4
Lagrange's Equations and the Variational Principle

FUNDAMENTAL equations in dynamics are based on Newton's second law of motion. When Newton's law is used to formulate a problem, an explicit expression of force or torque is required. Such expression may not be easy to obtain. An alternative approach is to employ Lagrange's equations. In the Lagrangian formulation for conservative systems, expressions for kinetic and potential energies are required, but knowledge of the force or torque is not needed.

There are different forms of Lagrange's equations. One form is for dynamic systems without constraints between generalized coordinates, which are coordinates based on configurations of the systems and are discussed in the next section. Another form is for systems with constraints. In this form, constraint relations are incorporated directly into Lagrange's equations as Lagrangian multipliers and constraint forces.

The Hamilton equations are discussed in Section 4.3. These equations are parallel to the Lagrangian equations for systems without constraints. Through this parallel approach, readers will become more familiar with the Lagrangian equations. The general form of Lagrangian equation is studied in Section 4.4. Different constraints are discussed, and Lagrangian multipliers are introduced for solving the problems. Note that Lagrangian multipliers are related to nonconservative forces. Many examples are given in this section.

In Section 4.5, the variational principle is introduced. The purpose of this principle is for optimization. It is discussed here because Lagrange's equations can be derived from the optimization of the Lagrangian function of dynamic systems. A case of optimum with a constraint condition also is studied. Examples are given for the application of the variational principle.

4.1 Generalized Coordinates, Velocities, and Forces

Generalized coordinates are the coordinates that must be specified in order to describe the configuration of a system. If a system of N particles is under consideration, three coordinates are needed to specify the position of one particle so that $3N$ coordinates are required for N particles. The system is said to have $3N$ degrees of freedom. If some coordinates are related by j equations or constraints, the degrees of freedom are reduced to $3N - j$.

For a particle traveling along a straight line, the only coordinate needed is the particle's traveling distance. For a wheel rotating on its fixed shaft, the coordinate describing the wheel is the rotating angular displacement. For a wheel with a shaft moving along a straight line, two coordinates must be specified: the distance traveled by the shaft and the angular displacement of the wheel. For a pair of long-nosed pliers lying on a table, four coordinates are needed to describe the system: (x, y) coordinates for the location of the center of pivot, α for the angle between the surface of the first jaw and the x axis, and β for the angle between the

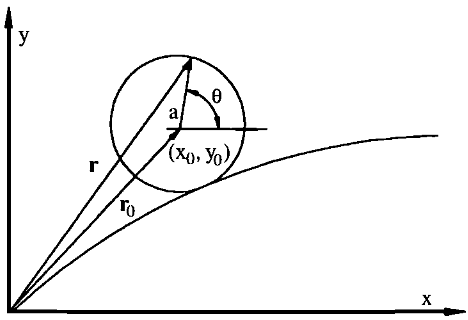

Fig. 4.1 Wheel rolling on a curved ground.

surfaces of two jaws. Because of the nature of generalized coordinates, the number of such coordinates is called the number of degrees of freedom of the system.

Usually, symbols $(q_1, q_2, \ldots, q_n)$ are used for generalized coordinates. A position vector $\boldsymbol{r}$ always can be expressed as a function of q, and we may write

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n)$$

or

$$\boldsymbol{r} = \boldsymbol{r}(q) \tag{4.1}$$

To illustrate the preceding statement, let us consider a point at the edge of a wheel rolling without slipping on a curved ground as shown in Fig. 4.1.

$$\boldsymbol{r} = \boldsymbol{r}_0 + a(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}) = (x_0 + a\cos\theta)\boldsymbol{i}$$
$$+ (y_0 + a\sin\theta)\boldsymbol{j} = \boldsymbol{r}(x_0, y_0, \theta) = \boldsymbol{r}(q_1, q_2, q_3)$$

where $q_1 = x_0$, $q_2 = y_0$, and $q_3 = \theta$. As the particle moves, we have

$$\dot{\boldsymbol{r}} = \sum_{\rho=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_\rho} \dot{q}_\rho \tag{4.2}$$

The quantities $\dot{q}_\rho \equiv \mathrm{d}q_\rho/\mathrm{d}t$ are called generalized velocities. Equation (4.2) suggests that

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}(q, \dot{q}) \tag{4.3}$$

Here q and $\dot{q}$ are considered independent variables.

Furthermore, a typical force F acts at a point (x, y, z). The virtual work produced by the force is

$$\delta W = \boldsymbol{F} \cdot \delta \boldsymbol{r} \tag{4.4}$$

where $\delta \boldsymbol{r}$ is the virtual displacement and can be expressed in terms of generalized coordinates as

$$\delta \boldsymbol{r} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_i} \delta q_i \tag{4.5}$$

With the use of Eq. (4.5), Eq. (4.4) becomes

$$\delta W = \sum_{i=1}^{n}\left(\boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_i}\delta q_i\right) = \sum_{i=1}^{n}(Q_i\delta q_i) \tag{4.6}$$

where $Q_i \equiv \boldsymbol{F}\cdot(\partial\boldsymbol{r}/\partial q_i) \equiv$ generalized force.

For a conservative force as defined in Section 2.5,

$$\boldsymbol{F} = -\nabla V$$

$$\boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial q_i} = -\nabla V\cdot\frac{\partial \boldsymbol{r}}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Hence,

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad i = 1, 2, \ldots, n \tag{4.7}$$

The generalized forces for a conservative system are the arithematic inverse of the partial derivatives of potential energy with respect to the generalized coordinates.

4.2 Lagrangian Equations

Consider a system of N particles with n degrees of freedom. A position vector $\boldsymbol{r}_i$ for the position of ith particle is, in general, a function of generalized coordinates and time.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots, q_n, t) = \boldsymbol{r}_i(q, t) \tag{4.8}$$

where q represents all the various q. In Eq. (4.8) q and t are independent variables, and the velocity of the ith particle is

$$\boldsymbol{v}_i = \boldsymbol{v}_i(q, \dot{q}, t) \tag{4.9}$$

where $\dot{q}$ is the generalized velocity representing $(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$. Certainly,

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{v}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \tag{4.10}$$

On the other hand, considering a virtual displacement

$$\delta\boldsymbol{r}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j \tag{4.11}$$

Note that the symbol δ is used for virtual displacement. No time is needed to reach $\delta\boldsymbol{r}_i$. Taking the partial derivative of $\boldsymbol{v}_i$ with respect to generalized velocity $\dot{q}_k$ from Eq. (4.10) gives

$$\frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\sum_{j=1}^{n}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}\right] = \frac{\partial \boldsymbol{r}_i}{\partial q_k} \tag{4.12}$$

Here we find that the partial derivative of the velocity of ith particle with respect to $\dot{q}_k$ equals the partial derivative of the position vector with respect to q_k. Differentiating $(\partial \boldsymbol{r}_i/\partial q_k)$ with respect to time yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) \tag{4.13}$$

Taking the partial derivative of $\dot{\boldsymbol{r}}_i$ with respect to q_k from Eq. (4.10), we have

$$\begin{aligned}\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k} &= \sum_{j=1}^{n} \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_i}{\partial t}\right) \\ &= \sum_{j=1}^{n}\left(\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k \partial q_j}\dot{q}_j\right) + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)\end{aligned} \tag{4.14}$$

