orbit around the sun to the elliptical transfer orbit at $r = r_{\text{Earth}}$, and $\Delta U_{\text{Mars}}$ is the increment of velocity as the vehicle moves from the elliptical transfer orbit at $r = r_{\text{Mars}}$ to the circular orbit of Mars around the sun.

In the Hohmann treatment, the energy per unit mass required to put the vehicle into the transfer orbit is

$$E_{\text{Hofmann}} = E_{\text{esc}} + \tfrac{1}{2}(\Delta U_T)^2 = \tfrac{1}{2}(\Delta U_T)^2 \tag{5.80}$$

Because $E_{\text{esc}} =$ the escape energy $= \frac{1}{2}U_{\text{esc}}^2 - (GM_e/R_e) = 0$, where $R_e =$ radius of the Earth.

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*Hohmann, W., "Die Erreichbarkeit der Himmelskorper (The Attainability of Heavenly Bodies)," NASA Technical Translations F-44, 1960.

Oberth* treated the problem slightly differently. He considered a higher terminal velocity for the space vehicle leaving the Earth, so that the vehicle can get into the transfer orbit directly from the first impulse of the rocket

$$E_{\text{Oberth}} = \frac{U_{\text{Oberth}}^2}{2} - \frac{GM_e}{R_e}$$

Therefore, the total energy per unit mass at the burning out time must equal to $(\Delta U_T)^2/2$, i.e.,

$$\frac{1}{2}U_{\text{Oberth}}^2 - \frac{GM_e}{R_e} = \frac{1}{2}(\Delta U_T)^2$$

because

$$\frac{1}{2}U_{\text{esc}}^2 = \frac{GM_e}{R_e}$$

Hence

$$U_{\text{Oberth}} = \sqrt{U_{\text{esc}}^2 + (\Delta U_T)^2} \tag{5.81}$$

where $\Delta U_T = U_T - U_e$, $U_T$ is the velocity of the spaceship on the transfer orbit, and $U_e$ is its velocity on the circular orbit of Earth around the sun. From the total energy of the spaceship and Eq. (5.49), we have

$$E = \frac{mU_T^2}{2} - \frac{GM_{\text{sun}}m}{r_{\text{Earth}}} = -\frac{GM_{\text{sun}}m}{2a}$$

Hence,

$$U_T = \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Earth}}} - \frac{1}{a}\right)}$$

With the velocity given in Eq. (5.81) as the first impulse and the increment of velocity from the elliptical transfer orbit at $r = r_{\text{Mars}}$ to the circular orbit of Mars $\Delta U_{\text{Mars}}$, the total increment for the whole journey is accomplished in two impulses and may be expressed as

$$\Delta u_{\text{Oberth}} = U_{\text{Oberth}} + \Delta U_{\text{Mars}} \tag{5.82}$$

Based on the treatment outlined, several trajectories are studied. The results are collected in Table 5.2. The trajectories are shown in Fig. 5.7. Details of two impulses for a space vehicle to reach Mars are given in Example 5.5.

## Example 5.5

Suppose that we send a space probe from Earth to Mars. When the probe reaches Mars, it will get into a spiral orbit around Mars to make close observations. The

*For Oberth's approach, see Hill, P. G., and Peterson, C. R., *Mechanics and Thermodynamics of Propulsion*, McGraw-Hill, New York, 1983.

**Table 5.2 Characteristics of different trajectories**

| Name of trajectory | Eccentricity, $\varepsilon$ | $U_{\text{Oberth}}$, km/s | $\Delta U_{\text{Mars}}$, km/s | Time required, days |
|---|---|---|---|---|
| Hohmann | 0.2075 | 11.6 | 2.6 | 256 |
| Ellipse 1 | 0.2525 | 11.7 | −2.6[a] | 175 |
| Ellipse 2 | 0.3418 | 12.1 | −5.0[a] | 135 |
| Parabolic | 1 | 16.7 | −16.9[a] | 70 |

[a]Note that the explanation of the case "Ellipse 1" is given in Example 5.5. The detailed expressions for the velocity vectors of the space vehicle and Mars for "Ellipse 2" and "Parabolic trajectories" are given at the end of Example 5.5. To verify these two cases, see the exercises in Problems 5.10 and 5.11.

length of the major axis is chosen as $4.0 \times 10^{11}$ m for the elliptic trajectory with the center of the sun as the focus. 1) Determine the impulse required for the probe leaving Earth. 2) Determine the required impulse to reduce the velocity of the space probe so that it will have the orbit spiraling down to the surface of Mars. 3) Find the traveling time for the probe from Earth to the Mars circular orbit around the sun.

