# 2
# Kinematics and Dynamics of a Particle

A *particle* is defined as a material point without dimensions but containing a definite quantity of matter. Strictly speaking, a particle cannot exist, because a definite amount of matter must occupy some space. When the size of a body is extremely small compared with its range of motion, however, it may be considered as a particle in certain cases. For example, although stars and planets are many thousands of miles in diameter, they are so small compared with their range of motion that they are often considered as particles in space.

This chapter covers material that should not be totally new to the reader. Coverage in some areas, such as kinematics of a particle in cylindrical and spherical coordinates, is more in depth than that given in an introductory course in dynamics. The relationship between curvilinear and rectangular coordinates for unit vectors is introduced in Section 2.1 so that velocities and accelerations in curvilinear coordinates are obtained easily. Some relatively modern examples illustrating particle dynamics are given in Section 2.2 although we expect the reader to have some familiarity with particle dynamics from studying elementary dynamics. Examples concerning missiles and space vehicles given here will be revisited in examples describing midair collisions of missiles in the next chapter. The change of angular momentum caused by applied moment is discussed in Section 2.3. Example 2.3 shows that the side force existing between a sliding block and rotating rod can be very significant. Work and conservative force are reviewed in Sections 2.4 and 2.5. They are useful for understanding the concept of potential energy used in Lagrangian equations.

## 2.1 Kinematics of a Particle

The location of a particle in three-dimensional space always can be specified by a position vector $\boldsymbol{r}$. Its velocity $\boldsymbol{v}$ is defined as

$$\boldsymbol{v} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{2.1}$$

Similarly the acceleration of the particle is defined as

$$\boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{2.2}$$

Now, let us develop expressions for velocity and acceleration of a particle in different coordinate systems.

### *Cartesian Coordinates*

The position vector of a particle is

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{2.3}$$

Note that $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are constant vectors. The velocity $\boldsymbol{v}$ is therefore

$$\boldsymbol{v} = \boldsymbol{i}\frac{\mathrm{d}x}{\mathrm{d}t} + \boldsymbol{j}\frac{\mathrm{d}y}{\mathrm{d}t} + \boldsymbol{k}\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z} \tag{2.4}$$

and the acceleration is

$$\boldsymbol{a} = \boldsymbol{i}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \boldsymbol{j}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \boldsymbol{k}\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z} \tag{2.5}$$

### *Cylindrical Coordinates*

The position vector of a particle in cylindrical coordinates is

$$\boldsymbol{r} = \rho\boldsymbol{e}_\rho + z\boldsymbol{k} \tag{2.6}$$

where $\rho$ is the projected length of $\boldsymbol{r}$ in the $x$–$y$ plane, as shown in Fig. 2.1.

The unit vector is $\boldsymbol{e}_\rho$ along $\rho$ in the $x$–$y$ plane and can be expressed in terms of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ as

$$\boldsymbol{e}_\rho = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j} \tag{2.7}$$

A unit vector that is perpendicular to $\boldsymbol{e}_\rho$ but lies in the $x$–$y$ plane is denoted by $\boldsymbol{e}_\phi$ as shown. It also can be expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ as

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j} \tag{2.8}$$

The velocity of a particle in cylindrical coordinates is

$$\begin{aligned}\boldsymbol{v} &= \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\boldsymbol{e}}_\rho + \dot{z}\boldsymbol{k}\\ &= \dot{\rho}[\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}] + \rho\dot{\phi}[-\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}] + \dot{z}\boldsymbol{k}\\ &= \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{k}\end{aligned} \tag{2.9}$$