Comparing Eq. (4.13) to Eq. (4.14), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right) = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k} \tag{4.15}$$

Now let us consider D'Alembert's principle for the ith particle of the system of N particles:

$$\boldsymbol{F}_i - \dot{\boldsymbol{P}}_i = 0 \tag{4.16}$$

where $\dot{\boldsymbol{P}}_i$ is the rate change of momentum of the ith particle. In addition, let us imagine a virtual displacement of $\delta \boldsymbol{r}_i$ for the ith particle. For the system we have

$$\sum_{i=1}^{N}(\boldsymbol{F}_i - \dot{\boldsymbol{P}}_i)\cdot \delta \boldsymbol{r}_i = 0 \tag{4.17}$$

Note that Eq. (4.17) is equivalent to Eq. (1.34). When D'Alembert's principle is considered, the inertia force is one of the applied forces. In Section 1.6, we reached the conclusion that the virtual work of applied forces in equilibrium is zero. Now, let us separately examine the two terms in detail as follows:

$$\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\delta \boldsymbol{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{n}\boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} Q_j \delta q_j \tag{4.18}$$

where

$$Q_j = \sum_{i=1}^{N}\boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j} \tag{4.19}$$

Q_j is the generalized force, and

$$\begin{aligned}\sum_{i=1}^{N}\dot{\boldsymbol{P}}_i\cdot\delta \boldsymbol{r}_i &= \sum_{i=1}^{N}\sum_{j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}(m_i \boldsymbol{v}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j \\ &= \sum_{i,j}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) - m_i \boldsymbol{v}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\right]\delta q_j\end{aligned}$$

Using Eqs. (4.12) and (4.15), we obtain

$$\sum_{i=1}^{N} \dot{\boldsymbol{P}} \cdot \delta \boldsymbol{r}_i = \sum_{i,j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \boldsymbol{\nu}_i \cdot \frac{\partial \boldsymbol{\nu}_i}{\partial \dot{q}_j} \right) - m_i \boldsymbol{\nu}_i \cdot \frac{\partial \boldsymbol{\nu}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \tag{4.20}$$

where $T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ = kinetic energy of the system. Combining Eqs. (4.18) and (4.20), we find that

$$\sum_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0 \tag{4.21}$$

Because all q_j are independent, the terms in the brackets must be zero, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \tag{4.22}$$

This is the first form of Lagrange's equations. For a conservative system,

$$Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_{i=1}^{N} (\nabla V)_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_i} = -\frac{\partial V}{\partial q_j}$$

where V is the potential energy of the system and is a function of generalized coordinates only. Now Eq. (4.22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0$$

Because potential energy is not a function of generalized velocity,

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

which can be subtracted from the first term. Thus, the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1, 2, \ldots, n \tag{4.23}$$

where the Lagrangian function $L = T - V$. Equation (4.23) is Lagrange's equation for a conservative system in which L is, in general, a function of q, $\dot{q}$, and t. For a nonconservative system, the generalized force can be expressed as a combination of conservative and nonconservative forces.

$$Q_j = -\frac{\partial V}{\partial q_j} + \mathcal{F}_j$$

where $\mathcal{F}_i$ is the nonconservative force. Therefore, in general, Lagrange's equation is in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \mathcal{F}_j \qquad j = 1, 2, \ldots, n \tag{4.24}$$

Example 4.1

Find the differential equation of motion for a simple pendulum of length L and finite angle of θ measured from the vertical as shown in Fig. 4.2.

Solution. Because the angle θ is sufficient to describe the configuration of the system, it is used as the generalized coordinate, and the system has only one degree of freedom.

Kinetic energy:

$$T = \tfrac{1}{2}m(L\dot{\theta}^2)$$

Potential energy:

$$V = mgL(1 - \cos\theta)$$

Lagrangian function:

$$L = T - V = \tfrac{1}{2}m(L\dot{\theta})^2 - mgL(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgL\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = mL^2\ddot{\theta} + mgL\sin\theta = 0$$

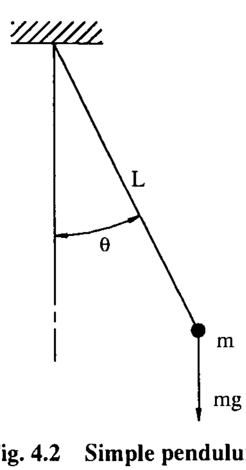

Fig. 4.2 Simple pendulum.

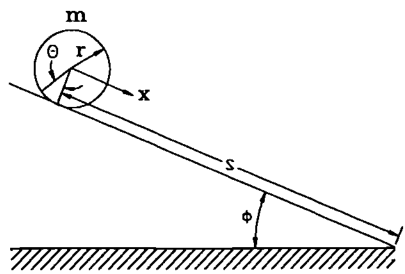

Fig. 4.3 Hoop rolling down an inclined plane.

Hence, the equation of motion is

$$\ddot{\theta} + (g/L)\sin\theta = 0 \tag{4.25}$$

Example 4.2

A hoop of radius r and mass m is rolling, without slipping, down an inclined plane at an angle ϕ. Find the equation of motion.

Solution. For the generalized coordinates, we choose the angle of rotation of the hoop θ and the distance x traveled by the center of the hoop.

Kinetic energy:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

where I is the mass moment of inertia of the hoop. Because $\dot{x} = r\dot{\theta}$ and $I = mr^2$, the kinetic energy, potential energy, and Lagrangian function of the hoop are

$$T = \tfrac{1}{2}m(r\dot{\theta})^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 = m(r\dot{\theta})^2$$

$$V = mg(s - x)\sin\phi = mg(s - r\theta)\sin\phi$$

$$L = T - V = m(r\dot{\theta})^2 - mg\sin\phi(s - r\theta)$$

Applying Lagrange's equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 2mr^2\ddot{\theta} - mgr\sin\phi = 0$$

Hence, the equation of motion is

$$\ddot{\theta} = (1/2r)g\sin\phi \tag{4.26}$$

Example 4.3

Two simple pendulums of length s and bob mass m swing in a common vertical plane and are attached to two different support points. If the masses are connected by a spring of constant k, use the Lagrangian approach to formulate the equations of motion. Assume small angles of oscillation.

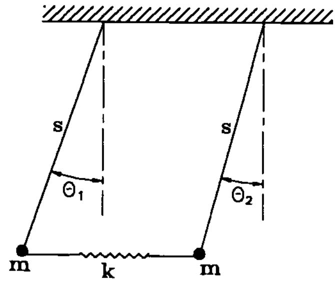

Fig. 4.4 Two simple pendulums.

Solution. θ_1 and θ_2 are the generalized coordinates.

$$T = \tfrac{1}{2}ms^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$V = mgs(1 - \cos\theta_1) + mgs(1 - \cos\theta_2) + \tfrac{1}{2}ks^2(\theta_1 - \theta_2)^2$$

$$L = T - V$$

$$L = \tfrac{1}{2}ms^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mgs(1 - \cos\theta_1) - mgs(1 - \cos\theta_2) - \tfrac{1}{2}ks^2(\theta_1 - \theta_2)^2$$

Working out the derivatives gives

$$\frac{\partial L}{\partial \dot{\theta}_1} = ms^2\dot{\theta}_1, \qquad \frac{\partial L}{\partial \theta_1} = -mgs\sin\theta_1 - ks^2(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = ms^2\dot{\theta}_2, \qquad \frac{\partial L}{\partial \theta_2} = -mgs\sin\theta_2 + ks^2(\theta_1 - \theta_2)$$

Hence the equations of motion are

$$ms^2\ddot{\theta}_1 + mgs\theta_1 + ks^2(\theta_1 - \theta_2) = 0 \tag{4.27}$$

$$ms^2\ddot{\theta}_2 + mgs\theta_2 - ks^2(\theta_1 - \theta_2) = 0 \tag{4.28}$$

Example 4.4

A solid cylinder of radius r and weight w rolls without slipping along a circular path of radius R as shown in Fig. 4.5. Determine the Lagrangian function and the equation of motion.

