1
Review of Fundamental Principles

THIS chapter reviews the fundamental principles necessary for the study of advanced dynamics. Although these principles may be familiar to students who have studied elementary mechanics, they are included here so that this book is reasonably self-contained.

The concepts of dimensions and units are reviewed in Section 1.1. Familiarity with these concepts will greatly facilitate formulating equations, checking dimensional homogeneity of an equation, and converting units. A brief review of vector analysis is given in Section 1.2. Formulas frequently used in this book are presented. Section 1.3 contains the definitions of statics and dynamics and a discussion of the difference between kinematics and kinetics. Section 1.4 presents Newton's laws of motion. The second law is written in an expanded form to include the effect of changing mass, which is essential for analyzing the dynamics of a rocket or any object with variable mass. D'Alembert's principle is presented in Section 1.5. Through the use of D'Alembert's principle, dynamic problems are simplified to static ones. Section 1.6 reviews the principles of virtual displacement and virtual work, which are the foundation for the derivation of Lagrange's equations discussed in Chapter 4.

1.1 Dimensions and Units

A dimension is the measure by which the magnitude of a physical quantity is expressed. In dynamics, there are usually four dimensions: mass, length, time, and force. A unit is a determinate quantity adopted as a standard of measurement. As shown in Table 1.1, the International System of Units (SI) specifies mass in kilograms (kg), length in meters (m), time in seconds (s), and force in newtons (N). In the British Gravitational System (BG), mass is measured in slugs, length in feet (ft), time in seconds (s), and force in pounds (lbf). It is important to mention that understanding dimensions and units will prevent errors from occurring when analyzing problems and converting units. The conversion factors for the two systems are given in Table 1.1.

Of the four dimensions mentioned in Table 1.1, mass, length, and time are considered as primary dimensions and force as a secondary dimension. Force can be expressed in terms of mass, length, and time as follows:

$$1\ \text{N} = 1\ \text{kg-m/s}^2 \tag{1.1}$$

$$1\ \text{lbf} = 1\ \text{slug ft/s}^2 \tag{1.2}$$

The following example illustrates the technique used in the conversion of units.

$$\begin{aligned} 1\ \text{km/s} &= 1000\frac{\text{m}}{\text{s}}\ \frac{1\ \text{ft}}{0.3048\ \text{m}}\ \frac{1\ \text{mile}}{5280\ \text{ft}}\ \frac{3600\ \text{s}}{1\ \text{h}} \\ &= 2236.94\ \text{mph} \end{aligned}$$

Table 1.1 Conversion factors

Dimensions	SI unit	BG unit	Conversion factor
Mass, M	Kilogram, kg	Slug	1 slug = 14.5939 kg
Length, L	Meter, m	Foot, ft	1 ft = 0.3048 m
Time, T	Second, s	Second, s	1 s = 1 s
Force, F	Newton, N	Pound, lbf	1 lbf = 4.4482 N

When discussing units and dimensions, it is worthwhile to mention that each term in an equation must have the same dimension, and the dimensions on both sides of the equal sign must be the same. This is known as the principle of dimensional homogeneity. Application of this principle will prevent algebraic errors from occurring in complicated manipulations of equations.

1.2 Elements of Vector Analysis

Physical quantities in mechanics can be expressed mathematically by means of scalars and vectors. A quantity characterized by magnitude only is called a scalar. Mass, length, time, and volume are scalar quantities. A vector is a quantity that has both a magnitude and direction and obeys the parallelagram law of addition. Force, velocity, acceleration, and position of a particle in space are vector quantities.

A vector can be broken down into several components according to convenience. In the Cartesian coordinate system, a vector $\boldsymbol{a}$ can be expressed in its components as

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

where a_x, a_y, and a_z are the components of the vector, and $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the corresponding unit vectors. Because vector analysis plays an important role in dynamics, fundamental mathematics of vectors is presented in this section. Note that throughout the book, vectors are denoted by bold letters.

