6 Structural Analysis

6.1 Overview

In this chapter, the basic theory needed for the determination of the stresses, strains, and deformations in fiber composite structures is outlined. Attention is concentrated on structures made in the form of laminates because that is the way composite materials are generally used.

From the viewpoint of structural mechanics, the novel features of composites (compared with conventional structural materials such as metals) are their marked anisotropy and, when used as laminates, their macroscopically heterogeneous nature.

There is a close analogy between the steps in developing laminate theory and the steps in fabricating a laminate. The building block both for theory and fabrication is the *single ply*, also referred to as the *lamina*. This is a thin layer of the material (typical thickness around 0.125 mm for unidirectional carbon/epoxy "tape" and 0.25 mm for a cross-ply fabric or "cloth") in which all of the fibers are aligned parallel to one another or in an orthogonal mesh. The starting point for the theory is the stress-strain law for the single ply referred to its axes of material symmetry, defined here as the 0–1, 2, 3 material axes. In constructing a laminate, each ply is laid-up so that its fibers make some prescribed angle with a reference axis fixed in the laminate. Here the laminate axes will be defined as the x-, y-, and z-axes.

All later calculations are made using axes fixed in the structure (the structural axes). In a finite element model, the material properties are usually entered in the material axes. The lay-up of the laminate is defined in the laminate axes. The laminate theory described in this chapter will indicate how the properties of the laminate are derived. The transformation from the laminate axes to the global structural axes is then completed during the solution process. Because the designer can select his own lay-up pattern (because the laminate stress-strain law will depend on that pattern), it follows that the designer can design the material (as well as the structure).

For more detailed discussions of the topics covered in this chapter, see Refs. 1–7. For background material on the theory of anisotropic elasticity, see Refs. 8–10.

6.2 Laminate Theory

Classical laminate theory defines the response of a laminate with the following assumptions:

- For two-dimensional plane stress analysis, the strain is constant through the thickness.
- For bending, the strain varies linearly through the thickness.
- The laminate is thin compared with its in-plane dimensions.
- Each layer is quasi-homogeneous and orthotropic.
- Displacements are small compared with the thickness.
- The behavior remains linear.

With these assumptions satisfied, the laminate theory allows the response of a laminate to be calculated, engineering constants to be determined to substitute into standard formulas for stresses and deflections, and material properties of the laminate to be defined for substitution into finite element analysis as described in Chapter 16.

6.2.1 Stress-Strain Law for a Single Ply in the Material Axes: Unidirectional Laminates

Consider a rectangular element of a single ply with the sides of the element parallel and perpendicular to the fiber direction (Fig. 6.1). Clearly, the direction of the fibers defines a preferred direction in the material; it is thus natural to introduce a cartesian set of material axes 0–1, 2, 3 with the *1*-axis in the fiber direction, the *2*-axis perpendicular to the fibers of the ply plane, and the *3*-axis perpendicular to the plane of the ply. Here, interest is in the behavior of the ply when subjected to stresses acting in its plane, in other words, under plane stress conditions. These stresses (also referred to the material axes) will be denoted by σ_1, σ_2, τ_{12} and the associated strains by ε_1, ε_2, and γ_{12}. (Note that in composite mechanics, it is standard practice to work with "engineering" rather than "tensor" shear strains.) Although a single ply is highly anisotropic, it is intuitively evident that the coordinate planes 012, 023, and 031 are those of material symmetry, there being a mirror image symmetry about these planes.

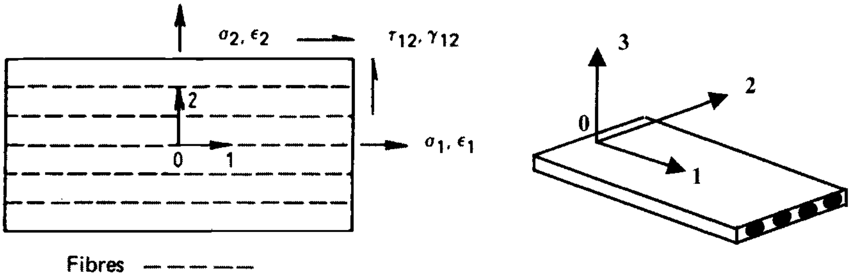


Fig. 6.1 Material axes for a single ply.

A material having three mutually orthogonal planes of symmetry is known as *orthotropic*. The stress-strain law for an orthotropic material under plane stress conditions, referred to the material axes, necessarily has the following form:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & \dfrac{-\nu_{21}}{E_2} & 0 \\ \dfrac{-\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{6.1}$$

where: E_1, E_2 = Young's moduli in the *1* and *2* directions, respectively; ν_{12} = Poisson's ratio governing the contraction in the *2* direction for a tension in the *1* direction; ν_{21} = Poisson's ratio governing the contraction in the *1* direction for a tension in the *2* direction; G_{12} = (in-plane) shear modulus.

There are five material constants in equation (*6.1*), but only four of these are independent because of the following symmetry relation[1]:

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \tag{6.2}$$

For unidirectional tape of the type being considered here, E_1 is much larger than either E_2 or G_{12} because the former is a "fiber-dominated" property, while the latter are "matrix dominated". For a bi-directional cloth, $E_1 = E_2$ and both are much larger than G_{12}. For tape, ν_{12} is matrix dominated and is of the order of 0.3, whereas the contraction implied in ν_{21} is resisted by the fibers and so is much smaller.

The above equations are all related to a single ply but, because the ply thickness does not enter into the calculations, they also apply to a "unidirectional laminate" that is simply a laminate in which the fiber direction is the same in all of the plies. In fact, most of the material constants for a single ply are obtained from specimen tests on unidirectional laminates, a single ply being itself too thin to test conveniently.

For much of the following analysis, it is more convenient to deal with the inverse form of equation (*6.1*), namely

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11}(0) & Q_{12}(0) & 0 \\ Q_{12}(0) & Q_{22}(0) & 0 \\ 0 & 0 & Q_{66}(0) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \tag{6.3}$$

where the $Q_{ij}(0)$, commonly termed the *reduced stiffness coefficients*, are given by

$$Q_{11}(0) = \frac{E_1}{1 - \nu_{12}\nu_{21}} \qquad Q_{22}(0) = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$
$$Q_{12}(0) = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \qquad Q_{66}(0) = G_{12} \tag{6.4}$$

It is conventional in composite mechanics to use the above subscript notation for Q, the point of which becomes evident only when three-dimensional anisotropic problems are encountered. The subscript 6 is for the sixth component of stress or strain that includes three direct terms and three shear terms.

6.2.2 Stress-Strain Law for Single Ply in Laminate Axes: Off-Axis Laminates

As already noted, when a ply is incorporated in a laminate, its fibers will make some prescribed angle θ with a reference axis fixed in the laminate. Let this be the x-axis, and note that the angle θ is measured from the x-axis to the *1*-axis and is positive in the counterclockwise direction; the y-axis is perpendicular to the x-axis and in the plane of the ply (See Fig. 6.2.). All subsequent calculations are made using the x-y, or "laminate" axes, therefore it is necessary to transform the stress-strain law from the material axes to the laminate axes. If the stresses in the laminate axes are denoted by σ_x, σ_y, and τ_{xy}, then these are related to the stresses referred to the material axes by the usual transformation equations,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{6.5}$$

where c denotes $\cos\theta$ and s denotes $\sin\theta$. Also, the strains in the material axes are related to those in the laminate axes, namely, ε_x, ε_y, and γ_{xy}, by what is essentially the strain transformation:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{6.6}$$

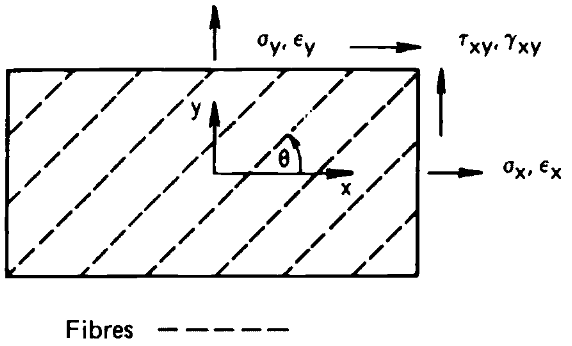


Fig. 6.2 Laminate axes for a single ply.

Now, in equation (*6.5*), substitute for σ_1, σ_2, and τ_{12} their values as given by equation (*6.3*). Then, in the resultant equations, substitute for ε_1, ε_2, and γ_{12} their values as given by equation (*6.6*). After some routine manipulations, it is found that the stress-strain law in the laminate axes has the form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{xx}(\theta) & Q_{xy}(\theta) & Q_{xs}(\theta) \\ Q_{xy}(\theta) & Q_{yy}(\theta) & Q_{ys}(\theta) \\ Q_{xs}(\theta) & Q_{ys}(\theta) & Q_{ss}(\theta) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{6.7}$$

where the $Q_{ij}(\theta)$ are related to the $Q_{ij}(0)$ by the following equations:

$$\begin{bmatrix} Q_{xx}(\theta) \\ Q_{xy}(\theta) \\ Q_{yy}(\theta) \\ Q_{xs}(\theta) \\ Q_{ys}(\theta) \\ Q_{ss}(\theta) \end{bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^3s & -cs(c^2-s^2) & -cs^3 & -2cs(c^2-s^2) \\ cs^3 & cs(c^2-s^2) & -c^3s & 2cs(c^2-s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \times \begin{bmatrix} Q_{11}(0) \\ Q_{12}(0) \\ Q_{22}(0) \\ Q_{66}(0) \end{bmatrix} \tag{6.8}$$

Observe that, in equation (*6.7*), the direct stresses depend on the shear strains (as well as the direct strains), and the shear stress depends on the direct strains (as well as the shear strain). This complication arises because, for non-zero θ, the laminate axes are not axes of material symmetry and, with respect to these axes, the material is not orthotropic; it is evident that the absence of orthotropy leads to the presence of the Q_{xs} and Q_{ys} terms in equation (*6.7*). Also, for future reference, note that the expressions for $Q_{xx}(\theta)$, $Q_{xy}(\theta)$, $Q_{yy}(\theta)$ and $Q_{ss}(\theta)$ contain only even powers of sin θ and therefore these quantities are unchanged when θ is replaced by $-\theta$. On the other hand, the expressions for Q_{xs} and Q_{ys} contain odd powers of sin θ and therefore they change sign when θ is replaced by $-\theta$.

Analogous to the previous section, the above discussion has been related to a single ply, but it is equally valid for a laminate in which the fiber direction is the same in all plies. A unidirectional laminate in which the fiber direction makes a non-zero angle with the x-laminate-axis is known as an "off-axis" laminate and is sometimes used for test purposes. Formulas for the elastic moduli of an off-axis laminate can be obtained by a procedure analogous to that used in deriving equation (*6.7*). Using equation (*6.1*) with the inverse forms of equations (*6.5*) and (*6.6*) leads to the inverse form of equation (*6.7*), in other words, with the strains expressed in terms of the stresses; from this result, the moduli can be written. Details can be found in most of the standard texts, for example page 54 of Ref. 3.

Only the result for the Young's modulus in the x direction, E_x, will be cited here:

$$\frac{1}{E_x} = \left(\frac{1}{E_1}\right)c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)c^2 s^2 + \left(\frac{1}{E_2}\right)s^4 \tag{6.9}$$

The variation of E_x with θ for the case of a carbon/epoxy off-axis laminate is shown in Figure 6.3. The material constants of the single ply were taken to be

$$E_1 = 137.44\,\text{GPa} \qquad E_2 = 11.71\,\text{GPa} \qquad G_{12} = 5.51\,\text{GPa}$$
$$\nu_{12} = 0.25 \qquad \nu_{21} = 0.0213$$

It can be seen that the modulus initially decreases quite rapidly as the off-axis angle increases from 0°; this indicates the importance of the precise alignment of fibers in a laminate.