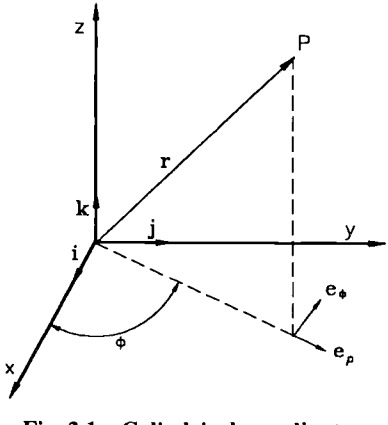


**Fig. 2.1 Cylindrical coordinates.**

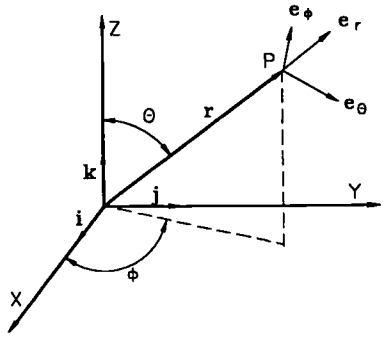


**Fig. 2.2 Spherical coordinates.**

Its acceleration in cylindrical coordinates is then

$$\boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = (\ddot{\rho} - \rho\dot{\phi}^2)\boldsymbol{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\boldsymbol{e}_\phi + \ddot{z}\boldsymbol{k} \tag{2.10}$$

### *Spherical Coordinates*

The unit vectors in spherical coordinates are denoted by $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$. The $\boldsymbol{e}_r$ is in the direction of position vector $\boldsymbol{r}$; hence

$$\boldsymbol{r} = r\boldsymbol{e}_r \tag{2.11}$$

The $\boldsymbol{e}_\theta$ is in the plane containing $\boldsymbol{r}$ and the $z$ axis, but is perpendicular to $\boldsymbol{e}_r$, as shown in Fig. 2.2. The $\boldsymbol{e}_\phi$ is perpendicular to both $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$. Therefore, they also can be expressed in terms of unit constant vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ as

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k} \tag{2.12}$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{i} + \cos\theta\sin\phi\boldsymbol{j} - \sin\theta\boldsymbol{k} \tag{2.13}$$

$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j} \tag{2.14}$$

During the differentiating of $\boldsymbol{r}$ with respect to time, Eqs. (2.12–2.14) are used. With some details omitted, the velocity of a particle in spherical coordinates is found to be

$$\begin{aligned}\boldsymbol{v} &= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \\ &= \dot{r}\boldsymbol{e}_r + r(\dot{\theta}\boldsymbol{e}_\theta + \dot{\phi}\sin\theta\boldsymbol{e}_\phi) \\ &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta\boldsymbol{e}_\phi\end{aligned} \tag{2.15}$$

Similarly, the acceleration of a particle can be obtained through the differentiation of $\boldsymbol{v}$ with respect to time and can be expressed as

$$\begin{aligned}\boldsymbol{a} \equiv\ & \boldsymbol{e}_r(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) \\ &+ \boldsymbol{e}_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) \\ &+ \boldsymbol{e}_\phi(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi})\end{aligned} \tag{2.16}$$

Note that with the use of Eqs. (2.12–2.14), Eq. (2.16) can be reduced to Eq. (2.5).

## 2.2 Particle Kinetics

In general, a force $\boldsymbol{F}$ acting on a point mass $m$ is a function of position, velocity, and time. The equation of motion for the particle with constant mass can be written simply as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{2.17}$$

Many cases are studied in introductory dynamics. Let us study a few special cases in the following examples.

### Example 2.1

Consider a missile moving in space as a particle with a mass decreasing constantly. The thrust applied is constant in magnitude and always in the direction of the particle's velocity. The coordinates are chosen such that the $x$–$z$ plane contains the trajectory with the $z$ axis perpendicular to the ground. Find the trajectories of the missile for thrust $\boldsymbol{F}$ = 14,500, 15,000, and 15,500 N, respectively. The initial conditions of the missile are $m_0 = 1000$ kg and $v_0 = 150$ m/s at an angle of 80 deg with the $x$ axis. The mass decreasing rate of $\dot{m} = 3$ kg/s.

