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**STATICALLY DETERMINATE STRUCTURES**

(Loads, Reactions, Stresses, Shears, Bending Moments, Deflections)

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CHAPTER A1
THE WORK OF THE
AEROSPACE STRUCTURES ENGINEER

A1.1 Introduction.

The first controllable human flight in a heavier than air machine was made by Orville Wright on December 17, 1903, at Kitty Hawk, North Carolina. It covered a distance of 120 feet and the duration of flight was twenty seconds. Today, this initial flight appears very unimpressive, but it comes into its true perspective of importance when we realize that mankind for centuries has dreamed about doing or tried to do what the Wright Brothers accomplished in 1903.

The tremendous progress accomplished in the first 50 years of aviation history, with most of it occurring in the last 25 years, is almost unbelievable, but without doubt, the progress in the second 50 year period will still be more unbelievable and fantastic. As this is written in 1954, jet airline transportation at 600 MPH is well established and several types of military aircraft have speeds in the 1200 to 2000 MPH range. Preliminary designs of a supersonic airliner with Mach 3 speed have been completed and the government is on the verge of sponsoring the development of such a flight vehicle, thus supersonic air transportation should become common in the early 1970’s. The rapid progress in missile design has ushered in the Space Age. Already many space vehicles have been flown in search of new knowledge which is needed before successful exploration of space such as landings on several planets can take place. Unfortunately, the rapid development of the missile and rocket power has given mankind a flight vehicle when combined with the nuclear bomb, the awesome potential to quickly destroy vast regions of the earth.

While no person at present knows where or what space exploration will lead to, relative to benefits to mankind, we do know that the next great aviation expansion besides supersonic airline transportation will be the full development and use of vertical take-off and landing aircraft. Thus persons who will be living through the second half century of aviation progress will no doubt witness even more fantastic progress than occurred in the first 50 years of aviation history.

A1.2 General Organization of an Aircraft Company Engineering Division.

The modern commercial airliner, military airplane, missile and space vehicle is a highly scientific machine and the combined knowledge and experience of hundreds of engineers and scientists working in close cooperation is necessary to insure a successful product. Thus the engineering division of an aerospace company consists of many groups of specialists whose specialized training covers all fields of engineering education such as Physics, Chemical and Metallurgical, Mechanical, Electrical and, of course, Aeronautical Engineering.

It so happens that practically all the aerospace companies publish extensive pamphlets or brochures explaining the organization of the engineering division and the duties and responsibilities of the many sections and groups and illustrating the tremendous laboratory and test facilities which the aerospace industry possesses. It is highly recommended that the student read and study these free publications in order to obtain an early general understanding of how the modern flight vehicle is conceived, designed and then produced.

In general, the engineering department of an aerospace company can be broken down into six large rather distinct sections, which in turn are further divided into specialized groups, which in turn are further divided into smaller working groups of engineers. To illustrate, the six sections will be listed together with some of the various groups. This is not a complete list, but it should give an idea of the broad engineering set-up that is necessary.

I. Preliminary Design Section.

II. Technical Analysis Section.

(1) Aerodynamics Group
(2) Structures Group
(3) Weight and Balance Control Group
(4) Power Plant Analysis Group
(5) Materials and Processes Group
(6) Controls Analysis Group

III. Component Design Section.

(1) Structural Design Group
   (Wing, Body and Control Surfaces)
(2) Systems Design Group
   (All mechanical, hydraulic, electrical and thermal installations)

IV. Laboratory Tests Section.
(1) Wind Tunnel and Fluid Mechanics Test Labs.
(2) Structural Test Labs.
(3) Propulsion Test Labs.
(4) Electronics Test Labs.
(5) Electro-Mechanical Test Labs.
(6) Weapons and Controls Test Labs.
(7) Analog and Digital Computer Labs.

V. Flight Test Section.

VI. Engineering Field Service Section.

Since this textbook deals with the subject of structures, it seems appropriate to discuss in some detail the work of the Structures Group. For the detailed discussion of the other groups, the student should refer to the various aircraft company publications.

Al.3 The Work of the Structures Group

The structures group, relative to number of engineers, is one of the largest of the many groups of engineers that make up Section II, the technical analysis section. The structures group is primarily responsible for the structural integrity (safety) of the airplane. Safety may depend on sufficient strength or sufficient rigidity. This structural integrity must be accompanied with lightest possible weight, because any excess weight has detrimental effect upon the performance of aircraft. For example, in a large, long range missile, one pound of unnecessary structural weight may add more than 200 lbs. to the overall weight of the missile.

The structures group is usually divided into sub-groups as follows:

(1) Applied Loads Calculation Group
(2) Stress Analysis and Strength Group
(3) Dynamics Analysis Group
(4) Special Projects and Research Group

THE WORK OF THE APPLIED LOADS GROUP

Before any part of the structure can be finally proportioned relative to strength or rigidity, the true external loads on the aircraft must be determined. Since critical loads come from many sources, the Loads Group must analyze loads from aerodynamic forces, as well as those forces from power plants, aircraft inertia; control system actuators; launching, landing and recovery gear; armament, etc. The effects of the aerodynamic forces are initially calculated on the assumption that the airplane structure is a rigid body. After the aircraft structure is obtained, its true rigidity can be used to obtain dynamic effects. Results of wind tunnel model tests are usually necessary in the application of aerodynamic principles to load and pressure analysis.

The final results of the work of this group are formal reports giving complete applied load design criteria, with many graphs and summary tables. The final results may give complete shear, moment and normal forces referred to a convenient set of XYZ axes for major aircraft units such as the wing, fuselage, etc.

THE WORK OF STRESS ANALYSIS AND STRENGTH GROUP

Essentially the primary job of the stress group is to help specify or determine the kind of material to use and the thickness, size and cross-sectional shape of every structural member or unit on the airplane or missile, and also to assist in the design of all joints and connections for such members. Safety with light weight are the paramount structural design requirements. The stress group must constantly work closely with the Structural Design Section in order to evolve the best structural over-all arrangement. Such factors as power plants, built in fuel tanks, landing gear retracting wells, and other large cut-outs can dictate the type of wing structure, as for example, a two spar single cell wing, or a multiple spar multiple cell wing.

To expedite the initial structural design studies, the stress group must supply initial structural sizes based on approximate loads. The final results of the work by the stress group are recorded in elaborate reports which show how the stresses were calculated and how the required member sizes were obtained to carry these stresses efficiently. The final size of a member may be dictated by one or more factors such as elastic action, inelastic action, elevated temperatures, fatigue, etc. To insure the accuracy of theoretical calculations, the stress group must have the assistance of the structures test laboratory in order to obtain information on which to base allowable design stresses.

THE WORK OF THE DYNAMICS ANALYSIS GROUP

The Dynamics Analysis Group has rapidly expanded in recent years relative to number of engineers required because supersonic airplanes, missiles and vertical rising aircraft have presented many new and complex problems in the general field of dynamics. In some aircraft companies the dynamics group is set up as a separate group outside the Structures Group.

The engineers in the dynamics group are responsible for the investigation of vibration and shock, aircraft flutter and the establishment of design requirements or changes for its control or correction. Aircraft contain dozens of mechanical installations. Vibration of any part of these installations or systems may be of such character as to cause faulty operation or danger of failure and therefore the dynamic
characteristics must be changed or modified in order to insure reliable and safe operation.

The major structural units of aircraft such as the wing and fuselage are not rigid bodies. Thus when a sharp air gust strikes a flexible wing in high speed flight, we have a dynamic load situation and the wing will vibrate. The dynamics must determine whether this vibration is serious relative to induced stresses on the wing structure. The dynamics group is also responsible for the determination of the stability and performance of missile and flight vehicle guidance and control systems. The dynamics group must work constantly with the various test laboratories in order to obtain reliable values of certain factors that are necessary in many theoretical calculations.

THE WORK OF THE SPECIAL PROJECTS GROUP

In general, all the various technical groups have a special sub-group which are working on design problems that will be encountered in the near or distant future as aviation progresses. For example, in the Structures Group, this sub-group might be studying such problems as: (1) how to calculate the thermal stresses in the wing structure at super-sonic speeds; (2) how to stress analyze a new type of wing structure; (3) what type of body structure is best for future space travel and what kind of materials will be needed, etc.

Chart 1 illustrates in general a typical make-up of the Structures Section of a large aerospace company. Chart 2 lists the many items which the structures engineer must be concerned with in insure the structural integrity of the flight vehicle. Both Charts 1 and 2 are from Chance-Vought Structures Design Manual and are reproduced with their permission.

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Chart 1. Structures Section Organization
Chance-Vought Corp.
THE LINKS TO STRUCTURAL INTEGRITY

...... ARE NO BETTER THAN THE WEAKEST LINK

MATERIALS OF CONSTRUCTION
- Fasteners
- Welding
- Bonding
- Plate and Bar
-Forgings
- Castings
- Extrusions
- Sheet Metal
- Sandwich
- Plastic Laminate
- Bearings

STIFFNESS CRITERIA
- Flutter
- Control System Stability
- Panel Flutter-Skin Contours
- Control System Deflections
- Thermal Effects
- Mechanical Vibrations
- Roll Power-Divergence
- Aerodynamic Center Shift
- Dynamic Response

STRESS ANALYSIS
- Skin Panels
- Beam Analysis
- Strain Compatibility
- Strain Concentration
- Joint Analysis
- Bearing Analysis
- Bulkhead Analysis
- Fitting Analysis
- Thermal Stress
- Mechanical Components
- Experimental Stress Analysis

LOADS AND ENVIRONMENT
- Flight Load Criteria
- Ground Load Criteria
- Flight Load Dynamics
- Launching Dynamics
- Landing Dynamics
- Dynamic Response
- Recovery Dynamics
- Flight Load Distributions
- Inertial Load Distributions
- Flexibility Effects
- Ground Load Distributions
- Repeated Load Spectrums
- Temperature Distributions
- Loads from Thermal
- Deformations
- Pressures-Impact

COMPONENT ANALYSIS
- Unit Solutions
- Ineiterminate Structures
- Wing Analysis
- Tail Analysis
- Fuselage Shell Analysis
- Thermal Analysis
- Deflection Analysis
- Stiffness

ALLOWABLES
- Yielding
- Fracture
- Fatigue
- Wear, Bending
- Creep
- Deflections
- Thermal Effects
- Stiffness
- Combined Loadings
- Buckling

Chart 2
From Chance-Vought Structures Design Manual
A2.1 Introduction. The equations of static equilibrium must constantly be used by the stress analyst and structural designer in obtaining unknown forces and reactions or unknown internal stresses. They are necessary whether the structure or machine be simple or complex. The ability to apply these equations is no doubt best developed by solving many problems. This chapter initiates the application of these important physical laws to the force and stress analysis of structures. It is assumed that a student has completed the usual college course in engineering mechanics called statics.

A2.2 Equations of Static Equilibrium.

To completely define a force, we must know its magnitude, direction and point of application. These facts regarding the force are generally referred to as the characteristics of the force. Sometimes the more general term of line of action or location is used as a force characteristic in place of point of application designation.

A force acting in space is completely defined if we know its components in three directions and its moments about 3 axes, as for example \( F_x, F_y, F_z \) and \( M_x, M_y, M_z \). For equilibrium of a force system, there can be no resultant force and thus the equations of equilibrium are obtained by equating the force and moment components to zero. The equations of static equilibrium for the various types of force systems will now be summarized.

**EQUILIBRIUM EQUATIONS FOR GENERAL SPACE (NON-COPLANAR) FORCE SYSTEM**

\[
\begin{align*}
Z_F &= 0 \\
Z_F &= 0 \\
Z_F &= 0 \\
\end{align*}
\]

**EQUILIBRIUM OF SPACE CONCURRENT FORCE SYSTEM**

Concurrent means that all forces of the force system pass through a common point. The resultant, if any, must therefore be a force and not a moment and thus only 3 equations are necessary to completely define the condition that the resultant must be zero. The equations of equilibrium available are therefore:

\[
\begin{align*}
Z_F &= 0 \\
Z_F &= 0 \quad \text{or} \quad Z_F &= 0 \\
Z_F &= 0 \\
\end{align*}
\]

A combination of force and moment equations to make a total of not more than 3 can be used. For the moment equations, axes through the point of concurrency cannot be used since all forces of the system pass through this point. The moment axes need not be the same as the directions used in the force equations but of course, they could be.

**EQUILIBRIUM OF SPACE PARALLEL FORCE SYSTEM**

In a parallel force system the direction of all forces is known, but the magnitude and location of each is unknown. Thus to determine magnitude, one equation is required and for location two equations are necessary since the force is not confined to one plane. In general the 3 equations commonly used to make the resultant zero for this type of force system are one force equation and two moment equations. For example, for a space parallel force system acting in the \( y \) direction, the equations of equilibrium would be:

\[
Z_F = 0, \quad Z_M = 0, \quad Z_M = 0
\]

**EQUILIBRIUM OF GENERAL COPLANAR FORCE SYSTEM**

In this type of force system all forces lie in one plane and it takes only 3 equations to determine the magnitude, direction and location of the resultant of such a force system. Either force or moment equations can be used, except that a maximum of 2 force equations can be used. For example, for a force system acting in the \( xy \) plane, the following combination of equilibrium equations could be used:

\[
\begin{align*}
Z_F &= 0 \\
Z_F &= 0 \\
Z_F &= 0 \\
\end{align*}
\]

(The subscripts 1, 2 and 3 refer to different locations for \( z \) axes or moment centers.)
A2.2 EQUILIBRIUM OF FORCE SYSTEMS. TRUSS STRUCTURES.

EQUILIBRIUM OF COPLANAR-CONCURRENT FORCE SYSTEM

Since all forces lie in the same plane and also pass through a common point, the magnitude and direction of the resultant of this type of force system is unknown but the location is known since the point of concurrency is on the line of action of the resultant. Thus only two equations of equilibrium are necessary to define the resultant and make it zero. The combinations available are,

\[
\begin{align*}
\Sigma F_x &= 0 \quad \Sigma F_y = 0 \quad \Sigma M_z &= 0 \\
\Sigma F_y &= 0 \quad \Sigma M_z = 0 \quad \Sigma M_z &= 0
\end{align*}
\]

(2.5)

(The z axis or moment center locations must be other than through the point of concurrency)

EQUILIBRIUM OF COPLANAR PARALLEL FORCE SYSTEM

Since the direction of all forces in this type of force system is known and since the forces all lie in the same plane, it only takes 2 equations to define the magnitude and location of the resultant of such a force system. Hence, there are only 2 equations of equilibrium available for this type of force system, namely, a force and moment equation or two moment equations. For example, for forces parallel to y axis and located in the xy plane the equilibrium equations available would be:

\[
\begin{align*}
\Sigma F_y &= 0 \quad \Sigma M_z &= 0 \\
\Sigma M_z &= 0 \quad \Sigma M_z &= 0
\end{align*}
\]

(2.6)

(The moment centers 1 and 2 cannot be on the same y axis)

EQUILIBRIUM OF COLINEAR FORCE SYSTEM

A colinear force system is one where all forces act along the same line or in other words, the direction and location of the forces is known but their magnitudes are unknown, thus only magnitude needs to be found to define the resultant of a colinear force system. Thus only one equation of equilibrium is available, namely

\[
\Sigma F = 0 \quad \Sigma M_1 = 0
\]

(2.7)

where moment center 1 is not on the line of action of the force system

A2.3 Structural Fitting Units for Establishing the Force Characteristics of Direction and Point of Application.

To completely define a force in space requires 5 equations and 3 equations if the force is limited to one plane. In general a structure is loaded by known forces and these forces are transferred through the structure in some manner of internal stress distribution and then reacted by other external forces, commonly referred to as reactions which hold the known forces in the structure in equilibrium. Since the static equations of equilibrium available for the various types of force systems are limited, the structural engineer resorts to the use of fitting units which establish the direction of an unknown force or its point of application or both, thus decreasing the number of unknowns to be determined. The figures which follow illustrate the type of fitting units employed or other general methods for establishing the force characteristics of direction and point of application.

Ball and Socket Fitting

For any space or coplanar force such as P and Q acting on the bar, the line of action of such forces must act through the center of the ball if rotation of the bar is prevented. Thus a ball and socket joint can be used to establish or control the direction and line of action of a force applied to a structure through this type of fitting. Since the joint has no rotational resistance, no couples in any plane can be applied to it.

Single Pin Fitting

For any force such as P and Q acting in the xy plane, the line of action of such a force must pass through the pin center since the fitting unit cannot resist a moment about a z axis through the pin center. Therefore, for forces acting in the xy plane, the direction and line of action are established by the pin joint as illustrated in the figure. Since a single pin fitting can resist moments about axes perpendicular to the pin axis, the direction and line of action of out of plane forces is therefore not established by single pin fitting units.

If a bar AB has single pin fittings at each end, then any force P lying in the xy plane and applied to end B must have a direction and line of action coinciding with a line joining the pin centers at and fittings A and B. Since the fittings cannot resist a moment about the z axis.
Double Pin – Universal Joint Fittings

Since single pin fitting units can resist applied moments about axes normal to the pin axis, a double pin joint as illustrated above is often used. This fitting unit cannot resist moments about y or z axes and thus applied forces such as P and Q must have a line of action and direction such as to pass through the center of the fitting unit as illustrated in the figure. The fitting unit can, however, resist a moment about the x axis or in other words, a universal type of fitting unit can resist a torsional moment.

Rollers

In order to permit structures to move at support points, a fitting unit involving the idea of rollers is often used. For example, the truss in the figure above is supported by a pin fitting at (A) which is further attached to a fitting portion that prevents any horizontal movement of truss at end (A), however, the other end (B) is supported by a nest of rollers which provide no horizontal resistance to a horizontal movement of the truss at end (B). The rollers fix the direction of the reaction at (B) as perpendicular to the roller bed. Since the fitting unit is joined to the truss joint by a pin, the point of application of the reaction is also known, hence only one force characteristic, namely magnitude, is unknown for a roller-pin type of fitting. For the fitting unit at (A), point of application of the reaction to the truss is known because of the pin, but direction and magnitude are unknown.