6.2.3 Plane Stress Problems for Symmetric Laminates

One of the most common laminate forms for composites is a laminated sheet loaded in its own plane, in other words, under plane stress conditions. In order for out-of-plane bending to not occur, such a laminate is always made with a lay-up that is symmetric about the mid-thickness plane. Just to illustrate the type of symmetry meant, consider an eight-ply laminate comprising four plies that are to be oriented at 0° to the reference (x) axis, two plies at +45°, and two plies at − 45°. An example of a symmetric laminate would be one with the following ply sequence:

$$0°/0°/+45°/-45°/-45°/+45°/0°/0°$$

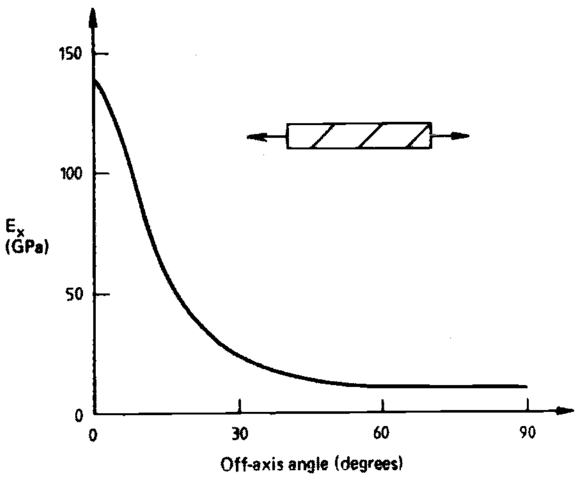


Fig. 6.3 Extensional modulus off-axis laminate.

On the other hand, an example of an unsymmetric arrangement of the same plies would be:

$$0^\circ/0^\circ/0^\circ/0^\circ/+45^\circ/-45^\circ/+45^\circ/-45^\circ$$

These two cases are shown in Figure 6.4 where z denotes the coordinate in the thickness direction.

6.2.3.1 Laminate Stiffness Matrix. Consider now a laminate comprising n plies and denote the angle between the fiber direction in the kth ply and the x laminate axis by θ_k (with the convention defined in Fig. 6.2). Subject only to the symmetry requirement, the ply orientation is arbitrary. It is assumed that, when the plies are molded into the laminate, a rigid bond (of infinitesimal thickness) is formed between adjacent plies. As a consequence of this assumption, it follows that under plane stress conditions the strains are the same at all points on a line through the thickness (i.e., they are independent of z). Denoting these strains by ε_x, ε_y, and γ_{xy}, it then follows from equation (6.7) that the stresses in the kth ply will be given by:

$$\begin{aligned}
\sigma_x(k) &= Q_{xx}(\theta_k)\varepsilon_x + Q_{xy}(\theta_k)\varepsilon_y + Q_{xs}(\theta_k)\gamma_{xy} \\
\sigma_y(k) &= Q_{xy}(\theta_k)\varepsilon_x + Q_{yy}(\theta_k)\varepsilon_y + Q_{ys}(\theta_k)\gamma_{xy} \\
\tau_{xy}(k) &= Q_{xs}(\theta_k)\varepsilon_x + Q_{ys}(\theta_k)\varepsilon_y + Q_{ss}(\theta_k)\gamma_{xy}
\end{aligned} \tag{6.10}$$

The laminate thickness is denoted by t and the thickness of the kth ply is $h_k - h_{k-1}$ with h_i defined in Figure 6.5. Assuming all plies are of the same thickness (which is the usual situation), then the thickness of an individual ply is simply t/n. Now consider an element of the laminate with sides of unit length parallel to the x- and y-axes. The forces on this element will be denoted by N_x, N_y,

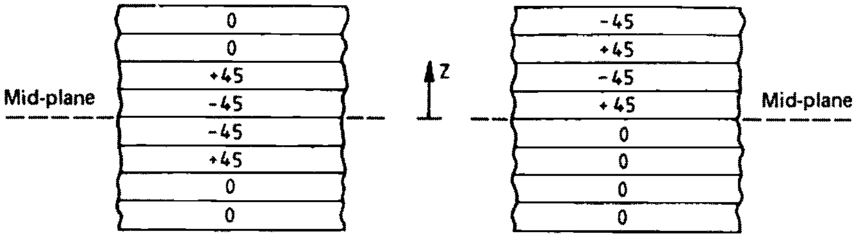


Fig. 6.4 Symmetric (*left*) and non-symmetric (*right*) eight-ply laminates.

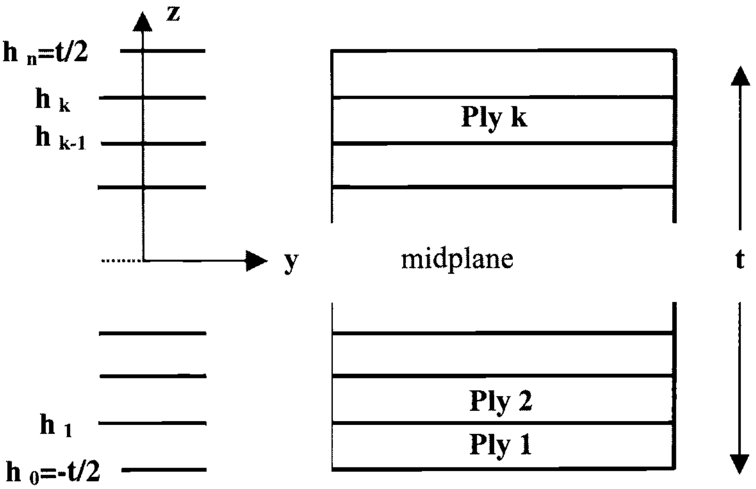


Fig. 6.5 Ply coordinates in the thickness direction, plies numbered from the bottom surface.

and N_s, (Figure 6.6); the N are generally termed *stress resultants* and have the dimension "force per unit length." Elementary equilibrium considerations give

$$N_x = \sum_{k=1}^{n} \sigma_x(k)(h_k - h_{k-1}), \quad N_y = \sum_{k=1}^{n} \sigma_y(k)(h_k - h_{k-1}),$$

$$N_s = \sum_{k=1}^{n} \tau_{xy}(k)(h_k - h_{k-1}) \tag{6.11}$$

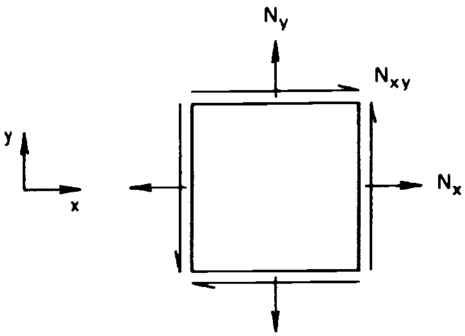


Fig. 6.6 Stress resultants.

Substituting from equations (*6.10*) into (*6.11*), and remembering that the strains are the same in all plies, the following result is readily obtained:

$$
\begin{aligned}
N_x &= A_{xx}\varepsilon_x + A_{xy}\varepsilon_y + A_{xs}\gamma_{xy} \\
N_y &= A_{xy}\varepsilon_x + A_{yy}\varepsilon_y + A_{ys}\gamma_{xy} \\
N_s &= A_{xs}\varepsilon_x + A_{ys}\varepsilon_y + A_{ss}\gamma_{xy}
\end{aligned}
\tag{6.12}
$$

where:

$$
A_{ij} = \sum_{k=1}^{n} Q_{ij}(\theta_k)(h_k - h_{k-1}) \tag{6.13}
$$

The quantities A_{ij} are the terms of the laminate "in-plane stiffness matrix." Given the single-ply moduli and the laminate lay-up details, they can be calculated routinely by using equations (*6.4*), (*6.8*), and (*6.13*). Equation (*6.12*) are generally taken as the starting point for any laminate structural analysis.

6.2.3.2 Laminate Stress—Strain Law. As was just implied, it seems to be the current fashion in laminate mechanics to work in terms of the stress resultants, rather than the stresses. However, for some purposes, the latter are more convenient. From the stress resultants, the average stresses (averaged through the thickness of the laminate) are easily obtained; writing these stresses simply as σ_x, σ_y, and τ_{xy} then:

$$
\sigma_x = \frac{N_x}{t}, \quad \sigma_y = \frac{N_x}{t}, \quad \tau_{xy} = \frac{N_s}{t} \tag{6.14}
$$

Hence, in terms of these average stresses, the stress-strain law for the laminate becomes:

$$
\begin{aligned}
\sigma_x &= A^*_{xx}\varepsilon_x + A^*_{xy}\varepsilon_y + A^*_{xs}\gamma_{xy} \\
\sigma_y &= A^*_{xy}\varepsilon_x + A^*_{yy}\varepsilon_y + A^*_{ys}\gamma_{xy} \\
\tau_{xy} &= A^*_{xs}\varepsilon_x + A^*_{ys}\varepsilon_y + A^*_{ss}\gamma_{xy}
\end{aligned}
\tag{6.15}
$$

where:

$$
A^*_{ij} = \frac{A_{ij}}{t} = \frac{1}{t}\sum_{k=1}^{n} Q_{ij}(\theta_k)(h_k - h_{k-1}) \tag{6.16}
$$

In some cases, equation (*6.15*) is more convenient than is equation (*6.12*).

6.2.3.3 Orthotropic Laminates. An orthotropic laminate, having the laminate axes as the axes of orthotropy, is one for which $A_{xs} = A_{ys} = 0$;

clearly, this implies that:

$$\sum_{k=1}^{n} Q_{xs}(\theta_k)(h_k - h_{k-1}) = 0, \quad \sum_{k=1}^{n} Q_{ys}(\theta_k)(h_k - h_{k-1}) = 0 \tag{6.17}$$

Thus, the stress-strain law for an orthotropic laminate reduces to:

$$\begin{aligned} \sigma_x &= A^*_{xx}\varepsilon_x + A^*_{xy}\varepsilon_y \\ \sigma_y &= A^*_{xy}\varepsilon_x + A^*_{yy}\varepsilon_y \\ \tau_{xy} &= A^*_{ss}\gamma_{xy} \end{aligned} \tag{6.18}$$

The coupling between the direct stresses and the shear strains and between the shear stresses and the direct strains, which is present for a general laminate, disappears for an orthotropic laminate. Most laminates currently in use are orthotropic.

It can be readily seen that the following laminates will be orthotropic:

(1) Those consisting only of plies for which $\theta = 0°$ or $90°$; here it follows from equation (*6.8*) that in either case $Q_{xs}(\theta) = Q_{ys}(\theta) = 0$.
(2) Those constructed such that for each ply oriented at an angle θ, there is another ply oriented at an angle $-\theta$; because, as already noted from the odd powers in equation (*6.8*),

$$Q_{xs}(-\theta) = -Q_{xs}(\theta), \quad Q_{ys}(-\theta) = -Q_{ys}(\theta)$$

There is then a cancellation of all paired terms in the summation of equation (*6.17*).
(3) Those consisting only of 0°, 90°, and matched pairs of $\pm\,\theta$ plies are also, of course, orthotropic.

An example of an orthotropic laminate would be one with the following ply pattern:

$$0°/+30°/-30°/-30°/+30°/0°$$

On the other hand, the following laminate (while still symmetric) would not be orthotropic:

$$0°/+30°/90°/90°/+30°/0°$$

6.2.3.4 Moduli of Orthotropic Laminates. Expressions for the moduli of orthotropic laminates can easily be obtained by solving equation (*6.18*) for simple loadings. For example, on setting $\sigma_y = \tau_{xy} = 0$, Young's modulus in the x direction, E_x, and Poisson's ratio ν_{xy} governing the contraction in the y direction for a stress in the x direction are then given by:

$$E_x = \frac{\sigma_x}{\varepsilon_x}, \qquad \nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x}$$

Proceeding in this way, it is found that:

$$E_x = A^*_{xx} - \frac{A^{*2}_{xy}}{A^*_{yy}} \qquad E_y = A^*_{yy} - \frac{A^{*2}_{xy}}{A^*_{xx}}$$
$$\nu_{xy} = \frac{A^*_{xy}}{A^*_{yy}} \qquad \nu_{yx} = \frac{A^*_{xy}}{A^*_{xx}} \qquad G_{xy} = A^*_{ss} \tag{6.19}$$

As illustrative examples of the above theory, consider a family of 24-ply laminates, symmetrical and orthotropic, and all made of the same material but with varying numbers of 0° and ±45° plies. (For the present purposes, the precise ordering of the plies is immaterial as long as the symmetry requirement is maintained; however, to ensure orthotropy, there must be the same number of +45° as −45° plies.) The single-ply modulus data (representative of a carbon/epoxy) are:

$$E_1 = 137.44\,\text{GPa} \quad E_2 = 11.71\,\text{GPa} \quad G_{12} = 5.51\,\text{GPa}$$
$$\nu_{12} = 0.2500 \quad \nu_{21} = 0.0213 \tag{6.20}$$

The lay-ups considered are shown in Table 6.1. The steps in the calculation are as follows:

(1) Calculate the $Q_{ij}(0)$ from equation (*6.4*).
(2) For each of the ply orientations involved here $\theta = 0°$, $+45°$, and $-45°$, calculate the $Q_{ij}(\theta)$ from equation (*6.8*). [Of course, here the $Q_{ij}(0)$ have already been obtained in step 1.]