Vector Algebra

Vector addition. The addition of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is computed as

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} + \boldsymbol{b} \\ &= a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} + b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k} \\ &= (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \end{aligned} \tag{1.3}$$

Vector subtraction. Vector subtraction, being a special case of vector addition, is performed as

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{a} - \boldsymbol{b} \\ &= (a_x - b_x)\boldsymbol{i} + (a_y - b_y)\boldsymbol{j} + (a_z - b_z)\boldsymbol{k} \end{aligned} \tag{1.4}$$

Scalar product of two vectors. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} \cdot \boldsymbol{b}$, which is a scalar quantity, and is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta \tag{1.5}$$

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{ab} = \frac{a_x b_x + a_y b_y + a_z b_z}{ab} \tag{1.6}$$

where a_x, a_y, a_z, b_x, b_y, and b_z are components of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and a is the magnitude of vector $\boldsymbol{a}$ and b the magnitude of vector $\boldsymbol{b}$.

Cross product of two vectors. The cross product of two vectors is written as $\boldsymbol{a} \times \boldsymbol{b}$, which is a vector, and is defined as

$$\boldsymbol{a} \times \boldsymbol{b} = (ab\sin\theta)\boldsymbol{e}$$

where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{e}$ is a unit vector perpendicular to the plane containing vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and in the direction according to right-hand rule. The mathematical operation of the cross product is performed as follows:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= \boldsymbol{i}(a_y b_z - a_z b_y) + \boldsymbol{j}(a_z b_x - a_x b_z) + \boldsymbol{k}(a_x b_y - a_y b_x)\end{aligned} \tag{1.7}$$

Triple scalar product. The triple scalar product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is defined as $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. The result is a scalar quantity and is obtained as

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.8}$$

Triple vector product. The triple vector product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is defined as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. The result is a vector quantity and is obtained as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{a})\boldsymbol{c} \tag{1.9}$$

Differentiation

The derivative of a vector, which is a function of time, is defined as

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{V}(t + \Delta t) - \boldsymbol{V}(t)}{\Delta t} \tag{1.10}$$

From the definition given in Eq. (1.10), the derivatives of the product of a scalar and vector, the scalar product of two vectors, and the cross product of two vectors are given in the following equations:

$$\frac{d}{dt}(\alpha V) = \frac{d\alpha}{dt}V + \alpha\frac{dV}{dt} \tag{1.11}$$

$$\frac{d}{dt}(a \cdot b) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt} \tag{1.12}$$

$$\frac{d}{dt}(a \times b) = \frac{da}{dt} \times b + a \times \frac{db}{dt} \tag{1.13}$$

where a, b, and V are vectors and α is a scalar. If V is expressed in its Cartesian components, then $V = V_1 i + V_2 j + V_3 k$, and its derivative is

$$\frac{dV}{dt} = \frac{dV_1}{dt}i + \frac{dV_2}{dt}j + \frac{dV_3}{dt}k \tag{1.14}$$

In a general case, the unit vectors e_1, e_2, and e_3 may change their orientations in space as time progresses; then $V = V_1 e_1 + V_2 e_2 + V_3 e_3$, and the derivative of V can be written as

$$\frac{dV}{dt} = \frac{dV_1}{dt}e_1 + \frac{dV_2}{dt}e_2 + \frac{dV_3}{dt}e_3 + V_1\frac{de_1}{dt} + V_2\frac{de_2}{dt} + V_3\frac{de_3}{dt} \tag{1.15}$$

or

$$\frac{dV}{dt} = \dot{V}_1 e_1 + \dot{V}_2 e_2 + \dot{V}_3 e_3 + V_1 \dot{e}_1 + V_2 \dot{e}_2 + V_3 \dot{e}_3$$

where $\dot{V}_i$ and $\dot{e}_i$ are the time derivatives.

Gradient, Divergence, and Curl Operations

The gradient of a scalar ϕ is defined as

$$\text{Gradient } \phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \tag{1.16}$$

The divergence of a vector F

$$\text{Div } F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.17}$$

The curl of a vector F is defined as

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + j\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \tag{1.18}$$