Lubricated Slot or Double Roller Type of Fitting Unit.

Another general fitting type that is used to establish the direction of a force or reaction is illustrated in the figure at the bottom of the first column. Any reacting force at joint (A) must be horizontal since the support at (A) is so designed to provide no vertical resistance.

Cables – Tie Rods

Since a cable or tie rod has negligible bending resistance, the reaction at joint B on the crane structure from the cable must be collinear with the cable axis, hence the cable establishes the force characteristics of direction and point of application of the reaction on the truss at point B.

A2.4 Symbols for Reacting Fitting Units as Used in Problem Solution.

In solving a structure for reactions, member stresses, etc., one must know what force characteristics are unknown and it is common practice to use simple symbols to indicate what fitting support or attachment units are to be used or are assumed to be used in the final design. The following sketch symbols are commonly used for coplanar force systems.

A small circle at the end of a member or on a triangle represents a single pin connection and fixes the point of application of forces acting between this unit and a connecting member or structure.

The above graphical symbols represent a reaction in which translation of the attachment point (b) is prevented but rotation of the attached structure about (b) can take place. Thus the reaction is unknown in direction and magnitude but the point of application is known, namely through point (b). Instead of using direction as an unknown, it is more convenient to replace the resultant reaction by two components at right angles to each other as indicated in the sketches.
The above fitting units using rollers fix the direction of the reaction as normal to the roller bed since the fitting unit cannot resist a horizontal force through point (b). Hence the direction and point of application of the reaction are established and only magnitude is unknown.

\[
\begin{align*}
\text{fixed} & \quad \rightarrow R_x \quad \rightarrow R_y \quad \rightarrow M_x \\
R & \quad \rightarrow \quad \text{(b) } \quad M_y
\end{align*}
\]

The graphical symbol above is used to represent a rigid support which is attached rigidly to a connecting structure. The reaction is completely unknown since all 3 force characteristics are unknown, namely, magnitude, direction and point of application. It is convenient to replace the reaction \( R \) by two force components referred to some point (b) plus the unknown moment \( M \) which the resultant reaction \( R \) caused about point (b) as indicated in the above sketch. This discussion applies to a coplanar structure with all forces in the same plane. For a space structure the reaction would have 3 further unknowns, namely, \( R_z \), \( M_x \) and \( M_y \).

A2.5 Statically Determinate and Statically Indeterminate Structures.

A statically determinate structure is one in which all external reactions and internal stresses for a given load system can be found by use of the equations of static equilibrium and a statically indeterminate structure is one in which all reactions and internal stresses cannot be found by using only the equations of equilibrium.

A statically determinate structure is one that has just enough external reactions, or just enough internal members to make the structure stable under a load system and if one reaction or member is removed, the structure is reduced to a linkage or a mechanism and is therefore not further capable of resisting the load system. If the structure has more external reactions or internal members than is necessary for stability of the structure under a given load system it is statically indeter-minate, and the degree of redundancy depends on the number of unknowns beyond that number which can be found by the equations of static equilibrium. A structure can be statically indeterminate with respect to external reactions alone or to internal stresses alone or to both.

The additional equations that are needed to solve a statically indeterminate structure are obtained by considering the distortion of the structure. This means that the size of all members, the material from which members are made must be known since distortions must be calculated. In a statically determinate structure this information on sizes and material is not required but only the configuration of the structure as a whole. Thus design analysis for statically determinate structure is straightforward whereas a general trial and error procedure is required for design analysis of statically indeterminate structures.

A2.6 Examples of Statically Determinate and Statically Indeterminate Structures.

The first step in analyzing a structure is to determine whether the structure as presented is statically determinate. If so, the reactions and internal stresses can be found without knowing sizes of members or kind of material. If not statically determinate, the elastic theory must be applied to obtain additional equations. The elastic theory is treated in considerable detail in Chapters A7 to A12 inclusive.

To help the student become familiar with the problem of determining whether a structure is statically determinate, several example problems will be presented.

Example Problem 1.

In the structure shown in Fig. A2.1, the known forces or loads are the distributed loads of 10 lb. per inch on member AB. The reactions at points A and C are unknown. The reaction at C has only one unknown characteristic, namely, magnitude because the point of application of \( R_C \) is through the pin center at C and the direction of \( R_C \) must be parallel to line CB because there is a pin at the other end of CB of member CB. At point A the reaction is unknown in direction and magnitude but the point of application must be through the pin center at A. Thus there are 2 unknowns at A and one unknown at C or a total
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

of 3. With 3 equations of equilibrium available for a coplanar force system, the structure is statically determinate. Instead of using an angle as an unknown at A to find the direction of the reaction, it is usually more convenient to replace the reaction by components at right angles to each other as $H_A$ and $V_A$ in the figure and thus the 3 unknowns for the structure are 3 magnitudes.

Example Problem 2.

Fig. 2.2 shows a structural frame carrying a known load system $P$. Due to the pins at reaction points A and B, the point of application is known for each reaction, however, the magnitude and direction of each is unknown making a total of 4 unknowns with only 3 equations of equilibrium available for a coplanar force system. At first we might conclude that the structure is statically indeterminate but we must realize this structure has an internal pin at C which means the bending moment at this point is zero since the pin has no resistance to rotation. If the entire structure is in equilibrium, then each part must likewise be in equilibrium and we can cut out any portion as a free body and apply the equilibrium equations. Fig. 2.3 shows a free body of the frame to left of pin at C. Taking moments about C and equating to zero gives us a fourth equation to use in determining the 4 unknowns, $H_A$, $V_A$, $V_B$ and $H_C$. The moment equation about C does not include the unknowns $V_C$ and $H_C$ since they have no moment about C because of zero arms. As in example problem 1, the reactions at A and B have been replaced by H and V components instead of using an angle (direction) as an unknown characteristic. The structure is statically determinate.

Example Problem 3.

Fig. 2.4 shows a straight member 1-2 carrying a known load system $P$ and supported by 5 struts attached to reaction points ABCD.

At reaction points A, B and D, the reaction is known in direction and point of application but the magnitude is unknown as indicated by the vector at each support. At point C, the reaction is unknown in direction because 2 struts enter joint C. Magnitude is also unknown but point of application is known since the reaction must pass through C. Thus we have 5 unknowns, namely, $R_C$, $R_B$, $R_P$, $V_C$ and $H_C$. For a coplanar force system we have 3 equations of equilibrium available and thus the first conclusion might be that we have a statically indeterminate structure to (5-3) = 2 degrees redundant. However, observation of the structure shows two internal pins at points E and F which means that the bending moment at these two points is zero, thus giving us 2 more equations to use with the 3 equations of equilibrium. Thus drawing free bodies of the structure to left of pin E and to right of pin F and equating moments about each pin to zero we obtain 2 equations which do not include unknowns other than the 5 unknowns listed above. The structure is therefore statically determinate.

Example Problem 4.

Fig. 2.5 shows a beam AB which carries a super-structure CED which in turn is subjected to the known loads P and Q. The question is whether the structure is statically determinate. The external unknown reactions for the entire structure are at points A and B. At A due to the roller type of action, magnitude is the only unknown characteristic of the reaction since direction and point of application are known. At B, magnitude and direction are unknown but point of application is known, hence we have 3 unknowns, namely, $R_B$, $V_B$ and $H_B$, and with 3 equations of equilibrium available we can find these reactions and therefore the structure is statically determinate with respect to external reactions. We now investigate to see if the internal stresses can be found by statics after having found the external reactions. Obviously, the internal stresses will be affected by the internal reactions at C and D, so we draw a free body of the super-structure as illustrated in Fig. 2.6 and consider the internal forces that existed at C and D as external reactions. In the actual structure the members are rigidly attached together at point C such as a welded or
multiple bolt connection. This means that all three force or reaction characteristics, namely, magnitude, direction and point of application are unknown, or in other words, 3 unknowns exist at C. For convenience we will represent these unknowns by three components as shown in Fig. 2.6, namely, \( H_c \), \( V_c \), and \( M_c \). At joint D in Fig. 2.5, the only unknown regarding the reaction is \( R_p \), a magnitude, since the pin at each end of the member DE establishes the direction and point of application of the reaction \( R_p \). Hence we have 4 unknowns and only 3 equations of equilibrium for the structure in Fig. 2.6, thus the structure is statically indeterminate with respect to all of the internal stresses. The student should observe that internal stresses between points AC, BD and FE are statically indeterminate, and thus the statically indeterminate portion is the structural triangle CED.

Example Problem 5

\[
\begin{align*}
\text{Fig. A2.7} & \quad \text{Fig. A2.8} & \quad \text{Fig. A2.9} \\
H_A & = A & H_B & = B & H_A & = A \\
V_A & = v_A & V_B & = v_B & V_A & = v_A \\
M_A & = M_A & M_B & = M_B & M_A & = M_B
\end{align*}
\]

Figs. 2.7, 2.8 and 2.9 show the same structure carrying the same known load system \( P \) but with different support conditions at points A and B. The question is whether each structure is statically indeterminate and if so, to what degree, that is, what number of unknowns beyond the equations of statics available. Since we have a coplanar force system, only 3 equations at statics are available for equilibrium of the structure as a whole.

In the structure in Fig. 2.7, the reaction at A and also at B is unknown in magnitude and direction but point of application is known, hence 4 unknowns and with only 3 equations of statics available, makes the structure statically indeterminate to the first degree. In Fig. 2.8, the reaction at A is a rigid one, thus all 3 characteristics of magnitude, direction and point of application of the reaction are unknown. At point B, due to pin only 2 unknowns, namely, magnitude and direction, thus making a total of 5 unknowns with only 3 equations of statics available or the structure is statically indeterminate to the second degree. In the structure of Fig. 2.9, both supports at A and B are rigid thus all 3 force characteristics are unknown at each support or a total of 6 unknowns which makes the structure statically indeterminate to the third degree.

Example Problem 6

\[
\begin{align*}
\text{Fig. A2.10} & \quad \text{Fig. A2.11} & \quad \text{Fig. A2.12} \\
& \quad \text{Fig. A2.13} & \quad \text{Fig. A2.14}
\end{align*}
\]

Fig. 2.10 shows a 2 bay truss supported at points A and B and carrying a known load system \( P, Q \). All members of the truss are connected at their ends by a common pin at each joint. The reactions at A and B are applied through fittings as indicated. The question is whether the structure is statically determinate. Relative to external reactions at A and B the structure is statically determinate because the type of support produces only one unknown at A and two unknowns at B, namely, \( V_A, V_B \) and \( H_A \) as shown in Fig. 2.10 and we have 3 equations of static equilibrium available.

We now investigate to see if we can find the internal member stresses after having found the values of the reactions at A and B. Suppose we cut out joint B as indicated by section 1-1 in Fig. 2.10 and draw a free body as shown in Fig. 2.11. Since the members of the truss have pins at each end, the loads in these members must be axial, thus direction and line of action is known and only magnitude is unknown. In Fig. 2.11 \( H_B \) and \( V_A \) are known but \( AB, \), \( CB, \), and \( DB \) are unknown in magnitude hence we have 3 unknowns but only 2 equations of equilibrium for a coplanar concurrent force system. If we cut through the truss in Fig. 2.10 by the section 2-2 and draw a free body of the lower portion as shown in Fig. 2.12, we have 4 unknowns, namely, the axial loads in \( CA, DA, CB, DB \) but only 3 equations of equilibrium available for a coplanar force system.

Suppose we were able to find the stresses in \( CA, DA, CB, DB \) in some manner, and we would now proceed to joint D and treat it as a free body or cut through the upper panel along section 4-4 and use the lower portion as a free body. The same reasoning as used above would show us we have one more unknown than the number of equilibrium equations available and thus we have the truss statically indeterminate to the second degree relative to internal member stresses.

Physically, the structure has two more members than is necessary for the stability of the structure under load, as we could leave out one diagonal member in each truss panel and
the structure would be still stable and all member axial stresses could be found by the equations of statical equilibrium without regard to their size of cross-section or the kind of material. Adding the second diagonal member in each panel would necessitate knowing the size of all truss members and the kind of material used before member stresses could be found, as the additional equations needed must come from a consideration involving distortion of the truss. Assume for example, that one diagonal in the upper panel was left out. We would then be able to find the stresses in the members of the upper panel by statics but the lower panel would still be statically indeterminate to 1 degree because of the double diagonal system and thus one additional equation is necessary and would involve a consideration of truss distortion. (The solution of statically indeterminate trusses is covered in Chapter A8.)

A.7 Example Problem Solutions of Statically Determinate Coplanar Structures and Coplanar Loadings.

Although a student has taken a course in statics before taking a beginning course in aircraft structures, it is felt that a limited review of problems involving the application of the equations of static equilibrium is quite justified; particularly if the problems are possibly somewhat more difficult than most of the problems in the usual beginning course in statics. Since one must use the equations of static equilibrium as part of the necessary equations in solving statically indeterminate structures and since statically indeterminate structures are beyond the scope of this book, only limited space will be given to problems involving statics in this chapter.

Example Problem 8.

Fig. A2.15 shows a free body of the wing spar to the right of hinge fitting at 0. In order to take moments, the distributed load on the spar has been replaced by the resultant load on each spar portion, namely, the total load on the portion acting through the centroid of the distributed load system. The strut reaction EA at A has been shown in phantom as it is more convenient to deal with its components YA and XA. The reaction at O is unknown in magnitude and direction and for convenience we will deal with its components X0 and Y0. The sense assumed is indicated on the figure.

The sense of a force is represented graphically by an arrow head on the end of a vector. The correct sense is obtained from the solution of the equations of equilibrium since, a force or moment must be given a plus or minus sign in writing the equations. Since the sense of a force or moment is unknown, it is assumed, and if the algebraic solution of the equilibrium equations gives a plus value to the magnitude then the true sense is as assumed, and opposite to that assumed if the solution gives a minus sign. If the unknown forces or axial loads in members is common practice to call tensile stress plus and compressive stress minus, thus if we assume the sense of an unknown axial load as tension, the solution of the equilibrium
equations will give a plus value for the magnitude of the unknown if the true stress is tension and a minus sign will indicate the assumed tension stresses should be reversed or compression, thus giving a consistency of signs.

To find the unknown $Y_A$ we take moments about point $O$ and equate to zero for equilibrium.

$$\Sigma M_O = -2460 \times 41 - 1013 \times 102 + 82Y_A = 0$$

Hence $Y_A = 204000/82 = 2480$ lb. The plus sign means that the sense as assumed in the figure is correct. By geometry $X_A = 2480 \times 117/56 = 4400$ lb. and the load in strut $EA$ equals $\sqrt{4400^2 + 2480^2} = 5060$ lb. tension or as assumed in the figure.

To find $X_O$ we use the equilibrium equation

$$\Sigma F_X = 0 = X_O - 4400 = 0,$$  whence $X_O = 4400$ lb.

To find $Y_O$ we use,

$$\Sigma F_Y = 0 = 2460 + 1013 - 2480 - Y_O = 0, $$  whence $Y_O = 993$ lb.

To check our results for equilibrium we will take moments of all forces about $A$ to see if they equal zero.

$$\Sigma F_A = 2460 \times 41 - 1013 \times 20 - 993 \times 82 = 0$$

On the spar portion $O'A'$, the reactions are obviously equal to 40/30 times those found for portion OA since the external loading is 40 as compared to 30.

Hence $A'E' = 6750, X_O' = 5880, Y_O' = 1325$

Fig. 2.16 shows a free body of the center spar portion with the reactions at $O$ and $O'$ as found previously. The unknown loads in the struts have been assumed tension as shown by the arrows.

To find the load in strut BC take moments about $B'$

$$\Sigma M_B' = 1325 \times 20 - 2000 \times 5 - 1500 \times 55 - 993 \times 80 + 60 (BC) 30/33.6 = 0$$

Whence, $BC = 2720$ lb. with sense as assumed.

To find strut load $B'C'$ take moments about point $C$.

$$\Sigma M_C = 1325 \times 65 + 2000 \times 40 + (5880 - 4400)$$

$$30 - 1500 \times 10 - 993 \times 55 - 30 \times 30/33.6 = 0$$

whence, $B'C' = 6000$ lb. with sense as shown.

To find load in member $B'C$ use equation

$$\Sigma F_Y = 0 = 1325 + 2000 + 1500 + 993 - 6000$$

$$(30/33.6) - 2720 (30/33.6) - B'C (30/54) = 0$$

whence, $B'C = -3536$ lb. The minus sign means it acts opposite to that shown in figure or is compression instead of tension.

The reactions on the spar can now be determined and shears, bending moments and axial loads on the spar could be found. The numerical results should be checked for equilibrium of the spar as a whole by taking moments of all forces about a different moment center to see if the result is zero.

---

**Example Problem 9.**

![Diagram](image)

Fig. 2.17 shows a simplified airplane landing gear unit with all members and loads confined to one plane. The brace struts are pinned at each end and the support at $C$ is of the roller type, thus no vertical reaction can be produced by the support fitting at point $C$. The member at $C$ can rotate on the roller but horizontal movement is prevented. A known load of 10,000 lb. is applied to axle unit at $A$. The problem is to find the load in the brace struts and the reaction at $C$.

**Solution:**

Due to the single pin fitting at each end of the brace struts, the reactions at $B$ and $D$
are collinear with the strut axis, thus direction and point of application are known for reaction in and out of Rg and Rg leaving only the magnitude of each as unknown. The roller type fitting at C fixes the direction and point of application of the reaction Rg, leaving magnitude as the only unknown. Thus there are 3 unknowns Rg, Rg and Rg and with 3 equations of static equilibrium available, the structure is statically determinate with respect to external reactions. The sense of each of the 3 unknown reactions has been assumed as indicated by the vector.

To find Rg take moments about point B:

\[
\Sigma M_B = -10000 \sin 30^\circ \times 36 - 10000 \cos 30^\circ \times 12 - Rg (12/17) \times 24 = 0
\]

whence, \( Rg = -16750 \) lb. Since the result comes out with a minus sign, the reaction Rg has a sense opposite to that shown by the vector in Fig. 2.17. Since the reaction Rg is collinear with the line DE because of the pin ends, the load in the brace strut DE is 16750 lb. compression. In the above moment equation about B, the reaction Rg was resolved into vertical and horizontal components at point D, and thus only the vertical component which equals (12/17) \( Rg \) enters into the equation since the horizontal component has a line of action through point B and therefore no moment. Rg does not enter in equation as it has zero moment about B.