Solution. From the conditions of no slippage, we have

$$(R - r)\dot{\theta} = r\dot{\phi}$$

The kinetic energy is

$$T = \frac{1}{2}\frac{w}{g}(R - r)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\phi}^2$$

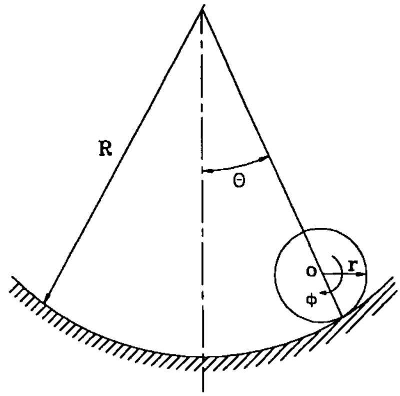

Fig. 4.5 Cylinder rolling on a circular path.

where $I_0 = \frac{1}{2}(w/g)r^2$. Therefore,

$$T = \frac{3}{4}\frac{w}{g}(R-r)^2\dot{\theta}^2$$

$$V = w(R-r)(1-\cos\theta)$$

The Lagrangian function is

$$L = T - V = \frac{3}{4}\frac{w}{g}(R-r)^2\dot{\theta}^2 - w(R-r)(1-\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2}\frac{w}{g}(R-r)^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -w(R-r)\sin\theta$$

Hence the equation of motion is

$$\frac{3}{2}\frac{w}{g}(R-r)^2\ddot{\theta} + w(R-r)\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{2g}{3(R-r)}\sin\theta = 0 \tag{4.29}$$

Example 4.5

Find the equations of motion for a particle with mass m in three-dimensional space for the following different coordinates: 1) rectangular, 2) cylindrical, and 3) spherical.

Solution. 1) Rectangular coordinates:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Using the first form of Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial x} = 0, \qquad Q_x = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = F_x$$

Hence we have

$$m\ddot{x} = F_x \tag{4.30}$$

Similarly,

$$m\ddot{y} = F_y \tag{4.31}$$

$$m\ddot{z} = F_z \tag{4.32}$$

2) Cylindrical coordinates:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$

$$\dot{x} = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi$$

$$\dot{y} = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi$$

$$\dot{z} = \dot{z}$$

Hence,

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$$

For the coordinate ρ, we have

$$\frac{\partial T}{\partial \dot{\rho}} = m\dot{\rho}, \qquad \frac{\partial T}{\partial \rho} = m\rho\dot{\phi}^2$$

$$Q_\rho = F_x\frac{\partial x}{\partial \rho} + F_y\frac{\partial y}{\partial \rho} + F_z\frac{\partial z}{\partial \rho}$$

$$= F_x\cos\phi + F_y\sin\phi = \boldsymbol{F}\cdot\boldsymbol{e}_\rho = F_\rho \tag{4.33}$$

where F_ρ is the component of force along direction ρ. Plugging into Eq. (4.22) gives

$$m\ddot{\rho} - m\rho\dot{\phi}^2 = F_\rho$$

For the coordinate ϕ,

$$\frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi}, \qquad \frac{\partial T}{\partial \phi} = 0$$

$$Q_\phi = F_x\frac{\partial x}{\partial \phi} + F_y\frac{\partial y}{\partial \phi} + F_z\frac{\partial z}{\partial \phi} = -F_x\rho\sin\phi + F_y\rho\cos\phi = \rho\boldsymbol{F}\cdot\boldsymbol{e}_\phi = \rho F_\phi$$

where $\boldsymbol{e}_\phi = -\sin\phi \boldsymbol{i} + \cos\phi \boldsymbol{j}$ as given in Eq. (2.8). Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\rho^2\dot{\phi}) = \rho F_\phi \tag{4.34}$$

For the coordinate z, the equation is the same as in the rectangular coordinates

$$m\ddot{z} = F_z \tag{4.35}$$

3) Spherical coordinates: In Chapter 2, the relationship between spherical coordinates and rectangular coordinates was already introduced. From Eqs. (2.11) and (2.12), we have

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = r\sin\theta\cos\phi\boldsymbol{i} + r\sin\theta\sin\phi\boldsymbol{j} + r\cos\theta\boldsymbol{k}$$

That is,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

We also have, from Eq. (2.15),

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta\boldsymbol{e}_\phi$$

Hence,

$$T = \tfrac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + (r\dot{\phi}\sin\theta)^2]$$

For the coordinate r,

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial T}{\partial r} = m(r\dot{\theta}^2 + r\dot{\phi}^2\sin^2\theta)$$

$$Q_r = F_x\frac{\partial x}{\partial r} + F_y\frac{\partial y}{\partial r} + F_z\frac{\partial z}{\partial r} = F_x\sin\theta\cos\phi$$
$$+ F_y\sin\theta\sin\phi + F_z\cos\theta = \boldsymbol{F}\cdot\boldsymbol{e}_r = F_r$$

With the use of Eq. (4.22), we obtain

$$m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) = F_r \tag{4.36}$$

for the equation of motion in the radial direction. For the coordinate θ,

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{\partial T}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta$$

$$Q_\theta = F_x r\cos\theta\cos\phi + F_y r\cos\theta\sin\phi - F_z r\sin\theta = r\boldsymbol{F}\cdot\boldsymbol{e}_\theta = rF_\theta$$

Hence the equation of motion in the direction of θ is

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\dot{\phi}^2 \sin\theta\cos\theta = rF_\theta \tag{4.37}$$

Note that the generalized force in θ direction is a torque. Similarly, for the coordinate ϕ

$$\frac{\partial T}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi}, \qquad \frac{\partial T}{\partial \phi} = 0$$

$$\begin{aligned} Q_\phi &= -F_x r \sin\theta \sin\phi + F_y r \sin\theta \cos\phi \\ &= r \sin\theta \boldsymbol{F} \cdot \boldsymbol{e}_\phi = r \sin\theta F_\phi \end{aligned}$$

The equation of motion in the direction of ϕ is, therefore,

$$\frac{d}{dt}(mr^2 \sin^2\theta\dot{\phi}) = r \sin\theta F_\phi \tag{4.38}$$

Equations (4.37) and (4.38) can be simplified to

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta) = F_\theta \tag{4.39}$$

$$m(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}) = F_\phi \tag{4.40}$$

Note that the acceleration terms on the left sides of Eqs. (4.36), (4.39), and (4.40) agree well with the expression in Eq. (2.16).

Example 4.6

Suppose that a person of mass M playing on a swing is modeled as a point mass $(M - m)$ at the end of the rope and a small mass m moving around $M - m$ at radius a and angular speed of ω as shown in Fig. 4.6. Find the equation of motion for this system.