*Solution.* The equation of motion for the missile is

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = F\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - mg\boldsymbol{k} \tag{2.18}$$

or

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = F\frac{v_x}{\sqrt{v_x^2 + v_z^2}} \tag{2.19}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = F\frac{v_z}{\sqrt{v_x^2 + v_z^2}} - mg \tag{2.20}$$

$$m = m_0 - \dot{m}t \tag{2.21}$$

Equations (2.19) and (2.20) are nonlinear and cannot be solved analytically. However, they can be integrated numerically by the Runge–Kutta method given in the Appendix A. The trajectory then can be obtained as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x \tag{2.22}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z \tag{2.23}$$

integrated together with Eqs. (2.19) and (2.21). Three trajectories are obtained for the three different values of thrust. The results are given in Fig. 2.3. In the numerical integration the increment of time used is 0.01 s and the total duration is more than 160 s. A convergence check is performed before the results are calculated.

### Example 2.2

Suppose that a space vehicle is moving from outer orbit into the atmosphere. The aerodynamic drag acting on the vehicle is proportional to the velocity squared.

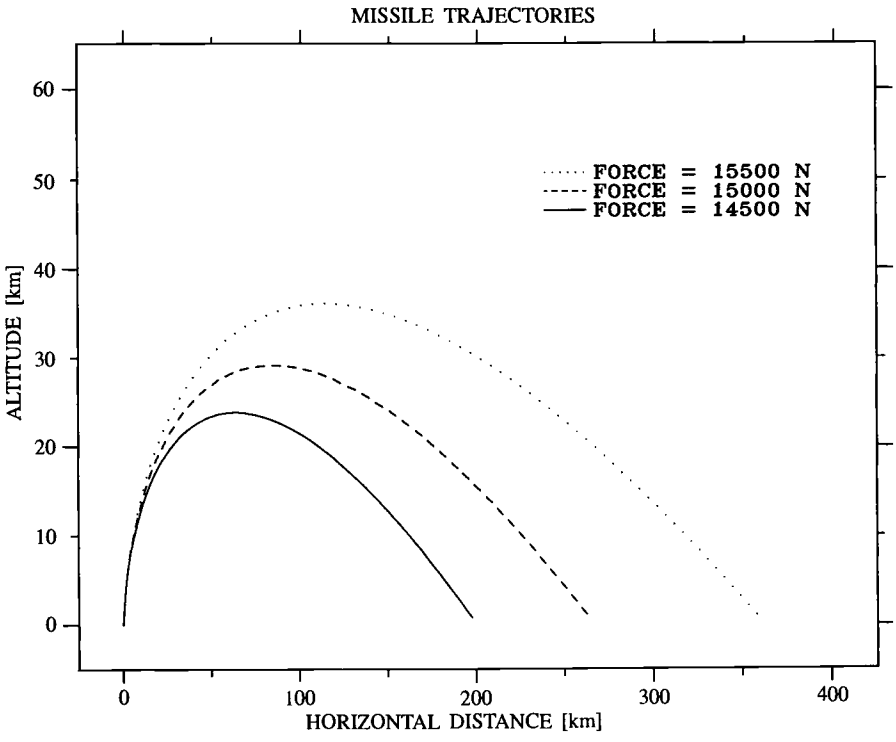


**Fig. 2.3 Trajectories of the missile.**

The coordinates are chosen such that the $x$–$y$ plane contains the trajectory and the $y$ axis is along $-\boldsymbol{g}$ as shown in Fig. 2.4. Determine the trajectories of the space vehicle as it descends with initial velocities of 7000, 8000 and 9000 m/s. The initial location of the vehicle is $x_0 = 0$, $y_0 = 20$ km. And its initial trajectory is always parallel to the ground.