To find Rg take \( \Sigma F_y = 0 \)

\[
\Sigma F_y = 10000 \times \cos 30^\circ + (16750)(12/17) + Rg (24/26.3) = 0
\]

whence, \( Rg = 3540 \) lb. Since sign comes out plus, the sense is the same as assumed in the figure. The strut load BF is therefore 3540 lb. tension, since reaction Rg is collinear with line BF.

To find Rg take \( \Sigma H = 0 \)

\[
\Sigma H = 10000 \sin 30^\circ - 3540 (12/26.8) + (16750) (12/17) + Rg = 0
\]

whence, \( Rg = 8407 \) lb. Result is plus and therefore assumed sense was correct.

To check the numerical results take moments about point A for equilibrium:

\[
\Sigma M_A = 8407 \times 36 + 3540 \times (24/26.3) 12 - 3540 (12/26.8) 36 + 16750 (12/17) (12 - 16750 (12/17) 36 = 303000 + 38100 - 57100 + 142000 - 425000 = 0 \quad \text{(check)}
\]

A2.8 Stresses in Coplanar Truss Structures Under Coplanar Loading.

In aircraft construction, the truss type of construction is quite common. The most common is the tubular steel welded trusses that make up the fuselage frame, and less frequently, the aluminum alloy tubular truss. Trussed type beams composed of closed and open type sections are also frequently used in wing beam construction. The stresses or loads in the members of a truss are commonly referred to as "primary" and "secondary" stresses. The stresses which are found under the following assumptions are referred to as primary stresses.

1. The members of the truss are straight, weightless and lie in one plane.

2. The members of a truss meeting at a point are considered as joined together by a common frictionless pin and all member axes intersect at the pin center.

3. All external loads are applied to the truss only at the joints and in the plane of the truss. Thus all loads or stresses produced in members are either axial tension or compression without bending or torsion.

Those trusses produced in the truss members due to the non-fulfillment of the above assumptions are referred to as secondary stresses. Most steel tubular trusses are welded together at their ends and in other truss types, the members are riveted or bolted together. This restraint at the joints may cause secondary stresses in some members greater than the primary stresses. Likewise it is common in actual practical design to apply forces to the truss members between their ends by supporting many equipment installations on these truss members. However, regardless of the magnitude of these so-called secondary loads, it is common practice to first find the primary stresses under the assumption outlined above.

GENERAL CRITERIA FOR DETERMINING WHETHER TRUSS STRUCTURES ARE STATICALLY DETERMINATE WITH RESPECT TO INTERNAL STRESSES.

The simplest truss that can be constructed is the triangle which has three members m and three joints j. A more elaborate truss consists of additional triangular frames, so arranged that each triangle adds one joint and two members. Hence the number of members to insure stability under any loading is:

\[ m = 2j - 3 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

Thus for j number of joints there are 2j
EQUILIBRIUM OF FORCE SYSTEMS. TRUSS STRUCTURES.

The static equations for the forces acting on joint $L_1$ are $EH$ and $ZV = 0$.

\[ SV = 500 - L_1U_s (40/50) = 0 \]  \hspace{1cm} \text{(a)}

whence, $L_1U_s = -1250$ lb. Since the sign came out minus the stress is opposite to that assumed in Fig. A2.19 or compression.

\[ EH = 500 - (-1250)(30/60) - L_1L_s = 0 \]  \hspace{1cm} \text{(b)}

whence, $L_1L_s = 250$ lb. Since sign comes out plus, sense is same as assumed in figure.

For equilibrium of joint $L_1$, $EH$ and $ZV = 0$.

\[ ZV = -500 - L_1U_s = 0, \text{ whence, } L_1U_s = 500 \text{ lb.} \]

Since the sign came out plus, the assumed sense in Fig. A2.20 was correct or compression.

\[ EH = 250 - L_1L_s = 0, \text{ whence } L_1L_s = 250 \text{ lb.} \]

Next consider joint $U_s$ as a free body cut out by section 3-3 in Fig. A2.18 and drawn as Fig. A2.21. The known member stresses are shown with their true sense as previously found. The two unknown member stresses $U_1L_1$ and $U_2U_1$ have been assumed as tension.

\[ SV = -500 - 1250 (40/50) + U_1L_1 (40/50) = 0 \]

whence, $U_1L_1 = 1875$ lb. (tension as assumed.)
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[ ZH = (-1280)(30/50) - 1875 (30/50) - U_1 U_a = 0 \]

whence, \( U_1 U_a = -1875 \) lb. or opposite in sense to that assumed and therefore compression.

Note: The student should continue with succeeding joints. In this example involving a cantilever truss it was not necessary to find the reactions, as it was possible to select joint \( L_2 \) as a joint involving only two unknowns. In trusses such as illustrated in Fig. A2.22 it is necessary to first find reactions \( R_4 \) or \( R_3 \) which then provides a joint at the reaction point involving only two unknown forces.

\[ \text{Fig. A2.22} \]

A2.10 Method of Moments.

For a coplanar-non-concurrent force system there are three equations of statics available. These three equations may be taken as moment equations about three different points. Fig. A2.22 shows a typical truss. Let it be required to find the loads in the members \( F_1, F_2, F_3, F_4, F_5 \) and \( F_6 \).

\[ \text{Fig. A2.22} \]

The first step in the solution is to find the reactions at points \( A \) and \( B \). Due to the roller type of support at \( B \) the only unknown element of the reaction force at \( B \) is magnitude. At point \( A \), magnitude and direction of the reaction are unknown giving a total of three unknowns with three equations of statics available. For convenience the unknown reaction at \( A \) has been replaced by its unknown \( H \) and \( V \) components.

Taking moments about point \( A \),

\[ ZV_A = 500 \times 30 + 1000 \times 60 + 1000 \times 90 + 500 \times 30 + 500 \times 120 - 150 \cdot V_B = 0 \]

Hence \( V_B = 1600 \) lb.

Take \( ZV = 0 \)

Take \( ZV = V_B = 1000-1000-600-500+1600 = 0 \) therefore \( V_B = 1400 \) lb.

Take \( ZH = 0 \)

\[ ZH = 500 - H_A = 0, \text{ therefore } H_A = 500 \text{ lb.} \]

The algebraic sign of all unknowns came out positive, thus the assumed direction as shown on Fig. A2.22 was correct.

Check results by taking \( ZM = 0 \)

\[ ZM = 1400 \times 150 + 500 \times 30 - 500 \times 120 - 500 \times 30 - 1000 \times 90 - 1000 \times 60 = 0 \] (Check)

To determine the stress in member \( F_1, F_2 \) and \( F_3 \) we cut the section 1-1 thru the truss (Fig. A2.22). Fig. A2.23 shows a free body diagram of the portion of the truss to the left of this section.

\[ \text{Fig. A2.23} \]

The truss as a whole was in equilibrium therefore any portion must be in equilibrium. In Fig. A2.23 the internal stresses in the members \( F_1, F_2, \) and \( F_3 \) which existed in the truss as a whole now are considered external forces in holding the portion of the truss to the left of section 1-1 in equilibrium in combination with the other loads and reactions. Since the members \( a \) and \( b \) in Fig. A2.23 have not been cut the loads in these members remain as internal stresses and have no influence on the equilibrium of the portion of the truss shown. Thus the portion of the truss to left of section 1-1 could be considered as a solid block as shown in Fig. A2.24 without affecting the values of \( F_1, F_2 \) and \( F_3 \). The method of moments as the name implies involves the operation of taking moments about a point to find the load in a particular member. Since there are three unknowns a moment center must be selected such that the moment of each of the two unknown stresses will have zero moment about the selected moment center, thus leaving only one unknown force or stress to enter into the equation for moments. For example to determine load \( F_1 \) in Fig. A2.24 we take moments about the intersection of forces \( F_2 \) and \( F_3 \) or point 0.

Thus \( ZM_0 = 1400 \times 30 - 18.97 F_3 = 0 \)

Hence \( F_3 = 42200 \) (18.97 lb. compression (or acting as assumed))

To find the arm of the force \( F_3 \) from the moment center 0 involves a small amount of calculation, thus in general it is simpler to resolve the unknown force into \( H \) and \( V \) components at a point on its line of action such that one of these components passes thru the moment center and the arm of the other component can usually be determined by inspection. Thus in
A2.12  EQUILIBRIUM OF FORCE SYSTEMS. TRUSS STRUCTURES.

Fig. A2.25 the force \( F_1 \) is resolved into its component \( F_{1v} \) and \( F_{1h} \) at point \( O' \). Then taking

\[
\sum M_0 = 1400 \times 30 - 20F_{1h} = 0
\]

whence, \( F_{1h} = 2100 \text{ lb.} \) and therefore

\( F_1 = 2100 (21.6/30) = 2215 \text{ lb.} \) as previously obtained.

The load \( F_1 \) can be found by taking moments about point \( m \), the intersection of forces \( F_a \) and \( F_1 \) (See Fig. A2.23).

\[
\sum M_m = 1400 \times 60 - 500 \times 30 - 500 \times 30 - 30F_1 = 0
\]

whence, \( F_1 = 2800 \text{ lb.} \) (Tension as assumed)

To find force \( F_a \) by using a moment equation, we take moments about point \( (r) \) the intersection of forces \( F_1 \) and \( F_a \) (See Fig. A2.26). To eliminate solving for the perpendicular distance from point \( (r) \) to line of action of \( F_a \), we resolve \( F_a \) into its \( H \) and \( V \) components at point 0 on its line of action as shown in Fig. A2.28.

\[
\sum M_r = -1400 \times 30 + 500 \times 60 \times 60 F_{av} = 0
\]

whence, \( F_{av} = 12000/60 = 200 \text{ lb.} \)

Therefore \( F_a = 200 \times \sqrt{2} = 282 \text{ lb.} \) compression

A2.11 Method of Shears

In Fig. A2.22 to find the stress in member \( F_a \) we cut the section 2-2 giving the free body for the left portion as shown in Fig. A2.27.

The method of moments is not sufficient to solve for member \( F_a \) because the intersection of the other two unknowns \( F_a \) and \( F_1 \) lies at infinity. Thus for conditions where two of the 3 cut members are parallel we have a method of solving for the web member of the truss commonly referred to as the method of shears, or the summation of all the forces normal to the two parallel unknown chord members must equal zero. Since the parallel chord members have no component in a direction normal to their line of action, they do not enter the above equation of equilibrium.

Referring to Fig. A2.27

\[
\sum V = 1400 - 500 - 1000 - F_a (1/\sqrt{2}) = 0
\]

whence \( F_a = -141 \text{ lb.} \) (tension or opposite to that assumed in the figure.

To find the stress in member \( F_r \), we cut section 3-3 in Fig. A2.22 and draw a free body diagram of the left portion in Fig. A2.28. Since \( F_1 \) and \( F_r \) are horizontal, the member \( F_a \) must carry the shear on the truss on this section 3-3, hence the name method of shears.

\[
\sum V = 1400 - 500 - 1000 + F_r = 0
\]

whence \( F_r = 100 \text{ lb.} \) (compression as assumed)

Note: The student should solve this example illustrating the methods of moments and shears using as a free body the portion of the truss to the right of the cut sections instead of the left portion as used in these illustrative examples. In order to solve for the stresses in the members of a truss most advantageously, one usually makes use of more than one of the above three methods, as each has its advantages for certain cases or members. It is important to realize that each is a method of sections and in a great many cases, such as trusses with parallel chords, the stresses can practically be found mentally without writing down equations of equilibrium. The following statements in general are true for parallel chord trusses:

(1) The vertical component of the stress in the panel diagonal members equals the vertical shear (algebraic sum of external forces to one side of the panel) on the panel, since the chord
members are horizontal and thus have zero vertical component.

(2) The truss verticals in general resist the vertical component of the diagonals plus any external loads applied to the end joints of the vertical.

(3) The load in the chord members is due to the horizontal components of the diagonal members and in general equals the summation of these horizontal components.

To illustrate the simplicity of determining stresses in the members of a parallel chord truss, consider the cantilever truss of Fig. A2.29 with supporting reactions at points A and J.

<table>
<thead>
<tr>
<th>H</th>
<th>1573</th>
<th>150</th>
<th>150</th>
<th>150</th>
<th>150</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>1573</td>
<td>150</td>
<td>150</td>
<td>150</td>
<td>150</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. A2.29

First, compute the length triangles in each panel of the truss as shown by the dashed triangles in each panel. The other triangles in each panel are referred to as load or index triangles and their sides are directly proportional to the length triangles.

The shear load in each panel is first written on the vertical side of each index triangle. Thus, in panel EFQR, considering forces to the right of a vertical section cut thru the panel, the shear is 100 lb., which is recorded on the vertical side of the index triangle.

For the second panel from the free end, the shear is 100 + 150 = 250 and for the third panel 100 + 150 + 150 = 400 lb., and in like manner 550 for fourth panel.

The loads in the diagonals as well as their horizontal components are directly proportional to the lengths of the diagonal and horizontal side of the length triangles. Thus the load in diagonal member DF = 100 (50/50) = 167 and for member DG = 250 (46.8/30) = 390. The horizontal component of the load in DF = 100 (40/30) = 133 and DG = 250 (56/30) = 300. These values are shown on the index triangles for each truss panel as shown in Fig. A2.29. We start our analysis for the loads in the members of the truss by considering joint E first.

Using \( \Sigma V = 0 \) gives \( EF = 0 \) by observation, since no external vertical load exists at joint E. Similarly, by the same reasoning for \( EH = 0 \), load in DE = 0. The load in the diagonal FD equals the value on the diagonal of the panel index triangle or 167 lb. It is tension by observation since the shear in the panel to the right is up and the vertical component of the diagonal FD must pull down for equilibrium.

Considering Joint F: \( EH = - FG - FD_H = 0 \), which means that the horizontal component of the load in the diagonal DF equals the load in FG, or is equal to the value of the horizontal side in the index triangle or - 133 lb. It is negative because the horizontal component of DF pulls on Joint F and therefore FG must push against the joint for equilibrium.

Considering Joint D:-

\[ \Sigma V = DF_Y + DG = 0 \]  But \( DF_Y = 100 \) (vertical side of index triangle)

\( . \) \( . \) \( DG = - 100 \)

\( EH = DE + DF_Y - DC = 0 \), but \( DE = 0 \) and \( DF_H = 133 \) (from index triangle)

\( . \) \( . \) \( DC = 133 \)

Considering Joint Q:-

\( EH = - GQ - GF - GC_H = 0 \)  But \( GF = - 133 \), and \( GC_H = 300 \) from index triangle in the second panel. Hence \( GQ = - 433 \) lb. Proceeding in this manner, we obtain the stress in all the members as shown in Fig. A2.29. All the equilibrium equations can be solved mentally and with the calculations being done on the slide rule, all member loads can be written directly on the truss diagram.

Observation of the results of Fig. A2.29 show that the loads in the truss verticals equal the values of the vertical sides of the index load triangle, and the loads in the truss diagonals equal the values of the index triangle diagonal side and in general the loads in the top and bottom horizontal truss members equal the summation of the values of the horizontal sides of the index triangles.

The reactions at A and J are found when the above general procedure reaches joints A and J. As a check on the work the reactions should be determined treating the truss as a whole.

Fig. A2.30 shows the solution for the stresses in the members of a simply supported Pratt Truss, symmetrically loaded. Since all panels have the same width and height, only one length triangle is drawn as shown. Due to symmetry, the index triangles are drawn for panels to only one side of the truss center line. First, the vertical shear in each panel is written on the vertical side of each index triangle. Due to the symmetry of the truss and
EQUILIBRIUM OF FORCE SYSTEMS. TRUSS STRUCTURES.

Loading, we know that one half of the external loads at joints $U_1$ and $L_4$ is supported at reaction $R_a$ and $1/2$ at reaction $R_e$, or shear in center panel = (100 + 50) L/2 = 75. The vertical shear in panel $U_1, U_4, L_1, L_4$ equals 75 plus the external loads at $U_4$ and $L_4$ or a total of 225 and similarly for the end panel shear = 225 + 50 = 275. With these values known, the other two sides of the index triangles are directly proportional to the sides of the length triangles for each panel, and the results are as shown in Fig. A2.30.

The reaction $R_a$ equals the value on the vertical side of our index triangle in the end panel, or 75. This should be checked using the truss as a whole and taking moments about $R_e$.

If a truss is loaded unsymmetrically, the reactions should be determined first, after which the index triangles can be drawn, starting with the end panels, since the panel shear is then readily calculated.

A2.12 Aircraft Wing Structure. Truss Type with Fabric or Plastic Cover

The metal covered cantilever wing with its better overall aerodynamic efficiency and sufficient torsional rigidity has practically replaced the externally braced wing except for low speed commercial or private pilot aircraft as illustrated by the aircraft in Figs. A2.31 and 32. The wing covering is usually fabric and therefore a drag truss inside the wing is necessary to resist loads in the drag truss direction. Figs. A2.33 and 34 shows the general structural layout of such wings. The two spars or beams are metal or wood. Instead of using double wires in each drag truss bay, a single diagonal strut capable of taking either tension or compressive loads could be used. The external brace struts are stream line tubes.

---

**Fig. A2.31 Piper Tri-Pacer**

**Fig. A2.32 Champion Traveler**
Fig. A2.34

Example Problem 10. Externally Braced Monoplane Wing Structure

Fig. A2.35 shows the structural dimensional diagram of an externally braced monoplane wing. The wing is fabric covered between wing beams, and thus a drag truss composed of struts and tie rods is necessary to provide strength and rigidity in the drag direction. The axial loads in all members of the lift and drag trusses will be determined. A simplified air loading will be assumed, as the purpose of this problem is to give the student practice in solving statically determinate space truss structures.

