3
Dynamics of a System of Particles

IN this chapter we shall study the motion of a system of n particles subjected to external and internal forces. These internal forces, which arise from the interaction between the particles, obey Newton's third law of motion. Therefore, when all of the particles are considered as a unit, the internal forces add up to zero. Next, we shall discuss the angular momentum of a system of n particles. This subject plays an important role in studying the rotational motion of a solid body later in this book.

The collision of missiles in midair is analyzed in Section 3.2. The example illustrates that as two missile sites are a few hundred kilometers apart, the spherical surface of the Earth must be considered in the determination of the launching angle. Otherwise the second missile will not collide with the first missile if the launching angle is set according to the flat ground formulation. The gravitational force studied in the missile-to-missile collision is approximated to be always parallel to the z axis. The gravitational force, however, is easily modeled toward the center of Earth with a major component in the $\boldsymbol{k}$ direction and a small component in $\boldsymbol{i}$ direction where $\boldsymbol{i}$ and $\boldsymbol{k}$ are along the Cartesian coordinates chosen at the missile site. To simplify calculation, each missile is modeled as a particle so that the effects of air drag and the thrust of side jets on the missile can be neglected. The thrust is treated as a constant in the section. Precise treatment of the gravitation force in this case is unnecessary. The computer program used to solve this example, however, is easily modified to handle forces in precise forms. In the study of missile collision, two missiles must be addressed in the same coordinate system. Based on the knowledge of vector algebra, the conversion of coordinates is formulated and discussed in Section 3.1.

In the presence of two particles, there exist gravitational force and potential between them. We shall discuss these concepts in Section 3.4. It is interesting to mention that the gravitational force outside a solid sphere, such as Earth, is equivalent to that of a point mass with the same mass occurring at the center of the solid sphere; on the other hand the gravitational force is zero for a point mass located at the center of the solid sphere.

The collisions of solid spheres are discussed in Section 3.5. Both elastic and inelastic collisions are considered. Special emphasis is placed on automobile collision, which is closely related to our daily life.

3.1 Conversion of Coordinates

Before studying the collision of two missiles in the next section, we need to discuss the conversion of coordinates. Because two missile sites are a few hundred kilometers apart, each missile may be described by its own coordinate system first; then they must be converted into one set of coordinates. The procedure of establishing the relationship between the two sets of coordinates is referred to as the conversion of coordinates.

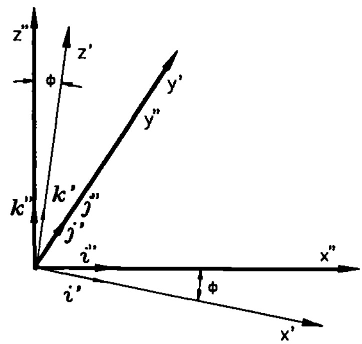

Fig. 3.1a $x''y''z''$ **rotated with respect to** j'' **by** ϕ.

Consider that the coordinate system XYZ is to exist permanently and the coordinate system xyz is to be converted. Starting from a general case, a system $x''y''z''$ is parallel to XYZ, i.e., $i''//\, i$, $j''//\, j$, $k''//\, k$. First, $x''y''z''$ is rotated with respect to the j'' axis by an angle of ϕ as shown in Fig. 3.1a. Then, the new coordinates $x'y'z'$ are rotated with respect to the k' axis by an angle of θ. After this rotation, the final coordinates are denoted by xyz as shown in Fig. 3.1b.

The relationship between XYZ and xyz is shown in Fig. 3.2. The position vector $\boldsymbol{R}$ locates the origin of xyz in XYZ. The position of a point P in xyz is denoted by the position vector ρ as

$$\rho = \boldsymbol{i}_\rho x + \boldsymbol{j}_\rho y + \boldsymbol{k}_\rho z$$

In terms of XYZ, the position vector of point P is $\boldsymbol{r}$ and we have

$$\boldsymbol{r} = \boldsymbol{R} + \rho \tag{3.1}$$

Writing in terms of their components, Eq. (3.1) becomes

$$X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} = X_0\boldsymbol{i} + Y_0\boldsymbol{j} + Z_0\boldsymbol{k} + x\boldsymbol{i}_\rho + y\boldsymbol{j}_\rho + z\boldsymbol{k}_\rho \tag{3.2}$$

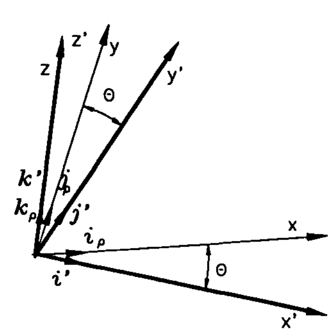

Fig. 3.1b $x'y'z'$ **rotated with respect to** k' **by** θ.

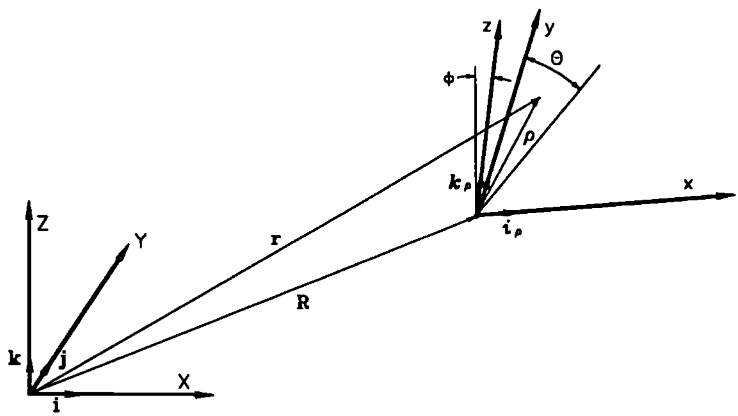

Fig. 3.2 Relationship between *XYZ* and *xyz* systems.

Note that in the preceding equation,

$$
\begin{aligned}
i_\rho &= \cos\theta i' + \sin\theta j' \\
&= \cos\theta(\cos\phi i - \sin\phi k) + \sin\theta j \\
&= \cos\theta\cos\phi i + \sin\theta j - \cos\theta\sin\phi k \\
j_\rho &= \cos\theta j' - \sin\theta i' \\
&= -\sin\theta\cos\phi i + \cos\theta j + \sin\theta\sin\phi k \\
k_\rho &= \sin\phi i + \cos\phi k
\end{aligned}
$$

In simplifying the preceding equations, we have used the relations $i'' = i$, $j'' = j$, $k'' = k$.

To obtain the X, Y, Z components of r, we take the scalar product of the unit vector with Eq. (3.2) as the following:

The scalar product of i with Eq. (3.2) gives

$$
\begin{aligned}
X &= X_0 + x\cos(i_\rho, i) + y\cos(j_\rho, i) + z\cos(k_\rho, i) \\
&= X_0 + x\cos\theta\cos\phi - y\sin\theta\cos\phi + z\sin\phi
\end{aligned}
\tag{3.3}
$$

The scalar product of j with Eq. (3.2) gives

$$
\begin{aligned}
Y &= Y_0 + x\cos(i_\rho, j) + y\cos(j_\rho, j) + z\cos(k_\rho, j) \\
&= Y_0 + x\sin\theta + y\cos\theta
\end{aligned}
\tag{3.4}
$$

Finally the scalar product of k with Eq. (3.2) gives

$$
\begin{aligned}
Z &= Z_0 + x\cos(i_\rho, k) + y\cos(j_\rho, k) + z\cos(k_\rho, k) \\
&= Z_0 - x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi
\end{aligned}
\tag{3.5}
$$

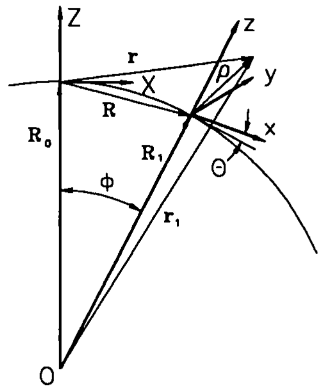

Fig. 3.3 Transfer of coordinates on spherical surface.