Solution. Velocity of $(M - m)$

$$V_{M-m} = s\dot{\theta}(\cos\theta i + \sin\theta j) \tag{4.41}$$

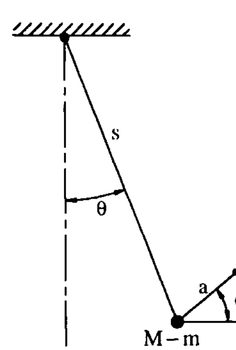

Fig. 4.6 Person playing on a swing.

Velocity of m is

$$V_m = s\dot{\theta}(\cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}) - a\omega\sin\omega t\,\boldsymbol{i} + a\omega\cos\omega t\,\boldsymbol{j} \tag{4.42}$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}(M - m)V_{M-m}^2 + \tfrac{1}{2}mV_m^2$$

$$T = \tfrac{1}{2}(M - m)(s\dot{\theta})^2 + \tfrac{1}{2}m[(s\dot{\theta}\cos\theta - a\omega\sin\omega t)^2 + (s\dot{\theta}\sin\theta + a\omega\cos\omega t)^2] \tag{4.43}$$

The potential energy is

$$V = (M - m)gs(1 - \cos\theta) + mg[s(1 - \cos\theta) + a\sin\omega t] = Mgs(1 - \cos\theta) + mga\sin\omega t \tag{4.44}$$

The Lagrangian function for the system is

$$L = T - V = \tfrac{1}{2}M(s\dot{\theta})^2 + \tfrac{1}{2}m[(a\omega)^2 - 2(a\omega\sin\omega t)(s\dot{\theta}\cos\theta) + 2(a\omega\cos\omega t)(s\dot{\theta}\sin\theta)] - Mgs(1 - \cos\theta) - mga\sin\omega t$$
$$= \tfrac{1}{2}M(s\dot{\theta})^2 + \tfrac{1}{2}m[(a\omega)^2 + 2as\omega\dot{\theta}\sin(\theta - \omega t)] - Mgs(1 - \cos\theta) - mga\sin\omega t \tag{4.45}$$

To find the equation of motion, we obtain

$$\frac{\partial L}{\partial\dot{\theta}} = Ms^2\dot{\theta} + mas\omega\sin(\theta - \omega t)$$

$$\frac{\partial L}{\partial\theta} = mas\omega\,\dot{\theta}\cos(\theta - \omega t) - Mgs\sin\theta$$

Substituting the preceding equations in Eq. (4.22) leads us to

$$Ms^2\ddot{\theta} + mas\omega\cos(\theta - \omega t)(\dot{\theta} - \omega) - mas\omega\,\dot{\theta}\cos(\theta - \omega t) + Mgs\sin\theta = 0$$

Rearranging, we obtain the equation of motion as

$$\ddot{\theta} + \frac{g}{s}\sin\theta = \frac{ma}{Ms}\omega^2\cos(\omega t - \theta) \tag{4.46}$$

Note that the term on the right-hand side is the force causing the swing to oscillate to a large angle. Resonance can take place as

$$\omega = \sqrt{g/s}$$

Through these examples it is easily seen that using the Lagrangian equation for deriving equations of motion for conservative systems is very simple and systematic. All we need are the expressions for potential and kinetic energy.

4.3 Hamilton's Principle

Hamilton's principle is an approach parallel to the Lagrangian equations. From here readers can get a deeper feeling about equations describing a dynamic system. Similar to Lagrange's approach, the Hamiltonian function H is defined as

$$H \equiv \sum_j \dot{q}_j p_j - L = H(p, q, t) \tag{4.47}$$

where p_j = the generalized momenta $= \partial L/\partial \dot{q}_j$. Taking the total derivative of Eq. (4.47) gives us

$$\begin{aligned} \mathrm{d}H &= \sum_j \dot{q}_j \mathrm{d}p_j + \sum_j p_j \mathrm{d}\dot{q}_j - \left[\sum_j \frac{\partial L}{\partial q_j}\mathrm{d}q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\mathrm{d}\dot{q}_j + \frac{\partial L}{\partial t}\mathrm{d}t\right] \\ &= \sum_j \dot{q}_j \mathrm{d}p_j - \sum_j \frac{\partial L}{\partial q_j}\mathrm{d}q_j - \frac{\partial L}{\partial t}\mathrm{d}t \end{aligned} \tag{4.48}$$

Also, we have

$$\mathrm{d}H = \sum_j \frac{\partial H}{\partial p_j}\mathrm{d}p_j + \sum_j \frac{\partial H}{\partial q_j}\mathrm{d}q_j + \frac{\partial H}{\partial t}\mathrm{d}t \tag{4.49}$$

Compare Eq. (4.48) to Eq. (4.49), we obtain

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{4.50a}$$

$$-\frac{\partial H}{\partial q_j} = \frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j \tag{4.50b}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{4.50c}$$

Equations (4.50a) and (4.50b) are called Hamilton's canonical equations for a conservative system because, in the intermediate step of deriving Eq. (4.50b), the conservative condition is used. Furthermore, for a conservative system

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_j \left(\frac{\partial H}{\partial q_j}\dot{q}_j + \frac{\partial H}{\partial p_j}\dot{p}_j\right) + \frac{\partial H}{\partial t} \\ &= \sum_j (-\dot{p}_j \dot{q}_j + \dot{q}_j \dot{p}_j) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \end{aligned} \tag{4.51}$$

To interpret the meaning of H, let us consider a case that happens often in dynamics; the position vectors are functions of q only:

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q) \quad i = 1, 2, \ldots, N$$

Then

$$v_i = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \sum_k \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k$$

$$T = \frac{1}{2}\sum_i m_i v_i \cdot v_i = \frac{1}{2}\sum_i m_i \sum_{k,l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_l}\right)\dot{q}_k\dot{q}_l$$

and

$$\sum_j \dot{q}_j p_j = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}$$

$$= \sum_j \dot{q}_j \left[\frac{1}{2}\sum_i m_i \sum_{k,l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_l}\right)\frac{\partial}{\partial \dot{q}_j}\dot{q}_k\dot{q}_l\right]$$

Looking into the details of the partial derivative in the last expression, we find

$$\sum_{k,l}\frac{\partial}{\partial \dot{q}_j}(\dot{q}_k\dot{q}_l) = \sum_{k,l}(\dot{q}_l\delta_{j,k} + \dot{q}_k\delta_{j,l}) = 2\dot{q}_j$$

Therefore,

$$\sum_j \dot{q}_j p_j = \sum_j \dot{q}_j \left[\frac{1}{2}\sum_i m_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_j}(2\dot{q}_j)\right]$$

$$= \sum_j \left[\sum_i m_i \left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j\right)^2\right] = 2T \tag{4.52}$$

Substituting Eq. (4.52) into Eq. (4.47), we find

$$H = 2T - L = 2T - (T - V) = T + V \tag{4.53}$$

Therefore H is the total energy of the system if the various $\boldsymbol{r}_i$ are functions of q only. For a conservative system

$$T + V = \text{const}$$

$$H = \text{const}$$

That means

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \qquad \text{and} \qquad \frac{\partial H}{\partial t} = 0 \tag{4.54}$$

For a nonconservative system, the Lagrangian equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \mathcal{F}_j \tag{4.55}$$

With this general expression, Eq. (4.50b) becomes

$$-\frac{\partial H}{\partial q_j} = \frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \mathcal{F}_j = \dot{p}_j - \mathcal{F}_j \tag{4.56}$$

Equation (4.51) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_j [(-\dot{p}_j + \mathcal{F}_j)\dot{q}_j + \dot{q}_j \dot{p}_j] + \frac{\partial H}{\partial t} = \sum_j \mathcal{F}_j \dot{q}_j + \frac{\partial H}{\partial t} \tag{4.57}$$

Therefore, Hamilton's canonical equations are true only for conservative systems. In general the total derivative of H with respect to time is not the partial derivative of H with respect to time.