Table 6.1 Moduli for Family of 24-ply 0°/±45° Laminates Constructed Using Unidirectional Tape

Lay-up							
No. 0° Plies	No. +45° Plies	No. −45° Plies	E_x, GPa	E_y, GPa	G_{xy}, GPa	ν_{xy}	ν_{yx}
24	0	0	137.4	11.7	5.51	0.250	0.021
16	4	4	99.4	21.1	15.7	0.578	0.123
12	6	6	79.5	24.5	20.8	0.647	0.199
8	8	8	59.6	26.4	25.8	0.693	0.307
0	12	12	19.3	19.3	36	0.752	0.752

(3) Calculate the A^*_{ij} from equation (*6.16*); in the present case, equation (*6.16*) becomes:

$$A^*_{ij} = [n_1 Q_{ij}(0) + n_2 Q_{ij}(+45) + n_3 Q_{ij}(-45)]/24$$

where n_1 is the number of 0° plies, n_2 of +45° plies and n_3 of −45° plies.

(4) Calculate the moduli from equation (*6.19*).

The results of the calculations are shown in Table 6.1.

The results in Table 6.1 have been presented primarily to exemplify the preceding theory; however, they also demonstrate some features that are important in design. The stiffness of a composite is overwhelmingly resident in the extensional stiffness of its fibers; hence, at least for simple loadings, if maximum stiffness is required, a laminate is constructed so that the fibers are aligned in the principal stress directions. Thus, for a member under uniaxial tension, a laminate comprising basically all 0° plies would be chosen; in other words, all fibers would be aligned parallel to the tension direction. As can be seen from Table 6.1, E_x decreases as the number of 0° plies decreases. On the other hand, consider a rectangular panel under shear, the sides of the panel being parallel to the laminate axes (Fig. 6.7a). The principal stresses here are an equal tension and compression, oriented at +45° and −45° to the x-axis. Thus, maximum shear stiffness can be expected to be obtained using a laminate comprising equal numbers of +45° and −45° plies; this is reflected in the high shear modulus G_{xy} for the all ±45° laminate of Table 6.1.

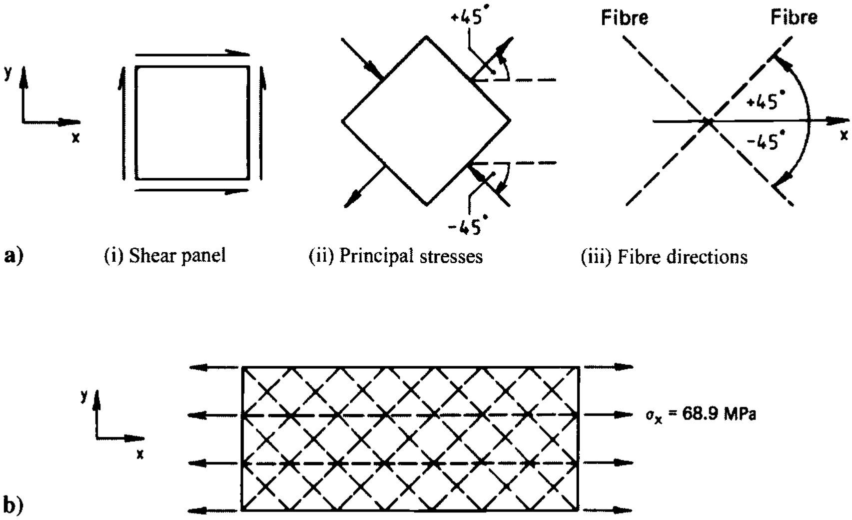


Fig. 6.7 Ply orientations for example problems: a) fiber orientations for a shear panel: b) 0° ± 45° laminate under uniaxial tension.

It should also be observed that, although for an isotropic material, Poisson's ratio cannot exceed 0.5, this is not the case for an anisotropic material.

6.2.3.5 Quasi-Isotropic Laminates. It is possible to construct laminates that are isotropic with regard to their in-plane elastic properties—in other words, they have the same Young's modulus E and same Poisson's ratio ν in all in-plane directions and for which the shear modulus is given by $G = E/2(1+\nu)$. One way of achieving this is to adopt a lay-up having an equal number of plies oriented parallel to the sides of an equilateral triangle. For example, a quasi-isotropic 24-ply laminate could be made with 8 plies oriented at each of 0°, +60°, and −60°. Using the same materials data (and theory) as were used in deriving Table 6.1, it will be found that such a laminate has the following moduli:

$$E = 54.2\,\text{Gpa}, \quad G = 20.8\,\text{Gpa}, \quad \nu = 0.305$$

Another way of achieving a quasi-isotropic laminate is to use equal numbers of plies oriented at 0°, +45°, −45°, and 90°. A quasi-isotropic 24-ply laminate (with, incidentally, the same values for the elastic constants as were just cited) could be made with 6 plies at each of 0°, +45°, −45°, and 90°. (For comparison, recall that Young's modulus and the shear modulus for a typical aluminum alloy are of the order of 72 and 27 GPa, respectively, and that the specific gravity of carbon/epoxy is about 60% that of aluminum.)

The term *quasi-isotropic* is used because, of course, such laminates have different properties in the out-of-plane direction. However, it is not usual practice to work with quasi-isotropic laminates; efficient design with composites generally requires that advantage be taken of their inherent anisotropy.

6.2.3.6 Stress Analysis of Orthotropic Laminates. The determination of the stresses, strains, and deformations experienced by symmetric laminates under plane stress loadings is carried out by procedures that are analogous to those used for isotropic materials. The laminate is treated as a homogeneous membrane having stiffness properties determined as described above. It should be noted, though, that while the strains and deformations so determined are the actual strains and deformations (within the limit of the assumptions), the stresses are only the average values over the laminate thickness.

If an analytical procedure is used, then generally a stress function F is introduced, this being related to the (average) stresses by

$$\sigma_x = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{xy} = \frac{-\partial^2 F}{\partial x \partial y} \qquad \text{(Eq. 6.21)}$$

It can be shown that F satisfies the following partial differential equation:

$$\left(\frac{1}{E_y}\right)\frac{\partial^4 F}{\partial x^4}+\left(\frac{1}{G_{xy}}-\frac{2\nu_{xy}}{E_x}\right)\frac{\partial^4 F}{\partial x^2\partial y^2}+\left(\frac{1}{E_x}\right)\frac{\partial^4 F}{\partial y^4}=0 \tag{6.22}$$

Solutions of equation (*6.22*) for several problems of interest can be found in Ref. 10.

Most structural analyses are now performed using finite element methods described in Chapter 16, and many general-purpose finite-element programs contain orthotropic membrane elements in their library. Once the laminate moduli are determined, they are used as input data for calculating the element stiffness matrix; the rest of the analysis proceeds as in the isotropic case.

As has already been emphasized, the stresses obtained from the above procedures are only the average stresses. To determine the actual stresses in the individual plies, it is necessary only to substitute the calculated values of the strains in equation (*6.10*). An elementary example may clarify this. Consider a rectangular strip under uniaxial tension (Fig. 6.7b) made of the 24-ply laminate considered earlier that had 12 plies at 0°, 6 plies at +45°, and 6 plies at −45°. Suppose the applied stress is $\sigma_x = 68.9$ MPa. The average stress here is uniform in the xy plane and given by:

$$\sigma_x = 68.9\,\text{MPa},\quad \sigma_y = \tau_s = 0$$

Using the values of E_x and ν_{xy} given in Table 6.1, it follows that the associated strains are:

$$\varepsilon_x = 0.8667\times10^{-3},\quad \varepsilon_y = -0.5607\times10^{-3},\quad \gamma_s = 0$$

The stresses in the individual plies can now be obtained by substituting these values into equation (*6.10*), with the appropriate values of the $Q_{ij}(\theta)$. The results of doing this are shown in Table 6.2.

Thus, the actual stress distribution is very different from the average one. In particular, note that transverse direct stresses and shear stresses are developed, even though no such stresses are applied; naturally, these stresses are self-equilibrating over the thickness. It follows that there is some "boundary layer" around the edges of the strip where there is a rapid transition from the actual stress boundary values (namely, zero on the longitudinal edges) to the calculated values shown in the table. This boundary layer would be expected to extend in from the edges a distance of the order of the laminate thickness (from the Saint-Venant principle). In the boundary

Table 6.2 Stresses in Individual Plies of a 24-Ply Laminate (12 at 0°, 6 at +45 and 6 at −45°) under Uniaxial Stress of 68.9 MPa

$\theta°$	$\sigma_x(\theta)$ MPa	σ_y (θ) Mpa	$\tau_{xy}(\theta)$ MPa
0	118.1	−4.1	0
+45	19.8	4.1	9.7
−45	19.8	4.1	−9.7

layer, the simple laminate theory presented above is not applicable and a three-dimensional analysis is required. The matter is of more than acadèmic interest since faults, such as delaminations, are prone to originate at the free edges of laminates because of the above effect. Edge effects will be considered further in Section 6.3.2.

Finally, when discussing allowable design values for composite structures, it is common to cite values of strain, rather than stress. Clearly, strain is the more meaningful quantity for a laminate.

6.2.3.7 Laminate Codes. Although the precise ordering of the plies of a laminate has not been of concern in the considerations of this section, generally there will be other factors that will determine such an ordering. In any case, when a laminate is being called up for manufacture, the associated engineering drawing should list the orientation of each ply. When referring to laminates in a text, some sort of abbreviated notation is necessary to specify the pattern. It is easiest to describe the code normally used with some examples.

Example 6.1

Consider an eight-ply uni-directional tape laminate with the following (symmetric) lay-up starting from the bottom ($z = t/2$) surface:

$$0°/0°/+45°/-45°/-45°/+45°/0°/0°$$

This is written in code form as:

$$[0_2/\pm 45]_s$$

Note that:

(1) Only half the plies in a symmetric laminate are listed, the symmetry being implied by the s outside the brackets.
(2) The degree signs are omitted from the angles.
(3) In a $+$ and $-$ combination, the upper sign is read first.

Example 6.2

Consider a four-ply fabric laminate with the following (symmetric) lay-up:

$$(0°,90°)/(\pm 45°)/(\pm 45°)/(0°,90°).$$

This is written in code form as

$$[(0, 90)/(\pm 45)/(0, 90)] \quad \text{or} \quad [(0, 90)/(\pm 15)].$$

Example 6.3

Consider a fifty-ply tape laminate containing repetitions of the ply sequence:

$$0°/0°/+45°/-45°/90°$$

The order in the sequence being reversed at the midplane to preserve the symmetry. This would be written in code form as:

$$[(0_2/\pm 45/90)_5]_s$$

Sometimes, in general discussions, a laminate is described by the percentages of its plies at various angles. Thus, the laminate of example (6.1) would be described as having "50% 0, 50% ± 45." Similarly, the laminate of example (6.2) would be described as having "40% 0, 40% ±45, 20% 90."

6.2.4 General Laminates Subjected to Plane Stress and Bending Loads

In this section the previous restriction to laminates that are symmetric about the mid-thickness plane will be dropped. It now becomes necessary to consider the plane stress and bending problems in conjunction, as in-plane loads can induce bending deformations and vice versa. Only the outline of the theory will be given below; for further details, including numerical examples, see Refs. 1–5.

6.2.4.1 General Theory. In contrast to the situation for symmetric laminates, the position of each ply in the laminate is now important. Thus, consider an n-ply laminate with z the coordinate in the thickness direction, measured from the mid-thickness plane. The kth ply lies between h_{k-1} and h_k (Fig. 6.5). As before, the total thickness of the laminate will be denoted by t.

It is assumed that when a laminate is subjected to in-plane and/or bending loads, the strain varies linearly through the thickness and can therefore be written in the form:

$$\varepsilon_x = \varepsilon_x^o + \kappa_x z \quad \varepsilon_y = \varepsilon_y^o + \kappa_y z \quad \gamma_{xy} = \gamma_{xy}^o + \kappa_s z \tag{6.23}$$

where the superscript ^{o} quantities are the mid-plane strains and the κ_l are the midplane curvatures (as in the bending of isotropic plates). Both these sets of quantities are independent of z.