In a special case, if there is no rotation with respect to the y axis, i.e., $\phi = 0$, Eqs. (3.3–3.5) reduce to

$$X = X_0 + x\cos\theta - y\sin\theta \tag{3.6}$$

$$Y = Y_0 + x\sin\theta + y\cos\theta \tag{3.7}$$

$$Z = Z_0 + z \tag{3.8}$$

On the other hand, when two coordinate systems are apart by an order of a few hundred kilometers on the surface of the Earth, the effect of the spherical surface must be taken into consideration. Consider that the coordinate systems are on the spherical surface of the Earth as shown in Fig. 3.3. The XYZ system is so chosen that the plane containing x and z axes is the same plane containing R_0, R_1, and R. The unit vector $\boldsymbol{k}$ is along the vector $\boldsymbol{R}_0$ that is pointing from the center of Earth radially to the origin of XYZ. $\boldsymbol{R}_1$ is the position vector of the origin of xyz. Hence

$$\boldsymbol{R}_0 = \boldsymbol{k}R_0$$

$$\boldsymbol{R}_1 = (\boldsymbol{i}\sin\phi + \boldsymbol{k}\cos\phi)R_1$$

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{R}_1 - \boldsymbol{R}_0 \\ &= \boldsymbol{i}R_1\sin\phi - \boldsymbol{k}(R_0 - R_1\cos\phi) \\ &= \boldsymbol{i}R_0\sin\phi - \boldsymbol{k}R_0(1-\cos\phi) \end{aligned} \tag{3.9}$$

In the preceding equation, it is assumed that the Earth is a perfect sphere, so $\boldsymbol{R}_1$ and $\boldsymbol{R}_0$ are equal. Applying Eqs. (3.3–3.5) with $\boldsymbol{R}$ given in Eq. (3.9), we have the scalar components of $\boldsymbol{r}$ as

$$X = R_0\sin\phi + x\cos\theta\cos\phi - y\sin\theta\cos\phi + z\sin\phi \tag{3.10}$$

$$Y = x\sin\theta + y\cos\theta \tag{3.11}$$

$$Z = -R_0(1-\cos\phi) - x\cos\theta\sin\phi + y\sin\theta\sin\phi + z\cos\phi \tag{3.12}$$

where R_0 is the average radius of Earth and its value is 6371.23 km.

3.2 Collision of Particles in Midair

Study of the collision of two missiles in midair is based on the motions of individual missiles. To simplify the problem let us model them as particles as in the example given in Section 2.2. Although it is known that the second missile is equipped with side jets for adjusting its course, these side thrusts are omitted here. The forces applied on each missile could be very complicated because of variable thrust and air drag. In addition, the mass of a missile is decreasing continuously. However, the model can be simplified greatly by considering that the force applied is constant and the mass ejected from the propulsion system is also at a constant rate. This is an approximate model. Let us study the collision of two missiles with the following example.

Example 3.1

Suppose that a missile is launched from the enemy side, which is designated as the first missile. Through the detection by a satellite, the trajectory can be simulated as given in Example 2.2 with the net thrust of $F = 14{,}500$ N. The coordinates are transferred. Because of the action taken for the determination of the trajectory of the first missile, the time for launching the second missile is delayed by 60 s. To simplify the calculation, the trajectories of the two missiles are assumed to be contained in the same plane, but the launching sites are 200 km apart. The data for the second missile are given as follows: initial mass $m_0 = 1000$ kg, thrust $F = 16{,}000$ N, initial velocity = 300 m/s, and the mass decreasing rate = 3 kg/s. The problem is to determine the launching angle of the second missile so that the two missiles are to collide high above the ground. The conversion of coordinates is treated in two different ways: 1) flat ground and 2) spherical ground.

Solution. 1) Consider that the two launching sites are on flat ground. Each missile is governed by the following equations:

$$m_i \frac{\mathrm{d}V_{xi}}{\mathrm{d}t} = F \frac{V_{xi}}{\sqrt{V_{xi}^2 + V_{zi}^2}} \qquad (i = 1, 2) \tag{3.13}$$

$$m_i \frac{\mathrm{d}V_{zi}}{\mathrm{d}t} = F \frac{V_{zi}}{\sqrt{V_{xi}^2 + V_{zi}^2}} - m_i g \qquad (i = 1, 2) \tag{3.14}$$

$$m_i = m_{i0} - \dot{m}_i t \qquad (i = 1, 2) \tag{3.15}$$

Equations (3.13) and (3.14) are nonlinear and are solved by numerical integration with

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = V_{xi}, \qquad \frac{\mathrm{d}z_i}{\mathrm{d}t} = V_{zi} \tag{3.16}$$

The conditions used for the first missile are

$$(m_1)_0 = 1000 \text{ kg}$$

$$\dot{m}_1 = 3 \text{ kg/s}$$

$$(V_1)_0 = 150 \text{ m/s}$$

$$\alpha_1 = 80 \text{ deg}$$

$$F_1 = 14{,}500 \text{ N}$$

where α is the launching angle measured from x axis. The coordinates are transferred simply by

$$X_1 = X_0 - x_1 \tag{3.17}$$

$$Z_1 = z_1 \tag{3.18}$$

The conditions used for the second missile are

$$(m_2)_0 = 1000 \text{ kg}$$

$$\dot{m}_2 = 3 \text{ kg/s}$$

$$(V_2)_0 = 300 \text{ m/s}$$

$$F_2 = 16{,}000 \text{ N}$$

The launching angle of the second missile is determined with a trial and error method performed on computer. In the calculation, the first number used is 1.00 rad with the increment of ± 0.01. To detect whether the collision is going to take place or not, the distance between the missiles is calculated. The unsuccessful simulation terminates as the distance between them increases. When the collision is nearly occurring, finer increments for the launching angle and the time step are used.

For the present study, the increments for the final step are $\Delta\alpha = 2.0E{-7}$ and $\Delta t = 5.0E{-5}$ s. The collision condition is reached when the distance between the two missiles is less than 8 cm. The launching angle for the second missile is found to be 0.982 145 4 rad. The collision is taking place at 144.8327 s after the launching of the first missile and is 84.8327 s after the launching of the second missile. The coordinates at the collision are $X = 66.82$ km, $Z = 16.26$ km. The missile shooting missile trajectories are shown in Fig. 3.4.