Example 4.7

Consider a spherical pendulum consisting of a point mass m that moves under gravity on a smooth spherical surface with radius a. The gravitational force is along the downward vertical. In terms of spherical angles θ and ϕ as shown in Fig. 2.2, except that θ is the angle between the position vector of mass m and the downward vertical axis, the kinetic and potential energies are

$$T = \tfrac{1}{2} m a^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$$V = -mga\cos\theta$$

Find the equations of motion for the mass m 1) from Lagrange's equation and 2) from Hamilton's principle.

Solution. 1) Lagrange's equation:

$$L = T - V = \tfrac{1}{2} m a^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mga\cos\theta$$

For the coordinate θ,

$$\frac{\partial L}{\partial \dot{\theta}} = m a^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m a^2 \dot{\phi}^2 \sin\theta\cos\theta - mga\sin\theta$$

Substituting the preceding expressions into Eq. (4.23), we find

$$m a^2 \ddot{\theta} - m a^2 \dot{\phi}^2 \sin\theta\cos\theta + mga\sin\theta = 0 \tag{4.58}$$

For the coordinate ϕ,

$$\frac{\partial L}{\partial \dot{\phi}} = m a^2 \dot{\phi}\sin^2\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m a^2 \dot{\phi}\sin^2\theta) = 0$$

$$\dot{\phi}\sin^2\theta = \text{const} \tag{4.59}$$

2) Hamilton's principle: In spherical coordinates

$$\boldsymbol{r} = r(\sin\theta\,\cos\phi\,\boldsymbol{i} + \sin\theta\,\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}) = \boldsymbol{r}(r, \theta, \phi)$$

Hence,

$$H = T + V = \tfrac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$$

In Hamilton's principle, however, H is to be expressed in terms of generalized coordinates q, generalized momenta p, and time t:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta} \tag{4.60}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ma^2\dot{\phi}\sin^2\theta \tag{4.61}$$

With the use of Eqs. (4.60) and (4.61), we have

$$H = \frac{1}{2}\frac{p_\theta^2}{ma^2} + \frac{1}{2}\frac{p_\phi^2}{ma^2\sin^2\theta} - mga\cos\theta \tag{4.62}$$

Taking the partial derivatives of H with respect to θ and ϕ, we have

$$\frac{\partial H}{\partial\theta} = -\frac{p_\phi^2}{ma^2\sin^3\theta}\cos\theta + mga\sin\theta$$

$$\frac{\partial H}{\partial\phi} = 0$$

Rewrite the canonical equations

$$\frac{\partial H}{\partial\theta} = -\dot{p}_\theta, \qquad \frac{\partial H}{\partial\phi} = -\dot{p}_\phi$$

With the help of Eqs. (4.60) and (4.61), and the canonical equations, we find

$$-\frac{p_\phi^2}{ma^2\sin^4\theta}\sin\theta\,\cos\theta + mga\sin\theta = -ma^2\ddot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\phi}\sin^2\theta) = 0$$

Further simplifying the preceding equation, we obtain

$$-ma^2\dot{\phi}^2\sin\theta\,\cos\theta + mga\sin\theta = -ma^2\ddot{\theta} \tag{4.63}$$

$$\dot{\phi}\sin^2\theta = \text{const} \tag{4.64}$$

Equations (4.63) and (4.64) are the same as Eqs. (4.58) and (4.59) obtained in part 1.

4.4 Lagrangian Equations with Constraints

In general there are two types of constraints in dynamics: holonomic and nonholonomic. When the relationship between generalized coordinates can be written as

$$f_i(q_1, q_2, \ldots, q_n, t) = 0 \quad i = 1, 2, \ldots, m \tag{4.65}$$

where $m < n$, the constraints of this form are known as holonomic constraints. Because of these m constraint equations, the various nq_j are not independent. In principle, there are only $(n - m)$ independent generalized coordinates, and $(n - m)$ Lagrangian equations for solving these q_i as functions of time. The remaining q_i can be obtained through Eqs. (4.65) already given.

Many problems, however, may be formulated differently such that the generalized coordinates can be reduced at the beginning. For example, let us consider the case of a double pendulum (Fig. 4.7). The two point masses m_1 and m_2 can be specified by (x_1, y_1) and (x_2, y_2) in the plane containing the double pendulum. The rods of length L_1 and L_2 are considered to be rigid and massless. The constraint equations are of the form

$$x_1^2 + y_1^2 = L_1^2$$

$$(x_2 - x_1)^1 + (y_2 - y_1)^2 = L_2^2$$

Because of these, we simply choose θ_1 and θ_2 as generalized coordinates and the equations of motion are simplified. On the other hand, when the constraint equations are written in the form

$$\sum_{j=1}^{n} C_{kj}\,dq_j + C_{kt}\,dt = 0 \quad k = 1, 2, \ldots, m \tag{4.66}$$

where the various C are, in general, functions of the generalized coordinates and time. Constraints of this form are known as nonholonomic constraints. While deriving the Lagrangian equation of the first form, there is a step written in

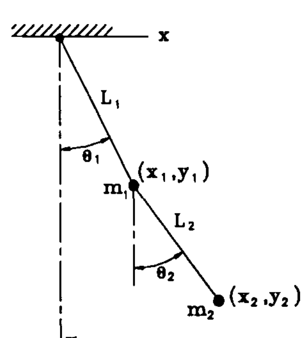

Fig. 4.7 Double pendulum.

Eq. (4.21) as

$$\sum_j \left[Q_j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

At that moment, because q_j is independent throughout, the terms in the brackets were set to zero. Now q_j is not independent and cannot be set to zero. The general expression for the generalized force, however, is still valid, i.e.,

$$Q_j = -\frac{\partial V}{\partial q_j} + \mathcal{F}_j$$

Furthermore, to broaden our considerations, the nonconservative forces may be treated as a combination of constraint force $\mathcal{F}_{cj}$ and the other nonconservative force $\mathcal{F}_{oj}$. Substituting this expression into Eq. (4.21), we have

$$\sum_j \left[\frac{\partial L}{\partial q_j} + \mathcal{F}_{cj} + \mathcal{F}_{oj} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] \delta q_j = 0 \tag{4.67}$$

Let Eq. (4.66) be multiplied by λ_k and summed over k throughout. Adding that to Eq. (4.67) gives

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} + \mathcal{F}_{cj} + \mathcal{F}_{oj} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \lambda_k C_{kj} \right] \delta q_j + \sum_{k=1}^{m} \lambda_k C_{kt}\, \mathrm{d}t = 0$$

Rearranging the equation leads to

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} \right] \mathrm{d}q_j$$

$$+ \sum_{j=1}^{n} \mathcal{F}_{cj}\, \mathrm{d}q_j + \sum_{k=1}^{m} \lambda_k C_{kt}\, \mathrm{d}t = 0$$