*Solution.* According to the given conditions, the equations of motion can be written as

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg\sin\alpha - H(v) \tag{2.24}$$

and

$$v^2/R = g\cos\alpha \tag{2.25}$$

where $R$ is the radius of curvature of the trajectory and $H(v)$ is the aerodynamic drag of the vehicle:

$$H(v) = mkv^2 \tag{2.26}$$

where $k$, which should be a function of altitude, is considered as a constant for

this example. Equations (2.24) and (2.26) lead us to

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g \sin\alpha - kv^2 \tag{2.27}$$

The value of $k$ is estimated to be $1.5 \times 10^{-6}(\mathrm{m}^{-1})$. However,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}, \qquad \frac{v^2}{R} = \frac{(\mathrm{d}s/\mathrm{d}t)^2}{(\mathrm{d}s/\mathrm{d}\alpha)} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = v\frac{\mathrm{d}\alpha}{\mathrm{d}t}$$

where $\mathrm{d}s$ is the infinitesimal displacement along the trajectory and $\alpha$ is the angle between the velocity and the horizontal line as shown in Fig. 2.4. Substituting the preceding equation into Eq. (2.25), we obtain

$$v\frac{\mathrm{d}\alpha}{\mathrm{d}t} = g\cos\alpha$$

or

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{g\cos\alpha}{v} \tag{2.28}$$

Note that we also have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x = v\cos\alpha \tag{2.29}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y = -v\sin\alpha \tag{2.30}$$

Equations (2.27–2.30) can be integrated by the Runge–Kutta method given in Appendix A to find $x(t)$, $y(t)$, which is the trajectory of the vehicle with time $t$ as the parameter. The result of numerical integration is given in Fig. 2.5.