Substituting from equation (*6.23*) into equation (*6.7*), it follows that the stresses in the kth ply will now be given by:

$$\begin{aligned}
\sigma_x(k) &= Q_{xx}(\theta_k)\left(\varepsilon_x^o + z\kappa_x\right) + Q_{xy}(\theta_k)\left(\varepsilon_y^o + z\kappa_y\right) + Q_{xs}(\theta_k)\left(\gamma_{xy}^o + z\kappa_s\right)\\
\sigma_y(k) &= Q_{xy}(\theta_k)\left(\varepsilon_x^o + z\kappa_x\right) + Q_{yy}(\theta_k)\left(\varepsilon_y^o + z\kappa_y\right) + Q_{ys}(\theta_k)\left(\gamma_{xy}^o + z\kappa_s\right)\\
\tau_{xy}(k) &= Q_{xs}(\theta_k)\left(\varepsilon_x^o + z\kappa_x\right) + Q_{ys}(\theta_k)\left(\varepsilon_y^o + z\kappa_y\right) + Q_{ss}(\theta_k)\left(\gamma_{xy}^o + z\kappa_s\right)
\end{aligned} \tag{6.24}$$

Now introduce the stress resultants (in the form of forces per unit length and moments per unit length) defined by:

$$N_x = \int \sigma_x \, dz \qquad N_y = \int \sigma_y \, dz \qquad N_s = \int \tau_{xy} \, dz$$
$$M_x = \int \sigma_x z \, dz \quad M_y = \int \sigma_y z \, dz \quad M_s = \int \tau_{xy} z \, dz \tag{6.25}$$

where all of the integrals are over the thickness of the laminate (i.e., from $z = -t/2$ to $z = t/2$); see Figure 6.8. Because each of the integrals in equation (*6.25*) can be written in forms such as:

$$N_x = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_x(k) dz, \quad M_x = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_x(k) z \, dz \tag{6.26}$$

it follows that substituting from equation (*6.24*) into equation (*6.26*), and performing some elementary integrations, leads to the result:

$$\begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{bmatrix} \tag{6.27}$$

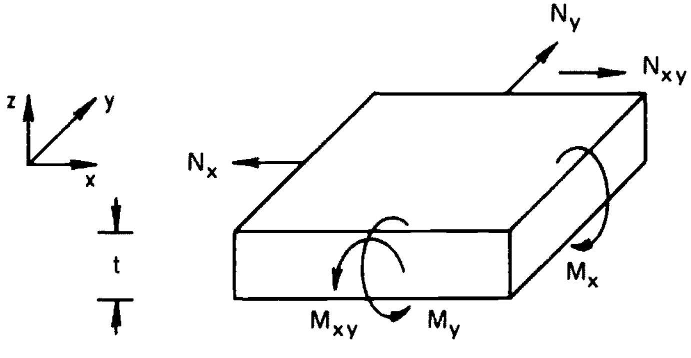


Fig. 6.8 Stress and moment resultants for a laminate.

The elements in the above combined "extensional-bending stiffness matrix" are given by:

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}(\theta_k)(h_k - h_{k-1}) \tag{6.28}$$

$$B_{ij} = \sum_{k=1}^{n} Q_{ij}(\theta_k)(h_k^2 - h_{k-1}^2)/2 \tag{6.29}$$

$$D_{ij} = \sum_{k=1}^{n} Q_{ij}(\theta_k)(h_k^3 - h_{k-1}^3)/3 \qquad i, j = x, y, s \tag{6.30}$$

[The above definition for A_{ij}, is, of course, equivalent to that of equation (*6.13*).] Equation (*6.27*) can be used to develop a theory for the stress analysis of general laminates, but this is quite formidable mathematically. It involves the solution of two simultaneous fourth order partial differential equations.

6.2.4.2 Uncoupling of the Stiffness Matrix. It can be seen from equation (*6.27*) that the plane stress and bending problems are coupled unless all the B_{ij}, are zero. It follows from equation (*6.29*) that the B_{ij} are indeed zero for a symmetric lay-up. [For a symmetric pair of plies, if that ply below the mid-plane has coordinates $h_{k-1} = -a$ and $h_k = -b$, then its mate above the mid-plane will have $h_{k-1} = b$ and $h_k = a$. Because each ply has the same Q_{ij}, there is a cancellation in the summation of equation (*6.29*).]

It is possible to achieve an approximate uncoupling of equation (*6.27*) for a multi-ply unsymmetric laminate by making it in the form of a large number of repetitions of a given ply grouping. It can be seen intuitively that such a laminate will be symmetric in the group of plies if not in the individual plies; as long as the number of plies in the group is small compared with the total number of plies in the laminate, the B_{ij}, will turn out to be small quantities. For example, a 48-ply laminate containing 24 groups of 45° plies laid up in the sequence $+/-/+/-$ etc. (without there being symmetry about the mid-plane) would be expected to behave much as a symmetrical laminate of the same plies.

6.2.4.3 Bending of Symmetric Laminates. The moment-curvature relation governing the bending of symmetric laminates out of their plane, as extracted from equation (*6.27*), is:

$$\begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{xy} & D_{yy} & D_{ys} \\ D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix}. \tag{6.31}$$

Analogously to the definition of orthotropy in plane stress, a laminate is said to be orthotropic in bending if $D_{xs} = D_{ys} = 0$. However, it is important to note that a laminate that is orthotropic in plane stress is not necessarily orthotropic in bending. For example, consider the four-ply laminate $[\pm 45]_s$. Here the coordinates for the $+45°$ plies may be written as $(-t/2, -t/4)$ and $(t/4, t/2)$, whereas those of the $-45°$ plies are $(-t/4, 0)$ and $(0, t/4)$; t, of course, is the laminate thickness. It is easy to establish that, while A_{xs} and A_{ys} are zero, D_{xs} and D_{ys} are not. On the other hand, a laminate, containing only 0° and 90° plies will be orthotropic in both plane stress and bending. These laminates are called specially orthotropic. (It is also worth noting that for multi-ply laminates made of groups of plies, if the group is orthotropic in plane stress, then the laminate will at least be approximately orthotropic in both plane stress and bending.)

As an example, consider the bending of the simply-supported beam shown in Figure 6.9a under a uniform distributed load. The laminate consists of a lay-up $[0_4\ 90_4]_s$. Using the single-ply modulus data given in equation (*6.20*) and a ply thickness of 0.125 mm, equation (*6.31*) becomes:

$$\begin{bmatrix} 81.584 & 1.962 & 0.0 \\ 1.962 & 18.382 & 0.0 \\ 0.0 & 0.0 & 3.673 \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} \text{Nm/m} \tag{6.32}$$

If the moment per unit width at the center of the beam is $M_x = 100$ N/m, $M_y = M_s = 0$, solving for the curvature gives $\kappa_x = 1.229$, $\kappa_y = -0.131$, and $\kappa_s = 0.0$, indicating the anti-elastic curvature effect typical of plates. Solving for strains in the laminate involves substituting in equations (*6.23*) and (*6.24*) with zero for the mid-plane strains. The stress and strain through the thickness are given in Figure 6.9b.

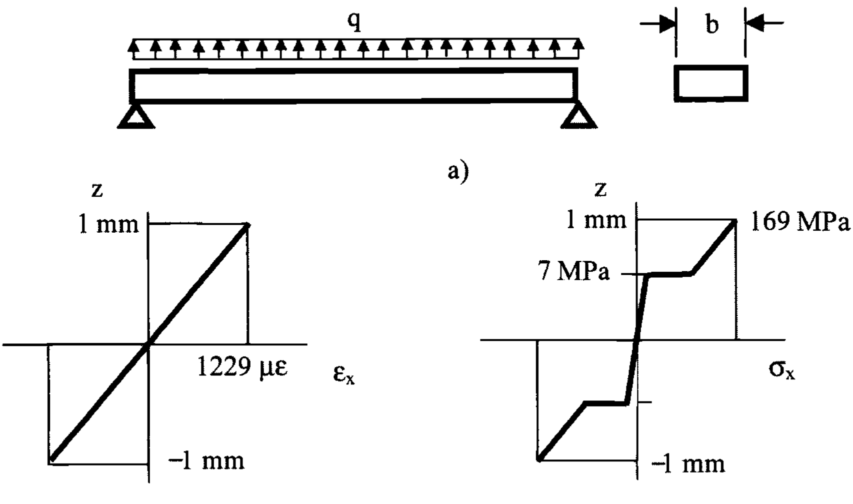


Fig. 6.9 Configuration and results for simply-supported beam.

To estimate the deflection at the center of the beam, the matrix of coefficients in equation (*6.32*) is inverted to give:

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} = \begin{bmatrix} d_{xx} & d_{xy} & d_{xs} \\ d_{xy} & d_{yy} & d_{ys} \\ d_{xs} & d_{ys} & d_{ss} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix}$$

$$= \begin{bmatrix} 0.01229 & -0.00131 & 0.0 \\ -0.00131 & 0.05454 & 0.0 \\ 0.0 & 0.0 & 0.27223 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} \text{rad/m}$$

In simple beam theory, a total moment M applied to a beam gives the curvature that satisfies the equation:

$$\kappa_x = -\frac{d^2w}{dx^2} = \frac{M}{EI}$$

Imposing a moment per unit width on the laminate $M_x = M/b$, $M_y = M_s = 0$ on the laminate of width b, then $d_{xx}/b = 1/(EI)_{\text{effective}}$. This effective bending stiffness can be substituted in the simple beam relation $\delta = 5qL^4/384EI$, where q is the load per unit length applied to the beam, to obtain a first order approximation of the deflection of the laminate.[4] For short beams, the simple bending theory is inaccurate, and shear deformation needs to be included.

Composite beams are often stiffened using a lightweight core of Nomex honeycomb or expanded foam.[11] The laminate can be analyzed by including a layer in the center of the laminate with negligible in-plane properties. Use of the laminate theory implies that the core material has sufficient transverse shear and through-thickness compression stiffness to ensure that the top and bottom skins and the core material are constrained to behave as an integral laminate. Repeating the calculation with a 10-mm core added at the center of the previous laminate gives the D_{ij} indicated in the following equation:

$$\begin{bmatrix} 4896.046 & 178.552 & 0.0 \\ 178.552 & 4200.829 & 0.0 \\ 0.0 & 0.0 & 334.273 \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_s \end{bmatrix} \text{Nm/m}$$

which represents an increase of bending stiffness approaching two orders of magnitude.

For an orthotropic plate in bending, the deflection w satisfies the following equation:

$$D_{xx}\frac{\partial^4 w}{\partial x^4} + 2(D_{xy} + 2D_{ss})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{yy}\frac{\partial^4 w}{\partial y^4} = q$$

where q is the applied pressure. Solutions of this equation can be found in Ref. 3. In this reference, the related problem of the buckling of laminated plates is also discussed.

6.3 Stress Concentration and Edge Effects

The strain variation through the thickness of the laminate defined by equation (*6.23*) is linear. Stresses vary discontinuously from ply to ply through the thickness as indicated in Figure 6.9. The determination of the stress in the *k*th ply requires the definition of the *z*-position of the ply in the laminate. Substitution into equation (*6.23*) defines the strains in the ply. These strains can then be substituted into equation (*6.24*) for the appropriate ply to define the stresses. These strains and stresses are defined in the laminate (x, y, z) coordinate system. When assessing failure in the ply, the stresses and strains are required in the material (1, 2, 3) axis system. Equation (*6.5*) for stress and equation (*6.6*) for strain can be implemented to achieve these transformations. Inverting the transformation in equation (*6.5*) gives the required relation:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

6.3.1 Stress Concentration Around Holes in Orthotropic Laminates

A common feature with isotropic materials is the stress-raising effect of holes and changes in geometry that modify the load paths. Several analytical solutions for the stresses around holes in (symmetric) orthotropic laminates are cited in Ref. 7. Details of the derivations of these are given in Ref. 10. It turns out that the value of the stress concentration factor (SCF) depends markedly on the relative values of the various moduli. This can be illustrated by considering the case of a circular hole in an infinite sheet under a uniaxial tension in the x-direction (Fig 6.10). Here, the stress concentration factor at point A in Figure 6.10 ($\alpha = 90°$), defined as the ratio of the average stress through the thickness of the laminate to the average applied stress $\bar{\sigma}_x$, is given by the following formula:

$$\mathrm{SCF} = K_T = \frac{\bar{\sigma}_{x\,\max}}{\bar{\sigma}_x} = 1 + \left\{ 2\left[\left(\frac{E_X}{E_Y} \right)^{1/2} - \nu_{xy} \right] + \frac{E_X}{G_{XY}} \right\}^{1/2} \tag{6.33}$$

The stress concentration factors for the laminates of Table 6.1 have been calculated from this formula, and the results are shown in Table 6.3.

For comparison, the SCF for an isotropic material is 3. As can be seen, when there is a high degree of anisotropy (e.g., an all 0° laminate), SCFs well in excess of that can be obtained. It should also be pointed out that, as the laminate pattern changes, not only does the value of the SCF change, but the point at which the SCF attains its maximum value can also change. Whereas for the first four laminates of Table 6.3, the maximum SCF does occur at point A, for the remaining laminate the maximum occurs at a point such as B in Figure 6.10.