The preceding equation can be considered a combination of two equations, which is proved here. The two equations are

$$\sum_{j=1}^{n} \left[\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} \right] \mathrm{d}q_j = 0 \tag{4.68}$$

and

$$\sum_{j=1}^{n} \mathcal{F}_{cj}\, \mathrm{d}q_j + \sum_{k=1}^{m} \lambda_k C_{kj}\, \mathrm{d}t = 0 \tag{4.69}$$

In Eq. (4.68), note that only $(n - m)$ q_j is independent, but there are m arbitrary λ_k values. Choose m λ_k values such that the sum of four terms in the bracket is zero for m brackets. These various $m q_j$ are presumed to be dependent coordinates. Then the remaining q_j are independent, and the sum of the four terms in the bracket are always zero, i.e.,

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \mathcal{F}_{oj} + \sum_k \lambda_k C_{kj} = 0 \quad j = 1, 2, \ldots, n \tag{4.70}$$

Now let us consider Eq. (4.69). When Eq. (4.66) is multiplied by λ_k and summed over k throughout, we have

$$\sum_k \lambda_k c_{kt}\,\mathrm{d}t = -\sum_k \sum_j \lambda_k C_{kj}\,\mathrm{d}q_j \tag{4.71}$$

Substitute this into Eq. (4.69), we find that

$$\sum_j \mathcal{F}_{cj}\,\mathrm{d}q_j - \sum_k \sum_j \lambda_k C_{kj}\,\mathrm{d}q_j = 0$$

or

$$\sum_j \left[\mathcal{F}_{cj} - \sum_k \lambda_k C_{kj}\right] \mathrm{d}q_j = 0 \tag{4.72}$$

But,

$$\mathcal{F}_{cj} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - \mathcal{F}_{oj}$$

With the use of this equation for the nonconservative force in Eq. (4.72), we obtain

$$\sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q_j}\right) - \frac{\partial L}{\partial q_j} - \mathcal{F}_{oj} - \sum_k \lambda_k C_{kj}\right] \mathrm{d}q_j = 0 \tag{4.73}$$

Equation (4.73) multiplied by (-1) is identical to Eq. (4.68), which has been proved to be true. Therefore, Eq. (4.69) is also true. Summarizing all the equations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_k \lambda_k C_{kj} + \mathcal{F}_{oj} \quad j = 1, 2, \ldots, n \tag{4.74}$$

$$\mathcal{F}_{cj} = \sum_k \lambda_k c_{kj} \qquad j = 1, 2, \ldots, n \tag{4.75}$$

$$\sum_j c_{kj}\,\mathrm{d}q_j + c_{kt}\,\mathrm{d}t = 0 \qquad k = 1, 2, \ldots, m \tag{4.76}$$

Totally, there are $2n + m$ equations for determining $n q_j$, $n\mathcal{F}_{cj}$ and $m\lambda_k$; λ_k is called the Lagrange multiplier, $\mathcal{F}_{cj}$ represents constraint forces, and $\mathcal{F}_{oj}$, the other nonconservative forces.

Example 4.8

A four-wheel wagon is modeled as a mass m in translational motion and four wheels in rotational motion (see Fig. 4.8). The mass m includes the four wheels. The moment of inertia for the four wheels with respect to the rotating axes is I. Determine the required coefficient of friction between tires and the pavement for the wagon to move without slipping down the slope inclined at angle ϕ.

Solution. Kinetic energy:

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

Potential energy:

$$V = mgx\sin\phi$$

Constraint equation:

$$\mathrm{d}x - r\mathrm{d}\theta = 0$$

where r is the radius of wheels. The Lagrangian function is

$$L = T - V = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2 - mgx\sin\phi$$

For the x coordinate,

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -mg\sin\phi$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + mg\sin\phi = \lambda = \mathcal{F}_x$$

or

$$m\ddot{x} = \mathcal{F}_x - mg\sin\phi \tag{4.77}$$

For the θ coordinate,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) = -\lambda r$$

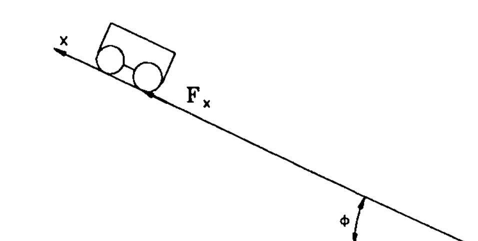

Fig. 4.8 Wagon rolling down inclined plane.

or

$$I\ddot{\theta} = -\lambda r = -\mathcal{F}_x r \tag{4.78}$$

From the constraint equation we have

$$\dot{x} = r\dot{\theta}$$

$$\ddot{x} = r\ddot{\theta}$$

Combining the preceding equation with Eqs. (4.77) and (4.78), we find

$$\mathcal{F}_x = \frac{g}{(1/m + r^2/I)} \sin\phi$$

Because $\mathcal{F}_x = \mu(mg\cos\phi)$

$$\mu = \frac{I}{I + mr^2} \tan\phi \tag{4.79}$$

where μ is the required frictional coefficient.

Example 4.9

Suppose that a car is just started and is to be driven without slipping on horizontal ground covered with ice. With the use of Lagrangian equations that are constrained, find the equations to describe the motion and find the required frictional coefficient between the tires and the ice. Explain why the driver should not attempt to accelerate rapidly. Assume that the mass of the car is M, the moment of inertia of wheels is I, and the torque exerted on the wheels is T_r. The weight of the car is distributed evenly on all four wheels, and this is a four-wheel-drive vehicle.

Solution. Kinetic energy:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

Potential energy:

$$V = 0$$

Constraint equation:

$$\mathrm{d}x - r\,\mathrm{d}\theta = 0$$

The nonconservative generalized force in the θ direction is T_r, and the Lagrangian function is

$$L = T - V = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2$$

For the x coordinate,

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x}, \qquad \frac{\partial L}{\partial x} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\dot{x}) = \lambda = \mathcal{F}_x$$

This is the equation of motion in the x direction. For the θ coordinate,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\dot{\theta}) = T_r - \lambda r = T_r - \mathcal{F}_x r$$

This is the equation of motion in the θ direction. From the constraint equation, we have

$$\ddot{x} = r\ddot{\theta}$$

Combining the equations of motion together with the preceding equation, we obtain

$$I\ddot{\theta} = I\frac{\ddot{x}}{r} = \frac{I}{r}\frac{\mathcal{F}_x}{M} = T_r - \mathcal{F}_x r$$

Rearranging, we find

$$\mathcal{F}_x\left(\frac{I}{Mr} + r\right) = T_r$$

Because friction can be expressed as the product of the frictional coefficient and its weight, the frictional coeffcient is determined as

$$\mu = \frac{T_r}{(Ig/r + Mgr)}$$

where g is gravitational acceleration. Hence the required frictional coefficient is higher as torque increases. The driver should not try to accelerate rapidly, because, as the torque increases, the required frictional coefficient to avoid spinning wheels on ice will exceed the actual frictional coefficient.

Example 4.10

Consider a block of mass m sliding on a straight rod without friction as a case for a time-dependent constraint. The rod is rotating in the x–y plane that is perpendicular to the gravitational force. The rod is rotating at constant velocity ω. Find 1) the radial position of the block as a function of time and 2) the constraint force from the rod on the block. A similar problem has been presented in Example 2.3. The physical conditions are shown in Fig. 2.7.