ASSUMED AIR LOADING:

1. A constant spanwise lift load of 46 lb/in from hinge to strut point and then tapering to 22.5 lb/in at the wing tip.

2. A forward uniform distributed drag load of 6 lb/in.

The above airloads represent a high angle of attack condition. In this condition a forward load can be placed on the drag truss as illustrated in Fig. A2.36. Projecting the air lift and drag forces on the drag truss direction, the forward projection due to the lift is greater than the rearward projection due to the air drag, which difference in our example problem has been assumed as 6 lb/in. In a low angle of attack the load in the drag truss direction would act rearward.

SOLUTION:

The running loads on the front and rear beams will be calculated as the first step in the solution. For our flight condition, the center of pressure of the airforces will be assumed as shown in Fig. A2.37.

The running load on the front beam will be 45 x 24.2/36 = 30.25 lb/in., and the remainder or 45 - 30.25 = 14.74 lb/in gives the load on the rear beam.

Fig. A2.37
To solve for loads in a truss system by a method of joints, all loads must be transferred to the truss joints. The wing beams are supported at one end by the fuselage and outboard by the two lift struts. Thus we calculate the reactions on each beam at the strut and hinge points due to the running lift load on each beam.

**Front Beam**

![Diagram of Front Beam](image)

\[ w = 30.26 \text{#/in.} \]

\[ R_a = 114.5' \]

\[ R_L = 70.5' \]

Taking moments about point (2)

\[ 114.5R_A - 114.5 \times 30.26 \times 114.5/2 = 15.13 \times 70.5 \times 149.75 - 15.13 \times 35.25 \times 138 = 0 \]

hence

\[ R_A = 3770 \text{ lb.} \]

Take \( ZV = 0 \) where \( V \) direction is taken normal to beam

\[ ZV = -R_A - 3770 + 30.26 + 114.5 = (30.26 + 15.13) \]

\[ 70.5 = 0 \]

hence \( R_A = 1295 \text{ lb.} \)

(The student should always check results by taking moments about point (1) to see if \( ZM_A = 0 \))

**Rear Beam**

![Diagram of Rear Beam](image)

\[ w = 14.74 \text{#/in.} \]

\[ R_A = 114.5' \]

\[ R_L = 70.5' \]

The rear beam has the same span dimensions but the loading is 14.74 lb/in. Hence beam reactions \( R_A \) and \( R_L \) will be 14.74/30.26 = .4875 times those for front beam.

hence \( R_A = .4875 \times 3770 = 1838 \text{ lb.} \)

\[ R_L = .4875 \times 1295 = 651 \text{ lb.} \]

The next step in the solution is the solving for the axial loads in all the members. We will use the method of joints and consider the structure made up of three truss systems as illustrated at the top of the next column, namely, a front lift truss, a rear lift truss, and a drag truss. The beams are common to both lift and drag trusses.

Table A2.1 gives the \( V \), \( D \) and \( S \) projections of the lift truss members as determined from information given in Fig. A2.35. The true member lengths \( L \) and the component ratios then follow by simple calculation.

<table>
<thead>
<tr>
<th>Member</th>
<th>Sym.</th>
<th>Y</th>
<th>D</th>
<th>L</th>
<th>V/L</th>
<th>D/L</th>
<th>S/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Front Beam</td>
<td>FB</td>
<td>5.99</td>
<td>144.34</td>
<td>114.50</td>
<td>.0523</td>
<td>0</td>
<td>.9986</td>
</tr>
<tr>
<td>Rear Beam</td>
<td>RB</td>
<td>1.99</td>
<td>144.34</td>
<td>114.50</td>
<td>.0523</td>
<td>0</td>
<td>.9986</td>
</tr>
<tr>
<td>Front Strut</td>
<td>Sp</td>
<td>57.99</td>
<td>11</td>
<td>144.34</td>
<td>128.79</td>
<td>4501</td>
<td>.0854</td>
</tr>
<tr>
<td>Rear Strut</td>
<td>Sr</td>
<td>57.49</td>
<td>0</td>
<td>144.34</td>
<td>128.00</td>
<td>.4486</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ V = \text{vertical direction}, \]
\[ D = \text{drag direction}, \]
\[ S = \text{side direction}, \]
\[ L = \sqrt{V^2 + D^2 + S^2} \]

We start the solution of joints by starting with joint (1). Free body sketches of joint (1) are sketched below. All members are considered two-force members or having pins at each end, thus magnitude is the only unknown characteristic of each member load. The drag truss members coming into joint (1) are replaced by a single reaction called \( D_1 \). After \( D_1 \) is found, its influence in causing loads in drag truss members can then be found when the drag truss as a whole is treated. In the joint solution, the drag truss has been assumed parallel to drag direction which is not quite true from Fig. A2.35, but the error on member loads is negligible.

**JOINT 1 (Equations of Equilibrium)**

\[ V = 3770 \times .9986 - .0523 \times 4501 \times S = 0 \quad (1) \]
\[ ES = -3770 \times .0523 - .9986 \times 4501 \times S = 0 \quad (2) \]
\[ ED = -.0854 \times S + D_1 = 0 \quad (3) \]

Solving equations 1, 2, and 3, we obtain

\[ FB = -8613 \text{ lb. (compression)} \]
\[ Sp = 9333 \text{ lb. (tension)} \]
\[ D_1 = -798 \text{ lb. (art)} \]
Joint (3) (Equations of equilibrium)

\[
\begin{align*}
\Sigma Y &= 1838 \times 0.9986 - 0.0523 RB - 0.4486 SR = 0 \quad (4) \\
\Sigma Z &= -1838 \times 0.0523 - 0.9986 RB - 0.8930 SR = 0 \quad (5) \\
\Sigma D &= D_a + 0 = 0 \quad (6)
\end{align*}
\]

Solving equations 4, 5 and 6, we obtain
\[\begin{align*}
RB &= -4189 \text{ lb. (compression)} \\
SR &= 4579 \text{ lb. (tension)} \\
D_a &= 0
\end{align*}\]

Fig. A2.38 shows the reactions of the lift struts on the drag truss at joints (1) and (3) as found above.

Fig. A2.38

Drag Truss Panel Point Loads Due to Air Drag Load.

It was assumed that the air load components in the drag direction were 6 lb./in. of wing acting forward.

The distributed load of 6 lb./in. is replaced by concentrated loads at the panel points as shown in Fig. A2.39. Each panel point takes one half the distributed load to the adjacent panel point, except for the two outboard panel points which are affected by the overhang tip portion.

Thus the outboard panel point concentration of 254 lb. is determined by taking moments about (3) of the drag load outboard of (3) as follows:

\[P = 70.5 \times 6 \times 35.25/58.5 = 254 \text{ lb.}\]

To simplify the drag truss solution, the drag strut and drag wires in the inboard drag truss panel have been modified to intersect at hinge points (2) and (4). In the design of the beam and fittings at this point, the effect of the actual conditions of eccentricity should of course be considered.

Combined Loads on Drag Truss

Adding the two load systems of Figs. A2.38 and A2.39, the total drag truss loading is obtained as shown in Fig. A2.40. The resulting member axial stresses are then solved for by the method of index stresses (Art. A2.9). The values are indicated on the truss diagram. It is customary to make one of the fittings attaching wing to fuselage incapable of transferring drag reaction to fuselage, so that the entire drag reaction from wing panel on fuselage is definitely confined to one point. In this example point (2) has been assumed as point where drag is resisted. Those drag wires which would be in compression are assumed out of action.

Fig. A2.40

Fuselage Reactions

As a check on the work as well as to obtain reference loads on fuselage from wing structure, the fuselage reactions will be checked against the externally applied air loads. Table A2.2 gives the calculations in table form.

**Table A2.2**

<table>
<thead>
<tr>
<th>Point</th>
<th>Member</th>
<th>Load</th>
<th>V</th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>RB_Drag Reaction</td>
<td>-13893</td>
<td>-726</td>
<td>0</td>
<td>-13870</td>
</tr>
<tr>
<td></td>
<td>R_2(Reaction)</td>
<td>-1295</td>
<td>0</td>
<td>-1908</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>RB_Drag Reaction</td>
<td>1191</td>
<td>62</td>
<td>0</td>
<td>1190</td>
</tr>
<tr>
<td></td>
<td>R_4(Reaction)</td>
<td>631</td>
<td>0</td>
<td>-33</td>
<td>-33</td>
</tr>
<tr>
<td>5</td>
<td>Fg</td>
<td>9332</td>
<td>4205</td>
<td>798</td>
<td>8290</td>
</tr>
<tr>
<td>6</td>
<td>Rs</td>
<td>4579</td>
<td>2055</td>
<td>0</td>
<td>4090</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td>7520</td>
<td>-1110</td>
<td>-400</td>
<td>-400</td>
</tr>
</tbody>
</table>

**Applied Air Loads:**

\[\begin{align*}
V \text{ component} &= (3770 + 1285 + 1838 + 631) \times 2 \times 0.9986 = 78523 \text{ lb. (error 3 lb.)} \\
D \text{ component} &= -165 \times 6 = 1110 \text{ lb. (error 0)} \\
S \text{ component} &= -(3770 + 1285 + 1838 + 631) \times 0.9986 = 394 \text{ lb. (error 6 lb.)}
\end{align*}\]
EQUILIBRIUM OF FORCE SYSTEMS. TRUSS STRUCTURES.

The wing beams due to the distributed lift air loads acting upon them, are also subjected to bending loads in addition to the axial loads. The wing beams thus act as beam-columns. The subject of beam-column action is treated in another chapter of this book.

If the wing is covered with metal skin instead of fabric, the drag truss can be omitted since the top and bottom skin act as webs of a beam which has the front and rear beams as its flange members. The wing is then considered as a box beam subjected to combined bending and axial loading.

Example Problem 11. 3-Section Externally Braced Wing.

Fig. A2.41 shows a high wing externally braced wing structure. The wing outer panel has been made identical to the wing panel of example problem 1. This outer panel attached to the center panel by single pin fittings at points (2) and (4). Placing pins at these points make the structure statically indeterminate, whereas if the beams were made continuous through all 3 panels, the reactions of the lift and cabane struts on the wing beams would be statically indeterminate since we would have a 3-span continuous beam resting on settling supports due to strut deformation. The fitting pin at points (2) and (4) can be made eccentric with the neutral axis of the beams, hence very little is gained by making beams continuous for the purpose of decreasing the lateral beam bending moments. For assembly, stowage and shipping it is convenient to build a wing in 3 portions. If a multiple bolt fitting is used at points (2) and (4) to obtain a continuous beam, not much is gained because the design requirements of the various governmental agencies specify that the wing beams must also be analyzed on the assumption that a multiple bolt fitting provides only 50 percent of the full continuity.

![Diagram of wing structure with points labeled](Diagram)

Solution of Center Panel

Center Rear Beam

Fig. A2.42 shows the lateral loads on the center rear beam. The loads consist of the distributed air load and the vertical component of the forces exerted by outer panel on center panel at pin point (4). From Table A2.2 of example problem 1, this resultant V reaction equals 650 + 62 = 692 lb.

The vertical component of the cabane reaction at joint (8) equals one half the total beam load due to symmetry of loading or 65 x 14.74 + 692 = 1650 lb.

Solution of force system at Joint 8

\[ \Sigma V = 1650 - .731 \ C_R = 0 \]

whence

\[ C_R = 1650/.731 = 2250 \text{ lb. (tension)} \]

\[ ZS = -CRB - 2250 \times .363 = 0 \]
whence

\[ CRB = -1510 \text{ lb. (compression)} \]
\[ XD = D_a - 2280 \times 0.1485 = 0 \]

whence

\[ D_a = 336 \text{ lb. drag truss reaction} \]

Center Front Beam

(Ref., Table A2.2)

\[ 568^* \quad (1294 - 726) = 568^* \]

\[ 20 \quad 90 \quad 720 \]

\[ R_a = 2535^* \quad R_e = 2535^* \]

Fig. A2.43

Fig. A2.43 shows the \( V \) loads on the center front beam and the resulting \( V \) component of the cabane reaction at joint (7).

Solution of force system at Joint 7

\[ CFB = 2535 \]
\[ C_D = 2535 \]
\[ C_F = 2535 \]

\[ ZV = 2535 - 0.721 C_F - 0.597 C_D = 0 \]
\[ ZS = -2535 - 0.648 C_F - 0.336 C_D = 0 \]
\[ XD = -0.240 C_F + 0.597 C_D = 0 \]

Solving the three equations, we obtain

\[ C_FB = -2281 \text{ (compression)} \]
\[ C_F = 2635 \]
\[ C_D = 1058 \]

Solution for Loads in Drag Truss Members

Fig. A2.44 shows all the loads applied to the center panel drag truss. The \( S \) and \( D \) reactions from the outer panel at joints (2) and (4) are taken from Table A2.2 of problem 1. The drag load of 336 lb. at (3) is due to the rear cabane strut, as is likewise the beam axial load of 1510 at (8). The axial beam load of 2281 lb. at (7) is due to reaction of front cabane truss. The panel point loads are due to the given running drag load of 6 lb./in. acting forward.

The reaction which holds all these drag truss loads in equilibrium is supplied by the cabane truss at point (7) since the front and diagonal cabane struts intersect to form a rigid triangle. Thus the drag reaction \( R \) equals one half the total drag loads or 2634 lb.

Solving the truss for the loading of Fig. A2.44 we obtain the member axial loads of Fig. A2.45.

Fig. A2.45

Loads in Cabane Struts Due to Drag Reaction at Point 7

\[ ZD = -2634 - 0.240 C_F + 0.597 C_D = 0 \]
\[ ZV = -0.721 C_F - 0.597 C_D = 0 \]

Solving for \( C_F \) and \( C_D \), we obtain

\[ C_F = -2740 \text{ lb. (compression)} \]
\[ C_D = 3310 \text{ (tension)} \]

adding these loads to those previously calculated for lift loads:

\[ C_F = -2740 + 2635 = -105 \]
\[ C_D = 1058 + 3310 = 4368 \text{ lb.} \]
\[ C_R = 2281 \text{ lb.} \]

Fuselage Reactions

As a check on the work the fuselage reactions will be checked against the applied loads. Table A2.3 gives the \( V \), \( D \) and \( S \) components of the fuselage reactions.
Table A2.3

<table>
<thead>
<tr>
<th>Point</th>
<th>Member</th>
<th>Load</th>
<th>V</th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>Front Strut Cp</td>
<td>-105</td>
<td>-74</td>
<td>-25</td>
<td>-68</td>
</tr>
<tr>
<td>10</td>
<td>Rear Strut Cp</td>
<td>2260</td>
<td>1510</td>
<td>335</td>
<td>1510</td>
</tr>
<tr>
<td></td>
<td>Dist. Strut Cp</td>
<td>4368</td>
<td>2610</td>
<td>-2610</td>
<td>2356</td>
</tr>
<tr>
<td>5</td>
<td>Frost Lift Strut Sp</td>
<td>9233</td>
<td>4205</td>
<td>798</td>
<td>8290</td>
</tr>
<tr>
<td>6</td>
<td>Rear Lift Strut Sp</td>
<td>4579</td>
<td>2535</td>
<td>0</td>
<td>4090</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td>10644</td>
<td>-1502</td>
<td></td>
<td>16178</td>
</tr>
</tbody>
</table>

Applied Air Loads

V component = 7523 (outer panel) + 65 x 45 = 10449 (check)

D component = -1110 (outer panel) - 65 x 6 = -1500 (error 2 lb.)

The total side load on a vertical plane thru centerline of airplane should equal the S component of the applied loads. The applied side loads = -394 lb. (see problem 1). The air load on center panel is vertical and thus has zero S component.

From Table A2.3 for fuselage reactions have ES = 16178. From Fig. A2.45 the load in the front beam at E of airplane equals -17308 and 568 for rear beam. The horizontal component of the diagonal drag strut at joints 11 equals 216 x 45/57.6 = 159 lb.

Then total S components = 16178 - 17308 + 568 - 159 = -393 lb. which checks the side component of the applied air loads.


In small wings or control surfaces, fabric is often used as the surface covering. Since the fabric cannot provide reliable torsional resistance, internal structure must be of such design as to provide torsional strength. A single spar plus a special type of truss system is often used to give a satisfactory structure. Fig. A2.46 illustrates such a type of structure, namely, a trussed single spar AEFN plus a triangular truss system between the spar and the trailing edge OS. Fig. A2.46 (a, b, c) shows the three projections and dimensions. The air load on the surface covering of the structure is assumed to be 0.5 lb./in. a intensity at spar line and then varying linearly to zero at the trailing edge (See Fig. d).

The problem will be to determine the axial loads in all the members of the structure. It will be assumed that all members are 2 force members as is usually done in finding the primary loads in trussed structures.

SOLUTION:

The total air load on the structure equals the average intensity per square inch times the surface area or (0.5)(.5)(36 x 94) = 786 lb. In order to solve a truss system by a method of joints the distributed load must be replaced by an equivalent load system acting at the joints of the structure. Referring to Fig. (a), the total air load on a strip 1 inch wide and 36 inches long is 36(0.5)/2 = 9 lb. and its c.g. or resultant location is 12 inches from line AE. In Fig. 46a this resultant load of 9 lb./in. is imagined as acting on an imaginary beam located along the line 1-1. This running load applied along this line is now replaced by an equivalent force system acting at joints OPQSERSTBCA. The results of this joint distribution are shown by the joint loads in Fig. A2.46. To illustrate how these joint loads were obtained, the calculations for loads at joints ESBR will be given.

Fig. A2.48 shows a portion of the structure to be considered. For a running load of 9 lb./in., along line 1-1, reactions will be for
simple beams resting at points 2, 3, 4, S, etc. The distance between 2-3 is 8 inches. The total load on this distance is 2 x 8 = 72 lb. One half or 36 lb. goes to point (2) and the other half to point (3). The 36 lb. at (2) is then replaced by an equivalent force system at E and S or (36)/3 = 12 lb. to S and (36)(2/3) = 24 to E. The distance between points (3) and (4) is 8 inches and the load is 8 x 9 = 72 lb. One half of this or 36 goes to point (3) and this added to the previous 36 gives 72 lb. at (3). The load of 72 is then replaced by an equivalent force system at S and D, or (72)(2/3) = 24 lb. to S and (72)(2/3) = 48 to D. The final load at S is therefore 24 + 12 = 36 lb. as shown in Fig. A2.46. Due to symmetry of the triangle CDB, one half of the total load on the distance CD goes to points (4) and (5) or (24 x 4)/2 = 108 lb. The distribution to D is therefore (108) (2/3) = 72 and (108)/3 = 36 to R. Adding 72 to the previous load of 48 at D gives a total load at D = 120 lb. as shown in Fig. A2.46. The 108 lb. at point (5) also gives (108)/3 = 36 to R or a total of 72 lb. at R. The student should check the distribution to other joints as shown in Fig. A2.46.