2) For a spherical surface, the equations governing the motions of missiles are the same as those used in part 1. Because the trajectories of the missiles are assumed to be in the same plane, the coordinates of the first missiles are transferred using Eqs. (3.10) and (3.12) with $y = 0$. These equations are as follows:

$$X = R_0 \sin\phi + x\cos\theta\cos\phi + z\sin\phi \tag{3.19}$$

$$Z = -R_0(1 - \cos\phi) - x\cos\theta\sin\phi + z\cos\phi \tag{3.20}$$

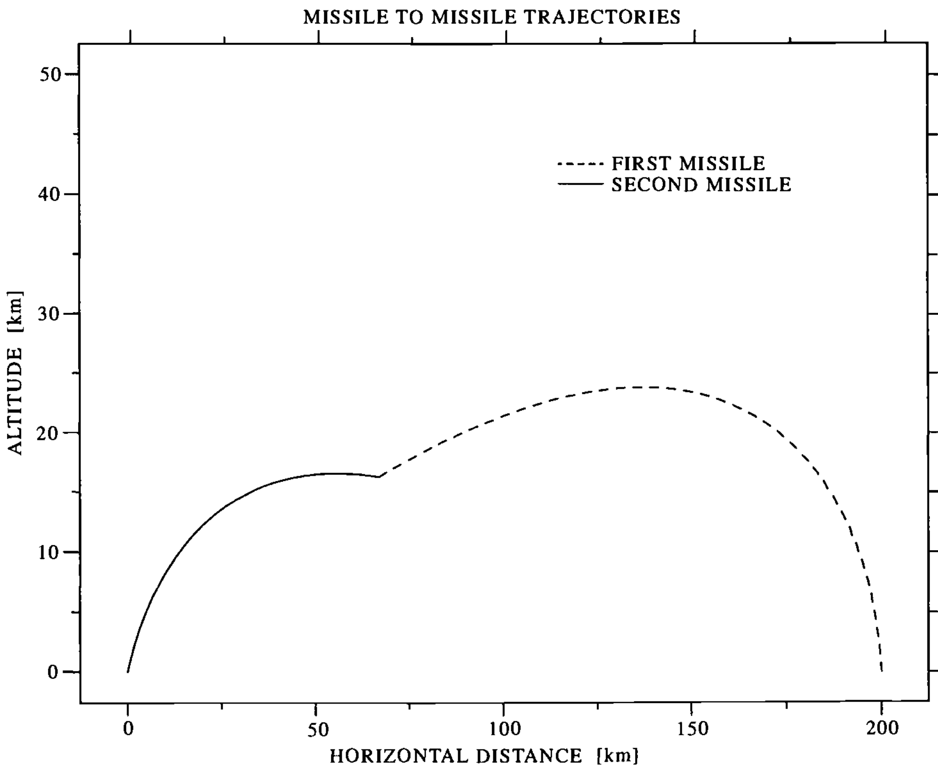

Fig. 3.4 Missile-to-missile trajectories on flat ground.

For the present case $R_0 = 6371.23$ km, $\theta = \pi$, and $\phi = 0.031391112$. Substituting these values into Eqs. (3.19) and (3.20), we have

$$X = 199{,}967.155 - 0.99950734x + 0.03138596z \qquad \text{(m)}$$

$$Z = -3138.8535 + 0.03138596x + 0.99950734z \qquad \text{(m)}$$

Note that the initial coordinates of the first missile are

$$X_0 = 199{,}967.1550 \qquad \text{(m)}$$

$$Z_0 = -3138.8535 \qquad \text{(m)}$$

The calculation procedure is the same as that used in part 1. The launching angle for the second missile is determined to be 0.9929676 rad, and the collision occurs 145.1400 s after the launching of the first missile and 85.1400 s after launching of the second missile. It is important to point out that the missiles will not collide if α is set as 0.9821454 rad, because the Earth's surface is actually spherical. The coordinates at the collision are $X = 66.64$ km and $Z = 17.18$ km. The missile

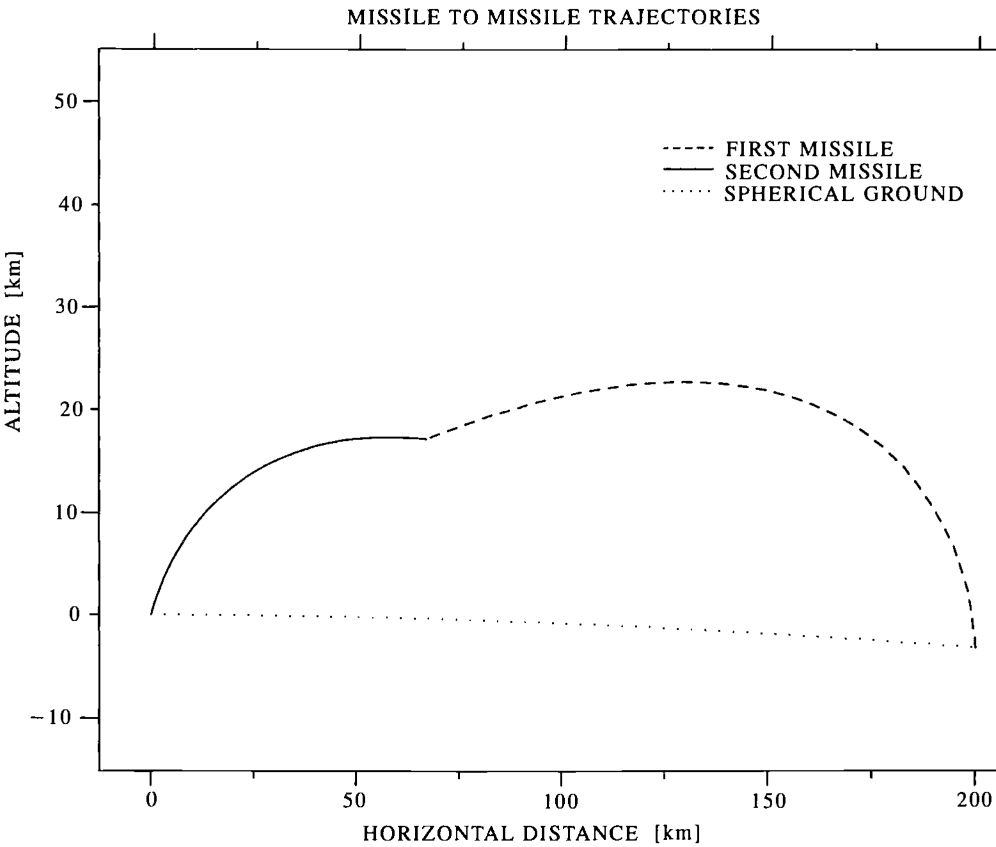

Fig. 3.5 Missile-to-missile trajectories on sphere.