Care should be taken when relating these results to the strength of the laminate. A laminate with all $\pm 45°$ plies has the lowest SCF at 2, but may also

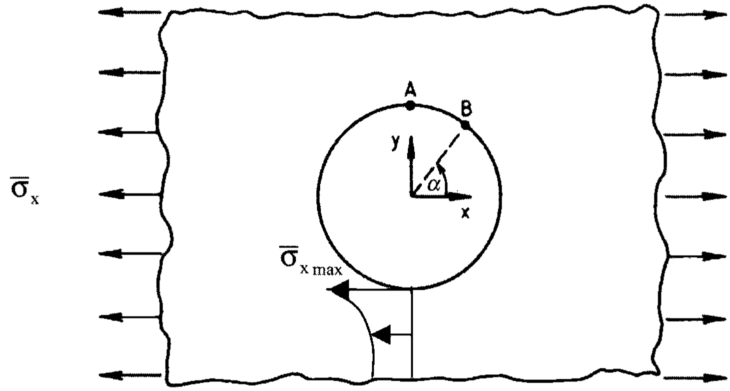


Fig. 6.10 Stress concentration on circular hole in infinite tension panel.

have the lowest strength due to the absence of 0° plies to carry the load. The best laminate has sufficient 0° plies to carry the load, but also ± 45° plies to reduce the stress concentration factor.

6.3.2 Edge Effects

Edge effects are caused by the requirement for strain compatibility between the plies in the laminate. They lead to interlaminar shear and through-thickness peel stresses near the free edges of the laminate. For example, if a laminate consisting of alternating 0° and 90° unidirectional plies is subject to a tensile load parallel to the 0° fibers, then the difference in Poisson's ratio leads to different transverse contraction, as indicated in Figure 6.11. However, the plies in the assembled laminate are forced to have the same transverse strain by the bonding provided by the resin. Therefore, an interlaminar shear develops between the plies, forcing the 0° ply to expand in the transverse direction and the 90° ply to contract. The shear stress is confined to the edge of the laminate because once the required tension σ_y is established in the 0° ply, for example, the compatibility will

Table 6.3 SCF at Circular Hole in Tension Panel (Laminate Data from Table 6.1)

Lay-up		SCF
No. 0° Plies	No. ± 45° Plies	Point A
24	0	6.6
16	8	4.1
12	12	3.5
8	16	3.0
0	24	2.0

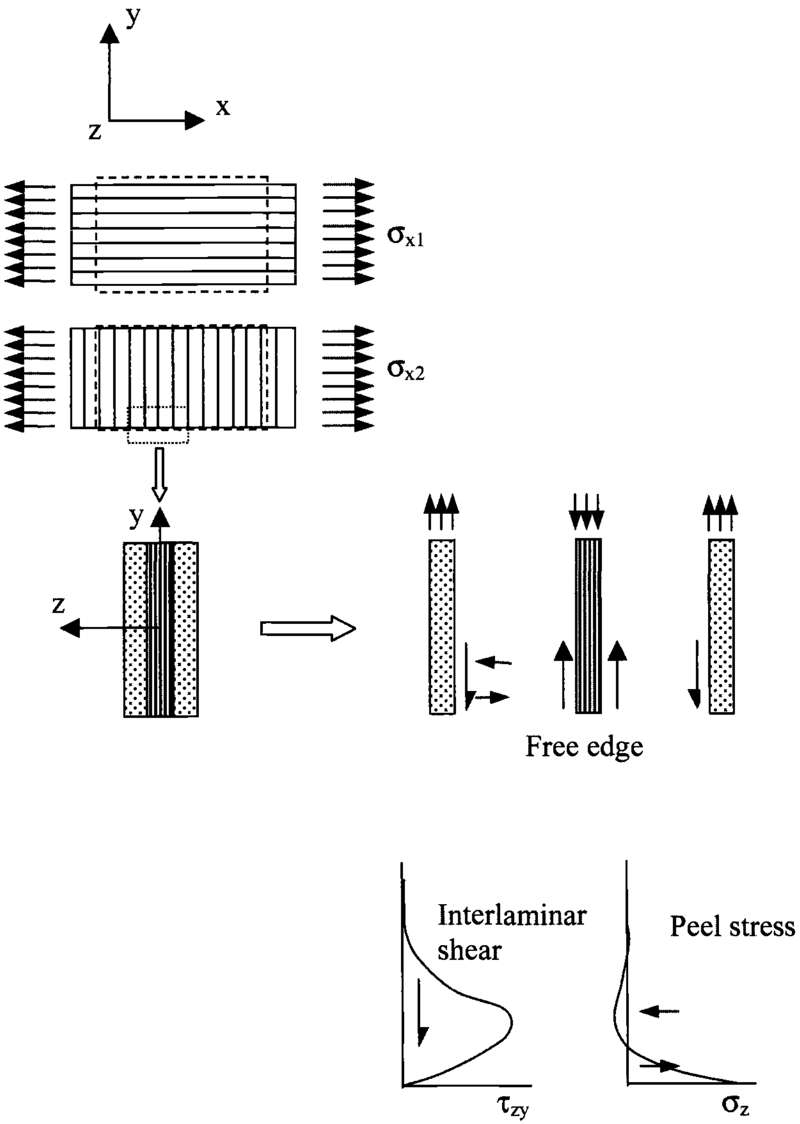


Fig. 6.11 Ply strain compatibility forced in a 0°/90° laminate and interlaminar shear and peel stresses at the edge of the laminate.

be ensured across the middle of the laminate. At the free edge, however, this tension stress must drop to zero if there is no applied edge stress.

The shear stress shown in Figure 6.11 is offset from the axis of the resultant of the stress σ_y, and therefore a turning moment is produced. To balance this moment, peel stresses σ_z develop in the laminate having the distribution indicated in Figure 6.11. These peel stresses can cause delamination at the edge. Similar stress distributions will be identified in Chapter 9 for bonded joints and bonded doubler plates.

Interlaminar shear stresses also occur in angle ply laminates as the individual plies distort differently under the applied loads. A more thorough treatment of edge effects is contained in Ref. 1.

6.4 Failure Theories

6.4.1 Overview-Matrix Cracking, First Ply Failure and Ultimate Load

The prediction of failure in laminates is complex. Failure is not only influenced by the type of loading, but also the properties of the fiber and properties of the resin, the stacking sequence of the plies, residual stresses, and environmental degradation. Failure will initiate at a local level in an individual ply or on the interface between plies but ultimate failure in multi-directional laminates may not occur until the failure has propagated to several plies.

Strains in the laminate are constant through the thickness for in-plane loading of symmetric laminates, or vary linearly if the laminates are subject to out-of-plane curvature. However, the stresses in each ply given by equations (*6.7*) and (*6.24*) depend on the modulus of the ply and vary discontinuously through the thickness of the laminate. Failure of the laminate described by a mean stress averaged through the thickness of the laminate will therefore apply only to a particular lay-up. The prediction of failure in multi-directional laminates usually requires the determination of strains and stresses for each ply in the material (1, 2, 3) axes for the ply. The prediction of ultimate failure then requires following the progression of failure through the laminate. A number of different types of failure therefore need to be assessed when evaluating the strength of a laminate:

(1) matrix cracking, which may have important implications for the durability of the laminate;
(2) first ply failure, where one of the plies in the laminate exceeds its ultimate stress or strain values;
(3) ultimate failure when the laminate fails; and
(4) transverse failure or splitting between the layers of the laminate.

Matrix cracking depends on the total state of stress or strain in the matrix. It depends on the residual stress in the matrix due to the curing processes as well as stress and strain due to mechanical loads. For example, in a thermoset laminate cured at elevated temperature, the resin can be considered to cure at or near the glass transition temperature. Because the thermal expansion coefficient of the matrix is much higher than that of the fibers, cooling to room temperature introduces tension into the matrix as it tries to shrink relative to the fibers. Matrix cracking under load then usually occurs at the interface between the most highly loaded ply aligned with the load direction and an off-axis ply.

To determine the load for first ply failure, the stress and strain in the principal material (1, 2, 3) axes in each ply are determined using the theory given in Section 6.2. Hence, the problem can be reduced to establishing a criterion for the ultimate strength of a single ply with the stresses or strains referred to the material axes. As before, these stresses will be denoted by σ_1, σ_2, and τ_{12}. A number of well-tried theories, including maximum stress and maximum strain, are discussed in the subsequent sections.

It is important to note that failure of the first ply does not necessarily constitute failure of the laminate. The stiffness of the failed ply can be reduced, say, to a defined percentage of its undamaged value, and the laminate re-analyzed to check whether the remaining plies can carry the load. If the load can be carried, the applied is increased until the next ply fails. When the load cannot be carried, ultimate failure has occurred.

The prediction of through-thickness failure has proved more difficult. This transverse failure occurs in the matrix. It is failure of the resin in either shear or tension. It depends on the total state of stress or strain in the matrix, including the stresses introduced by manufacturing of the component. Several approaches will be discussed in the following sections. When a flaw is present, a fracture mechanics approach is used to predict the growth of the flaw leading to delamination and structural failure. In the fracture mechanics approach, the strain energy release rate is determined and compared with the critical value for the matrix material. The approach has proved useful for predicting stiffener separation and delamination growth.[12]

6.4.2 Stress-Based Failure Theory

Stress-based failure theory can be classified into two categories: maximum stress theories[13–16] and quadratic stress failure theories.[17,18] The stresses are first determined for each ply and transformed to the material (1, 2, 3) axes.

Maximum stress theory directly compares the maximum stress experienced by the material with its strength. The maximum stress across a number of failure modes is compared with the strength in each failure mode. First ply failure will not occur if:

$$\max\left(\frac{\sigma_1}{X_T}, \left|\frac{\sigma_1}{X_C}\right|, \frac{\sigma_2}{Y_T}, \left|\frac{\sigma_2}{Y_C}\right|, \left|\frac{s_{12}}{S_{12}}\right|\right) \prec 1 \tag{6.34}$$

The quadratic failure criteria include the affect of biaxial (multiaxial) load. The most used quadratic failure theories are:

Tsai-Hill Theory

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1\sigma_2}{X^2} + \frac{\sigma_2^2}{Y^2} + \left(\frac{\tau_{12}}{S_{12}}\right)^2 \prec 1 \tag{6.35}$$

If $\sigma_1 > 0$, $X = X_T$; otherwise, $X = X_C$. If $\sigma_2 > 0$, $Y = Y_T$; otherwise, $Y = Y_C$. This failure criterion is a generalization of the von Mises yield criterion for isotropic ductile metals. When plotted with σ_1 and σ_2 as the axes and with constant values of τ_{12}/S_{12}, this equation defines elliptical arcs joined at the axes.[1]

A more general theory can be developed based on an interactive tensor polynomial relationship. The Tsai-Wu criterion is invariant for transformation between coordinate systems and is capable of accounting for the difference between tensile and compressive strengths.[1]

Tsai-Wu Theory[17]

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 \prec 1 \tag{6.36}$$

The Tsai-Wu coefficients are defined as follows:

$$F_1 = \frac{1}{X_T} + \frac{1}{X_C}$$

$$F_2 = \frac{1}{Y_T} + \frac{1}{Y_C}$$

$$F_{11} = -\frac{1}{X_T X_C}$$

$$F_{22} = -\frac{1}{Y_T Y_C}$$

$$F_{66} = \frac{1}{S_{12}^2}$$

The coefficient F_{12} requires biaxial testing. Let σ_{biax} be the equal biaxial tensile stress ($\sigma_1 = \sigma_2$) at failure. If it is known, then:

$$F_{12} = \frac{1}{2\sigma_{\text{biax}}^2}\left(1 - \left(\frac{1}{X_T} + \frac{1}{X_C} + \frac{1}{Y_T} + \frac{1}{Y_C}\right)\sigma_{\text{biax}} + \left(\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C}\right)\sigma_{\text{biax}}^2\right)$$

otherwise,

$$F_{12} = f^1\sqrt{F_{11}F_{22}}$$

where $-1.0 \leq f^1 \leq 1.0$. The default value of f^1 is zero.

The maximum stress theories define simple regions in stress space. Stresses lying inside the limits defined by the solid lines in Figure 6.12 will not cause failure. Failure occurs for combinations of stresses that lie outside the failure envelope. The polynomial theories define elliptical regions. The plot appearing as a dashed line in Figure 6.12 is for the Tsai-Wu criterion for combinations of direct stress with zero shear stress. In this case, the ellipse crosses the axes at the four points corresponding to X_T, X_C, Y_T, and Y_C.