To check the equivalence of the derived joint load system with the original air load system, the magnitude and moments of each system must be the same. Adding up the total joint loads as shown in Fig. A2.46 gives a total of 756 lb. which checks the original air load. The moment of the total air load about an x axis at left end of structure equals 756 x 42 = 31752 in. lb. The moment of the joint load system in Fig. A2.46 equals (66 x 12) + (72 x 36) + (72 x 50) + (56 x 84) + 144 (24 + 40) + (120 x 72) + (24 x 84) = 31752 in. lb. or a check. The moment of the total air load about line AE equals 756 x 12 = 9072 in. lb. The moment of the distributed joint loads equals (6 + 66 + 72 + 72 + 36)36 = 9072 or a check.

Calculation of Reactions

The structure is supported by single pin fittings at points A, N and O, with pin axes parallel to x axis. It will be assumed that the fitting at N takes off the spar load in z direction. Fig. A2.46 shows the reactions Oy, Oy, Ay, Ny, Nz. To find O2, take moments about y axis along spar AEFN.

\[ \Sigma M_y = (6 + 66 + 72 + 72 + 36)36 - 36 O_2 = 0 \]
whence \( O_2 = 252 \) lb. acting down as assumed.

To find Oy take moments about z axis through point (A).

\[ \Sigma M_z = 0 + 36 O_y = 0, \quad O_y = 0 \]

To find Ay take moments about x axis through point N. The moment of the air loads was previously calculated as - 31752, hence,

\[ \Sigma M_x = - 31752 + 9 A_y = 0, \quad A_y = 3528 \] lb.

To find Ny take \( \Sigma F_y = 0 \)

\[ \Sigma F_y = 3528 - N_y = 0, \quad N_y = 3528 \] lb.

To find Nz take \( \Sigma F_z = 0 \)

\[ \Sigma F_z = 252 + 756 - N_z = 0, \quad N_z = 504 \] lb.

The reactions are all recorded on Fig. A2.46.

Solution of Truss Member Loads

For simplicity, the load system on the structure will be considered separately as two load systems. One system will include only those loads acting along the line AE and the second load system will be the remaining loads which act along line OS. Since no bending moment can be resisted at joint O, the external load along spar AE will be reacted at A and N entirely or in other words, the spar alone resists the loads on line AE.

Fig. A2.49 shows a diagram of this spar with its joint external loading. The axial loads produced by this loading are written on the truss members. (The student should check these member loads.)

\[ \begin{align*}
\text{TRIANGULAR TRUSS SYSTEM} \\
72 & \quad 144 \\
2336 & \quad -1184 & \quad -1184 & \quad -416 & \quad -416 & \quad -32 & \quad 0 \\
2336 & \quad \text{N} & \quad 1760 & \quad 1760 & \quad 800 & \quad 800 & \quad 224 & \quad 224 & \quad 32 \\
& \quad 504 & \quad \text{Fig. A2.49} \\
\end{align*} \]
load of 36 lb. at Joint S in order to be transmitted to point O through the diagonal truss system must follow the path SDRCQBPAOG. In like manner the load of 72 at R to reach O must take the path RCQBPAO, etc.

Calculation of Loads in Diagonal Truss Members:

<table>
<thead>
<tr>
<th>Member</th>
<th>z</th>
<th>y</th>
<th>x</th>
<th>L</th>
<th>z/L</th>
<th>y/L</th>
<th>x/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Diagonal Truss Members</td>
<td>4.5</td>
<td>12</td>
<td>36</td>
<td>38.2</td>
<td>.118</td>
<td>.314</td>
<td>.943</td>
</tr>
<tr>
<td>AO, NO</td>
<td>4.5</td>
<td>0</td>
<td>36</td>
<td>37.5</td>
<td>.120</td>
<td>0</td>
<td>.960</td>
</tr>
</tbody>
</table>

Consider Joint S

The triangular truss SEF cannot assist in transferring any portion of the 36 lb. load at S because the reaction of this truss at SE would put torsion on the spar and the spar has no appreciable torsional resistance.

Considering Joint S as a free body and writing the equilibrium equations:

\[
\begin{align*}
Z_{F_X} &= -0.943 DS - 0.943 GS = 0 \\
& \text{whence, } DS = -GS \\
Z_{F_Z} &= 35 - 0.118 DS - 0.118 GS = 0
\end{align*}
\]

Subt. DS = -GS and solving for GS, gives GS = 159 lb. (tension), DS = -159 (compression)

Consider Joint D

Let Ty and Tz be reactions of diagonal truss system on spar truss at Joint D.

\[
\begin{align*}
Z_{F_X} &= -159 x 0.943 + 0.943 DR = 0, \text{ hence DR} = 159 \text{ lb.} \\
Z_{F_Z} &= -159 x 0.118 + 159 x 0.118 - T_z = 0
\end{align*}
\]

whence Tz = 0, which means the diagonal truss produces no Z reaction or shear load on spar truss at D.

\[
\begin{align*}
Z_{F_Y} &= -0.314 x 159 - 0.314 x 159 - Ty = 0
\end{align*}
\]

whence Ty = -100 lb.

The results at Joint D shows that the rear diagonal truss system produces no shear load on the spar but does produce a couple reaction on the spar in the Y direction which produces compression in the top chord of the spar truss and tension in the bottom chord.

Consider Joint R

The load to be transferred to truss RCJR is equal to the 72 lb. at R plus the 36 lb. at S which comes to Joint R from truss DRG.

\[
\begin{align*}
\text{Hence load in RC} &= (72 + 36)0.5 \times (1/1.118) \\
&= -457 \text{ lb.}
\end{align*}
\]

\[
\begin{align*}
\text{Hence load in RJ} &= 457 \text{ lb.}
\end{align*}
\]

Consider Joint Q

Load to be transferred to truss QBL = 72 + 72 + 36 = 180 lb.

\[
\begin{align*}
\text{Hence load in QB} &= (180 x 0.5)(1/1.118) = -762
\end{align*}
\]

\[
\begin{align*}
\text{Hence load in QL} &= 762, BP = 762, LP = -762
\end{align*}
\]

Consider Joint P as a free body.

\[
\begin{align*}
Z_{F_X} &= -1040 x 0.943 + 0.960 AO = 0, AO = 1022 \text{ lb.}
\end{align*}
\]

In like manner, considering Joint N, gives NO = -2022 lb.

Couple Force Reactions on Spar

As pointed out previously, the diagonal torsion truss produces a couple reaction on the spar in the Y direction. The magnitude of the force of this couple equals the Y component of the load in the diagonal truss members meeting at a spar joint. Let Ty equal this reaction load on the spar.

At Joint C:

\[
\begin{align*}
Ty &= -(457 + 457) \times 0.314 = -287 \text{ lb.}
\end{align*}
\]

Likewise at Joint J, Ty = 287

At Joint S:

\[
\begin{align*}
Ty &= -(762 + 762) \times 0.314 = -479
\end{align*}
\]

Likewise at Joint L, Ty = 479

At Joint A:

\[
\begin{align*}
Ty &= -(1040 x 0.314) = -326
\end{align*}
\]

Likewise at Joint N, Ty = 326
These reactions of the torsion truss upon the spar truss are shown in Fig. A2.50. The loads in the spar truss members due to this loading are written adjacent to each truss member. Adding these member loads to the loads in Fig. A2.49, we obtain the final spar truss member loads as shown in Fig. A2.51.

If we add the reactions in Figs. A2.50 and A2.51, we obtain 3528 and 504 which check the reactions obtained in Fig. A2.46.

A2.13 Landing Gear Structure

The airplane is both a landborne and air-borne vehicle, and thus a means of operating the airplane on the ground must be provided which means wheels and brakes. Furthermore, provision must be made to control the impact forces involved in landing or in taxiing over rough ground. This requirement requires a special energy absorption unit in the landing gear beyond that energy absorption provided by the tires. The landing gear thus includes a so-called shock strut commonly referred to as an oleo strut, which is a member composed of two telescoping cylinders. When the strut is compressed, oil inside the air tight cylinders is forced through an orifice from one cylinder to the other and the energy due to the landing impact is absorbed by the work done in forcing this oil through the orifice. The orifice can be so designed as to provide practically a uniform resistance over the displacement or travel of the oleo strut.

An airplane can land safely with the airplane in various attitudes at the instant of ground contact. Fig. A2.52 illustrates the three attitudes of the airplane that are specified by the government aviation agencies for design of landing gear. In addition to these symmetrical unbraked loadings, special loadings, such as a braked condition, landing on one wheel condition, side load on wheel, etc., are required. In other words, a landing gear can be subjected to a considerable number of different loadings under the various landing conditions that are encountered in the normal operation of an airplane.

An aerial view of the tail down landing is shown in Fig. A2.52.

A2.14 Example Problems of Calculating Reactions and Loads on Members of Landing Gear Units

In its simplest form, a landing gear could consist of a single oleo strut acting as a cantilever beam with its fixed end being the upper end which would be rigidly fastened to the supporting structure. The lower cylinder of the oleo strut carries an axle at its lower...
Fig. A2.33  Main Landing Gear Illustrations (One Side)
Fig. A2.54 Nose Wheel Gear Installations
end for attaching the wheel and tire. This cantilever beam is subjected to bending in two directions, torsion and also axial loads. Since the gear is usually made retractable, it is difficult to design a single fitting unit at the upper end of the oleo strut that will resist this combination of forces and still permit movement for a simple retracting mechanism. Furthermore, it would be difficult to provide carry-through supporting wing or fuselage structure for such large concentrated load systems.

Thus to decrease the magnitude of the bending moments and also the bending flexibility of the cantilever strut and also to simplify the retracting problem and the carry-through structural problem, it is customary to add one or two braces to the oleo strut. In general, effort is made to make the landing gear structure statically determinate by using specially designed fittings at member ends or at support points in order to establish the force characteristics of direction and point of application.

Two example problem solutions will be presented, one dealing with a gear with a single wheel and the other with a gear involving two wheels.

Example Problem 13

Fig. A2.55 shows the projections of the landing gear configuration on the VS and VD planes. Fig. A2.56 is a space dimensional diagram. In landing gear analysis it is common to use V, D and S as reference axes instead of the symbols Z, X and Y. This gear unit is assumed as representing one side of the main gear on a tricycle type of landing gear system. The loading assumed corresponds to a condition of nose wheel up or tail down. (See lower sketch of Fig. A2.52). The design load on the wheel is vertical and its magnitude for this problem is 15000 lb.

The gear unit is attached to the supporting structure at points F, H and G. Retraction of the gear is obtained by rotating gear rearward and upward about axis through F and H. The fittings at F and H are designed to resist no bending moment hence reactions at F and H are unknown in magnitude and direction. Instead of using the reaction and an angle as unknowns, the resultant reaction is replaced by its V and D components as shown in Fig. A2.56. The reaction at G is unknown in magnitude only since the pin fitting at each end of member GC fixes the direction and line of action of the reaction at G. For convenience in calculations, the reaction G is replaced by its components Gv and Gd. For a side load on the landing gear, the reaction in the S direction is taken off at point F by a special designed unit.

SOLUTION

The supporting reactions upon the gear at points F, H, and G will be calculated as a beginning step. There are six unknowns, namely Fs, Fy, Fd, Hv, Hg and G (See Fig. A2.56). With 6 equations of static equilibrium available for a space force system, the reactions can be found by statics. Referring to Fig. A2.56:-

To find Fs take ES = 0

Fs = 0 = 0, hence Fs = 0

To find reaction Gv take moments about an S axis through points F, H.

\[ \sum Gv = 3119 \times 50 - 24 \times Gv = 0 \]

Whence, Gv = 6500 lb, with sense as assumed. (The wheel load of 15000 lb. has been resolved into V and D components as indicated in Fig. A2.55).

With Gv known, the reaction G equals (6500) \((31.8/24) = 8610 \text{ lb, and similarly the component } Gd = (6500)(21/24) = 5690 \text{ lb.} \]
To find \( F_V \), take moments about a D axis through point H.

\[ \sum M_H(D) = 16 \, G_y + 14672 \times 8 - 22 \, F_V = 0 \]
\[ = 16 \times 5600 + 14672 \times 8 - 22 \, F_V = 0 \]

W hence \( F_V = 10083 \) lb. with sense as assumed.

To find \( H_D \), take moments about V axis through F.

\[ \sum M_F(V) = -6 \, G_D - 22 \, H_D + 3119 \times 14 = 0 \]
\[ = -6 \times 5690 - 22 \, H_D + 3119 \times 14 = 0 \]

W hence, \( H_D = 433 \) lb.

To find \( F_D \), take CD = 0

\[ \sum Z = -F_D + H_D + G_P = 3119 = 0 \]
\[ = -F_D + 433 + 5690 - 3119 = 0 \]

W hence, \( F_D = 3004 \) lb.

To find \( H_V \) take ZV = 0

\[ \sum F_V = -F_V + \, G_V - H_V + 14672 = 0 \]
\[ = -10083 + 5600 - H_V + 14672 = 0 \]

W hence, \( H_V = 11109 \) lb.

F i g. A 2.57 summarizes the reactions as found. The results will be checked for equilibrium of the structure as a whole by taking moments about D and V axes through point A.

\[ \sum M_A(D) = -10063 \times 14 + 6500 \times 8 + 11109 \times 8 \]
\[ = -140882 + 52000 + 88882 = 0 \text{(check)} \]

\[ \sum M_A(V) = 5690 \times 8 - 453 \times 8 - 3004 \times 14 \]
\[ = 45520 - 3464 - 42065 = 0 \text{ (check)} \]

T he next step in the solution will be the calculation of the forces on the oleo strut unit. F i g. A 2.58 shows a free body of the oleo-strut-axis unit. The brace members BI and CG are two force members due to the pin at each end, and thus magnitude is the only unknown reaction characteristic at points B and C. The fitting at point E between the oleo strut and the top cross member FH is designed in such a manner as to resist torsional moments about the oleo strut axis and to provide D, V and S force reactions but no moment reactions about D and S axes. The unknowns are therefore BI, CG, \( E_g \), \( E_V \), \( E_D \) and \( T_F \) or a total of 6 and therefore statically determinate. The torsional moment \( T_F \) is represented in F i g. A 2.58 by a vector with a double arrow. The vector direction represents the moment axis and the sense of rotation of the moment is given by the right hand rule, namely, with the thumb of the right hand pointing in the same direction as the arrows, the curled fingers give the sense of rotation.

To find the resisting torsional moment \( T_F \) take moments about V axis through E.

\[ \sum M_E(V) = -3119 \times 8 + T_F = 0 \text{, hence } T_F = 24952 \text{ in.lb.} \]

To find CG take moments about S axis through E.

\[ \sum M_E(S) = 3119 \times 50 - (24/31.8) \times CG \times 3 - 24 \times (21/31.8) \times CG = 0 \]

W hence, \( CG = 8610 \) lb.

T his checks the value previously obtained when the reaction at G was found to be 8610. The D and V components of CG thus equal,

\[ CG_D = 8610 \times (21/31.8) = 5590 \text{ lb.} \]
\[ CG_V = 8610 \times (24/31.8) = 6500 \text{ lb.} \]

To find load in brace strut BI, take moments about D axis through point E.

\[ \sum M_E(D) = -14672 \times 8 + 3 \times (BI) \times 22.5/24.6 + 24 \times (BI) \times 11/24.6 = 0 \]

W hence, \( BI = 6775 \) lb.

a nd thus, \( B_Iy = (6775)(22.5/24.6) = 7840 \) lb.
\[ B_Iy = (6775)(11/24.6) = 3020 \text{ lb.} \]

To find \( E_g \) take \( ZS = 0 \)

\[ ZS = E_g - 3520 = 0 \text{, hence } E_g = 3520 \]
To find $E_D$ take $ZD = 0$

$ZD = 5630 - 3119 - E_D = 0$, hence $E_D = 2571$

To find $F_Y$ take $EY = 0$

$EY = - F_Y + 14672 - 7840 + 6500 = 0$, hence $F_Y = 13332$ lb.

Fig. A2.59 shows a free body of the top member FH. The unknowns are $F_Y$, $F_D$, $F_S$, $H_Y$ and $H_D$. The loads or reactions as found from the analysis of the oleo strut unit are also recorded on the figure. The equations of equilibrium for this free body are:

$ZS = 0 = - 3920 + 3920 + F_S = 0$, or $F_S = 0$

$EMF(y) = 22 H_Y - 3920 x 2 - 7840 x 20 - 13332 x 6 = 0$

Whence, $H_Y = 11110$ lb. This check value obtained previously, and therefore is a check on our work.

$EMF(y) = 24952 - 2571 x 6 - 22 H_D = 0$

whence, $H_D = 433$ lb.

$ZV = - F_Y + 13332 + 7840 - 11110 = 0$

whence, $F_Y = 10063$

$ZD = - F_D + 2571 + 433 = 0$

whence, $F_D = 3004$ lb.

Thus working through the free bodies of the oleo strut and the top member FH, we come out with same reactions at $F$ and $H$ as obtained when finding these reactions by equilibrium equation for the entire landing gear.

The strength design of the oleo strut unit and the top member FH could now be carried out because with all loads and reactions on each member known, axial, bending and torsional stresses could now be found.

The loads on the brace struts GC and BI are axial, namely, 8610 lb. tension and 5775 lb. compression respectively, and thus need no further calculation to obtain design stresses.