trajectories are shown in Fig. 3.5. For completeness, the computer program written in Fortran is included in this section.

```
C     PROGRAM MISSILE TO MISSILE FOR EXAMPLE 3-1 ON SPHERICAL
C     SURFACE
      REAL T(18001),X1(18001),X2(18001),Z1(18001),Z2(18001),M1,M2
      OPEN (2,FILE='MSLTMSLS.FIL')
      X1(1) = 0.0
      Z1(1) = 0.0
      X = X1(1)
      Z = Z1(1)
      VX10 = 26.0472
      VZ10 = 147.7212
      M1 = 1000.0
      G = 9.81
      DM = 3.0
      VX1 = VX10
      VZ1 = VZ10
      DO 100 N = 1,18000
      VXN = VX1
      VZN = VZ1
```

```
      X2(N) = 0.0
      Z2(N) = 0.0
      AH = 0.01
      AM = M1-AH*3.0*FLOAT(N-1)
      F1 = 14500.
      F = F1
      IF (N .LT. 14513) GO TO 90
      AH = 0.00005
      AM = 564.64-AH*3.0*FLOAT(N-14512)
 90   CALL RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      X1(N+1) = X
      Z1(N+1) = Z
      VX1 = VXN
      VZ1 = VZN
100   CONTINUE
      X10 = 199967.155
      Z10 = -3138.8535
      ALP = 0.9929676
      M2 = 1000.
      V2 = 300.0
      F2 = 16000.
      F = F2
      WRITE (2,8) M1,F1,AH,M2,F2
      WRITE (2,9) X10,Z10,VX10,VZ10
      NN = 1
120   VX2 = V2*COS(ALP)
      VZ2 = V2*SIN(ALP)
      VXN = VX2
      VZN = VZ2
      WRITE (2,10) X2(1),Z2(1),VXN,VZN
      AH = 0.01
      ALPOLD = ALP
      X = X2(1)
      Z = Z2(1)
      DO 200 N = 6001,18000
      AM = M2-AH*3.0*FLOAT(N-6001)
      IF (N .LT. 14513) GO TO 150
      AH = 0.00005
      AM = 564.64-AH*3.0*FLOAT(N-14512)
150   CONTINUE
      BX0 = X
      BZ0 = Z
      CALL RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      X2(N+1) = X
      Z2(N+1) = Z
      VX2 = VXN
      VZ2 = VZN
      AX0 = X10-X1(N)*0.99950734+Z1(N)*0.03138596
      AZ0 = Z10+Z1(N)*0.99950734+X1(N)*0.03138596
      AX1 = X10-X1(N+1)*0.99950734+Z1(N+1)*0.03138596
      AZ1 = Z10+Z1(N+1)*0.99950734+X1(N+1)*0.03138596
      BX1 = X2(N+1)
      BZ1 = Z2(N+1)
      D0 = SQRT((BX0-AX0)**2+(BZ0-AZ0)**2)
      D1 = SQRT((BX1-AX1)**2+(BZ1-AZ1)**2)
      IF (D1 .GT. D0) GO TO 190
```

```
      IF (D1 .LT. 0.080) GO TO 220
      GO TO 200
  190 WRITE (2,22) N,D1,D0
      GO TO 210
  200 CONTINUE
  210 ALP = ALPOLD-0.00000004
      NN = NN+1
      IF (NN .GT. 10) GO TO 240
      GO TO 120
  220 WRITE (2,21) ALP
      WRITE (2,11)
      DO 236 I = 1,N
      AH = 0.01
      IF (I .LT. 14513) GO TO 230
      AH = 0.00005
      T(I) = 145.12+AH*FLOAT(I-14512)
      GO TO 234
  230 T(I) = AH*FLOAT(I-1)
  234 XX = (X10-X1(I)*0.99950734+Z1(I)*0.03138596)/1000.
      Z1(I) = (Z10+Z1(I)*0.99950734+X1(I)*0.03138596)/1000.
      X1(I) = XX
      X2(I) = X2(I)/1000.
      Z2(I) = Z2(I)/1000.
  236 CONTINUE
      DO 238 I = 1,N,100
      WRITE (2,20) T(I),X1(I),Z1(I),X2(I),Z2(I),D1
  238 CONTINUE
      WRITE (2,20) T(N),X1(N),Z1(N),X2(N),Z2(N),D1
      GO TO 250
  240 WRITE (2,25)
    8 FORMAT ('M1 = ',F5.0,' kg  F1 = ',F6.0,' N  AH = ',F8.6,' S
     *M2 = ',F5.0,' kg  F2 = ',F6.0,' N')
    9 FORMAT (' X1o = ',F7.0,' m  Z1o = ',F8.2,' m  VX1o =
     * ',F8.2,'m/s  VZ1o = ',F8.2,' m/s ')
   10 FORMAT ('X2o = ',F7.0,' m  Z2o = ',F8.2,' m  VX2o =
     *',F8.2,'m/s  VZ2o = ',F8.2,' m/s ')
   11 FORMAT (3X,' T(s) ',6X,' X1(km)',7X,' Z1(km)',7X,'
     * X2(km)',7X,' Z2  (km)',7X, ' D1(m)')
   20 FORMAT (1X,F9.5,5(2X,E12.4))
   21 FORMAT ('MISSILES COLLIDED WITH ALPHA = ',F10.8)
   22 FORMAT ('MISSILES ARE NOT COLLIDING  N = ',I6,'  D1
     * = ',F8.4,'m  D0 = ',F8.4,'m')
   25 FORMAT ('MAXIMUM ITERATIONS EXCEEDED')
  250 STOP
      END
      SUBROUTINE RK (X,Z,VXN,VZN,AH,AM,DM,F,G)
      AK1 = AH*(F/AM)*VXN/SQRT(VXN**2+VZN**2)
      BK1 = AH*((F/AM)*VZN/SQRT(VXN**2+VZN**2)-G)
      XK1 = AH*VXN
      ZK1 = AH*VZN
      AM = AM-DM*AH/2.
      AK2 = AH*(F/AM)*(VXN+AK1/2.)/SQRT((VXN+AK1/2.)**2+
     C (VZN+BK1/2.)**2)
      BK2 = AH*((F/AM)*(VZN+BK1/2.)/SQRT((VXN+AK1/2.)**2+
     C (VZN+BK1/2.)**2)-G)
```

```
XK2 = AH*(VXN+AK1/2.)
ZK2 = AH*(VZN+BK1/2.)
AK3 = AH*(F/AM)*(VXN+AK2/2.)/SQRT((VXN+AK2/2.)**2+
C (VZN+BK2/2.)**2)
BK3 = AH*((F/AM)*(VZN+BK2/2.)/SQRT((VXN+AK2/2.)**2+
C (VZN+BK2/2.)**2)-G)
XK3 = AH*(VXN+AK2/2.)
ZK3 = AH*(VZN+BK2/2.)
AM = AM-DM*AH/2.
AK4 = AH*(F/AM)*(VXN+AK3)/SQRT((VXN+AK3)**2+
C (VZN+BK3)**2)
BK4 = AH*((F/AM)*(VZN+BK3)/SQRT((VXN+AK3)**2+
C (VZN+BK3)**2)-G)
XK4 = AH*(VXN+AK3)
ZK4 = AH*(VZN+BK3)
VXN1 = VXN+(AK1+2.*AK2+2.*AK3+AK4)/6.
VZN1 = VZN+(BK1+2.*BK2+2.*BK3+BK4)/6.
XX = X+(XK1+2.*XK2+2.*XK3+XK4)/6.
ZZ = Z+(ZK1+2.*ZK2+2.*ZK3+ZK4)/6.
VXN = VXN1
VZN = VZN1
X = XX
Z = ZZ
RETURN
END
```

3.3 General Motion of a System of Particles

Consider a system of n particles. For each particle there are two kinds of forces acting on it. One is the resultant of the external forces, and the other is the internal forces between particles. The mass of each particle is fixed. For the ith particle, the equation of motion is

$$m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \boldsymbol{F}_i + \sum_{\substack{j=1 \\ i \neq j}}^{n} \boldsymbol{f}_{ij} \tag{3.21}$$

where $\boldsymbol{f}_{ij}$ is the internal force exerted on the particle i by the particle j. $\boldsymbol{F}_i$ is the resultant force acting on particle i from the forces external to the system of particles. Because there are n particles in the system, the equation of motion for the system is

$$\sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \sum_{i=1}^{n} \boldsymbol{F}_i + \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \boldsymbol{f}_{ij}$$

According to Newton's third law, the internal forces exerted by two particles i and j on each other are equal in magnitude and opposite in direction, that is $\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$. Therefore, the sum of the internal forces is zero and we obtain

$$\boldsymbol{F} = \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{i=1}^{n} m_i \boldsymbol{r}_i \tag{3.22}$$

where $\boldsymbol{F}$ is the vector sum of all the external forces acting on all the particles. To simplify this equation, let us recall the method for locating the center of mass for the system:

$$\boldsymbol{r}_c \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \boldsymbol{r}_i$$

$$\boldsymbol{r}_c = \frac{\sum_{i=1}^{n} m_i \boldsymbol{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_i m_i \boldsymbol{r}_i \tag{3.23}$$

where $\boldsymbol{r}_c$ is the position vector of the center of mass. With the use of Eq. (3.23), Eq. (3.22) becomes

$$\boldsymbol{F} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} M \boldsymbol{r}_c = M \frac{\mathrm{d}^2 \boldsymbol{r}_c}{\mathrm{d}t^2} \tag{3.24}$$

Therefore, we can conclude that the motion of a system of particles is equivalent to that of a single particle with mass M located at the mass center of the system.