*Solution.* 1) Take the radius of the Earth's circular orbit as the $r_{\min}$ of the elliptic orbit. It is known that $r_{\text{Earth}} = 1.495 \times 10^{11}$ m. Therefore,

$$\begin{aligned} c &= a - r_{\text{Earth}} \\ &= (2.0 - 1.495) \times 10^{11} = 0.505 \times 10^{11} \qquad \text{(m)} \end{aligned}$$

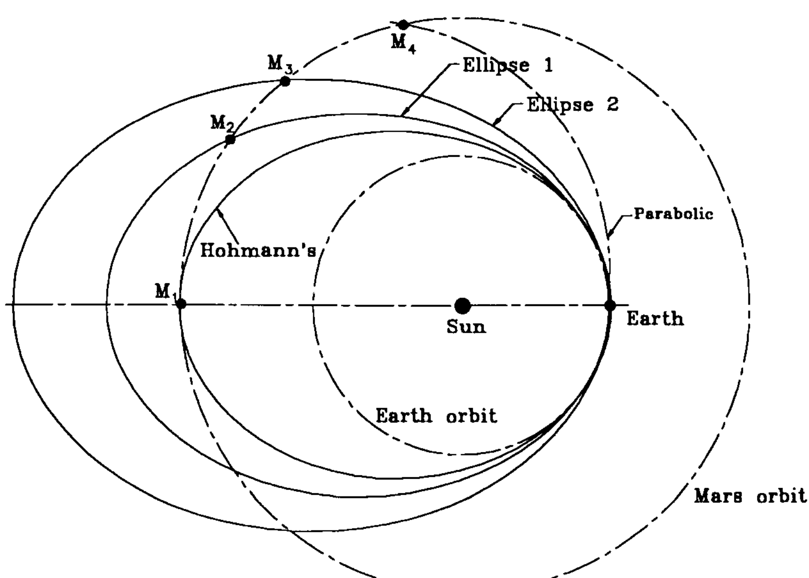


**Fig. 5.7 Four different trajectories for Earth–Mars journey.**

and

$$\varepsilon = \frac{c}{a} = 0.2525.$$

On the transfer orbit, the velocity of the probe at the surface of Earth is

$$\begin{aligned} U_T &= \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Earth}}} - \frac{1}{a}\right)} \\ &= \sqrt{(6.67 \times 10^{-11} \times 1.9866 \times 10^{30})\left(\frac{2}{1.495 \times 10^{11}} - \frac{1}{2 \times 10^{11}}\right)} \\ &= 3.3343 \times 10^4 \qquad \text{(m/s)} \end{aligned}$$

The speed of Earth in the circular orbit around the sun is

$$U_e = \sqrt{\frac{GM_{\text{sun}}}{r_{\text{Earth}}}} = 2.98 \times 10^4 \qquad \text{(m/s)}$$

and the escape velocity of the space probe leaving the Earth is

$$U_{\text{esc}} = \sqrt{\frac{2GM_e}{R_e}} = 1.118 \times 10^4 \qquad \text{(m/s)}$$

where $R_e$ is the radius of the Earth.

To launch the space probe into the transfer orbit directly from the surface of Earth, the required impulse is

$$U_{\text{Oberth}} = \sqrt{U_{\text{esc}}^2 + (U_T - U_e)^2} = 1.1724 \times 10^4 \qquad \text{(m/s)}$$

2) To find the required impulse to reduce the velocity of the space probe so that it can spiral down to Mars, we must determine first the intersection point between the elliptic transfer orbit and the Mars circular orbit, then the velocity of the space probe and the relative velocity between the probe and the Mars at that point. From the study of the auxiliary circle of elliptic orbit, we have

$$r = a(1 - \varepsilon \cos\phi)$$

At the intersection point $r = r_{\text{Mars}} = 2.278 \times 10^{11}$ m, we find

$$\phi = 2.15375 \qquad \text{(rad)}$$

and

$$\cos\theta = (1/r)(a\cos\phi - c) = -0.7050$$

$$\theta = 2.3532 \qquad \text{(rad)}$$

Hence the intersection point is at $r = 2.278 \times 10^{11}$ m and $\theta = 2.3532$ rad. On the transfer orbit with $r = r_{\text{Mars}}$, we have

$$U_T = \sqrt{GM_{\text{sun}}\left(\frac{2}{r_{\text{Mars}}} - \frac{1}{a}\right)} = 2.2395 \times 10^4 \qquad \text{(m/s)}$$