**TORQUE LINK**

The oleo strut consists of two telescoping tubes and some means must be provided to transmit torsional moment between the two tubes and still permit the lower cylinder to move upward into the upper cylinder. The most common way of providing this torque transfer is to use a double-cantilever-nut cracker type of structure. Fig. A2.60 illustrates how such a torque length could be applied to the oleo strut in our problem.

The torque to be transferred in our problem is 24952 in. lb.

The reaction $R_1$ between the two units of the torque link at point (2), see Fig. A2.50, thus equals $24952/2 = 2773$ lb.

The reactions $R_3$ at the base of the link at point (3) = $2773 x 8.5/2.75 = 6660$ lb. With these reactions known, the strength design of the link units and the connections could be made.

**Example Problem 14**

The landing gear as illustrated in Fig. A2.61 is representative of a main landing gear which could be attached to the under side of a wing and retract forward and upward about line AB into a space provided by the lower portion of the power plant nacelle structure. The oleo strut CE has a sliding attachment at E, which prevents any vertical load to be taken by member AB at E. However, the fitting at E does transfer shear and torque reactions between the oleo strut and member AB. The brace struts GB, FD and UC are pinned at each end and will be assumed as 2 force members.

An airplane level landing condition with unsymmetrical wheel loading has been assumed as shown in Fig. A2.61.

**SOLUTION**

The gear is attached to supporting structure at points A, B and C. The reactions at these points will be calculated first, treating the entire gear as a free body. Fig. A2.62
shows a space diagram with loads and reactions. The reactions at A, B and C have been replaced by their V and D components.

To find reaction Cy take moments about an S axis through points AB.

\[ \Sigma M_{AB} = -(15000 + 10000) \times 64 + 24 Cy = 0 \]

Whence \( Cy = 66666 \) lb. With sense as assumed in Fig. A2.62.

The reaction at C must have a line of action along the line CD since member CD is pinned at each end, thus the drag component and the load in the strut CD follow as a matter of geometry. Hence, \( Cp = 66666 \times (24/28) = 57142 \) lb.

\[ Cp = 66666 \times (36.93/28) = 87900 \] lb. tension

To find By take moments about a drag axis through point (A).

\[ \Sigma M_A(D) = -60000 \times 9 - 40000 \times 29 - 66666 \times 19 + 38 By = 0 \]

whence, \( By = 78070 \) lb.

To find \( Ay \), take \( ZV = 0 \)

\[ \Sigma V = -78070 + 60000 + 40000 + 66666 - V_A = 0 \]

whence, \( Ay = 88596 \) lb.

To find \( Bd \) take moments about V axis through point (A).

\[ \Sigma M_A(V) = 57142 \times 19 - 15000 \times 9 - 10000 \times 29 - 38 Bd = 0 \]

whence, \( Bd = 17386 \) lb.

To find \( Ap \) take \( ZD = 0 \)

\[ \Sigma D = -57142 + 15000 + 10000 + 17386 + Ap = 0 \]

whence \( Ap = 14756 \) lb.

To check the results take moments about V and D axes through point O.

\[ \Sigma M_O(V) = 5 \times 10000 + 14756 \times 19 - 17386 \times 19 = 0 \] (check)

\[ \Sigma M_O(D) = 20000 \times 10 - 88596 \times 19 + 78070 \times 19 = 0 \] (check)

**REACTIONS ON OLEO STRUT OE**

Fig. A2.63 shows a free body of the oleo-strut OE. The loads applied to the wheels at

---

The axle centerlines have been transferred to point (O). Thus the total V load at (O) equals 60000 + 40000 = 100000 and the total D load equals 15000 + 10000 = 25000. The moment of these forces about V and D axes through (O) are

\[ Mo(V) = (15000 - 10000) \times 10 = 50000 \text{ in.lb.} \]

\[ Mo(D) = (60000 - 40000) \times 10 = 200000 \text{ in.lb.} \]

These moments are indicated in Fig. A2.63 by the vectors with double arrows. The sense of the moment is determined by the right-hand thumb
and finger rule.

The fitting at point E is designed to resist a moment about V axis or a torsional moment on the olio strut. It also can provide shear reactions Eq and Ep but no bending resistance about S or D axes.

The unknowns are the forces Eq, Ep, DF, DG and the moment Tg.

To find Tg take moments about axis CE.
\[ \Sigma M_{CE} = -50000 + Tg = 0, \text{ whence } Tg = 50000 \text{ in.lb} \]

To find Eq take moments about D axis through point D.
\[ \Sigma M(D) = 200000 - 28 Eq = 0, \text{ whence } Eq = 7143 \text{ lb} \]

To find force DFy take moments about D axis through point G.
\[ \Sigma M(G) = 200000 - 100000 \times 17 - 66666 \times 17 + 34 DFy = 0 \]
whence, \[ DFy = 77451 \text{ lb} \]

Then \[ DGs = 77451 (17/28) = 47023 \text{ lb} \]
and \[ DF = 77451 (32.72/28) \]
\[ = 90508 \text{ lb} \]

To find DGy take \( ZV = 0 \)
\[ ZV = 100000 - 77451 - 66666 = DGy = 0, \]
or \[ DGy = 89215 \]
Then, \[ DGs = 89215 (17/28) = 54164 \text{ lb} \]
\[ DG = 89215 (32.73/28) = 104190 \text{ lb} \]

To find Ep take moments about S axis through point D.
\[ \Sigma M(S) = -25000 \times 35 + 28 Ep = 0, \]
\[ Ep = 32143 \text{ lb} \]

The results will be checked for static equilibrium of strut. Take moments about D axis through point (0).
\[ \Sigma M(0) = 200000 - 54146 \times 36 - 47023 \times 36 - 7145 \times 64 + 200000 + 1949904 - 1692828 - 457150 = 0 \text{ (check)} \]
\[ \Sigma M(S) = 32143 \times 64 - 57145 \times 36 = 0 \text{ (check)} \]

**REATIONS ON TOP MEMBER AB**

Fig. A2.64 shows a free body of member AB with the known applied forces as found from the previous reactions on the olio strut.

The unknowns are \( AD, BD, AP, \) and \( By \). To find By take moments about B axis through A.

\[ E\text{MA}(D) = -89215 \times 2 - 77451 \times 36 - 33 By = 0 \]
whence, \[ By = 78070 \text{ lb} \]

To find Ay take \( ZV = 0 \)
\[ ZV = 89215 + 77451 - 78070 - Ay = 0 \]
whence, \[ Ay = 88586 \text{ lb} \]

To find Bp take moments V axis through A.
\[ E\text{MY}(D) = 50000 + 32143 \times 19 - 33 Bp = 0 \]
whence, \[ Bp = 17386 \text{ lb} \]

To find Ap take \( ZD = 0 \)
\[ ZD = 17386 - 32143 - 33 = 0, \text{ or } Ap = 14757 \text{ lb} \]

These four reactions check the reactions obtained originally when gear was treated as a free body, thus giving a numerical check on the calculations.

With the forces on each part of the gear known, the parts could be designed for strength and rigidity. The olio strut would need a torsion link as discussed in example problem A3 and Fig. A2.60.

**A2.15 Problems**

1. For the structures numbered 1 to 10 determine whether structure is statically determine with respect to external reactions and internal stresses.
(2) Find the horizontal and vertical components of the reactions on the structures illustrated in Figs. 11 to 15.

(3) Find the axial loads in the members of the trussed structures shown in Figs. 16 to 18.

(4) Determine the axial loads in the members of the structure in Fig. 19. The members are pinned to supports at A, B and C.

(5) Fig. 20 shows a tri-pod frame for hoisting a propeller for assembly on engine. Find the loads in the frame for a load of 1000 lb. on hoist.

(6) Fig. 21 shows the wing structure of an externally braced monoplane. Determine the axial loads in all members of the lift and drag trusses for the following loads.
Front beam lift load = 30 lb./in. (upward)
Rear beam lift load = 24 lb./in. (upward)
Wing drag load = 8 lb./in. acting aft

(7) Fig. 22 shows a braced monoplane wing. For the given air loading, find axial loads in lift and drag truss members. The drag reaction on drag truss is taken off at point A.
The fitting at points A and B for the landing gear structure in Fig. 23 provides resistance to V, D and S reactions and moments about D and V axes. Find the reactions at A and B and the load in member CD for given wheel loading.

In the landing gear of Fig. 24, the brace members BC and BF are two force members. The fitting at E provides resistance to V, D and S reactions but only moment resistance about V axis. Find reactions at E and loads in members BF and BC under given wheel loading.
CHAPTER A3
PROPERTIES OF SECTIONS - CENTROIDS
MOMENTS OF INERTIA ETC.

A3.1 Introduction  In engineering calculations, two terms, center of gravity and moment of inertia, are constantly being used. Thus, a brief review of these terms is in order.

A3.2 Centroids, Center of Gravity. The centroid of a line, area, volume, or mass is that point at which the whole line, area, volume, or mass may be conceived to be concentrated and have the same moment with respect to an axis as when distributed in its true or natural way. This general relationship can be expressed by the principle of moments, as follows:

**Lines:** \( \bar{x} = \frac{\sum xL}{\sum L} \)

**Areas:** \( \bar{x} = \frac{\sum xA}{\sum A} \)

**Volumes:** \( \bar{V} = \frac{\sum VX}{\sum V} \)

**Masses:** \( \bar{M} = \frac{\sum MX}{\sum M} \)

If a geometrical figure is symmetrical with respect to a line or plane, the centroid of the figure lies in the given line or plane. This is obvious from the fact that the moments of the parts of the figure on opposite sides of the line or plane are numerically equal but of opposite sign. If a figure is symmetrical to two lines or planes, the centroid of the figure lies at the intersection of the two lines or the two planes, and likewise, if the figure has 3 planes of symmetry, the centroid lies at the intersection of the 3 planes.

A3.3 Moment of Inertia. The term moment of inertia is applied in mechanics to a number of mathematical expressions which represents second moments of areas, volumes and masses, such as \( \int y^2 \, dA, \int r^2 \, dV, \int r^2 \, dM \) etc.

A3.4 Moment of Inertia of an Area. As applied to an area, the term moment of inertia has no physical significance, but represents a quantity entering into a large number of engineering problems or calculations. However, it may be considered as a factor which indicates the influence of the area itself in determining the total rotating moment of uniformly varying forces applied over an area.

Let Fig. A3.1 be any plane area referred to three coordinate axes, ox, oy, and oz; ox and oy being the plane of area.

Let \( dA \) represent an elementary area, with coordinates \( x, y, \) and \( r \) as shown.

Then

\[ I_x = \int y^2 \, dA, \quad I_y = \int x^2 \, dA, \quad I_z = \int r^2 \, dA \]

where \( I_x, I_y, \) and \( I_z \) are moments of inertia of the area about the axes xx, yy, and zz respectively.

![Fig. A3.1](image)

A3.5 Polar Moment of Inertia. In Fig. A3.1, the moment of inertia \( I_z = \int r^2 \, dA \) about the Z axis is referred to as the polar moment of inertia and can be defined as the moment of inertia of an area with respect to a point in its surface. Since \( r^2 = x^2 + y^2 \) (Fig. A3.1)

\[ I_z = \int (y^2 + x^2) \, dA = I_x + I_y \]

or, the polar moment of inertia is equal to the sum of the moments of inertia with respect to any two axes in the plane of the area, at right angles to each other and passing thru the point of intersection of the polar axis with the plane.

A3.6 Radius of Gyration. The radius of gyration of a solid is the distance from the inertia axis to that point in the solid at which, if its entire mass could be concentrated, its moment of inertia would remain the same.

Thus, \( \int r^2 \, dM = \rho^2 M \), where \( \rho \) is the radius of gyration

Since, \( \int r^2 \, dM = I, \) then \( I = \rho^2 M \) or \( \rho = \sqrt{\frac{I}{M}} \)

By analogy, in the case of an area,

\[ I = \rho^2 A \] or \( \rho = \sqrt{\frac{I}{A}} \)

A3.7 Parallel Axis Theorem. In Fig. A3.2 let \( I_y \) be the moment of inertia of the area referred to the centroidal axis \( y'y' \) be required. \( y'y' \) is parallel to \( yy \). Consider the elementary area \( dA \) with distance \( x + d \) from \( y'y' \).

Then,

\[ I_{y'} = \int (d + x)^2 \, dA = \int x^2 \, dA + 2d \int x \, dA + d \int dA \]
The first term, \( \int x^2 dA \), represents the moment of inertia of the body about its centroidal axis \( y-y \) and will be given the symbol \( I_y \). The second term is zero because \( \int x dA \) is zero since \( y \) is the centroidal axis of the body. The last term, \( d^2 dA = Ad^2 \) or area of body times the square of the distance between axes \( yy \) and \( YY \).

Thus in general, \( I = I_y + Ad^2 \).

This expression states that the amount of inertia of an area with respect to any axis in the plane of the area is equal to the moment of inertia of the area with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the two axes.

**Parallel Axis Theorem for Masses.** If instead of area the mass of the body is considered, the parallel axis can be written:

\[ I = I_y + Md^2 \], where \( M \) refers to the mass of the body.

### A3.7a Mass Moments of Inertia

The product of the mass of a particle and the square of its distance from a line or plane is referred to as the moment of inertia of the mass of the particle with respect to the line or plane. Hence,

\[ I = 2Mr^2 \]. If the summation can be expressed by a definite integral, the expression may be written \( I = \int r^2 dm \).

**Moments of Inertia of Airplanes.** In both flying and landing conditions the airplane may be subjected to angular accelerations. To determine the magnitude of the accelerations as well as the distribution and magnitude of the mass inertia resisting forces, the moment of inertia of the airplane about the three coordinate axes is generally required in making a stress analysis of a particular airplane.

The mass moments of inertia of the airplane about the coordinate \( X, Y \) and \( Z \) axes through the center of gravity of the airplane can be expressed as follows:

\[
\begin{align*}
I_x &= 2wz^2 + 2wx^2 + \Sigma I_{x_i} \\
I_y &= 2wx^2 + 2wy^2 + \Sigma I_{y_i} \\
I_z &= 2wx^2 + 2wz^2 + \Sigma I_{z_i}
\end{align*}
\]
### Table 1 - Continued

<table>
<thead>
<tr>
<th>Shape</th>
<th>Area</th>
<th>Moment of Inertia about X-axis</th>
<th>Polar Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptical Ring</td>
<td>( \pi (a_1 b_1 - a_2 b_2) )</td>
<td>( \frac{\pi}{4} (a_2^3 - a_1^3 b_1^2 - a_2^3 b_2^2) )</td>
<td>( \frac{1}{\text{Area}} )</td>
</tr>
<tr>
<td>Circular Fillet</td>
<td>( \frac{\pi}{4} a^4 )</td>
<td>( \frac{1}{4} a^4 )</td>
<td>( \frac{1}{2} a^4 )</td>
</tr>
<tr>
<td>Parabolic</td>
<td>( \frac{1}{5} x_1 y_1 )</td>
<td>( \frac{3}{4} x_1 )</td>
<td>( \frac{3}{7} y_1 )</td>
</tr>
<tr>
<td>Parabolic</td>
<td>( \frac{1}{5} x_1 y_1 )</td>
<td>( \frac{3}{4} x_1 )</td>
<td>( \frac{3}{7} y_1 )</td>
</tr>
<tr>
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<td>( \frac{1}{5} x_1 y_1 )</td>
<td>( \frac{3}{4} x_1 )</td>
<td>( \frac{3}{7} y_1 )</td>
</tr>
</tbody>
</table>

### Properties of Solids

<table>
<thead>
<tr>
<th>Solid Cylindrical</th>
<th>Volume</th>
<th>Mass (M)</th>
<th>Moment of Inertia about X-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1905 m</td>
<td>( \frac{27}{4} )</td>
<td>( \frac{27}{4} r^2 )</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Hollow Cylindrical</th>
<th>Volume</th>
<th>Mass (M)</th>
<th>Moment of Inertia about X-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{5} L(r_1^3 - r_2^3) )</td>
<td>( \frac{M(r_1^3 + r_2^3)}{2} )</td>
<td>( \frac{M(r_1^3 + r_2^3)}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

### Rectangular Prism

<table>
<thead>
<tr>
<th>Volume</th>
<th>Mass (M)</th>
<th>Moment of Inertia about X-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( abL )</td>
<td>( \frac{12}{2} )</td>
<td>( \frac{12}{2} )</td>
</tr>
</tbody>
</table>

### Sphere

<table>
<thead>
<tr>
<th>Volume</th>
<th>Mass (M)</th>
<th>Moment of Inertia about X-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{4}{3} \pi r^3 )</td>
<td>( \frac{24}{5} )</td>
<td>( \frac{24}{5} r^5 )</td>
</tr>
</tbody>
</table>

### Thin Hollow Sphere

<table>
<thead>
<tr>
<th>Volume</th>
<th>Mass (M)</th>
<th>Moment of Inertia about X-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{4}{3} \pi r^3 )</td>
<td>( \frac{24}{5} )</td>
<td>( \frac{24}{5} r^5 )</td>
</tr>
</tbody>
</table>
### TABLE 2 - Continued

<table>
<thead>
<tr>
<th>Ring with Circular Section</th>
<th>$M = \omega$ = mass per unit volume ( \frac{z}{2} ) of body.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_{xx} = \frac{m}{z} R z^{2} \left( \frac{z^{2}}{4} + \left( \frac{R}{z} \right)^{2} \right)$</td>
</tr>
<tr>
<td></td>
<td>$I_{yy} = \frac{m}{z} R z^{2} \left( 4R^{2} + 3z^{2} \right)$</td>
</tr>
</tbody>
</table>

### TABLE 3

**Section Properties of Lines (l is small in comparison to radius)**

<table>
<thead>
<tr>
<th>Arc Type</th>
<th>Area ( = \pi rt )</th>
<th>$I_{1-1} = \pi r^{3}t$</th>
<th>$I_{polar} = 2 \pi r^{3}t$</th>
<th>$\rho_{x} = 0.707r$</th>
<th>$\rho_{polar} = r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular Arc</td>
<td>$\frac{\pi r^{2}}{2}$</td>
<td>$\frac{\pi r^{3}t}{2}$</td>
<td>$\frac{\pi r^{4}t}{2}$</td>
<td>$0.707r$</td>
<td>$r$</td>
</tr>
<tr>
<td>Semi-circle Arc</td>
<td>$\frac{\pi r^{2}}{2}$</td>
<td>$\frac{\pi r^{3}t}{2}$</td>
<td>$\frac{\pi r^{4}t}{2}$</td>
<td>$0.707r$</td>
<td>$r$</td>
</tr>
<tr>
<td>Quarter-circular Arc</td>
<td>$\frac{\pi r^{2}}{2}$</td>
<td>$\frac{\pi r^{3}t}{2}$</td>
<td>$\frac{\pi r^{4}t}{2}$</td>
<td>$0.707r$</td>
<td>$r$</td>
</tr>
</tbody>
</table>

### CIRCULAR ARC

- **Area** = \( \pi rt \)  
- **\( \bar{x} \)** = \( r \sin \alpha \) \( \bar{x} \) in Radians
  - **\( \bar{y} \)** = \( \bar{y} = \bar{x} \cos \alpha \)  
- **\( I_{yy} \)** = \( \frac{r^{3}t}{2} \) \( \bar{x} + \sin \frac{2\alpha}{2} \)
- **\( I_{NAy} \)** = \( \frac{r^{3}t}{2} \left( \frac{\bar{x} + \sin \frac{2\alpha}{2}}{2} \right) \)
- **\( \bar{y} \)** = \( \left( \frac{1 - \cos \alpha}{2} \right) \)  
- **\( M_{xx} \)** = \( A \bar{y} = r^{3}t \left( (1 - \cos \alpha) \right) \)
- **\( I_{xx} \)** = \( \frac{r^{3}t}{2} \left( \bar{x} - \sin \frac{2\alpha}{2} \right) \)

### TABLE 4

<table>
<thead>
<tr>
<th>Centroids of Trapezoidal Areas</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Ratio ( \frac{a}{h} )</th>
<th>Distance ( x )</th>
<th>Distance ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.01</td>
<td>0.4992</td>
<td>0.5008</td>
</tr>
<tr>
<td>1.02</td>
<td>0.4994</td>
<td>0.5016</td>
</tr>
<tr>
<td>1.03</td>
<td>0.4996</td>
<td>0.5024</td>
</tr>
<tr>
<td>1.04</td>
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### Diagram

[Diagram of CIRCULAR ARC with labels and equations]
where $I_x$, $I_y$, and $I_z$ are generally referred to as the rolling, pitching, and yawing moments of inertia of the airplane.