Solution. The r and θ are the generalized coordinates. The constraint equation is

$$\theta = \omega t$$

or

$$\mathrm{d}\theta - \omega\,\mathrm{d}t = 0 \tag{4.80}$$

so that

$$C_r = 0, \qquad C_\theta = 1, \qquad C_t = -\omega$$

The kinetic energy is

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

The potential energy is a constant that is set to zero, i.e.,

$$V = 0$$

Therefore,

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

1) For the equation in the r direction,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\omega^2 = 0$$

$$\ddot{r} - \omega^2 r = 0$$

$$\begin{aligned} r &= A\cosh\omega t + B\sinh\omega t \\ &= r_0\cosh\omega t + (\dot{r}_0/\omega)\sinh\omega t \end{aligned} \tag{4.81}$$

where r_0 and $\dot{r}_0$ are the initial position and velocity of the block along the r direction.

2) For the constraint force,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 2m\omega r\dot{r} = \lambda \tag{4.82}$$

$$\mathcal{F}_\theta = 2m\omega r\dot{r}$$

Here, the generalized constraint force is a torque. The force between the rod and the block is $2m\omega\dot{r}$.

4.5 Calculus of Variations

The calculus of variations is a totally different approach from Lagrangian equations. It is a method for us to determine conditions under which the integral of a given function will reach a maximum or minimum. But it can also reach Lagrange's equation for a conservative system. Because of that it is included in this chapter.

To understand the method, let us consider a function f that is to be integrated over a path $y(x)$. The starting point of the path is (x_1, y_1), and the end point is (x_2, y_2) as shown in Fig. 4.9. Assume that the function f can be written as

$$f = f(y, y', x)$$

where y and y' and x are independent variables, although y and y' are functions

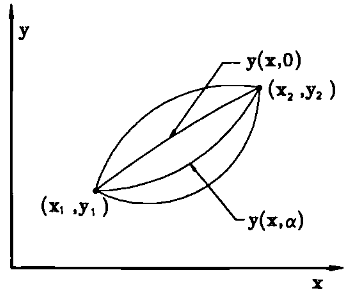

Fig. 4.9 Paths for line integration.

of x. The integral of f is then

$$I = \int_{x_1}^{x_2} f(y, y', x)\mathrm{d}x \tag{4.83}$$

Clearly, the result of the integral depends on the path $y(x)$ chosen. Here we want to determine a particular path $y(x)$, so that it makes the integral to reach the extremum. To reach that goal, we let

$$y(x, \alpha) = y(x, 0) + \alpha g(x) \tag{4.84}$$

where $g(x) = \partial y/\partial\alpha$ and $g(x_1) = g(x_2) = 0$. This means that the path is varied from $y(x)$ to $y(x, \alpha)$. The condition for the extremum of the integral is then

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0 \tag{4.85}$$

From Eq. (4.83), we have

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)\mathrm{d}x \tag{4.86}$$

In the preceding equation, the second term on the right can be simplified with the use of integration by parts, i.e.,

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha}\mathrm{d}x = \frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x = \frac{\partial f}{\partial y'}\bigg|_{x=x_2} g(x_2)$$
$$- \frac{\partial f}{\partial y'}\bigg|_{x=x_1} g(x_1) - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x = - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\mathrm{d}x$$

Substituting the preceding equation into Eq. (4.86), we obtain

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}\mathrm{d}x$$

Now multiplying the equation by $\mathrm{d}\alpha$ and setting α to 0 and writing

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} \mathrm{d}\alpha = \delta I$$

$$\left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} \mathrm{d}\alpha = \delta y$$

we find

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y \mathrm{d}x = 0$$

Because δy is arbitrary and not zero as $x_1 < x < x_2$, the terms in the brackets must be zero, i.e.,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{4.87}$$

This equation is known as the Euler–Lagrange equation. Note that if we change symbols, $f \to L$, $y' \to \dot{q}$, $y \to q$, and $x \to t$, we can write Eq. (4.87) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

which is Lagrange's equation for a conservative system. Equation (4.87) is the tool for us to find $y(x)$ for I to become the extremum. It is similar to Lagrange's equation, from which we find $q(t)$. For a special case, when f is not an explicit function of x, Eq. (4.87) can be further simplified. Multiplying Eq. (4.87) by y', we have

$$y'\frac{\partial f}{\partial y} - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Adding and subtracting $(\partial f/\partial y')y''$ and also adding $\partial f/\partial x$, which is zero anyway, we obtain

$$\frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y'}y'' - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Rewrite the first three terms as $\mathrm{d}f/\mathrm{d}x$ and the last two terms as $(\mathrm{d}/\mathrm{d}x)[y'(\partial f/\partial y')]$; we find

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) = 0$$

or

$$f - y'\frac{\partial f}{\partial y'} = \text{const} \tag{4.88}$$

which is even simpler than Eq. (4.87) for finding $y(x)$. It will become clear after studying a few examples later.

On the other hand, sometimes we like to have

$$I = \int_{x_1}^{x_2} f(y, y', x)\,dx \tag{4.89}$$

to reach the extremum, but notice a condition is imposed, such as,

$$\int_{x_1}^{x_2} \sigma(y, y', x)\,dx = C_0 \tag{4.90}$$

To treat this type of problem, we multiply Eq. (4.90) by λ and add that to Eq. (4.89), then we have

$$\begin{aligned} I + \lambda c_0 = I' &= \int_{x_1}^{x_2} (f + \lambda\sigma)\,dx \\ &= \int_{x_1}^{x_2} F(y, y', x)\,dx \end{aligned}$$

$$\delta I' = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y\,dx = 0$$

in which $F(y, y', x) = f + \lambda\sigma$. Similar to the way we find Eq. (4.87), we obtain

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{4.91}$$

From this equation, $y(x, \lambda)$ will be found. The constant λ then is determined by Eq. (4.90), which is equivalent to the constraint equation already discussed.

Example 4.11

A geodesic on a given surface is a curve, lying on that surface, along which the distance between two points is shortest. Determine the equation of geodesic on a right circular cylinder.

Solution. The radius of the cylinder is a. Take the z axis along the axis of the cylinder. The two points on the cylindrical surface are (z_1, θ_1) and (z_2, θ_2). The distance between two points is

$$S = \int_{\theta_1}^{\theta_2} \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\,d\theta$$

Therefore,

$$f(z, z', \theta) = \sqrt{a^2 + z'^2}$$

$$\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{a^2 + z'^2}}$$

Using Eq. (4.88), we have

$$f = \sqrt{a^2 + z'^2} = \text{const}$$

or

$$z' = \frac{\mathrm{d}z}{\mathrm{d}\theta} = \text{const}$$

$$z = c_0\theta + c_1$$

Therefore, the equation for the geodesic on a circular cylinder is found to be

$$z = \frac{z_2 - z_1}{\theta_2 - \theta_1}\theta + \frac{z_1\theta_2 - z_2\theta_1}{\theta_2 - \theta_1} \tag{4.92}$$

Example 4.12

Just to illustrate the point that the calculus of variations also leads to the Lagrangian equation for a conservative system, let us consider a particle of mass m freely falling under gravity. Find the equation of motion by considering

$$I = \int_{t_1}^{t_2} L(y, \dot{y}, t)\mathrm{d}t$$

Solution. The energies of the system are

$$T = \tfrac{1}{2}m\dot{y}^2, \qquad V = mg(y - y_0)$$

$$L = T - V = \tfrac{1}{2}m\dot{y}^2 - mg(y - y_0)$$

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 = -mg - m\ddot{y}$$

The equation of motion is

$$\ddot{y} = -g$$

Note that the y axis is taken vertically upward.