From the orbital equation, we obtain

$$\frac{\mathcal{L}^2}{mk} = \frac{(r^2\dot{\theta})^2}{GM_{\text{sun}}} = r(1 + \varepsilon\cos\theta)$$
$$= 2.278 \times 10^{11}(1 + 0.2525\cos 2.3532) = 1.8725 \times 10^{11} \qquad \text{(m)}$$
$$r\dot{\theta} = 21{,}882 \qquad \text{(m/s)}$$
$$\dot{r} = \sqrt{U_T^2 - (r\dot{\theta})^2} = 4767 \qquad \text{(m/s)}$$

Therefore,

$$\boldsymbol{U}_T = \dot{r}\boldsymbol{e}_r + (r\dot{\theta})\boldsymbol{e}_\theta = 4767\boldsymbol{e}_r + 21{,}882\boldsymbol{e}_\theta \qquad \text{(m/s)}$$

On the other hand, the velocity of Mars on its circular orbit is $24{,}100\boldsymbol{e}_\theta$ m/s Hence, the relative velocity between the space probe and Mars is

$$\boldsymbol{U}_{T-M} = \boldsymbol{U}_T - \boldsymbol{U}_M = 4767\boldsymbol{e}_r - 2218\boldsymbol{e}_\theta \qquad \text{(m/s)}$$

Transform this velocity to an observer on the surface of Mars with the unit vectors denoted by $(\boldsymbol{i}_r, \boldsymbol{i}_\theta)$ on Mars. They are related to the unit vectors in the transfer orbit by $\boldsymbol{i}_r = -\boldsymbol{e}_\theta$, $\boldsymbol{i}_\theta = \boldsymbol{e}_r$. To that observer, he finds that the velocity of the probe at the surface of Mars is

$$v_p = 2218\boldsymbol{i}_r + 4767\boldsymbol{i}_\theta \qquad \text{(m/s)}$$

With this velocity the space probe will have a hyperbolic orbit around Mars. However, if a proper impulse is applied, the probe can stay in the vicinity of Mars. We determine the required reduction of velocity by setting the tangential velocity less than the tangential velocity needed to balance the centrifugal force on a circular orbit and the radial component zero. For a circular orbit of radius of 3500 km, which is slightly greater than the radius of Mars (3332 km), the tangential velocity is

$$V_\theta = \sqrt{\frac{GM_{\text{Mars}}}{R_P}} = \sqrt{\frac{6.67 \times 10^{-11} \times 0.63873 \times 10^{24}}{3{,}500{,}000}} = 3489 \qquad \text{(m/s)}$$

where $R_P$ is the radius of the probe position measured from the center of Mars. From this calculation we determine the required reduction in velocity by choosing $v_p' = 3480\boldsymbol{i}_\theta$ (m/s),

$$v_P' - v_p = -1287\boldsymbol{i}_\theta - 2218\boldsymbol{i}_r$$

and

$$\Delta v = |v'_P - v_P| = \sqrt{1287^2 + 2218^2}$$
$$= 2564 \qquad (\text{m/s})$$

Therefore, the velocity of the space probe is reduced to the velocity less than the velocity for circular orbit. With this velocity, the space probe will stay in the vicinity of Mars. The radius of the orbit is expected to decrease gradually, spiraling down to the surface of Mars because its centrifugal force is slightly less than the gravitational force for a circular orbit.

3) The traveling time of the space probe from the surface of Earth to the Mars circular orbit around the sun is calculated as follows.