Now let us consider the angular momentum or the moment of momentum of a system of n particles. Taking the cross product of $\boldsymbol{r}_i$ with Eq. (3.21) leads to

$$\boldsymbol{r}_i \times m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{r}_i \times \sum_{\substack{j=1 \\ j \neq i}}^{n} \boldsymbol{f}_{ij}$$

Looking into details in the preceding equation, we find

$$\begin{aligned} \boldsymbol{r}_i \times m_i \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} &= \boldsymbol{r}_i \times \frac{\mathrm{d}}{\mathrm{d}t}(m_i \dot{\boldsymbol{r}}_i) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{P}_i) = \frac{\mathrm{d}\boldsymbol{H}_i}{\mathrm{d}t} \end{aligned}$$

$$\dot{\boldsymbol{H}} = \sum_i \frac{\mathrm{d}\boldsymbol{H}_i}{\mathrm{d}t}$$

$$\sum_{i,j}^{n} \boldsymbol{r}_i \times \boldsymbol{f}_{ij} = 0 \qquad \text{because } \boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$$

Therefore, we obtain

$$\dot{\boldsymbol{H}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{M} \tag{3.25}$$

Thus, the time rate of change of angle momentum is equal to the total moment of external forces acting on the particles with respect to a fixed point. This equation is the same as Eq. (2.32) for a single particle.

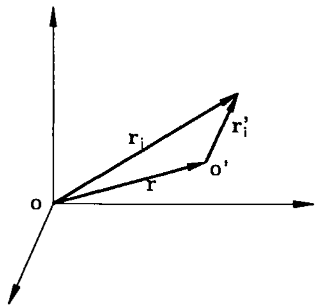

Fig. 3.6 Particles with the center of mass at 0′.

We may express Eq. (3.25) from a different perspective by considering that the center of mass is located at the origin 0′ of another coordinate system $x'y'z'$. Then, as shown in Fig. 3.6, we have

$$\boldsymbol{r}_i = \boldsymbol{r} + \boldsymbol{r}'_i$$

where $\boldsymbol{r}$ is the position vector of the center of mass for the system of n particles, $\boldsymbol{r}'_i$ is the position vector of ith particle in $x'y'z'$. Taking the time derivative of the position vector equation, we obtain

$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}} + \dot{\boldsymbol{r}}'_i$$

The angular momentum of the n particles about 0 is

$$\begin{aligned}\boldsymbol{H} &= \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{P}_i = \sum_{i=1}^{n} \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i = \sum_i (\boldsymbol{r} + \boldsymbol{r}'_i) \times m_i(\dot{\boldsymbol{r}} + \dot{\boldsymbol{r}}'_i) \\ &= \sum_i m_i \boldsymbol{r} \times \dot{\boldsymbol{r}} + \sum_i \boldsymbol{r} \times m_i \dot{\boldsymbol{r}}'_i + \sum_i \boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}} + \sum_i \boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}}'_i \\ &= \boldsymbol{r} \times M\dot{\boldsymbol{r}} + \sum_i \boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}}'_i \end{aligned} \tag{3.26}$$

Simplifying Eq. (3.26) is based on the fact that, because 0′ is the center of mass, the following expressions are true:

$$\sum_i m_i \boldsymbol{r}'_i = 0,$$

$$\sum_i \boldsymbol{r} \times m_i \dot{\boldsymbol{r}}'_i = \boldsymbol{r} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \boldsymbol{r}'_i = 0$$

$$\sum_i \boldsymbol{r}'_i \times m_i \dot{\boldsymbol{r}} = \sum_i \boldsymbol{r}'_i m_i \times \dot{\boldsymbol{r}} = 0$$

Equation (3.26) states that the angular momentum of the system with respect to point 0 equals the sum of the angular momentum of total mass M at point 0′ with respect to 0 and the angular momentum of the system with respect to the center of mass. Furthermore, from the right hand of Eq. (3.25), we have

$$\begin{aligned} \boldsymbol{M} &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_i (\boldsymbol{r} + \boldsymbol{r}_i') \times \boldsymbol{F}_i \\ &= \boldsymbol{r} \times \boldsymbol{F} + \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i = \boldsymbol{r} \times \boldsymbol{F} + \boldsymbol{M}' \end{aligned} \tag{3.27}$$

$$\begin{aligned} \boldsymbol{M}' &= \sum_i \boldsymbol{r}_i' \times m_i \ddot{\boldsymbol{r}}_i = \sum_i \boldsymbol{r}_i' \times m_i (\ddot{\boldsymbol{r}} + \ddot{\boldsymbol{r}}_i') \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}}_i'\right] + \left(\sum_i m_i \boldsymbol{r}_i'\right) \times \ddot{\boldsymbol{r}} = \dot{\boldsymbol{H}}' \end{aligned} \tag{3.28}$$

Differentiating Eq. (3.26) with respect to time and using Eqs. (3.27) and (3.28), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times M\dot{\boldsymbol{r}}) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \boldsymbol{r}_i' \times m_i \dot{\boldsymbol{r}}_i'\right] \\ &= \boldsymbol{r} \times M\ddot{\boldsymbol{r}} + \dot{\boldsymbol{H}}' = \boldsymbol{r} \times \boldsymbol{F} + \boldsymbol{M}' = \boldsymbol{M} \end{aligned} \tag{3.29}$$

From Eq. (3.29), we see that the total moment acting on the system with respect to point 0 equals the sum of the moment produced by the total external force with respect to point 0 and the moment of the system with respect to point 0′.

3.4 Gravitational Force and Potential Energy

When a point mass is in the vicinity of a large mass, such as Earth, it experiences a gravitational force directed toward the mass center of the large mass and possesses a potential energy with respect to the large mass. If there are two point masses placed side by side, an attractive force will exist between them. According to Newton's law of universal gravitation, the magnitude of this attractive force can be expressed as

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant and is 6.67×10^{-11} N-m²/kg², and r is the distance between the two point masses m_1 and m_2. Suppose that the position vector of m_1 is $\boldsymbol{r}_1$ and that of m_2 is $\boldsymbol{r}_2$ as shown in Fig. 3.7. Then the force acting on m_1 is

$$\boldsymbol{F}_1 = G\frac{m_1 m_2}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}(\boldsymbol{r}_2 - \boldsymbol{r}_1)$$

When m_2 is a distributed mass, however, the gravitational force on m_1 becomes

$$\boldsymbol{F}_1 = m_1 \int_v G\frac{\rho(\boldsymbol{r}')\mathrm{d}v'}{|\boldsymbol{r}' - \boldsymbol{r}|^3}(\boldsymbol{r}' - \boldsymbol{r})$$

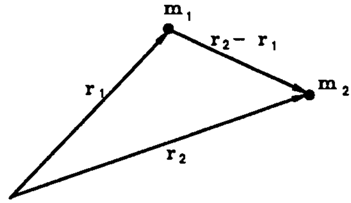

Fig. 3.7 Relationship between two point masses.

where $\rho(\boldsymbol{r}')$ is the density of the distributed mass. The relationship between m_1 and a distributed mass is shown in Fig. 3.8. Rearranging the preceding equation gives the gravitational intensity as

$$\frac{\boldsymbol{F}_1}{m_1} = \boldsymbol{g} = \int_v G\frac{\rho(\boldsymbol{r}')\mathrm{d}v'}{|\boldsymbol{r}' - \boldsymbol{r}|^3}(\boldsymbol{r}' - \boldsymbol{r}) \tag{3.30}$$

Gravitational force is a typical conservative force that can be expressed as

$$\boldsymbol{g} = -\nabla V \tag{3.31}$$

where V = potential energy per unit mass or gravitational potential. In terms of the gradient of a scalar function, Eq. (3.30) can be written as

$$\boldsymbol{g} = G\int_v \nabla\left(\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}\right)\rho\,\mathrm{d}v' \tag{3.32}$$

Therefore, the gravitational potential is

$$V = -G\int_v \frac{\rho\,\mathrm{d}v'}{|\boldsymbol{r}' - \boldsymbol{r}|} \tag{3.33}$$

Example 3.2

Derive expressions for the gravitational force and the potential energy for a point mass under the following two circumstances: 1) outside a uniform solid sphere and 2) inside the solid sphere.