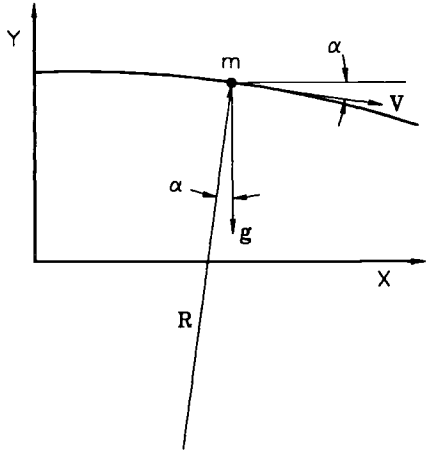


**Fig. 2.4 Coordinates of the space vehicle.**

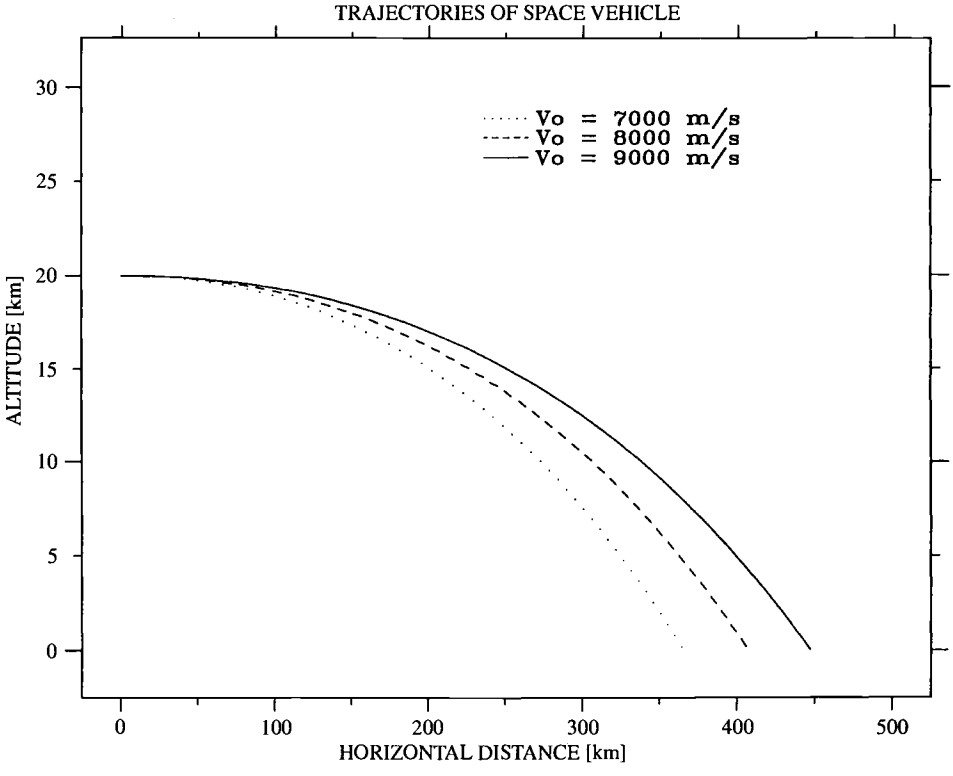


**Fig. 2.5 Trajectory of the space vehicle.**

## 2.3 Angular Momentum (Moment of Momentum) of a Particle

Another aspect of the particle dynamics is the change of angular momentum with respect to a certain axis when an external moment is applied. The angular momentum or moment of momentum of a particle is defined as

$$\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v} \tag{2.31}$$

where $\boldsymbol{r}$ is the position vector from the axis to the particle. The relationship between angular and linear momentum is shown in Fig. 2.6. The moment produced by the force applied to the particle is

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{F} = m(\mathrm{d}\boldsymbol{v}/\mathrm{d}t)$. Differentiating Eq. (2.31) leads to

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times m\boldsymbol{v} + \boldsymbol{r} \times m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}$$

The term $(\mathrm{d}\boldsymbol{r}/\mathrm{d}t) \times m\boldsymbol{v}$ is dropped because $(\mathrm{d}\boldsymbol{r}/\mathrm{d}t) = \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = 0$. Hence

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} \tag{2.32}$$

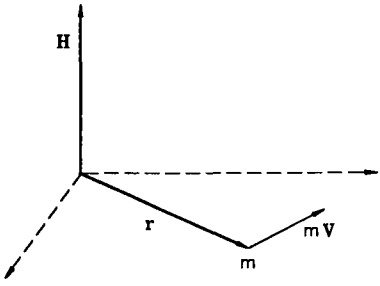


**Fig. 2.6 Relationship between angular and linear momentums.**

## Example 2.3

To illustrate the meaning of Eq. (2.32), let us consider a block as a particle sliding on a straight rod without friction at a uniform velocity of 30 ft/s, as shown in Fig. 2.7. The rod is in the $x$–$y$ plane, which is perpendicular to the gravitational force. The angular velocity of the rod is 50 rad/s. The position of the block is 6 in. away from the rotating axis. Determine the force between the block and the rod if the mass of the block is 1/30 slug.

*Solution.* Rewrite Eq. (2.31) as

$$\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v}$$

For this example, it is convenient to use cylindrical coordinates. The position vector of the particle at time $t$ is $\boldsymbol{r} = r\boldsymbol{e}_\rho$. Its velocity is

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_\rho + r\omega\boldsymbol{e}_\phi$$

Hence

$$\boldsymbol{H} = r\boldsymbol{e}_\rho \times m(\dot{r}\boldsymbol{e}_\rho + r\omega\boldsymbol{e}_\phi) = mr^2\omega\boldsymbol{k}$$

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = rF\boldsymbol{k} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} = 2mr\dot{r}\omega\boldsymbol{k}$$

$$F = 2m\dot{r}\omega = 2(1/30)(30)(50) = 100 \qquad \text{(lbf)}$$

Therefore, the force between the block and the rod is 100 lbf.

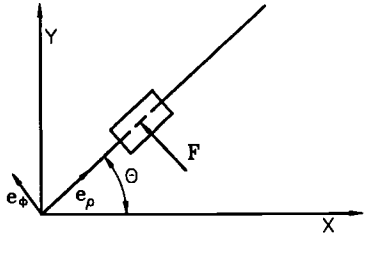


**Fig. 2.7 Block sliding on a rotating rod.**

## 2.4 Work and Kinetic Energy

Work usually is defined as a force $F$ acting through a displacement $x$ with the displacement occurring in the direction of the force. That is,

$$W = \int_1^2 F \cdot \mathrm{d}x$$

Using vector notation, the equivalent expression is

$$W = \int_1^2 \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

In general, if $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ are not in the same direction, only the component of $\mathrm{d}\boldsymbol{r}$ along $\boldsymbol{F}$ will contribute to the work. If the force is applied to a particle with a constant mass, then