6.4.2.1 Stress-Based Theories: Considering Actual Failure Modes. The Tsai-Hill and Tsai-Wu criteria do not identify which mode of failure has

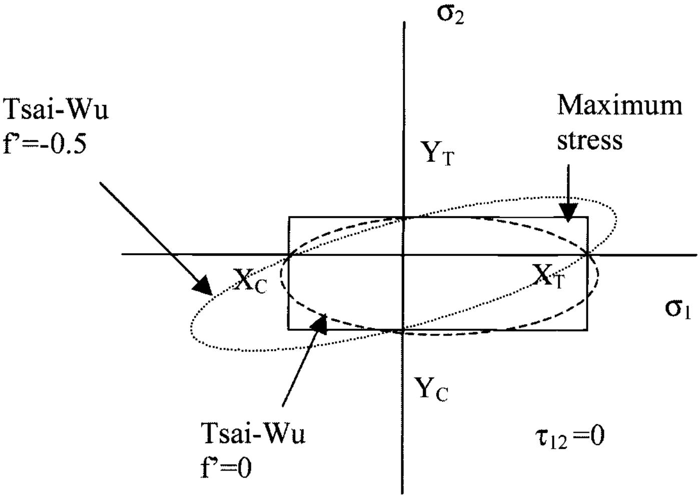


Fig. 6.12 Stress failure envelopes for a typical unidirectional carbon fiber. (X_T = 1280 MPa, X_C = 1440 MPa, Y_T = 57 MPa, Y_C = 228 MPa)

become critical. Failure theories in the second category treat the separate failure modes inde-pendently. The maximum stress criteria and the Hashin-Rotem failure criterion treat the separate failure modes independently.[15,19]

The two-dimensional Hashin-Rotem failure criterion has the following components for unidirectional material. For tensile fiber failure ($\sigma_{11} > 0$):

$$\left(\frac{\sigma_{11}}{X_T}\right)^2 + \left(\frac{\tau_{12}}{S_{12}}\right)^2 = 1 \tag{6.37a}$$

For compressive fiber failure ($\sigma_{11} < 0$):

$$\left(\frac{\sigma_{11}}{X_C}\right)^2 = 1 \tag{6.37b}$$

For tensile matrix failure ($\sigma_{22} > 0$):

$$\left(\frac{\sigma_{22}}{Y_T}\right)^2 + \left(\frac{\tau_{12}}{S_{12}}\right)^2 = 1 \tag{6.37c}$$

For compressive matrix failure ($\sigma_{22} < 0$):

$$\left(\frac{\sigma_{22}}{2S_{23}}\right)^2 + \left[\left(\frac{Y_C}{2S_{23}}\right)^2 - 1\right]\frac{\sigma_{22}}{Y_C} + \left(\frac{\tau_{12}}{S_{12}}\right)^2 = 1 \tag{6.37d}$$

For through-thickness failure, the combination of through-thickness stresses predicts delaminations will initiate when:

$$\left(\frac{\sigma_{33}}{Z_T}\right)^2+\left(\frac{\sigma_{23}}{S_{23}}\right)^2+\left(\frac{\sigma_{31}}{S_{31}}\right)^2=1$$

Here σ_{11}, σ_{22} and τ_{12}, are the longitudinal stress, transverse stress, and shear stress, respectively, and X, Y, Z, and S are the longitudinal strength, transverse strengths (Z is through thickness), and shear strength, respectively, that are obtained from testing. The subscripts T and C refer to tension and compression mode, respectively.

If the matrix failure parameter is satisfied first, then an initial crack forms in the matrix.

Zhang[20] separated the through thickness mode into stear and tension.

For interlaminar shear failure occurs when:

$$\sqrt{\tau_{13}^2+\tau_{23}^2} \geq \text{interlaminar shear strength}$$

For interlaminar tension:

$$\frac{\sigma_{\text{peel}}}{Z_T}=1 \tag{6.38}$$

6.4.2.2 Stress-Based Methods: Application to Laminates. To predict ultimate failure, the lamina failure criterion is applied to examine which ply undergoes initial failure. The stiffness of this ply is then reduced and the load is increased until the second ply fails. The process is repeated until the load cannot be increased indicating the ultimate failure load has been reached. Residual stresses can be taken into account at the laminate level.

Hashin and Rotem[15] used a different stiffness reduction method for predicting ultimate failure. A stiffness matrix represents the laminate where the stiffness of the cracked lamina decreases proportionally to the logarithmic load increase in the laminate. Residual stresses are considered, and a non-linear analysis is used.

Liu and Tsai[18] use the Tsai-Wu[17] linear quadratic failure criterion. The failure envelope is defined by test data. The data are obtained from uniaxial and pure shear tests. After initial matrix failure, cracking in the matrix occurs. The stiffness is reduced for the failed lamina by using a matrix degradation factor that is computed from micromechanics. This process is repeated until the maximum load is reached. Thermal residual stresses, along with moisture stresses, are estimated using a linear theory of thermoelasticity. This assumes that the strains are proportional to the curing temperature.

Through-thickness failure is failure in the matrix caused by tensile stresses perpendicular to the plies. A typical example is shown in Figure 6.13. The delamination can be predicted by the interlaminar tension criterion of Zhang.[20]

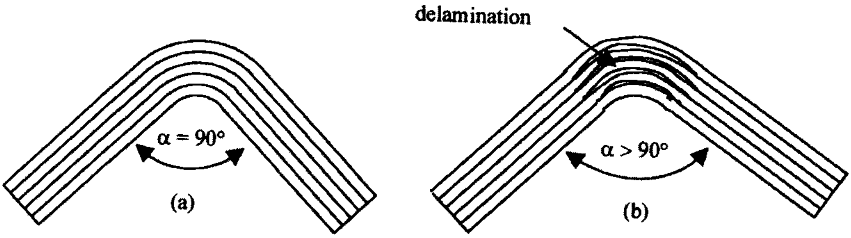


Fig. 6.13 Interlaminar failure—splitting in a curved beam.

Wisnom et al.[21,22] use a stress-based failure criterion to predict through-thickness failure in composite structures. This matrix failure criterion uses an equivalent stress σ_e, calculated from the three principal stresses:

$$\sigma_e^2 = \frac{1}{2.6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 0.6\sigma_e(\sigma_1 + \sigma_2 + \sigma_3)] \tag{6.39}$$

Interlaminar strength is considered to be related to the volume of stressed material. Therefore, the stressed volume is taken into account using a Weibull statistical strength theory. The criterion can be applied to three-dimensional geometric structures with any lay-up. Residual stresses and the effect of hydrostatic stress are accounted for.

6.4.3 Strain-Based Failure Theories

The simplest strain-based failure theories compare strains in the laminate with strain limits for the material.[23,24] Failure occurs if

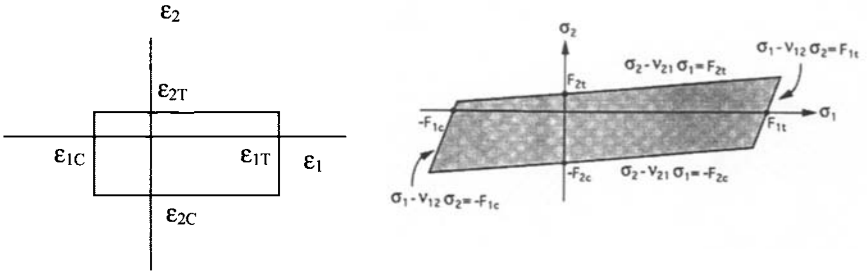
$$\max\left(\frac{\varepsilon_1}{\varepsilon_{1T}}, \left|\frac{\varepsilon_1}{\varepsilon_{1C}}\right|, \frac{\varepsilon_2}{\varepsilon_{2T}}, \left|\frac{\varepsilon_2}{\varepsilon_{2C}}\right|, \left|\frac{\varepsilon_{12}}{\gamma_{12}}\right|\right) = 1 \tag{6.40}$$

where the subscripts T and C, as before, refer to critical strains for tension and compression and γ_{12} to critical shear strain.

The failure envelope for this failure criterion is sketched in Figure 6.14a. If the Poisson's ratio for the laminate is non-zero, tensile strain of the laminate in the 1-direction will be accompanied by contraction in the 2-direction. Transforming the failure envelope from strain axes to stress axes therefore leads to the distorted failure envelope shown in Figure 6.14b.

Puck and Schurmann[25] have developed a strain-based theory for fiber failure in tension, including deformation of fiber in the transverse direction.

$$\frac{1}{\varepsilon_{1T}}\left(\varepsilon_1 + \frac{\nu_{f12}}{E_{f1}} m_{\sigma f}\sigma_2\right) = 1$$



a) b)

Fig. 6.14 Maximum strain failure envelope: *a*) maximum strain failure envelope with strain axes; *b*) maximum strain failure envelope on stress axes.

where ε_{1T} is tensile failure strain of a unidirectional layer, and m_{of} is mean stress magnification factor for the fiber in y-direction, due to the difference between the transverse modulus of the fiber and the modulus of the matrix. For carbon fiber, m_{of} is equal to about 1.1.

6.4.3.1 Strain Invariant Failure Theory. Although the strain invariant failure theory (SIFT) is not new for isotropic metallic materials,[26] its application to the failure of the matrix in composite materials is a development initiated by Gosse and Christensen.[27] Theoretically, for an isotropic material like steel, yield under complex stress must directly result from the magnitude of stress or strain and must be independent of the direction of the coordinate system defining the stress field. Similarly, a strain-base criterion not linked to a location and direction in the laminate must be a function of invariant strains so that it is unaffected by a transformation of the coordinates.

Under complex stress, the strain invariants can be determined from the following cubic characteristic equation determined from the strain tensor.[26]

$$\varepsilon^3 - (\varepsilon_x + \varepsilon_y + \varepsilon_z)\,\varepsilon^2 + \left(\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\varepsilon_{xy}^2 - \frac{1}{4}\varepsilon_{yz}^2 - \frac{1}{4}\varepsilon_{zx}^2\right)\varepsilon$$
$$- \left(\varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} - \frac{1}{4}\varepsilon_x\varepsilon_{yz}^2 - \frac{1}{4}\varepsilon_y\varepsilon_{zx}^2 - \frac{1}{4}\varepsilon_z\varepsilon_{xy}^2\right) = 0$$

The coefficients of the cubic equation are invariant to transformation of coordinates, designated as invariant strains. We can write:

$$\varepsilon^3 - J_1\varepsilon^2 + J_2\varepsilon - J_3 = 0$$

where:

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{6.41}$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\varepsilon_{xy}^2 - \frac{1}{4}\varepsilon_{yz}^2 - \frac{1}{4}\varepsilon_{zx}^2$$
$$= \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \tag{6.42}$$

$$J_3 = \varepsilon_x\varepsilon_y\varepsilon_z + \frac{1}{4}\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} - \frac{1}{4}\varepsilon_x\varepsilon_{yz}^2 - \frac{1}{4}\varepsilon_y\varepsilon_{zx}^2 - \frac{1}{4}\varepsilon_z\varepsilon_{xy}^2 = \varepsilon_1\varepsilon_2\varepsilon_3 \tag{6.43}$$

6.4.3.2 First Invariant Strain Criterion for Matrix Failure. Obviously, the simplest criterion of such a kind is a function of the first invariant strain J_1, which indicates the part of the state of strain corresponding to change of volume. However, it is well known that a material would not yield under compressive hydrostatic stress; consequently, this first invariant strain criterion is applicable only to tension-tension load cases experiencing volume increases.

6.4.3.3 Second Deviatoric Strain Criterion. To consider material yielding by the part of the state of strain causing change of shape (distortion) and to exclude the part of state of strain causing change of volume, a function of the second deviatoric invariant strain J_2' has been suggested where:

$$J_2' = \frac{1}{6}((\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2)$$
$$- \frac{1}{4}\varepsilon_{xy}^2 - \frac{1}{4}\varepsilon_{yz}^2 - \frac{1}{4}\varepsilon_{zx}^2 \tag{6.44}$$

A more convenient form for use as a failure criterion is:

$$\gamma_{crit} = \sqrt{3J_2'}$$
$$= \sqrt{\frac{1}{2}((\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2) - \frac{3}{4}\varepsilon_{xy}^2 - \frac{3}{4}\varepsilon_{yz}^2 - \frac{3}{4}\varepsilon_{zx}^2} \tag{6.45}$$

This criterion can be simplified using principal strains to:

$$\varepsilon_{\text{eqv}} = \sqrt{((\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2)/2}$$

This equivalent strain, often referred to the Von Mises shear strain, is a combination of invariants:

$$\varepsilon_{\text{eqv}} = \sqrt{J_1^2 - 3J_2}$$

Consequently, it is also invariant to any transformation of axis.