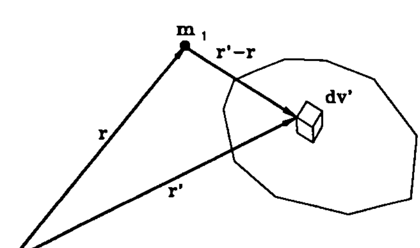

Fig. 3.8 Relationship between m_1 and a distributed mass.

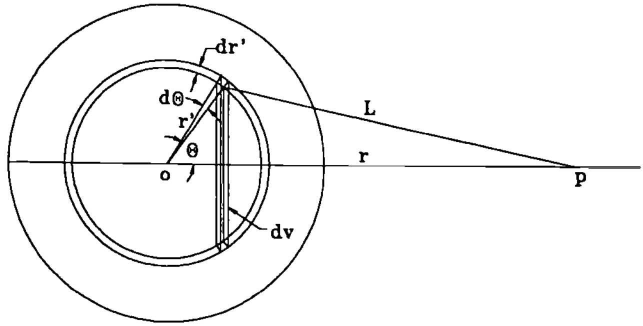

Fig. 3.9 Point P located outside a uniform solid sphere.

Solution. 1) Outside a uniform solid sphere, consider that a particle with unit mass is located at P; the distance between the center of the sphere and the particle is r as shown in Fig. 3.9. The infinitesimal volume under consideration is

$$\mathrm{d}v = (2\pi r'^2 \sin\theta)\mathrm{d}\theta\,\mathrm{d}r'$$

The distance between p and $\mathrm{d}v$ is

$$L = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

Using Eq. (3.33), the potential energy is

$$\begin{aligned} V &= -G\rho\int_v \frac{\mathrm{d}v}{L} = -G\rho\int_0^R\int_0^\pi \frac{2\pi r'^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} \\ &= -\frac{G\rho}{r}\frac{4\pi}{3}R^3 = -\frac{GM}{r} \end{aligned} \tag{3.34}$$

where M is the mass of the solid sphere. This result states that the potential of unit mass outside a solid sphere is equivalent to that of a point mass with the same mass concentrated at the center of the sphere. From the result of Eq. (3.34), we find the gravitational force as

$$\boldsymbol{g} = -\nabla V = \boldsymbol{e}_r\frac{\partial}{\partial r}\left(\frac{GM}{r}\right) = -\boldsymbol{e}_r\frac{GM}{r^2} \tag{3.35}$$

where $\boldsymbol{e}_r$ is the unit vector along r. This expression is used for calculating the gravitational acceleration.

2) Inside a uniform solid sphere, consider that the point mass is at P located inside the sphere. The infinitesimal volume is a ring and can be expressed as

$$\mathrm{d}v = (2\pi r'\sin\theta)\mathrm{d}r'(r'\mathrm{d}\theta)$$

The distance between P and $\mathrm{d}v$ is L:

$$L = [(r'\sin\theta)^2 + (R\cos\theta_1 - r'\cos\theta)^2]^{1/2}$$

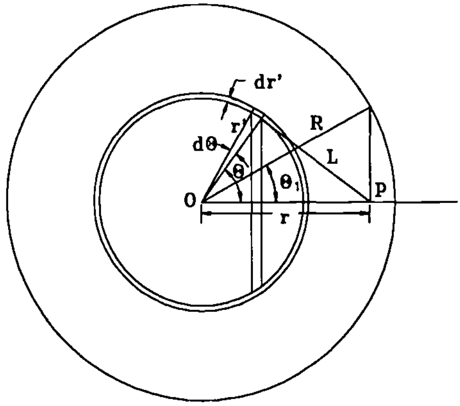

Fig. 3.10 Point *P* located inside a uniform solid sphere.

Therefore, the potential energy is

$$V = -G\rho \int_0^R \int_0^\pi \frac{2\pi r'^2 \sin\theta \, d\theta \, dr'}{\sqrt{r'^2 + R^2\cos^2\theta_1 - 2Rr'\cos\theta_1\cos\theta}}$$

$$= -G\rho\frac{2\pi}{3}\left[3R^2 - (R\cos\theta_1)^2\right] = -\frac{2}{3}\pi G\rho(3R^2 - r^2) \qquad (3.36)$$

where r is the distance of op. In the integration, it ought to be pointed out that when the integrand is integrated with respect to θ, the term for the lower integral limit is always kept to be positive so that the result is the subtraction of the upper limit term by the lower limit term. And the gravitational intensity is found:

$$g = -\nabla V = -\frac{4\pi}{3}G\rho r \boldsymbol{e}_r \qquad (3.37)$$

Note that the g approaches zero as r reaches zero and the potential energy reaches minimum at the center of the sphere.

Example 3.3

Suppose that a homogeneous right circular cylinder of radius R, height L, and mass M is placed along the z axis between $z = 0$ and $z = L$ as shown in Fig. 3.11. Find the gravitational intensity and potential of the cylinder on the axis at distance h from the origin with $h > L$.

Solution. Consider that the infinitesimal element in the cylinder is a ring with the cross section area of $dr\,dz$ and with

$$dv = 2\pi r \, dr \, dz$$

The distance from the ring to point P is

$$S = \sqrt{r^2 + (h - z)^2}$$

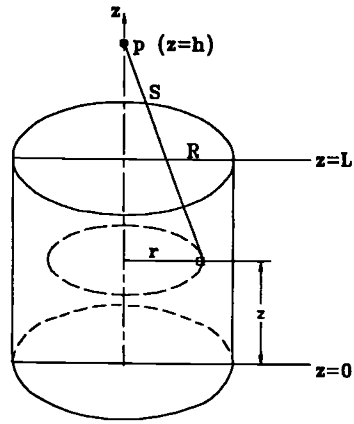

Fig. 3.11 Point P located on the axis of the cylinder.

Hence the gravitational potential is

$$V = -G\rho \int_0^L \int_0^R \frac{2\pi r\,\mathrm{d}r\,\mathrm{d}z}{\sqrt{r^2 + (h-z)^2}}$$

$$= -2\pi G\rho \int_0^L [r^2 + (h-z)^2]^{1/2}|_0^R\,\mathrm{d}z$$

$$= -2\pi G\rho \int_0^L \{[R^2 + (h-z)^2]^{1/2} - (h-z)\}\mathrm{d}z$$

$$= \pi G\rho \Bigg\{ [(h-L)\sqrt{(h-L)^2 + R^2} - h\sqrt{h^2 + R^2}]$$

$$+ [h^2 - (h-L)^2] + R^2 \ln \left[\frac{(h-L) + \sqrt{(h-L)^2 + R^2}}{h + \sqrt{h^2 + R^2}} \right] \Bigg\}$$

And the gravitational intensity is

$$g = -\nabla V = -k\frac{\partial}{\partial h}V$$

$$= -k2\pi G\rho\left[\sqrt{(h-L)^2 + R^2} - \sqrt{h^2 + R^2} + L\right]$$

3.5 Collision of Two Spheres on a Plane

The action of two bodies colliding with a large inertial force in a short time interval is called impact. Depending upon the material properties of the bodies, collision can be elastic or inelastic. We shall discuss these two kinds of collision in this section.