While discussing the curl of a vector, it is interesting to examine the physical meaning of the curl of the velocity vector of a rotating body, $\boldsymbol{V}$. To do this, $\boldsymbol{V}$ is expressed in terms of rotating velocity $\boldsymbol{\omega}$ and position vector $\boldsymbol{r}$, then

$$\begin{aligned}\boldsymbol{V} &= \boldsymbol{\omega} \times \boldsymbol{r} \\ &= \boldsymbol{i}(\omega_2 z - \omega_3 y) + \boldsymbol{j}(\omega_3 x - \omega_1 z) + \boldsymbol{k}(\omega_1 y - \omega_2 x)\end{aligned}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$, and x, y, and z are the components of $\boldsymbol{r}$. The computation of curl $\boldsymbol{V}$ gives

$$\begin{aligned}\nabla \times \boldsymbol{V} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix} \\ &= \boldsymbol{i}2\omega_1 + \boldsymbol{j}2\omega_2 + \boldsymbol{k}2\omega_3 = 2\boldsymbol{\omega} \qquad (1.19)\end{aligned}$$

Therefore $\nabla \times \boldsymbol{V}$ is related with rotational velocity and is known as vorticity in fluid mechanics.

1.3 Statics and Dynamics

Statics is the study of objects at rest or in equilibrium under the actions of forces and/or torques. The equations of statics for different dimensions of space are summarized as follows.

For a one-dimensional problem,

$$\sum F = 0 \qquad (1.20)$$

For a two-dimensional problem,

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum M_o = 0 \qquad (1.21)$$

where F_x, F_y are the components of force in the x and y axes, respectively, and M_o is the moment with respect to a reference axis o perpendicular to the x–y plane.

For a three-dimensional problem,

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum F_z = 0 \qquad (1.22)$$

$$\sum M_{xx} = 0, \qquad \sum M_{yy} = 0, \qquad \sum M_{zz} = 0 \qquad (1.23)$$

where M_{xx}, M_{yy}, and M_{zz} are the moments with respect to the x, y, and z axes, respectively. Therefore, in general, there are six unknowns to be determined by six equations for the three-dimensional problem.

Dynamics is the branch of science that studies the physical phenomena of a body or bodies in motion. Dynamics usually includes kinematics and kinetics. Kinematics concerns only the space–time relationship of a given motion of a body, not the forces that cause the motion. Kinetics concerns finding the motion that a given body or bodies will have under the action of given forces, or finding what forces must be applied to produce a prescribed motion.

1.4 Newton's Laws of Motion

Dynamics is based on Newton's laws of motion, which were written by Sir Isaac Newton in the 17th century; however, before stating his laws we must introduce the concept of a "frame of reference." The position, velocity, and acceleration of a particle in space must be described relative to other points within the space; that is, there must exist a frame of reference in the space. Newton's laws of motion apply only when the frame of reference is either fixed in space or moving with constant velocity. Such a frame of reference is called an inertial frame of reference. An Earth-fixed reference frame usually is acceptable as an inertial reference frame for solving many engineering problems even though the Earth is moving relative to the sun with a speed of 29.8 km/s and a radius of curvature of 1.495×10^8 km. Newton's laws of motion are stated as follows:

First law (law of inertia): A particle remains at rest or at a constant velocity if the resultant force acting on the particle is zero.

Second law (the basic equation of motion): The rate of change of a particle's linear momentum is proportional to the force applied to the particle and occurs in the direction of the force.

Third law (law of action and reaction): For every force a particle exerts on another particle, there exists a reaction force back on the first particle; these two forces are equal in magnitude and opposite in direction.

There are advantages to stating the second law as just shown. For example, a body with changing mass with respect to time can accelerate without any external force applied. To substantiate this statement, the equation of motion is written as

$$\begin{gathered}\frac{\mathrm{d}mV}{\mathrm{d}t} = m\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \\ ma = -\dot{m}V\end{gathered} \tag{1.24}$$

This result shows that, if the body is a rocket, the thrust of a rocket is the product of the mass flow rate and its velocity, and the direction of thrust is opposite to the velocity. Because of the way the second law is stated, the equation of motion for a particle with constant mass can be written as

$$\boldsymbol{F} = (1/g_c)m\boldsymbol{a}$$

or

$$w = (1/g_c)mg \tag{1.25}$$

In the preceding equation, if the unit of mass is pounds of mass and that of the force is pounds of force,

$$g_c = 32.174\frac{\text{lbm} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}$$

However, for the International System of Units (SI) and British Gravitational System (BG) units, g_c is reduced to unity and can be omitted in Eq. (1.25).