$w = \text{weight of the items in the airplane}$

$x$, $y$, and $z$ equal the distances from the axes thru the center of gravity of the airplane and the weights $w$. The last term in each equation is the summation of the moments of inertia of the various items about their own $X$, $Y$, and $Z$ centroidal axes.

If $w$ is expressed in pounds and the distances in inches, the moment of inertia is expressed in units of pound-inches squared, which can be converted into slug feet squared by multiplying by $1/32.16 \times 144$.

Example Problem 1. Determine the gross weight center of gravity of the airplane shown in Fig. A3.3. The airplane weight has been broken down into the 10 items or weight groups, with their individual c.g. locations denoted by the symbol $w$.

Solution. The airplane center of gravity will be located with respect to two rectangular axes. In this example, a vertical axis thru the centerline of the propeller will be selected as a reference axis for horizontal distances, and the thrust line as a reference axis for vertical distances. The general expressions to be solved are:

$$x = \frac{\Sigma w x}{\Sigma w} \quad \text{distance to airplane c.g. from ref. axis A-B}$$

$$y = \frac{\Sigma w y}{\Sigma w} \quad \text{distance to airplane c.g. from ref. axis X-X}$$

Table 4 gives the necessary calculations, whence

$$\hat{x} = 417100 = 123.3$" aft of $\text{propeller}$

$$\hat{y} = 5480 = -1.74" \quad \text{(below thrust line)}$$

Example Problem 2. Determine the moment of inertia about the horizontal centroidal axis for the area shown in Fig. A3.4.

Solution. We first find the moment of inertia for the horizontal reference axis. In this solution, this arbitrary axis has been taken as axis $x'x'$ thru the base as shown. Having this moment of inertia, a transfer to the centroidal axis can be made. Table 5 gives the detailed calculations for the moment of inertia about axis $x'x'$.

$$I_{xx} = \text{moment of inertia about centroidal axis}$$

$$I_{xx} = I_{xx'} - A \bar{y} = 79.47 - 6.18 \times 2.91" = 27.2 \text{ in}^4$$

Radius of Gyration, $R_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{27.2} = 2.1"$

Fig. A3.4

Example Problem #3. Determine the moment of inertia of the stringer cross section shown in Fig. A3.5 about the horizontal centroidal axis.

Solution. A horizontal reference axis $x'x'$ is assumed as shown. The moment of inertia is first calculated about this axis and then transferred to the centroidal axis $xx$. See Table 6.
Table 5

<table>
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<tr>
<th>Portion</th>
<th>Area A</th>
<th>y</th>
<th>Ay</th>
<th>Ay^2</th>
<th>I_{cx}</th>
<th>( I_{cx} \times \frac{x}{y} )</th>
<th>Ay^2 \times \frac{x}{y}</th>
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<tr>
<td>C</td>
<td>2.12</td>
<td>3.75</td>
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<td>0.375 x 3/36 x 1/12 x 1/12 x 5 = .07</td>
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<td>79.47 in^4</td>
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Table 6

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<th>Ay</th>
<th>Ay^2</th>
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<th>( I_{cx} \times \frac{x}{y} )</th>
<th>Ay^2 \times \frac{x}{y}</th>
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\[ \bar{y} = \frac{\sum A y}{\sum A} = .069 = 0.0465^* \]

\[ I_{xx} = I_{x'}x' - A\overline{y}^2 = .05787 - .1933 \times 0.0465^2 = .06745 \text{ in}^4 \]

Radius of gyration, \( r_{x-x} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{0.06745}{0.1933}} = .55^* \)

Detailed explanation of Table 6: -

Portion 1 + 1'
\[ I_{cx} = \frac{1}{12} bd^3 = \left(\frac{.04 \times .75^3}{12}\right) \times 2 = .00281 \text{ in}^4 \]

Portion 2 (ref. table 1)
\[ y = .375 + \frac{4}{3\pi} \left(R + R + R^*\right) = .375 + \frac{4}{3\pi} \left(5.5^2 + 5.5 x .54 + .54^2\right) = .375 + 331 = .706^* \]

By approx. method see Table 1.
\[ y = .375 + 2 \cdot \frac{r}{\pi} = .375 + 2 \times \left(\frac{.82}{\pi}\right) = .706^* \]

\[ I_{cx} = \frac{.1098 \times (R + R^*)^2 (R + R^*) - .283 R^*^2 R^*^2}{R + R^*} = \frac{.1098 \times .2916 + .25 \times .04 \times 1.04 - .588 \times .2916 \times .25 \times .04}{1.04} = .000475 - .000793 = .001682 \text{ in}^4 \]

Approx. \( I_{cx} = .3 \text{ in}^4 \times .04 = .001668 \text{ in}^4 \).

Portion 3 + 3' (ref. Table 1)
\[ y = .375 + \frac{4}{3\pi} \left(\frac{.0641 + .0725 + .0835}{3}\right) = .375 + \frac{2.26}{3\pi} = .172 \text{ in}^4 \]

\[ I_{cx} = \left[ .1098 \times (.0641 + .0625 + .04 \times .54 - .283 \times .0641 \times .0625 \times .04 \times .54) \right] = .002375 \text{ in}^4 \]

approx. \( I_{cx} = .3 \times .04 \times .27^2 = .002365 \text{ in}^4 \)

Problem 4. Determine the moment of inertia of the flywheel in Fig. A3.5a about axis of rotation. The material is aluminum alloy casting (weight = .1 lb. per cu. inch.)

Solution: The spokes may be treated as slender round rods and the rim and hub as hollow cylinders. (Refer to Table 2)

Rim,
weight \( W = \pi (R^2 - r^2) x 3 x .1 = \pi (5^2 - 4^2) .3 = 9.48 \text{ lb.} \)

\[ I = \frac{\pi}{2} (R^2 + r^2) = \frac{\pi}{2} (5^2 + 4^2) = 173.7 \text{ lb. in}^2 \]

Hub,
weight \( W = \pi (1^2 - .5^2) x 3 x .1 = .707 \text{ lb.} \)

\[ I = \frac{\pi}{2} (1^2 + .5^2) = .44 \text{ lb. in}^2 \]

Spokes,
Length of spoke = 3''
Weight of spoke = 3 x .25'' x .1 = .886''

I of one spoke = \( \frac{W L^2}{12 + \frac{W}{60}} = .588 \times 3^2 / 12 + .588 \times 2.5^2 = 4.10 \text{ lb. in}^2 \)

I for 4 spokes = 4 x 4.10 = 16.40

Total I of wheel = 16.40 + .44 + 173.7 = 190.54 lb. in^2

I = 190.54/32.16 x 144 = .0411 slug ft.

Example Problem 5. Moment of Inertia of an Airplane.

To calculate the moment of inertia of an airplane about the coordinate axes through the gross weight center of gravity, a break down of the airplane weight and its distribution is
necessary, which is available in the weight and balance estimate of the airplane.

Table 6A shows the complete calculation of the moments of inertia of an airplane. This table is reproduced from N.A.A.O.A. Technical note #575, "Estimation of moments of inertia of airplanes from Design Data."

**Explanation of Table**

Fig. A3.5b shows the reference of planes and axes which were selected. After the moments of inertia have been determined relative to these axes the values about parallel axes through the center of gravity of the airplane are found by use of the parallel axis theorem.

Column (1) of Table 6A gives breakdown of airplane units or items.

Column (2) gives the weight of each item.

Columns (3), (4) and (5) give the distance of the c.g. of the items from the reference planes or axes.

Columns (6) and (7) give the first moments of the item weights about the y' and x' reference axes.

Columns (8), (9) and (10) give the moment of inertia of the item weights about the reference axes.

Columns (11), (12) and (13) give the moments of inertia of each item about its own centroidal axis parallel to the reference axes. Such items as the fuselage skeleton, wing panels and engine have relatively large values for their centroidal moments of inertia.

The last values in Columns (3) and (5) give the distances from the reference planes to the center of gravity of the airplane.

\[
X_{c.g.} = \frac{\Sigma X}{\Sigma W} = 417.024 \text{ (col. 6)} = 115.9 \text{ in.}
\]

\[
Z_{c.g.} = \frac{\Sigma Z}{\Sigma W} = 414.848 \text{ (col. 7)} = 77.8 \text{ in.}
\]

The last values in columns (8), (9) and (10) were obtained by use of the parallel axis theorem, as follows:

\[
Z_{w} \text{ about c.g. of airplane} = 97,391,586 - 5325.3 \times 115.9 = 26,691,595
\]

\[
Z_{w} \text{ at c.g. of airplane} = 33,252,035 - 5325.3 \times 77.8 = 992,035
\]

The third from the last value in columns (11), (12) and (13) give the moments of inertia of the airplane about the x, y and z axes through the c.g. of the airplane. The values are obtained as follows:

\[
l_{y} = \Sigma Z_{w}^{2} + \Sigma Z_{w}^{2} + 31,120,364 = 30,304,014 \text{ lb. in.}
\]

\[
l_{x} = \Sigma Z_{w}^{2} + \Sigma Z_{w}^{2} + 10,287,522 = 14,179,027 \text{ lb. in.}
\]

\[
l_{z} = \Sigma Z_{w}^{2} + \Sigma Z_{w}^{2} + 2,890,470 = 42,136,503 \text{ lb. in.}
\]

In order to determine the principal inertia axes, the product of inertia of the weight about the reference axes is necessary. Column (14) gives the values about the reference axes. To transfer the product of inertia to the c.g. axes of the airplane, we make use of the parallel axis theorem. Thus:

\[
\Sigma Z_{w} \times Z_{c.g.} = \Sigma Z_{w} \text{ (Ref. axes)} - 2 Z_{w} Z_{c.g.} = 48,657,558 - 5325.3 \times 115.9 \times 77.8 = 839,253 \text{ lb. in.}
\]

To reduce all values to slug ft., multiply \[
\frac{1}{32.17}
\]

\[
1 \quad 144
\]

Hence \[l_{x} = 3061, l_{y} = 6680, l_{z} = 9096, l_{xz} = 161\]

Having the inertia properties about the coordinate c.g. axes, the moments of inertia about the principal axes are determined in a manner as explained for areas. (See A3.13).

The angle \(\phi\) between the X and Z axes and the principal axes is given by,

\[
\tan 2 \phi = \frac{2 Z_{w} Z_{c.g.}}{Z_{w}^{2} - Z_{c.g.}^{2}} = 2 \times 191 = .05998 \text{ hence } \phi = \frac{1}{2} 45^\circ
\]
### CENTROIDS, CENTER OF GRAVITY, MOMENTS OF INERTIA

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*(Table 6a From N.A.C.A. Tech. Note #75)*
The principal moments of inertia are given by following equation.
\[ I_{xp} = I_y \cos^2 \theta + I_z \sin^2 \theta - I_{xz} \sin 2 \theta. \] (See Art. A3.11)
\[ I_{yp} = I_y \]
\[ I_z = I_x \sin^2 \theta + I_z \cos^2 \theta + I_{xz} \sin 2 \theta \]
Substituting
\[ I_{xp} = 3096 x (0.9996)^2 + 9096 x (0.0300)^2 - 181 x \]
\[ 0.0599 = 3096 \]
\[ I_{zp} = 3096 x (0.0300)^2 + 9096 x (0.9996)^2 + 181 x \]
\[ 0.0599 = 9192 \]
\[ I_{yp} = 6680 \]

A3.7b Problems

(1) Determine the moment of inertia about the horizontal centroidal axis for the beam section shown in Fig. A3.6.

(2) For the section as shown in Fig. A3.7 calculate the moment of inertia about the centroidal Z and X axes.

(3) Determine the moment of inertia about the horizontal centroidal axis for the section shown in Fig. A3.8.

(4) In the beam cross-section of Fig. A3.9 assume that the four corner members are the only effective material. Calculate the centroidal moments of inertia about the vertical and horizontal axes.

A3.8 Product of Inertia

In various engineering problems, particularly those involving the calculation of the moments of inertia of unsymmetrical sections, the expression \( I_{xy} \) is used. This expression is referred to as the product of inertia of the area with respect to the rectangular axes \( x \) and \( y \). The term, product of inertia of an area, will be given the symbol \( I_{xy} \), hence
\[ I_{xy} = \int xy \, dA \] (1)
The unit, like that of moment of inertia, is expressed in inches or feet to the 4th power. Since \( x \) and \( y \) may be either positive or negative, the term \( I_{xy} \) may be zero or either positive or negative.

Product of Inertia of a Solid. The product of inertia of a solid is the sum of the products obtained by multiplying the weight of each small portion in which it may be assumed to be divided by the product of its distances from two of the three coordinate planes through a given point.

Thus with respect to planes \( X \) and \( Y \)
\[ I_{xy} = \int xy \, dW \]
\[ I_{xz} = \intxz \, dW \]
\[ I_{yz} = \intyz \, dW \]

A3.9 Product of Inertia for Axes of Symmetry.

If an area is symmetrical about two rectangular axes, the product of inertia about these axes is zero. This follows from the fact that symmetrical axes are centroidal \( x \) and \( y \) axes.

If an area is symmetrical about only one of two rectangular axes, the product of inertia, \( I_{xy}dA \), is zero because for each product \( xydA \) for an element on one side of the axis of symmetry, there is an equal product of opposite sign for the corresponding element of area on the opposite side of the axis, thus making the expression \( /ydA \) equal to zero.

A3.10 Parallel Axis Theorem

The theorem states that, "the product of inertia of an area with respect to any pair of co-planar rectangular axes is equal to the product of inertia of the area with respect to a pair of parallel centroidal axes plus the product of the area and the distances of the centroid of the total area from the given pair of axes". Or, expressed as an equation,
\[ I_{xy} = I_{xy} + A(X^2 - y) \]
(2)

This equation is readily derivable by referring to Fig. A3.10. \( Y \) and \( Z \) are centroidal axes for a given area. \( Y \) and \( Z \) are parallel axes passing through point 0.

The product of inertia about axes \( Y \) and \( X \) is
\[ I_{xy} = \int (x + X)(y + Y) \, dA \]
\[ = \int xydA + XYZ / dA + X / y dA + Y / x dA \]

The last two integrals are each equal to zero, since \( /ydA \) and \( /xdA \) refer to centroidal axes. Hence, \( I_{xy} = \int xydA + XYZ / dA \), which can be written in the form of equation (2).
A3.11 Moments of Inertia with Respect to Inclined Axes

Unsymmetrical beam sections are very common in aircraft structure, because the airfoil shape is generally unsymmetrical. Thus, the general procedure with such sections is to first find the moment of inertia about some set of rectangular axes and then transfer to other inclined axes. Thus, in Fig. A3.11 the moment of inertia of the area with respect to axis X1 Y1 is:

\[ I_{X1} = \int \int (y_x^2 \sin \theta - x_x^2 \cos \theta) \, dA \]

and likewise in a similar manner, the following equation can be derived:

\[ I_{Y1} = \int \int (y_x^2 \cos \theta + x_x^2 \sin \theta - 2 I_{XY} \sin \theta \cos \theta) \, dA \]

By adding equations (3) and (4), we obtain

\[ I_{X1} + I_{Y1} = I_X + I_Y + \int \int \frac{y \sin \theta}{x \cos \theta + y \sin \theta} \, dA \]

or the sum of the moments of inertia of an area with respect to all pairs of rectangular axes, thru a common point of intersection, is constant.

A3.12 Location of Axes for which Product of Inertia is Zero.