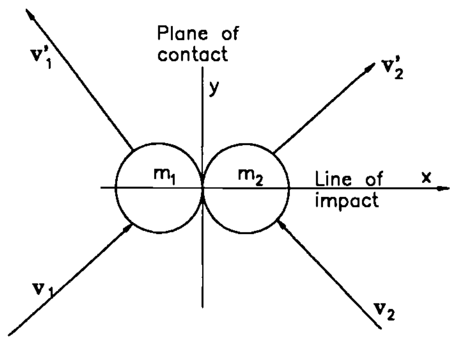

Fig. 3.12 Oblique central impact.

For the present study, the bodies in collision are modeled as two spheres with identical diameter and with the centers of mass at the centers of spheres. This, however, does not imply that the mass must be the same. The system may be pictured as two balls of different mass colliding on a frictionless table, which forms a perfect plane perpendicular to the gravitational force. For a general two-dimensional case, the collision is named an oblique central impact. The plane tangent to two spheres at the contact point is called the plane of contact. The line perpendicular to the plane of contact is termed the line of impact that goes through two centers of spheres. For an oblique central impact, the velocities of spheres are at angles away from the line of impact as shown in Fig. 3.12. When the velocities are on the line of impact, the action is called central impact. Therefore, a central impact is a special case of oblique central impact. The method of analysis for oblique central impact can be applied easily to the central impact.

Automobile accidents are common occurrences in this country. Every day there are thousands of car collisions with hundreds of injuries and deaths. As we study the collision of bodies, it is interesting to try to answer two questions arising from car collisions. In a collision, is the driver of a heavier car safer than the driver of a lighter car? As an unavoidable head-on collision is about to happen, should the drivers accelerate their cars as much as possible to protect themselves? These questions will be answered in the examples. Although cars are in complicated shapes, the modeling of cars as spheres is only the first step in studying car collisions.

Certainly the application of the collision of two spheres is not limited to billiards and automobiles. It can be applied also to the collision of molecules in chemical reactions or in turbulent flows. Let us study the collisions in two different conditions, elastic and inelastic, as follows.

Elastic Collision

During the collision there is an impulsive force between two masses. If the force is not very large, the stresses developed on the spheres are below the yielding points. Then the shapes of the two spheres are restored completely to their original forms without any permanent deformation as shown in Fig. 3.13. Such a collision is called an elastic collision.

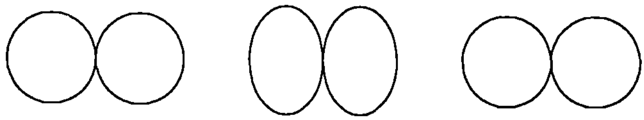

Fig. 3.13 Deformation and restitution during elastic impact.

Because there is no external force involved during collision, the total momentum of the two spheres is conserved, and we have

$$m_1 \boldsymbol{V}_1 + m_2 \boldsymbol{V}_2 = m_1 \boldsymbol{V}_1' + m_2 \boldsymbol{V}_2' \tag{3.38}$$

where $\boldsymbol{V}_i$ and $\boldsymbol{V}_i'$ are velocities of m_i before and after impact respectively.

For the elastic collision, because the shapes of the spheres are completely restored, the kinetic energy of the system is conserved and we have

$$\tfrac{1}{2} m_1 V_1^2 + \tfrac{1}{2} m_2 V_2^2 = \tfrac{1}{2} m_1 V_1'^2 + \tfrac{1}{2} m_2 V_2'^2 \tag{3.39}$$

Equation (3.38) is a vector equation that may be considered as two equations in terms of x and y directions. Hence there are three equations, but, in general, there are four unknowns: V_{1x}', V_{1y}', V_{2x}', V_{2y}', the velocity components after the collision. To determine them, additional conditions must be specified. In the collision process, no coordinate system exists in the space. Without loss of generality, we choose x axis along the line of impact and y axis along the plane of impact. With the frictionless model, it is reasonable to accept that the velocity components in the y direction are not changed, i.e.,

$$V_{1y}' = V_{1y}$$

$$V_{2y}' = V_{2y}$$

Then the momentum and energy equations become

$$m_1 V_{1x} + m_2 V_{2x} = m_1 V_{1x}' + m_2 V_{2x}' \tag{3.40}$$

$$m_1 V_{1x}^2 + m_2 V_{2x}^2 = m_1 V_{1x}'^2 + m_2 V_{2x}'^2 \tag{3.41}$$

Now V_{1x}' and V_{2x}' can be determined by Eqs. (3.40) and (3.41).

Inelastic Collision

During collision, sometimes the stresses produced by the impact force are much higher than the yielding strength of the materials, and permanent deformation results, as shown in Fig. 3.14. Such a collision is called an inelastic collision. With permanent deformations, some energy is dissipated by the stress-strain energy so that the conservation of energy is no longer true. Therefore, additional information will be needed to predict the velocities after the impact. To do this, we will define

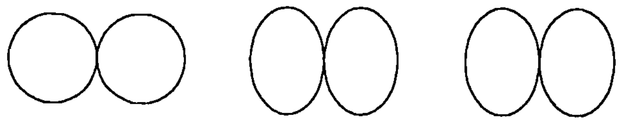

Fig. 3.14 Deformation and restitution during inelastic impact.

the coefficient of restitution ϵ, as the ratio of the impulse during the restitution period to the impulse during the deformation period, i.e.,

$$\epsilon = \frac{\text{impulse during restitution}}{\text{impulse during deformation}} = \frac{\int R\,\mathrm{d}t}{\int D\,\mathrm{d}t} \tag{3.42}$$

where R and D are the impact forces during restitution and deformation periods, respectively. The deformation period is the interval between the beginning of contact of the spheres and the instant of the maximum deformation, and the restitution period is the interval between the instant of maximum deformation and the moment that the spheres just separate. Thus, the changes of momentum of m_1 in these periods can be written as

$$\int D\,\mathrm{d}t = -[(m_1 V_{1x}) - (m_1 V_{1x})_D]$$

$$\int R\,\mathrm{d}t = -[(m_1 V_{1x})_D - (m_1 V'_{1x})]$$

Therefore,

$$\epsilon = \frac{(V_{1x})_D - V'_{1x}}{V_{1x} - (V_{1x})_D} \tag{3.43}$$

where $(V_{1x})_D$ is the velocity component in the x direction of m_1 at the maximum deformation. Similarly, for mass m_2,

$$\epsilon = \frac{(V_{2x})_D - V'_{2x}}{V_{2x} - (V_{2x})_D} \tag{3.44}$$

Note that, at the moment of maximum deformation, the two masses are in contact, and their velocities are the same along the line of impact, $(V_{1x})_D = (V_{2x})_D$. Thus Eqs. (3.43) and (3.44) can be combined to become

$$\epsilon = -\frac{V'_{2x} - V'_{1x}}{V_{2x} - V_{1x}} \tag{3.45}$$

The values of ϵ presumably are known for some common materials. With the use of the momentum equation in x direction, Eq. (3.40) together with Eq. (3.45), V'_{1x} and V'_{2x} can be predicted.

The following are a few remarks about the significance of the coefficient of restitution. During a perfectly elastic collision, the impulse for the period of restitution equals the impulse for the period of deformation, so that the coefficient of restitution is unity for this case. For inelastic collisions, the coefficient of restitution is less than unity because the impulse is diminished on restitution as a result of failure of the spheres to resume their original shapes. For a perfectly plastic collision, $\epsilon = 0$ (i.e., $V'_{2x} = V'_{1x}$) and the spheres remain in contact.

Example 3.4

Two billiard balls of the same size and mass collide with the velocities of approach shown in Fig. 3.15. What are the final velocities of the balls directly after an elastic collision?