1.5 D'Alembert's Principle

In statics, we are familiar with

$$\sum \boldsymbol{F} = 0$$

From this we can solve for the three unknowns in three-dimensional space. In dynamics, the equation of motion for a particle with constant mass is written as

$$\sum \boldsymbol{F} = m\boldsymbol{a} \tag{1.26}$$

where $\sum \boldsymbol{F}$ is the sum of the external forces acting on the particle, m is the particle mass, and $\boldsymbol{a}$ is the acceleration of the particle relative to an inertial reference frame. Now, we rewrite the equation as

$$\sum \boldsymbol{F} - m\boldsymbol{a} = 0 \tag{1.27}$$

and consider the term $-m\boldsymbol{a}$ to represent another force known as an inertia force, then Eq. (1.27) simply states that the vector sum of all forces, external and inertial, is zero. Thus, the dynamics problem has been reduced to a statics problem. This conversion in concept is known as D'Alembert's principle. Similarly, for a body in rotation, the equation of motion is

$$\sum \boldsymbol{T} = I\alpha \tag{1.28}$$

where $\sum \boldsymbol{T}$ is the sum of external torques applying on the body, I is the mass moment of inertia of the body with respect to the rotating axis, and α is the angular acceleration of the body. Equation (1.28) also can be written as

$$\sum \boldsymbol{T} - I\alpha = 0 \tag{1.29}$$

Similar to Eq. (1.27), Eq. (1.29) states that the vector sum of all torques, external and inertial, is zero. Furthermore, the combination of Eqs. (1.27) and (1.29) can be applied to solve problems for a body simultaneously undergoing translation and rotation. In conclusion, this change of concept from dynamics to statics greatly simplifies complicated dynamic problems in mechanics.

1.6 Virtual Work

Consider a system of N particles whose positions are specified by Cartesian coordinates $x_1, x_2, \ldots, x_{3N}$. Suppose that there are $3N$ forces $F_1, F_2, \ldots, F_{3N}$ applied to the particles in the direction of each coordinate. The forces are in static equilibrium. Now imagine that at a given instant the system is given arbitrary and small displacements $\delta x_1, \delta x_2, \ldots, \delta x_{3N}$ in the direction of each coordinate. The work done by the applied forces is

$$\delta w = \sum_{i=1}^{3N} F_i \delta x_i \tag{1.30}$$

δw is known as virtual work and the small displacements δx_i are called virtual displacements. Equation (1.30) can be written in vector notation for the virtual work as

$$\delta w = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \tag{1.31}$$

where $\boldsymbol{F}_i$ is the force applied to particle i and $\delta \boldsymbol{r}_i$ is the virtual displacement.

Similar to particles in a solid, if particles in space are in static equilibrium, they do not move relative to each other. Total force applied to particle i is the combination of the applied force $\boldsymbol{F}_i$ and the internal force

$$\sum_{j=1}^{N} \boldsymbol{F}_{ij} \qquad (j \neq i)$$

due to other particles. Therefore the equation for the total force is

$$(\boldsymbol{F}_T)_i = \boldsymbol{F}_i + (\boldsymbol{F}_c)_i = 0 \tag{1.32}$$

where

$$(\boldsymbol{F}_c)_i = \sum_{j=1}^{N} \boldsymbol{F}_{ij} \qquad (j \neq i)$$

and $(\boldsymbol{F}_T)_i = 0$ because of equilibrium.

Because the total force is zero, the work done by the total force must be zero, that is, $(\boldsymbol{F}_T)_i \cdot \delta \boldsymbol{r}_i = 0$. The virtual work of all the forces as a result of the virtual displacement $\delta \boldsymbol{r}_i$ is

$$\sum_{i=1}^{N} (\boldsymbol{F}_i + \boldsymbol{F}_{ci}) \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_{ci} \cdot \delta \boldsymbol{r}_i = 0 \tag{1.33}$$