The period for the whole elliptic trajectory is

$$T = 2\pi\sqrt{\frac{a^3}{GM_{\text{sun}}}} = \frac{2\pi(2\times 10^{11})^{1.5}}{\sqrt{1.352\times 10^{20}}} = 565.06 \text{ days}$$

The time required for traveling from $\phi = 0$ to $\phi = 2.15375$ is

$$t = \frac{T}{2\pi}(\phi - \varepsilon \sin\phi) = 174.7 \text{ days}$$

Note that in Table 5.2, the values of $\Delta U_{\text{Mars}}$ for the cases of ellipse 2 and parabolic trajectories are computed with the considerations of spiraling orbits around Mars as given in this example. Detailed expressions between the velocity vectors of the space vehicle and Mars are given as follows:

$$(\Delta U_{\text{Mars}})_{\text{ell},2} = 8.236\boldsymbol{e}_r - 1.451\boldsymbol{e}_\theta \qquad (\text{km/s})$$
$$(\Delta U_{\text{Mars}})_{\text{parab}} = 20.01\boldsymbol{e}_r - 3.551\boldsymbol{e}_\theta \qquad (\text{km/s})$$

## Problems

**5.1.** A single-stage rocket is launched vertically from the surface of the Earth. The velocity and position of the rocket at the burnout are predetermined. Suppose that the mass ratio $(m_0/m_b)$ is also given. Find the required mass flow rate and the exhaust velocity at the nozzle exit to launch such a rocket.

**5.2.** Compare the terminal payload velocities between a single-stage rocket and a two-stage rocket with same payload ratio of $m_L/m_{01}$, structure coefficient, and the exhaust velocity. Suppose that the initial mass $m_{01} = 100{,}000$ kg, payload $m_L = 2000$ kg, the structure coefficient $\epsilon = 0.15$, and the exhaust velocity $v_e =$ 3500 m/s. Neglect the gravity and the air drag.

**5.3.** A satellite is launched from the surface of the Earth. At the time of burnout, the satellite is located at altitude of 1000 km with the radial velocity of $v_r = 500$ m/s. Determine the required tangential velocity such that the minimum radius of the orbit is 7000 km.

**5.4.** A particle moves in an elliptical orbit of major axis $2a$ and minor axis $2b$, with the origin at the center of the ellipse. If the radius vector to the particle sweeps out area at a constant rate as usual, find the law of force in terms of the mass $m$ and period $P$ of the motion. If the minor axis $2b$ approaches zero under the same force law, what kind of motion would result?

**5.5.** The gravitational potential for the inverse square law of force is $-k/r$. Suppose a small variation $\delta/r^2$ is added to the potential. Find the general orbital equation. Show that, if $\delta$ is a constant and $\delta \ll \mathcal{L}^2/2m$, the orbit is given by an ellipse with major axis precessing slowly, having angular velocity of precession given by $\delta/(\mathcal{L}a^2\sqrt{1-\varepsilon^2})$.

**5.6.** Take the speed of a planet (or satellite) in an elliptical orbit.

(a) Prove that the speed at the point when the planet is at its maximum distance from the major axis is equal to the geometric mean of the maximum and minimum orbital speeds.

(b) Show that the ratio of extreme orbital speeds (at perihelion and aphelion) is $(1+\varepsilon)/(1-\varepsilon)$.

(c) Take the Earth's eccentricity as 0.0167 and that of Halley's comet as 0.967; calculate the ratio in part (b) for each.

**5.7.** With the use of the Runge–Kutta method, find the trajectory of an electrical propulsion rocket with the initial conditions in dimensionless form $\rho = 1$, $\dot{\rho} = \ddot{\rho} = 0$, and the parameter $\upsilon = 0.001$. Plot the computed results.

**5.8.** Prove that the solutions obtained from the small perturbation method satisfy the differential equations and the initial conditions for the electrical propulsion rocket.

**5.9.** A satellite is launched from the surface of the Earth. At the time of burnout, the satellite is located at an altitude of 700 km with velocity of $\upsilon = 1000\boldsymbol{e}_r + 5000\boldsymbol{e}_\theta$ (m/s). Determine the impulse required to increase the velocity in the tangential direction when $\upsilon_r$ is zero, so that the orbit of satellite is circular around the Earth.

**5.10.** Verify the results of the ellipse 2 trajectory in Table 5.2 for a space vehicle from the surface of Earth to Mars' orbit. The length of major axis $2a$ is $4.5 \times 10^{11}$ m.

**5.11.** Verify the results of the parabola trajectory in Table 5.2 for a space vehicle from the surface of Earth to Mars's orbit.