6.4.3.4 SIFT Applied to Laminates. Use of the strain invariant failure criteria for composite laminates[30] is a break from traditional methods because it considers three planes of strain as opposed to the conventional maximum principal strain. Two mechanisms for matrix failure are considered. These are dilatational failure, characterized by the first strain invariant, J_1, and distortional failure, characterized by a function of an equivalent shear strain, ε_{eqv}. Initial failure occurs when either of these parameters exceeds a critical value. The calculation of strain includes micromechanical models that take into account residual stresses and a strain amplification factor that includes strain concentration around the fiber. The criterion is a physics-based strain approach, based on properties at the lamina level. It can be applied to three-dimensional laminates with any lay-up, boundary, and loading conditions.

Gosse and Christensen[27] undertook several test cases on laminates, each with different lay-ups, for verification of the strain invariant failure criterion. These tests gave a good correlation between interlaminar cracking and the first strain invariant. Hart-Smith and Gosse[28] extended the SIFT approach to map matrix damage to predict final failure. This is done using a strain-energy approach.

6.4.4 Matrix Failure Envelopes

The matrix failure envelope[27] for the SIFT criterion can be seen in Figure 6.15. When the cylindrical section is cut by the plane formed by the first two principal axes, an ellipsoid is formed. The ellipsoidal region of the envelope is governed by shear components of strain characterized by the equivalent strain, ε_{eqv}. The 45° cut-off plane, dominated by transverse tensile strain, is characterized by the first strain invariant, J_1.

The insert in Figure 6.15, developed by Sternstein and Ongchin,[29] is a failure envelope for polymers constrained within glass fibers, where the strain in the 3-direction is equal to zero. When the SIFT failure envelope is transferred to stress axes both sections become segments of an ellipse as indicated in Figure 6.16.

6.4.5 Comparison of Failure Prediction Models

Soden et al.[31] have compared the failure predictions achieved by several theories and compared them to experimental results.[31,32,34,35] Almost all of the failure envelopes presented give an ellipsoidal shape in the negative stress region corresponding to shear loading. Transverse tensile loading causes failure in the

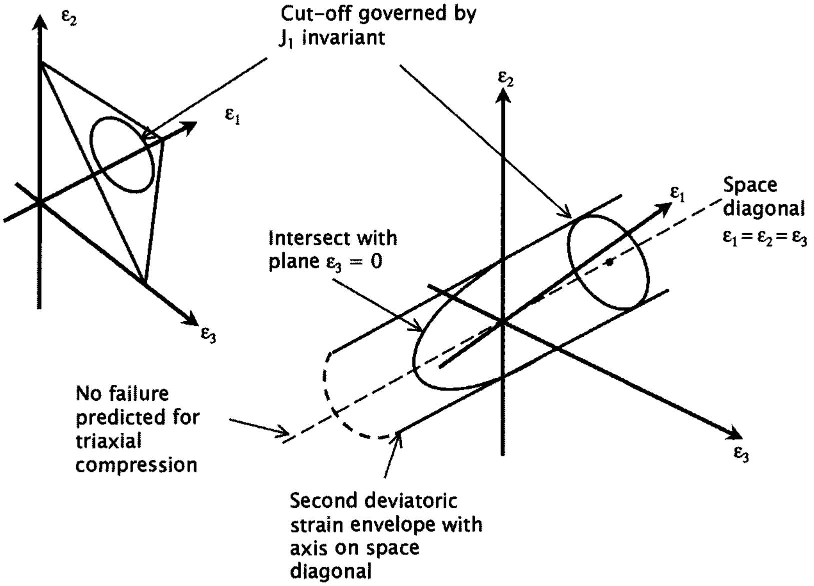


Fig. 6.15 The SIFT matrix failure envelope.

positive stress-strain region.[30] SIFT represents failure in this region with a cut-off plane characterized by the first strain invariant J_1, whereas most of the other failure theories continue with either an ellipsoidal shape or a curved, irregular shape.

6.5 Fracture Mechanics

In the fracture mechanics approach, failure is predicted to occur when a crack grows spontaneously from an initial flaw.[33] This approach has found application in predicting stiffener debonding[36] and for predicting interply failures.

Fracture toughness and crack growth was discussed in Chapter 2. It is apparent from that discussion that cracks are likely to grow parallel to the fibers (splitting) and parallel to the plies (delamination). Failure can also occur in assembled structures in adhesive bonds between the components. The prediction of when a delamination or disbond will grow is important for the assessment of damage tolerance where the initial defect can be assumed to arise due to manufacturing processes or due to impact or other damage mechanisms for the structure. The size of the defect is usually linked to the limits of visual inspection or the inspections that follow manufacture because known defects will be repaired.

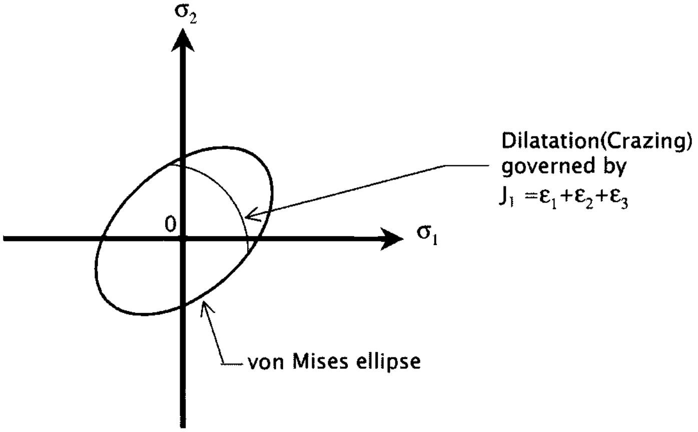


Fig. 6.16 SIFT failure envelope for polymeric materials.

The requirement is that damage that cannot be detected should not grow under operational loads.

The prediction of the growth of interlaminar splitting and disbonds in bonded joints is based on three modes of crack opening. Mode I is crack opening due to interlaminar tension, mode II due to interlaminar sliding shear, and mode III due to interlaminar scissoring shear. The components G_I, G_{II}, and G_{III} of the strain energy release rate, G, can be determined using a virtual crack closure technique in which the work done by forces to close the tip of the crack are calculated. If mode I and mode II crack opening is contributing to the growth, the delamination is predicted to grow when:

$$\left(\frac{G_I}{G_{IC}}\right)^m + \left(\frac{G_{II}}{G_{IIC}}\right)^n = 1 \qquad \text{(Eq. 6.46)}$$

where m and n are empirically defined constants. The finite element analysis depicted in Figure 16.11 identifies the role that buckling of a panel and flanges of a stiffener can play in driving a disbond in a bonded joint.

6.6 Failure Prediction Near Stress Raisers and Damage Tolerance

The behavior of carbon fiber laminates with epoxy resins is predominantly linearly elastic. However, some stress relief occurs near stress concentrations that is similar to the development of a plastic zone in ductile metals. Microcracking in the laminate softens the laminate in the vicinity of the notch. In the case of

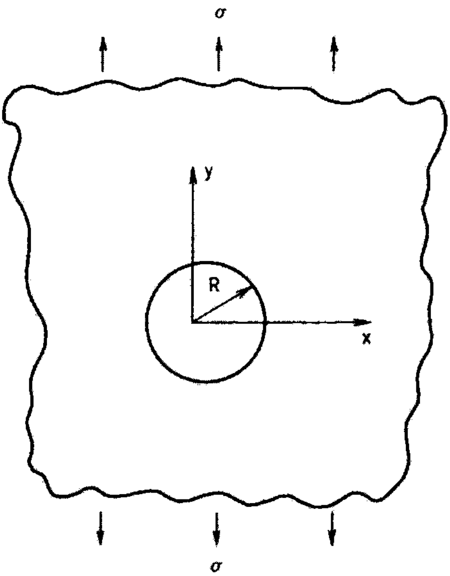


Fig. 6.17 Reference axes for a hole in an orthotropic panel under uniform tension.

holes in composite laminates, two different approaches have been successfully developed to predict failure based on the stress distribution. These are the average stress failure criterion and the point stress failure criterion.

Consider a hole of radius R in an infinite orthotropic sheet (Fig. 6.17). If a uniform stress σ is applied parallel to the y-axis at infinity, then, as shown in Ref. 21, the normal stress σ_y along the x-axis in front of the hole can be approximated by:

$$\sigma_y(x, 0) = \frac{\sigma}{2}\left\{2 + \left(\frac{R}{x}\right)^2 + 3\left(\frac{R}{x}\right)^4 - (K_T - 3)\left[5\left(\frac{R}{x}\right)^6 - 7\left(\frac{R}{x}\right)^8\right]\right\} \qquad (6.47)$$

where K_T is the orthotropic stress concentration factor given by equation (*6.33*).

The average stress failure criterion[40,41] then assumes that failure occurs when the average value of σ_y over some fixed length a_o ahead of the hole first reaches the un-notched tensile strength of the material. That is, when:

$$\frac{1}{a_o}\int_R^{R+a_o} \sigma_y(x, 0)\mathrm{d}x = \sigma_o$$

Using this criterion with equation (*6.47*) gives the ratio of the notched to un-notched strength as:

$$\frac{\sigma_N}{\sigma_o} = \frac{2(1-\phi)}{2-\phi^2-\phi^4+(K_T-3)(\phi^6-\phi^8)}$$

where:

$$\phi = \frac{R}{R+a_o}$$

In practice, the quantity a_o is determined experimentally from strength reduction data.

The point stress criterion assumes that failure occurs when the stress σ_y at some fixed distance d_o ahead of the hole first reaches the un-notched tensile stress,

$$\sigma_y(x,0)|_{x=R+d_o} = \sigma_o$$

It was shown in Ref. 40 that the point stress and average stress failure criteria are related, and that:

$$a_o = 4d_o$$

The accuracy of these methods, in particular the average stress method, can be seen in Figure 6.18, where a_o was taken as 3.8 mm. The solid lines represent predicted strength using the average stress criterion, and the dotted lines are the predicted strengths from the point stress method. Tests in Ref. 41 were carried out on various 16-ply carbon/epoxy laminates (AS/3501-5) with holes. The laminates were: $(0/\pm45_2/0/\pm45)_s$, $(0_2/\pm45/0_2/90/0)_s$, and $(0/\pm45/90)_{2s}$. The results are shown in Table 6.4 and are compared with predicted values using the average stress method with $a_o = 2.3$ mm.

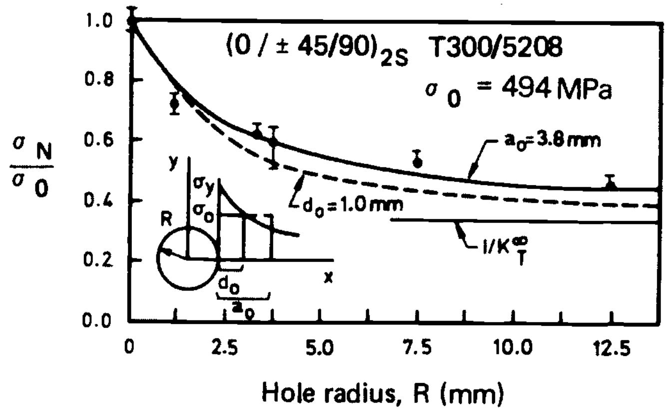
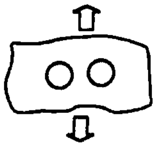
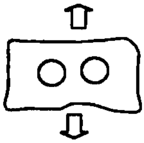


Fig. 6.18 Comparison of predicted and experimental failure stresses for circular holes in $(0/\pm45/90)_{2s}$, T300/5208.

Table 6.4 Static Strength Predictions[45]

Number of Holes, Hole Size and Placement		Laminate No.	% of Unnotched Strength	
			Test	Avg Stress
2 4.8-mm diameter countersunk.		1	58.9	53.6
		2	48.1	51.4
		3	51.8	53.2
2 4.8-mm diameter countersunk.		3	48.7	45.9
2 4-mm diameter countersunk.		3	53.1	50.3
1 4-mm diameter noncountersunk		2	54.0	52.6

As can be seen from the examples given, the average stress criterion provides accurate estimates of the strength reduction due to the presence of holes. This method is widely used in the aerospace industry[42] and has been applied to biaxial stress problems,[43] to the estimation of strength reduction due to battle damage,[44] and to problems in which the stress is compressive.[45]

Damage tolerance in laminates is considered in Chapters 8 and 12. The analysis requirements include prediction of the growth of defects caused by impact and the determination of the compressive strength after impact. The analysis of the growth of disbonds can be based on fracture mechanics approaches. The compression after impact strength has often been analyzed by approximating the damaged zone as an open hole and assessing the strength of the laminate under compressive loads using the procedures described above.