The second term of the preceding equation is further explored as follows:

$$\begin{aligned}
\sum_{i=1}^{N} (\boldsymbol{F}_c)_i \cdot \delta \boldsymbol{r}_i &= \sum_{i,j}^{N} (\boldsymbol{F}_{ij}) \cdot \delta \boldsymbol{r}_i \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_k + \boldsymbol{F}_{\ell k} \cdot \delta \boldsymbol{r}_\ell + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_k - \boldsymbol{F}_{k\ell} \cdot \delta \boldsymbol{r}_\ell + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot (\delta \boldsymbol{r}_k - \delta \boldsymbol{r}_\ell) + \cdots \\
&= \cdots \boldsymbol{F}_{k\ell} \cdot \delta (\boldsymbol{r}_k - \boldsymbol{r}_\ell) + \cdots
\end{aligned}$$

in which $i = k, j = \ell$ is considered in the first term and $i = \ell, j = k$ in the second term. The symbol $\delta(\boldsymbol{r}_k - \boldsymbol{r}_\ell)$ is the change of $\boldsymbol{r}_k - \boldsymbol{r}_\ell$ in the solid and can occur only in the direction perpendicular to $\boldsymbol{r}_k - \boldsymbol{r}_\ell$, but $\boldsymbol{F}_{k\ell}$ is along $\boldsymbol{r}_k - \boldsymbol{r}_\ell$, hence the dot product must be zero. Therefore,

$$\sum_{i=1}^{N} (\boldsymbol{F}_c)_i \cdot \delta \boldsymbol{r}_i = 0$$

$$\delta w = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0 \tag{1.34}$$

i.e., the virtual work of applied forces is zero. The concept of virtual work will be used for the derivation of Lagrange's equations.

Example 1.1

Using the method of virtual work, determine the relationship between the torque T applied to the crank R and the force F applied to the slider in the mechanism to be shown in Fig. 1.1.

Solution. According to the conditions given in Fig. 1.1, the vector forms of torque, force, and displacements can be written as

$$\boldsymbol{T} = -\boldsymbol{k}T, \qquad \delta\boldsymbol{\theta} = \boldsymbol{k}\delta\theta$$

$$\boldsymbol{F} = -\boldsymbol{i}F, \qquad \delta\boldsymbol{x} = \boldsymbol{i}\delta x$$

In static equilibrium, the total virtual work δw is zero, and its equation is

$$\delta w = -T\delta\theta - F\delta x = 0 \tag{1.35}$$

From the given geometry, we have

$$x = R\cos\theta + L\cos\phi$$

$$R\sin\theta = h = L\sin\phi$$

Solving the two equations, we obtain

$$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - (R/L)^2\sin^2\theta}$$

$$x = R\cos\theta + L\sqrt{1 - (R/L)^2\sin^2\theta}$$

Differentiating the equation for x, we have

$$\delta x = -R\sin\theta\,\delta\theta - (R^2/L)\frac{\sin\theta\cos\theta}{\sqrt{1 - (R/L)^2\sin^2\theta}}\delta\theta \tag{1.36}$$

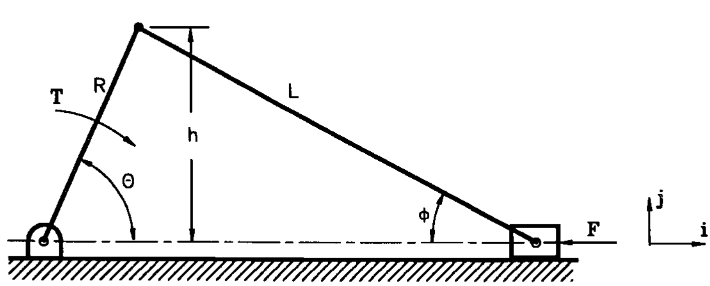

Fig. 1.1 Crank–slider mechanism.

Substituting Eq. (1.36) into Eq. (1.35) and simplifying, we find the required relationship between the torque and the force acting on the slider as

$$T = FR\sin\theta\left\{1 + \frac{R\cos\theta}{L\sqrt{1-(R/L)^2\sin^2\theta}}\right\} \tag{1.37}$$

Problems

1.1. Determine a unit vector perpendicular to the plane passing through (a, 0, 0), (0, b, 0), and (0, 0, c).

1.2. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined as follows:

$$\boldsymbol{a} = 2\boldsymbol{i} - 4\boldsymbol{k}, \qquad \boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

(a) Find the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.
(b) Find the angle between the positive directions of the vectors.