Example 4.13

The surface area for a body revolving with the x axis can be expressed as

$$I = 2\pi \int_{x_1}^{x_2} y(1 + y'^2)^{\frac{1}{2}}\,\mathrm{d}x$$

Determine the function $y(x)$ that minimizes the integral I.

Solution. Rewrite the integral as

$$\frac{I}{2\pi} = \int_{x_1}^{x_2} y(1 + y'^2)^{\frac{1}{2}}\,\mathrm{d}x$$

Here the function f is

$$f(y, y', x) = y(1 + y'^2)^{\frac{1}{2}}$$

which is not an explicit function of x. Using Eq. (4.88), we find

$$y(1 + y'^2)^{\frac{1}{2}} - y' \frac{yy'}{(1 + y'^2)^{\frac{1}{2}}} = c_1$$

or

$$y(1 + y'^2) - yy'^2 = c_1(1 + y'^2)^{\frac{1}{2}}$$

Simplifying leads to

$$y = c_1(1 + y'^2)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y^2}{c_1^2} - 1\right)^{\frac{1}{2}}$$

Integrating yields

$$y = c_1 \cosh\left(\frac{x}{c_1} + c_2\right)$$

where c_1, c_2 are integral constant and can be determined if the two end points are specified.

Example 4.14

Determine the equation for the shortest arc that passes through the points (0, 0) and (1, 0) and encloses a prescribed area A with the x axis (Fig. 4.10).

Solution. According to the given conditions, we have

$$I = \int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \tag{4.93}$$

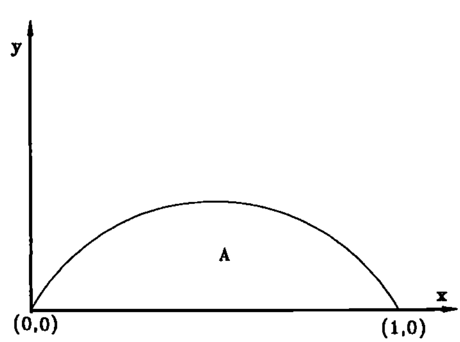

Fig. 4.10 Shortest arc between (0, 0) and (1, 0) but enclosing A.

and

$$A = \int_0^1 y \mathrm{d}x \tag{4.94}$$

Hence,

$$\begin{aligned} f &= \sqrt{1+y^2}, \qquad \sigma = y \\ F &= f + \lambda\sigma = \sqrt{1+y^2} + \lambda y \\ \frac{\partial F}{\partial y'} &= \frac{y'}{\sqrt{1+y'^2}}, \qquad \frac{\partial F}{\partial y} = \lambda \end{aligned} \tag{4.95}$$

Using Eq. (4.91), we have

$$\lambda - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \tag{4.96}$$

Integrating leads to

$$\frac{y'}{\sqrt{1+y'^2}} = \lambda x + c_1$$

$$y' = \pm\frac{\lambda x + c_1}{\sqrt{1-(\lambda x + c_1)^2}}$$

Integrating again, we find

$$y = \mp\frac{1}{\lambda}\sqrt{1-(\lambda x + c_1)^2} + c_2 \tag{4.97}$$

Applying the boundary conditions (0, 0) and (1, 0), we find

$$c_1 = -\frac{\lambda}{2}, \qquad c_2 = -\frac{1}{\lambda}\sqrt{1-\frac{\lambda^2}{4}} \tag{4.98}$$

Substituting c_1 and c_2 into Eq. (4.97) and using Eq. (4.94), we obtain

$$\begin{aligned} A &= \int_0^1 y\mathrm{d}x = \int_0^1 \frac{1}{\lambda}\sqrt{1-\left(\lambda x - \frac{\lambda}{2}\right)^2}\mathrm{d}x + c_2 x\Big|_0^1 \\ &= \frac{1}{\lambda^2}\left[\frac{\lambda}{2}\sqrt{1-\frac{\lambda^2}{4}} + \sin^{-1}\frac{\lambda}{2}\right] + c_2 \\ \frac{\lambda}{2} &= \sin\left[\lambda^2 A + \frac{\lambda}{2}\sqrt{1-\frac{\lambda^2}{4}}\right] \end{aligned} \tag{4.99}$$

The value of λ is determined by this equation. And $y(x)$ is written as

$$y = \frac{1}{\lambda}\sqrt{1 - \left(\lambda x - \frac{\lambda}{2}\right)^2} - \frac{1}{\lambda}\sqrt{1 - \frac{\lambda^2}{4}} \tag{4.100}$$

which can be rewritten in a familiar form as

$$\left(x - \frac{1}{2}\right)^2 + (y - c_2)^2 = \frac{1}{\lambda^2}$$

Therefore the curve is a circular arc with the center at $(\frac{1}{2}, c_2)$ and a radius of $1/\lambda$.

Problems

4.1. Derive the equations of motion for Example 2.1 with the use of Lagrangian equations.

4.2. Derive the equations of motion for Example 2.2 with the use of Lagrangian equations. Make some necessary assumptions to simplify the problem.

4.3. Develop the Lagrangian equation for the momentum equation of the incompressible fluid flow in fluid mechanics.

4.4. Use the result of Problem 4.3 to find the momentum equations in cylindrical and spherical coordinates for incompressible fluid flow in fluid mechanics.

4.5. Suppose a point mass m is attached to one end of a horizontal spring with spring constant k, the other end of which is fixed on a cart that is being moved uniformly in a horizontal plane by an external device with speed v_0. If we take as a generalized coordinate the position x of the mass particle in the stationary system, find the equation of the motion for m, from the following:
(a) The Lagrangian equation.
(b) Hamilton's canonical equations.

4.6. A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrangian multipliers and Lagrange's equations. Find the height at which the particle falls off.

4.7. Consider a car that is driven up an inclined slope (Fig. P4.7). With the use of constrained Lagrangian equations, find the equations of motion, and also find the power required to drive the car at the minimum speed. Make assumptions necessary to simplify the problem.

4.8. A circular loop of wire is located in the x–y plane, with one point on it fixed at the origin and its center on the y axis; the radius varies in time according to $r = a + bt^2$, where a and b are constants. Find the equations of motion for a bead of mass m sliding smoothly on the wire and the normal force of wire on the bead (expressed as a function of an appropriate angular coordinate and its time derivative).

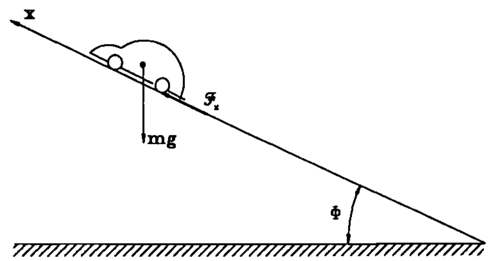

Fig. P4.7

4.9. Find the geodesic of a sphere.

4.10. The ends of a uniform inextensible string of length ℓ are connected to two points fixed at the same level, a distance $2a$ apart. Find the curve along which the string must hang if it is to have its center of mass as low as possible.