$$\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}}$$

and the work done by the force is

$$\begin{aligned} W &= \int_1^2 m\dot{\boldsymbol{v}} \cdot \mathrm{d}\boldsymbol{r} = \int_1^2 m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r} \\ &= \int_1^2 m\mathrm{d}\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \int_1^2 m\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} \\ &= \tfrac{1}{2}m\left(v_2^2 - v_1^2\right) = T_2 - T_1 \end{aligned} \tag{2.33}$$

where $T$ is the kinetic energy of the particle. Equation (2.33) says that the change in kinetic energy of a particle moving from one point to another is equal to the work done by the force acting on the particle.

## 2.5 Conservative Forces

Suppose that a particle $m$ moves from $A$ to $B$ as shown in Fig. 2.8 and a force $\boldsymbol{F}$ is applied to the particle during the process. Then the work is

$$W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

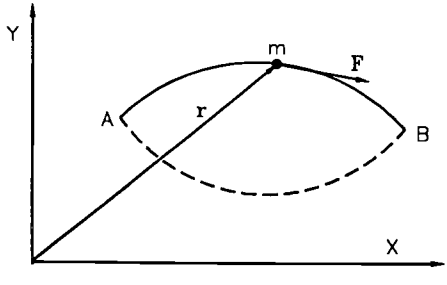


**Fig. 2.8 Moving paths of a particle.**

which is the line integral from $A$ to $B$ and may be represented by the solid line $AB$ in Fig. 2.8.

On the other hand, if

$$\int_B^A \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

where the line integral from $B$ to $A$ may be represented by the dotted line $BA$ in the figure, then

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$$

This means that the line integral of $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ over a closed path is zero. According to Stoke's theorem given in Appendix B,

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_s \nabla \times \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} \tag{2.34}$$

where $s$ is the area bounded by the closed path in the line integral. If the closed path is arbitrarily chosen, then

$$\nabla \times \boldsymbol{F} = 0$$

is true everywhere. According to vector analysis, the force $\boldsymbol{F}$ must be a gradient of a scalar function, i.e.,

$$\boldsymbol{F} = \nabla\phi$$

where $\phi$ is a scalar function to be identified. Force with this property is called a conservative force. Work done by such a force is

$$\begin{aligned} w &= \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B \nabla\phi \cdot \mathrm{d}\boldsymbol{r} \\ &= \int_A^B \left( \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z \right) = \int_A^B \mathrm{d}\phi = \phi_B - \phi_A \end{aligned} \tag{2.35}$$

Combining the preceding equation with Eq. (2.33) gives

$$\phi_B - \phi_A = T_B - T_A$$

or

$$T_A - \phi_A = T_B - \phi_B \tag{2.36}$$

To identify $\phi$, let us recall the principle of conservation of mechanical energy, which states that the sum of kinetic and potential energies is constant for a conservative system. Put in equational form,

$$T_A + V_A = T_B + V_B \tag{2.37}$$

where $V$ is the potential energy of the particle. Comparing Eqs. (2.36) and (2.37), we find

$$\phi = -V$$

Therefore $\boldsymbol{F} = -\nabla V$. A conservative force is equal to a gradient of potential energy with a change of sign.

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## Problems

**2.1.** Prove that the velocity expressed in cylindrical coordinates

$$\boldsymbol{v} = \dot{\rho}\boldsymbol{e}_\rho + \rho\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{k}$$

can be converted to the expression of velocity in Cartesian coordinates.

**2.2.** Prove that the expression of acceleration in spherical coordinates, Eq. (2.16), can be converted to

$$\boldsymbol{a} = \boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z}$$

**2.3.** The position vector of a moving particle is

$$\boldsymbol{r} = \boldsymbol{i}a\cos\omega t + \boldsymbol{j}b\sin\omega t$$

where $a$, $b$, and $\omega$ are constants.

(a) Find the velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and prove that $\boldsymbol{r} \times \boldsymbol{v}$ is constant

(b) Show that the acceleration is directed toward the origin and is proportional to the distance from the origin.