1.3. Find the moment of the force $\boldsymbol{F} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, acting at the point (1, 1, 2), about the z axis in arbitrary units.

1.4. Prove that $\boldsymbol{u} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{u}\cdot\boldsymbol{v}) - \boldsymbol{u}\cdot\nabla\boldsymbol{v}$, if $\boldsymbol{u}$ is constant.

1.5. Determine a unit vector in the plane of the vectors $\boldsymbol{i} + \boldsymbol{k}$, and $\boldsymbol{j} + \boldsymbol{k}$, and perpendicular to vector $\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$.

1.6. Let $\boldsymbol{r}$ represent the position vector of a moving point mass M, subject to a force $\boldsymbol{F}$. If $\boldsymbol{L}$ denotes the moment of the momentum $m\boldsymbol{v}$ about 0, prove that

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}$$

where $\boldsymbol{M}$ is the moment of the force $\boldsymbol{F}$ about 0.

1.7. Do the following:
(a) Find the unit vector normal to the plane $Ax + By + Cz = D$.
(b) Prove that the shortest distance from the point $P_0(x_0, y_0, z_0)$ to the plane $Ax + By + Cz = D$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

where the point P_0 is located above the plane. **HINT:** Let $P_1(x_1, y_1, z_1)$ be any point on the plane and determine the distance by letting P_0P_1 along the normal from the plane.

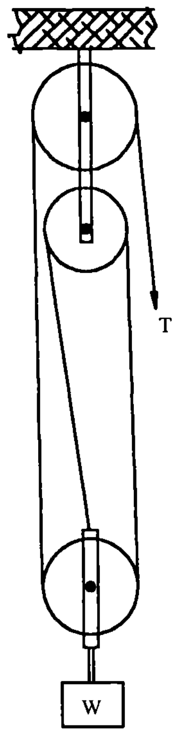

Fig. P1.8

1.8. A light cable passes around a pulley mounted on smooth bearings as shown in Fig. P1.8. The tension on both sides of the pulley is equal. Using the method of virtual work, find the displacement of the cable with tension T in terms of the vertical displacement of weight W. Assume that the pulleys and cable are light and the distance between the upper and lower pulleys is so great that the cables may be regarded as vertical.

1.9. A framework $ABCD$ consists of four equal, light rods smoothly joined together to form a square. It is suspended from a peg at A, and a weight W is attached to C. Further, the framework is kept in shape by a light rod connecting B and D. Determine the force exerted in this rod. **HINT:** The method of virtual work may be applied if the rod BD is removed and external forces are supplied to the joint B and D.

1.10. Consider a U-joint connecting two shafts that are not along a straight line as shown in Fig. P1.10. AB is a shaft, branching into the fork BCD; $A'B'$ is another axis, with fork $B'C'D'$. These forks are connected by a rigid body composed of two bars CD, $C'D'$, joined perpendicularly at their common center O. The lines AB, $A'B'$ meet at O and are perpendicular to CD, $C'D'$, respectively. There are smooth bearings at CD, $C'D'$ and the axes AB, $A'B'$ are free to turn in

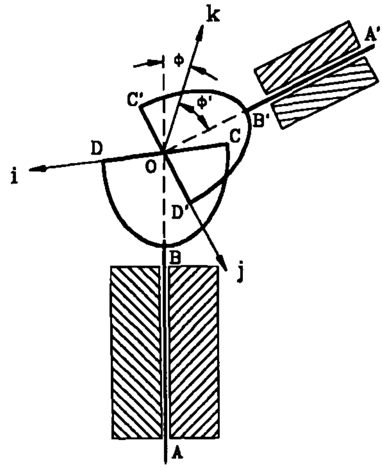

Fig. P1.10

smooth bearings. With the use of the method of virtual work, determine the torque transmitted through the joint. **HINT:** The velocity at the point D must be the same as rotated from two rigid bodies $ABCD$ and $CDC'D'$. Similarly, the velocity at D' must be the same from $A'B'C'D'$ and $CDC'D'$. Establish the virtual angular displacements from two shafts by equating the rotational displacements of CD and $C'D'$.