Solution. The initial velocities of the balls are

$$V_1 = 5\boldsymbol{i} \ \text{m/s}$$

$$V_2 = -7.07\boldsymbol{i} + 7.07\boldsymbol{j} \ \text{m/s}$$

Because there is no friction, the velocities after the impact in y direction are

$$V'_{1y} = 0$$

$$V'_{2y} = 7.07 \ \text{m/s}$$

With $m_1 = m_2$, the momentum equation in the x direction gives

$$\begin{aligned} V_{1x} + V_{2x} &= V'_{1x} + V'_{2x} \\ 5 + (-7.07) &= V'_{1x} + V'_{2x} = -2.07 \end{aligned} \tag{3.46}$$

The energy equation (3.41) leads to

$$V_{1x}^2 + V_{2x}^2 = V_{1x}'^2 + V_{2x}'^2 \tag{3.47}$$

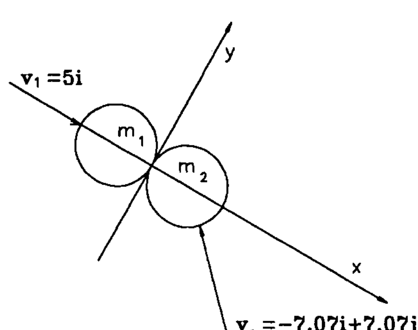

Fig. 3.15 Initial condition of the impact of balls.

Solving Eq. (3.46) and Eq. (3.47) simultaneously leads to

$$V'_{1x} - V'_{2x} = V_{2x} - V_{1x} = -7.07 - 5.0 = -12.07$$

Therefore, in the x direction, the velocities after the impact are

$$V'_{1x} = -7.07 \text{ m/s}$$
$$V'_{2x} = 5.00 \text{ m/s}$$

In the vector form, the velocities after the impact are

$$\boldsymbol{V}'_1 = -7.07\boldsymbol{i} \ \text{ m/s}$$
$$\boldsymbol{V}'_2 = 5.0\boldsymbol{i} + 7.07\boldsymbol{j} \ \text{ m/s}$$

Example 3.5

Prove that in a case of a two-car, head-on collision, the driver of a heavier car is usually less severely injured.

Solution. Let m_1, m_2 represent the mass of the two cars. Assume that car 2 is heavier than car 1, i.e., $m_2 > m_1$, and the collision is elastic. Then, from the momentum equation, we have

$$m_1(V'_1 - V_1) = m_2(V_2 - V'_2)$$
$$m_1 \Delta V_1 = m_2 \Delta V_2$$

The preceding result says that the change of momentum of car 1 equals that of car 2. Becasue $m_2 > m_1$, we conclude $|\Delta V_2| < |\Delta V_1|$, that is, the change of velocity for car 2 is less than for car 1.

Let m_d be the mass of the driver, and Δt be the time interval of the impact. Assume that the drivers have the same mass. Thus the inertial force acting on the driver is

$$m_d \frac{\Delta v}{\Delta t}$$

Comparing the inertial force acting on the two drivers, we have

$$m_d \left| \frac{\Delta V_2}{\Delta t} \right| < m_d \left| \frac{\Delta V_1}{\Delta t} \right| \text{ as } m_2 > m_1$$

Because the inertial force on the driver in car 2 is less than that on the driver in car 1, the injury to the driver in a heavier car is less than that in the lighter car.

Example 3.6

Estimate the difference in impact force for the following two cases: 1) Two cars have the same constant velocity of 50 mph but in opposite direction, and 2) one of the cars is accelerating at 5 ft/s^2 although at the time of collision the cars' velocities are the same as case 1. The mass of the cars are 100 slugs, and the duration of impact is 0.020 s. The two cars are stopped after the collision.

Solution. 1) Let F be the impact force

$$F\,\Delta t = m\Delta V$$

$$m = 100\,\text{slug}$$

Because the cars are stopped after the collision, their final velocities are zero. Therefore,

$$\Delta V = \frac{50 \times 5280}{3600} = 73.5\,\text{ft/s}$$

$$F = m\frac{\Delta V}{\Delta t} = 100\frac{73.5}{0.020} = 367.5 \times 10^3\,\text{lbf}$$

2) With the acceleration in one car, additional external force due to friction must be considered. The total impact force is

$$F' = m\frac{\Delta V}{\Delta t} + F_f$$

However,

$$F_f = ma = 100 \times 5 = 500\,\text{lbf}$$

$$F' = 367.5 \times 10^3 + 0.5 \times 10^3 = 368.0 \times 10^3\,\text{lbf}$$

The result shows that the impact force due to the acceleration of one car is very small compared with the total impact force.

Problems

3.1. Find the transformation of coordinates for the trajectory of the enemy missile. The enemy's missile site is 1000 km away from ours and is on a mountain 5 km above the surface of the average radius of the Earth. Assume that for the missile-to-missile collision, two trajectories are contained in the same plane.

3.2. Consider that the gravitational force always is pointing toward the center of the Earth. Suppose that the enemy's missile is launched from the site as given in Problem 3.1. What are the components of the gravitational force in the (x, z) directions?

3.3. Suppose that a rocket is launched vertically, and at the time of burnout the speed of the rocket is v_0 at the altitude of h_0 above the surface of the Earth. Use the expression $g = k/r^2$ for the gravitational acceleration, where k is a constant and r is the distance from the center of Earth to the rocket. Find the maximum height the rocket can reach. Also find the escape velocity for a rocket launched in a vertical position.

3.4. Show that the gravitational attraction due to a homogeneous circular disk at a point on the axis of the disk is

$$\frac{2MG}{a^2}\left[1 - \frac{h}{\sqrt{h^2 + a^2}}\right]$$

where M is the mass of the disk, a is the radius of the disk, and h is the height of the point above the center of the disk.

3.5. A uniform sphere of mass M is embedded in a hole of radius R in an infinite thin plane having mass per unit area σ. Find the gravitational field intensity and the potential energy per unit mass at a distance d above the center of the sphere.

3.6. In introductory dynamics, the potential energy of a mass m at z above the ground is always expressed as mgz. Now we have learned that the potential energy of mass m outside the spherical Earth is $-GmM/r$. What is the relationship between them?

3.7. Explain that, in the oblique impact, the coefficient of restitution cannot be defined in the direction that is not perpendicular to the plane of contact.

3.8. Two spherical balls of the same size and mass are in a head-on collision. Because of a manufacturing defect, the center of mass of one ball is not at the center of the sphere. Formulate the equations governing this impact. Predict the motions of the balls after the impact.

3.9. Suppose that a hard, small ball m drops vertically at a point on a hard, solid spherical surface as shown in Fig. P3.9, with mass $M \gg m$. The initial height of the ball is h_0.

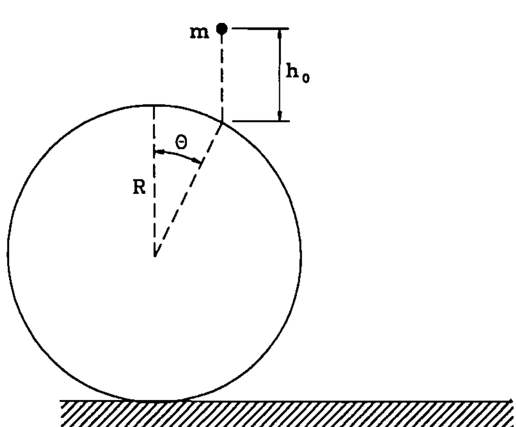

Fig. P3.9

(a) What is the velocity of the ball immediately after the impact for a coefficient of restitution $e = 0.85$?

(b) What is the trajectory of the ball after the impact but before it lands on the floor?

3.10. A ball is dropped from a height of 3 m onto a level floor. If the coefficient of restitution $e = 0.9$, how long will it take the ball to come to rest? What is the total distance traveled by the ball?