**2.4.** At a certain instant, a particle of mass $m$ moving freely in a vertical plane under a constant gravity is at a height $h$ above the ground and has a speed $v$. Use the principle of energy to find its speed when it strikes the ground.

**2.5.** Two masses, $m_1$ and $m_2$, are connected by a massless, inextensible rope that passes over a pulley, as shown in Fig. P2.5. Neglecting the mass and the bearing friction of the pulley, find the acceleration of $m_1$ as the system moves under the action of gravity.

**2.6.** A constant force is applied to a point mass so that the mass is accelerating. Two frames of reference are chosen for consideration. One is a fixed reference frame; the $x$ axis is oriented along the acceleration. The other is moving with a constant velocity along the negative $x$ direction of the fixed reference frame. However, they coincide at the beginning of observation.

(a) Find the velocity and position of the particle as a function of time in both reference frames.

(b) Find the work done by the force during a time interval $t$ in both frames.

(c) Are the results of (b) different in the two frames? If so, are the laws of mechanics different in the two inertial frames of reference? Explain your answer.

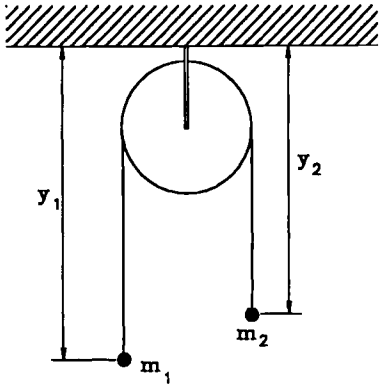


Fig. P2.5

**2.7.** Suppose that a missile is launched with the initial conditions: constant thrust, constant mass flow rate at the nozzle exit, and a proper launch angle. What will be the force exerting on the missile after the propellant is burned. Formulate the equations for describing the trajectory of the missile.

**2.8.** Find the best launch angle for a missile to reach the maximum horizontal distance through numerical integration. The fourth-order Runge–Kutta method is to be used for integration. The initial conditions are $F = 15{,}000$ N, $M_0 = 1000$ kg, $V_0 = 150$ m/s, and $\dot{m} = 3$ kg/s. At the time of burnout, the mass of missile is $M_f = 300$ kg. Plot the trajectory of the missile at the best launch angle.

**2.9.** Do the following:

(a) Using Green's theorem, prove that

$$\frac{1}{2}\oint_c (x\,\mathrm{d}y - y\,\mathrm{d}x) = A$$

where $A$ is the area enclosed by the curve $c$.

(b) Find the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**2.10.** Show that

(a)

$$\frac{1}{3}\oint_c (xy\,\mathrm{d}y - y^2\mathrm{d}x) = A\bar{y}$$

(b)

$$\frac{1}{4}\oint_c (xy^2\mathrm{d}y - y^3\mathrm{d}x) = I_x$$

where $A$ is the area bounded by $C$, $(\bar{x}, \bar{y})$ is its centroid, and $I_x$ its moment of inertia about $x$ axis.

**2.11.** If $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_\theta$ are the unit vectors in spherical coordinates, show that the unit vectors in Cartesian coordinates can be written as

$$\boldsymbol{i} = (\boldsymbol{e}_r \sin\theta + \boldsymbol{e}_\theta \cos\theta)\cos\phi - \boldsymbol{e}_\phi \sin\phi$$

$$\boldsymbol{j} = (\boldsymbol{e}_r \sin\theta + \boldsymbol{e}_\theta \cos\theta)\sin\phi + \boldsymbol{e}_\phi \cos\phi$$

$$\boldsymbol{k} = \boldsymbol{e}_r \cos\theta - \boldsymbol{e}_\theta \sin\theta$$

**2.12.** A particle of mass moves in a plane under the action of a force with components

$$F_x = -K^2(2x + y), \qquad F_y = -K^2(x + 2y)$$

where $K$ is a constant. Consider that the force is conservative. What is the potential energy?