6.7 Buckling

In laminated composites, buckling can occur at the laminate or fiber level.

6.7.1 Buckling of the Laminate

Buckling loads for the laminate can be estimated using classical analysis for orthotropic plates. In general, buckling loads are increased by arranging the lay-up with plies aligned with the compressive load in the outer layers. The effect is to increase the bending stiffness of the laminate. Data sheets for the buckling coefficient for specially orthotropic laminates are presented in the ESDU data sheets.[46] Data for plates loaded either uniaxially or bi-axially is presented in terms of the coefficients K_o and C for a variety of edge conditions. The buckling load is given by

$$N_{xb} = \frac{K_o(D_{11}D_{22})^{1/2}}{b^2} + \frac{C\pi^2 D_o}{b^2} \quad \text{where} \quad D_o = D_{12} + 2D_{33}$$

and the coefficients D_{ij} are the coefficients derived in Section 6.2 and equations (*6.30*) and (*6.31*).

6.7.2 Buckling of the Fibers

Buckling failures can also be associated with lamina. These failures are identified by kink zones that form normal to the layers. Typical kink bands are shown in Figure 6.19. In the most common mode of buckling, the fibers buckle in an in-phase or shear mode. Fiber buckling has been discussed in Chapter 2.

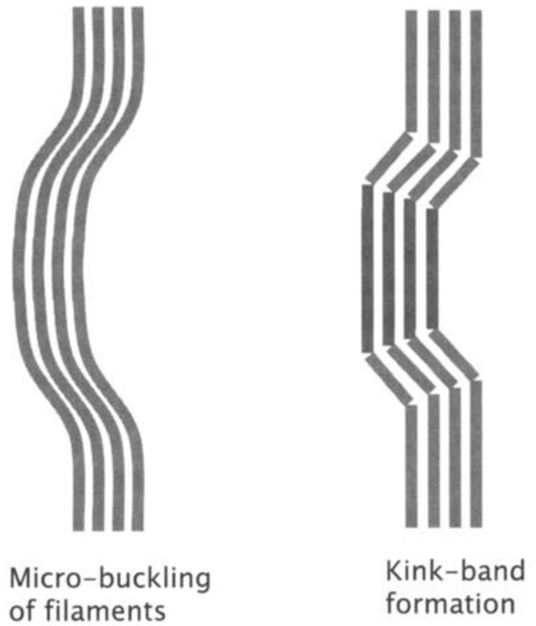


Fig. 6.19 Microbuckling of fibers. Taken from Ref. 1.

6.8 Summary

Classical laminate analysis has been introduced in this chapter. The analysis gives a relationship between in-plane load resultants and strain, and between out-of-plane bending moments and curvatures for panels consisting of layers or plies of unidirectional and fabric material. Once these relationships have been derived, the analysis of composites becomes equivalent to the classical analysis of anisotropic materials. A laminate analysis, for example, precedes a finite element analysis in which the algorithm calculates the equivalent plate properties from the A, B, and D matrices. Once this step is completed, the full power and versatility of finite element analysis (See Chapter 16) can be applied to the analysis and design of composite structures. Data sheets for design using composite panels can also be based on stiffness and strains produced by a laminate analysis as indicated for the case of buckling in Section 6.7. Finally the laminate analysis can be used to define the equivalent stiffness of the panel enabling the application of standard formulas to check the stiffness of plate and beam structures.

The prediction of failure in composites is a difficult problem. The materials consist of both fibers and a matrix—both of which exhibit distinct failure modes. In addition the interface between the fibers and the resin, the ply stacking sequence and the environmental conditions all contribute to failure. To compound the problem, the manufacturing processes introduce significant residual stresses into the resins. These residual stresses become apparent when the structure distorts due to the phenomenon called spring-in and when the matrix cracks after cure even before the structures are loaded. The failure theories discussed in this chapter are still being developed. Tension fiber failures are generally well predicted, and design margins of safety can be small. However, much still remains to be done to improve the reliability of the techniques for predicting matrix failures and the growth of delaminations.

References

[1]Daniel, I. M., and Ishai, O., *Engineering Mechanics of Composite Materials*, Oxford University Press, 1994.

[2]Staab, G. H., *Laminated Composites*, Butterworth-Heinemann, 1999.

[3]Jones, R. M., *Mechanics of Composite Materials*, Tokyo, McGraw-Hill, Kogakusha, 1975.

[4]Tsai, S. W., and Hahn, H. T., *Introduction to Composite Materials*, Technomic Publishing, Westport, CT, 1980.

[5]Agarwal, B. D., and Broutman, L. J., *Analysis and Performance of Fiber Composites*, Wiley, New York, 1980.

[6]Broutman, L. J., and Krock, R. H., (eds.), *Composite Materials*, Vols. 7 and 8 "Laminate Design and Analysis", Parts I and II, edited by C. C. Chamis, New York, Academic Press, 1975.

[7]"Plastics for Aerospace Vehicles," *Pt.* 1, *Reinforced Plastics*, MIL-HDBK-17A, Washington, DC, U.S. Department of Defense, 1971.

[8]Hearmon, R. F. S., *An Introduction to Applied Anisotropic Elasticity*, London, Oxford University Press, 1961.

[9]Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Body*, Moscow, Mir Publishers, 1981.

[10]Lekhnitskii, S. G., *Anisotropic Plates*, Gordon and Breach, New York, 1968.

[11]Vinson, J. R., *The Behaviour of Sandwich Structures of Isotropic and Composite Materials*, Technomic, 1999.

[12]Davies, G. A. O., and Zhang, X., "Impact Damage Predictions, in Carbon Composite Structures," *International Journal of Impact Engineering*, Vol. 16, No. 1, 1995, pp. 149–170.

[13]Sun, C. T., and Tao, J. X., "Prediction Failure Envelopes and Stress/Strain Behaviour of Composite Laminates," *Composites Science and Technology*, Vol. 58, 1998, pp. 1125–1113.

[14]Zinoviev, P., Grigoriev, S. V., Labedeva, O. V., Tairova, L. R., "The Strength of Multilayered Composites under Plane Stress State," *Composites Science and Technology*, Vol. 58, 1998, pp. 1209–1223.

[15]Hashin, Z., and Rotem, A., "A Fatigue Failure Criterion for Fibre Reinforced Materials," *Journal of Composite Materials*, Vol. 7, 1973, pp. 448–464.

[16]Edge, E. C., "Stress Based Grant-Sanders Method for Predicting Failure of Composite Laminates," *Composites Science and Technology*, Vol. 58, 1998, pp. 1033–1041.

[17]Tsai, S. W., and Wu, E. M., "A General Theory of Strength for Anisotropic Materials" *Journal of Composite Materials*, Vol. 5, 1971, pp. 58–80.

[18]Liu, K. S., and Tsai, S. W., "A Progressive Quadratic Failure Criterion for a Laminate," *Composites Science and Technology*, Vol. 58, 1998, pp. 1023–1032.

[19]Rotem, A., "The Rotem Failure Criterion Theory and Practice," *Composites Science and Technology*, Vol. 62, 2002, pp. 1663–1671.

[20]Zhang, X., "Impact Damage in Composite Aircraft Structures—Experimental Testing and Numerical Simulation," *Journal of Aerospace Engineering*, Vol. 212, No. 4, 1998, pp. 245–259.

[21]Wisnom, M. R., Hill, G. F. J., Jones, M., *Through-Thickness Failure Prediction of Composite Structural Elements*, 2001, 13th Inter. Conference of Composite Materials, Beijing, 1623.

[22]Wisnom, M. R., Hill, G. F. J., Jones, M., "Interlaminar Tensile Strength of Carbon Fiber-Epoxy Specimens Size, Lay-up and Manufacturing Effects," *Advanced Composites. Letters*, Vol. 10, No. 4, 2001.

[23]Eckold, G. G., "Failure Criteria for Use in Design Environment," *Composites Science and Technology*, Vol. 58, 1998, pp. 1095–1105.

[24]Hart-Smith, L. J., "Predictions of the Original and Truncated Maximum Strain-Strain Failure Models for Certain Fibrous Composite Laminates," *Composites Science and Technology*, Vol. 58, 1998, pp. 1151–1178.

[25]Puck, A., and Schurmann, H., "Failure Analysis of FRP Laminates by Means of Physically Based Phenomenological Models," *Composites Science and Technology*, Vol. 58, 1998, pp. 1045–1067.

[26]Ford, H., and Alexander, J., *Advanced Mechanics of Materials*, 2nd Ed, Ellis Horwood, Chichester, UK, 1977.

[27]Gosse, J. H., and Christensen, S., *Strain Invariant Failure Criteria for Polymers in Composite Materials*, Paper AIAA-2001-1184, Phantom Works, Seattle, 2001.

[28]Hart-Smith, L. J., Gosse, J. H., Christensen, S., *Characterizing the Strength of Composite Materials from the Perspective of Structural Design*, Boeing, Paper MDC00K0103, Seattle, WA, 2000.

[29]Sternstein, S., and Ongchin, L., *Yield Criteria for Plastic Deformation of Glassy High Polymers in General Stress Fields, Polymer Preprints*, Vol. 10, 1969, pp. 1117–1124.

[30]Li, R., Kelly, D., and Ness, R., "Application of a First Invariant Strain Criterion for Matrix Failure in Composite Materials," *Journal of Composite Materials*, Vol. 37, No. 22, 2003, pp. 1977–2000.

[31]Soden, P. D., Hinton, M. J., Kaddour, A. S., "A Comparison of the Predictive Capabilities of Current Failure Theories for Composite Laminates," *Composites Science and Technology*, Vol. 58, 1998, pp. 1225–1254.

[32]Soden, P. D., Hinton, M. J., Kaddour, A. S., "Lamina Properties, Lay-up Configurations and Loading Conditions for a Range Reinforced Composite Laminates," *Composites Science and Technology*, Vol. 58, 1998, pp. 1011–1022.

[33]Gotsis, P. K., Chamis, C. C., Minnetyan, L., "Prediction of Composite Laminate Fracture: Micromechanics and Progressive Fracture," *Composites Science and Technology*, Vol. 58, 1998, pp. 1137–1149.

[34]Rotem, A., "Prediction Laminate Failure with Rotem Failure Criterion," *Composites Science and Technology*, Vol. 58, 1998, pp, 1083–1094.

[35]Wolfe, W. E., and Butalia, T. S., "A Strain-Energy Based Failure Criterion for Nonlinear Analysis of Composite Laminates Subjected Biaxial Loading," *Composites Science and Technology*, Vol. 58, 1998, pp. 1107–1124.

[36]Yap, J., Scott, M., Thomson, R., Hachenberg, D., "The Analysis of Skin-to-Stiffener Debonding in Composite Aerospace Structures," *ICCS-11*, Monash University, 2001.

[37]Sih, G., Paris, P. C., Irwin, G. R., "On Cracks in Rectilinearly Anisotropic Bodies," *International Journal of Fracture Mechanics*, Vol. 1, 1965, p. 189.

[38]Kelly, A., *Strong Solids*, Oxford University Press, London, 1973.

[39]Harrison, N. L., "Splitting of Fibre-Reinforced Materials," *Fibre Science and Technology*, Vol. 6, 1973, p. 25.

[40]Nuismer, R. J., and Whitney, J. M., "Uniaxial Failure of Composite Laminates Containing Stress Concentrations," ASTM STP 593, 1975, pp. 117–142.

[41]Nuismer, R. J., and Labor, J. D., "Applications of the Average Stress Failure Criterion, Part 1: Tension," *Journal of Composite Materials*, Vol. 12, 1978, p. 238.

[42]Pimm, J. H., *Experimental Investigation of Composite Wing Failure*, AIAA Paper 78–509, 1978.

[43]Daniel, I. M., "Behaviour of Graphite Epoxy Plates with Holes under Biaxial Loading," *Experimental Mechanics*, Vol. 20, 1980, pp. 1–8.

[44]Husman, G. E., Whitney, J. M., Halpin, J. C., *Residual Strength Characterization of Laminated Composites Subjected to Impact Loading*, ASTM STP 568, 1975, pp. 92–113.

[45]Nuismer, R. J., and Labor, J. D., "Application of the Average Stress Failure Criterion: Part 2-Compression," *Journal of Composite Materials*, Vol. 13, 1979, pp. 49–60.

[46]Engineering Science Data Unit (ESDU) Data Item 80023 *Buckling of Rectangular Specially Orthotropic Plates*, ESDU International London.