In Fig. A3.11

\[ I_{X1} Y_1 = \int \int (x \cos \theta + y \sin \theta) (y \cos \theta - x \sin \theta) \, dA \]

\[ = \left( \cos \theta \sin \theta / x \cos \theta + y \sin \theta \right) \, dA \]

\[ = I_{XY} \cos 2 \theta + \frac{1}{2} \left( I_X - I_Y \right) \sin 2 \theta \]

Therefore, \( I_{X1} Y_1 \) is zero when

\[ \tan 2 \theta = \frac{2 I_{XY}}{I_Y - I_X} \]

A3.13 Principal Axes.

In problems involving unsymmetrical bending, the moment of an area is frequently used with respect to a certain axis called the principal axis. A principal axis of an area is an axis about which the moment of inertia of the area is either greater or less than for any other axis passing thru the centroid of the area.

Axes for which the product of inertia is zero are principal axes.

Since the product of inertia is zero about symmetrical axes, it follows that symmetrical axes are principal axes.

The angle between a set of rectangular centroidal axes and the principal axes is given by equation (6).

Example Problem 4.

Determine the moment of inertia of the angle as shown in Fig. A3.12 about the principal axes passing through the centroid. Solution: Reference axes X and Y are assumed as shown in Fig. A3.12 and the moment of inertia is first calculated about these axes. Table 8 gives the calculations. The angle is divided into the two portions (1) and (2).

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<th>x</th>
<th>y</th>
<th>Ax</th>
<th>Ay</th>
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<th>( I_y )</th>
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<td>( 12 x 1.5^{1/2} = 0.0012 )</td>
<td>( 12 x 1.5^{1/2} = 0.0070 )</td>
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<td>2</td>
<td>0.500</td>
<td>1.25</td>
<td>1.25</td>
<td>0.625</td>
<td>0.0625</td>
<td>0.7800</td>
<td>0.0078</td>
<td>0.0781</td>
<td>( \frac{1}{12} x^{2/3} \times 167 )</td>
<td>( \frac{1}{12} x^{2/3} \times 167 )</td>
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<td>0.875</td>
<td>0.918</td>
<td>0.3435</td>
<td>0.7858</td>
<td>0.2168</td>
<td>0.1132</td>
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</table>
Location of centroidal axes:

\[ \bar{y} = \frac{ZAy}{2A} = 0.8719 = 0.757^\circ \]

\[ \bar{x} = \frac{ZAx}{2A} = 0.2435 = 0.392^\circ \]

Transfer moment of inertia and product of inertia from reference X and Y axes to parallel centroidal axes:

\[ I_x = I_{cx} - A\bar{x}^2 = 0.955 - 0.875 \times 0.757^2 = 0.440 \]

\[ I_y = I_{cy} - A\bar{y}^2 = 0.291 - 0.875 \times 0.392^2 = 0.157 \]

\[ I_{xy} = I_{cyx} - A\bar{x}\bar{y} = 0.1123 - 0.875 \times 0.392 = -0.150 \]

Calculate angle between centroidal X and Y axes and principal axes through centroid as follows:

\[ \tan 2\theta = \frac{2I_{xy}}{I_x - I_y} = 2(-0.150) = -0.30 = 1.06 \]

\[ \theta = 46^0 - 40^0 = 23^0 \pm 20^0 \]

Calculate moments of inertia about centroidal principal axes as follows:

\[ I_{xp} = I_x \cos^2 \theta - I_y \sin^2 \theta - 2I_{xy} \sin \theta \cos \theta = 0.44 \times 0.318^2 + 0.157 \times 0.3955^2 - 2(-0.150) \times 0.3965 \times 0.918 = 504 \text{ in}^4 \]

\[ I_{yp} = I_x \sin^2 \theta - I_y \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta = 0.44 \times 0.3965 + 0.157 \times 0.318^2 + 2(-0.150) \times 0.3965 \times 0.918 = 0.092 \text{ in}^4 \]

Example Problem 5:

Fig. A3.13 shows a typical distributed flange - 2 cell - wing beam section. The upper and lower surface is stiffened by Z and bulb angle sections. Determine the moment of inertia of the section about the principal axes.

Solution:

The properties of the cross-section depend upon the effective material which can develop resisting axial stresses. The question of effective material is taken up in later chapter. Table 9 shows the calculations for the moment of inertia about the assumed rectangular reference axes XX and YY (see Fig. A3.13). The cross-sectional has been broken down into 16 stringers as listed in column 1. For the top surface, a width of 30 thicknesses of the 0.032 skin is assumed to act with the stringers and a width of 25 thicknesses of the 0.04 skin (see Col. 2). On the lower surface, the skin half way to adjacent stringers is assumed acting with each stringer, or the entire skin is effective. Column 4 gives the combined area of each stringer unit and is considered as concentrated at the centroid of the stringer and effective skin. All distances, x and y, columns 5 and 8, have been scaled from a large drawing.

![Fig. A3.12]

![Fig. A3.13]

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Location of centroidal axes with respect to ref. axes:

\[ \bar{y} = \frac{ZAy}{2A} = 1.465 = -0.396^\circ \]

\[ \bar{x} = \frac{ZAx}{2A} = 58.238 = -15.74^\circ \]

\[ I_x = 187.04 - 3.70 x 0.358^3 = 186.5 \text{ in}^4 \]

\[ I_y = 1248.36 - 3.70 x 15.74^3 = 431.7 \text{ in}^4 \]

\[ I_{xy} = -13.35 - (3.70 x 0.396 x 15.74) = 36.41 \text{ in}^4 \]

\[ \tan 2\theta = \frac{2I_{xy}}{I_x - I_y} = 2(-36.41) = -25696 \\
I_{xp} = I_x \cos^2 \theta - I_y \sin^2 \theta - 2I_{xy} \sin \theta \cos \theta = 186.46 x 0.358^2 - 431.7 x 14.53^2 = -36.41 x 0.396 x (-1.438) = 181.2 \text{ in}^4 \]
### A3.12 Centroids, Center of Gravity, Moments of Inertia

\[ I_{yx} = I_x \sin \theta + I_y \cos \theta \cdot \theta + 2 I_{xy} \sin \theta \cos \theta \]

\[ \cos \theta = 186.46 \times 1.4332 \times 0.317 \times 0.0986 \times 2 = 36.41 \times 36.59 \times (-1.433) = 437 \text{ in}^4 \]

### A3.14 Section Properties of Typical Aircraft Structural Sections

Table A3.10 through A3.15 and Chart A3.1 give the section properties of a few structural shapes common to aircraft. Use of these tables will be made in later chapters of this book.

### Table A3.10 Properties of Zee Sections

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### Table A3.11 Properties of Channel Sections

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**Note:** The table continues with more entries.
### Table A3.12
Properties of Extruded Aluminum
Alloy Equal Leg Angles. (Ref. 1938
Alcoa Handbook)

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### Table A3.14
Properties of Unequal Angles

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A3.15 Problems

Figure A3.14

Figure A3.15

(1) For the section of Fig. A3.14 determine the moment of inertia about each of the principal axes $X_p$ and $Z_p$.

(2) Calculate the moment of inertia of the section in Fig. A3.15 about the principal axes.

Figure A3.16

Figure A3.17

(3) Fig. A3.16 illustrates a box type beam section with six longitudinal stringers. Determine the moment of inertia of the beam section about the principal axes for the following assumptions:

(a) Assume the beam is bending upward putting the top portion in compression and the lower portion in tension. Therefore, neglect the lower side since it has very little resistance to compressive stresses. The sheet on the bottom side is effective since it is in tension. For simplicity neglect the vertical webs in the calculations.

(b) Reverse the conditions in (a) thus placing top side in tension and lower side in compression.

(4) For the three stringer single cell box beam section in Fig. A3.17, calculate the moments of inertia about the principal axes. Assume all web or wall material ineffective.

Figure A3.18

Figure A3.19

(5) For the beam section in Fig. A3.16, calculate the moment of inertia about the principal axes assuming the four stringers as the only effective material.

(6) Fig. A3.19 shows a wing beam section with a cut-out on the lower surface. Determine the moments of inertia about the principal axes assuming the eight stringers are the only effective material.

Figure A3.20

(7) Fig. A3.20 shows a 3 cell multiple flange beam. The 7 flange members on the upper face of beams have an area of .3 sq. in. each and those on the bottom skin .2 sq. in. each. The bottom skin is .03 inches in thickness. Compute the moments of inertia about the principle axes assuming that the flange members and the bottom skin comprise the effective material.

Figure A3.21

(8) Fig. A3.21 shows the cross-section of a small fuselage. The dashed line represents a cut-out in the structure due to a door. Assume each of the 13 stringers have an area of .01 sq. in. Consider fuselage skin ineffective. Calculate the moment of inertia of the effective section about the principal axes.
CHAPTER A4
GENERAL LOADS ON AIRCRAFT

A4.1 Introduction.

Before the structural design of an airplane can be made, the external loads acting on the airplane in flight, landing and take-off conditions must be known. The complete determination of the air loads on an airplane requires a thorough theoretical knowledge of aerodynamics, since modern aircraft fly in sub-sonic, trans-sonic and super-sonic speed ranges. Furthermore, there is a wide range of wing configurations, such as the straight tapered wing, the swept wing and the delta wing, and many of these wings often include leading and trailing edge devices for promoting better lift or control characteristics. The presence of power plant nacelle units, external fuel tanks, etc. are units that effect the airflow around the wing and thus effect the magnitude and distribution of the air forces on the wing. Likewise, the fuselage or airplane body itself influences the airflow over the wing. The theoretical calculation of the airloads on the airplane is too large a subject to be covered in a structures book and it is customary in college aeronautical curricula to provide a separate course for this subject.

In most airplane companies the loads on the airplane are determined by a group of engineers assigned to the Structures Analysis Section and this group is often referred to as the Aircraft Load Calculation group. While the work of this group is primarily based on the use of aerodynamics, it is that phase of aerodynamics which is concerned with determining the magnitude and distribution of the air loads on the airplane so that the airplane structure can be properly designed to support these air forces safely and efficiently. The engineering department of an airplane company has a distinct or separate aerodynamics section, but in general their responsibility is the use of the subject of aerodynamics to insure or guarantee the performance, stability and control of the airplane.

A basic general over-all knowledge of the loads on aircraft is desirable in the study of aircraft structural theory, and hence this chapter attempts to give this information. In a later chapter dealing with wing design, this subject will be further expanded.


Because an airplane is designed to carry out a definite job, there result many types of aircraft relative to size, configuration and performance. For example, a commercial transport like the Douglas DC-8 is designed to do a job of transporting a certain number of passengers safely, efficiently and comfortably over various distances between airports. On the other hand the Air Force Fighter type of aircraft has a job of shooting down enemy aircraft or protecting slower friendly aircraft. To do this job efficiently requires a far different configuration as compared to the DC-8 transport. Furthermore the Fighter type airplane must be maneuvered far more sharply to do its required job as compared to the DC-8 in doing its required job.

In general the magnitude of the air forces on an airplane depend on the velocity of the airplane and the rate at which this velocity is changed in magnitude and direction (acceleration). The magnitude of the flight acceleration factor may be governed by the capacity of the human body to withstand these acceleration inertia forces without injury which is the situation in a fighter type of airplane. On the other hand the maneuvering accelerations for the DC-8 are not dictated by what the human body can withstand, but are determined by what is necessary to safely transport passengers from one airport to another.

Designing the airplane structure for loads greater than the airplanes suffers in the performance of its required job, obviously will add considerable weight to the airplane and decrease its performance or over-all efficiency relative to the job it is designed to do.

To particularly insure safety in the air-transportation, along with uniformity and efficiency of design, the government aeronautical agencies (civil and military) have definite requirements for the various types of aircraft relative to the magnitude of loads to be used in the structural design of aircraft. In referring in general to these specified aircraft loads two terms are used as follows:-

Limit or Applied Loads.

The terms limit and applied refer to the same loads with the civil agencies (C.A.A.) using the term limit and the military agencies using the term applied.

Limit loads are the maximum loads anticipated on the airplane during its lifetime of service.

The airplane structure shall be capable of supporting the limit loads without suffering detrimental permanent deformations. At all loads up to the limit loads the deformation of the structure shall be such as not to interfere with the safe operation of the airplane.

Ultimate or Design Loads.

These two terms are used in general to mean
the same thing. Ultimate or Design Loads are equal to the limit loads multiplied by a factor of safety (F.S.) or

Design Loads = Limit or Applied Loads times F.S.

In general, the over-all factor of safety is 1.5. The government requirements also specify that these design loads be carried by the structure without failure.

Although aircraft are not supposed to undergo greater loads than the specified limit loads, a certain amount of reserve strength against complete structural failure of a unit is necessary in the design of practically any machine or structure. This is due to many factors such as:—

- (1) The approximations involved in aerodynamic theory and also structural stress analysis theory;
- (2) Variation in physical properties of materials;
- (3) Variation in fabrication and inspection standards. Possibly the most important reason for the factors of safety for airplanes is due to the fact that practically every airplane is limited to the maximum velocity it can be flown and the maximum acceleration it can be subjected to in flight or landing. Since these are under the control of the pilot it is possible in emergency conditions that the limit loads may be slightly exceeded but with a reserve factor of safety against failure this exceeding of the limit load should not prove serious from an airplane safety standpoint, although it might cause permanent structural deformations that might require repair or replacements of small units or portions of the structure.

Loads due to airplane gusts, are arbitrary in that the gust velocity is assumed. Although this gust velocity is based on years of experience in measuring and recording gust forces in flight all over the world, it is quite possible that during the lifetime of an airplane, turbulent conditions near storm areas or over mountains or water areas might produce air gust velocities slightly greater than that specified in the load requirements, thus the factor of safety insures safety against failure if this situation would arise.

The broad general category of external loads on conventional aircraft can be broken down into such classifications as follows:—

(1) Air Loads

- Due to Airplane Maneuvers. (under the control of the pilot).
- Due to Air Gusts. (not under control of pilot).

   - Landing on Land. (wheel or ski type).

(2) Landing Loads

- Landing on Water.
- Arresting. (Landing on Aircraft Carriers).

(3) Power Plant Loads

- Thrust.
- Torque.

(4) Take off Loads

- Catapulting.
- Assisted take off with auxiliary short period thrust units.

(5) Special Loads

- Hoisting Airplane.
- Towing Airplane.
- Beaching of Hull type Airplanes
- Fuselage Pressurizing.

(6) Weight and Inertia Loads.

In resolving external loads for stress analysis purposes, it is convenient to have a set of reference axes. The reference axes X Y Z passing through the center of gravity of the airplane as illustrated in Fig. A4.1 are those normally used in stress analysis work as well as for aerodynamic calculations. For convenience the reference axes are often referred to another origin other than the airplane c.g.

![Figure A4.1](image)

**A4.4 Weight and Inertia Forces.**

The term weight is that constant force, proportional to its mass, which tends to draw every physical body toward the center of the earth. An airplane in steady flight (uniform velocity) is acted upon by a system of forces in equilibrium, namely, the weight of the airplane, the air forces on the complete airplane, and the power plant forces. The pilot can change this balanced steady flight condition by changing the engine power or by operating the surface controls to change the direction of the airplane velocity. These unbalanced forces thus cause the airplane to accelerate or de-accelerate.

**Inertia Forces For Motion of Pure Translation of Rigid Bodies.**

If the unbalanced forces acting on a rigid body cause only a change in the magnitude of the velocity of the body, but not its direction, the motion is called translation, and from basic Physics, the accelerating force \( F = Ma \), where \( M \) is the mass of the body or W/g. In Fig. A4.1 the unbalanced force system \( F \) causes the rigid body to accelerate to the right. Fig. A4.2 shows the effect of this unbalanced force in producing
A force on each mass particle of $m_1a$, $m_2a$, etc., thus the total effective force is $\Sigma ma = Ma$. If these effective forces are reversed they are referred to as inertia forces. The external forces and the inertia forces therefore form a force system in equilibrium.

From basic Physics, we have the following relationships for a motion of pure translation if the acceleration is constant:

1. $\mathbf{v} = \mathbf{v}_0 + \mathbf{at}$
2. $\mathbf{s} = \mathbf{v}_0t + \frac{1}{2}\mathbf{at}^2$
3. $\mathbf{v}^2 = \mathbf{v}_0^2 + 2\mathbf{as}$

where,

- $\mathbf{s}$ = distance moved in time $t$
- $\mathbf{v}_0$ = initial velocity
- $\mathbf{v}$ = final velocity after time $t$

Inertia Forces on Rotating Rigid Bodies.

A common airplane maneuver is a motion along a curved path in a plane parallel to the ZE plane of the airplane, and generally referred to as the pitching plane. A pull up from steady flight or a pull out from a dive causes an airplane to follow a curved path. Fig. A4.3 shows an airplane following a curved path. If at point A the velocity is increasing along its path, the airplane is being subjected to two accelerations, namely, $a_n$, tangential to the curve at point A and equal in magnitude to $a_n = \frac{\mathbf{F}_n}{m}$. If $a_n = \frac{\mathbf{F}_n}{m}$ is an acceleration normal to the flight path at A and directed toward the center of rotation (o). From Newton's Law the effective forces due to these accelerations are:

4. $\mathbf{F}_n = m\mathbf{r}\omega^2 = \frac{\mathbf{MV}^2}{T}$
5. $\mathbf{F}_t = m\mathbf{r}\omega^2$

where $\omega$ = angular velocity at the point A.
$g$ = angular acceleration at point A.
$T$ = radius of curvature of flight path at point A.

The inertia forces are equal and opposite to these effective forces as indicated in Fig. A4.3. These inertia forces can then be considered as part of the total force system on the airplane which is in equilibrium.

If the velocity of the airplane along the path is constant, then $a_n$ = zero and thus the inertia force $\mathbf{F}_t = 0$, leaving only the normal inertia force $\mathbf{F}_n$.

If the angular acceleration is constant, the following relationships hold:

6. $\omega = \omega_0 + \frac{1}{2}\mathbf{a}\mathbf{t}^2$
7. $\omega = \omega_0 + \frac{1}{2}\mathbf{a}\mathbf{t}^2$
8. $\omega^2 = \omega_0^2 + 2\mathbf{a}\mathbf{t}$

where $\omega$ = angle of rotation in time $t$
$\omega_0$ = initial angular velocity in rad/sec.
$\omega$ = angular velocity after time $t$.

In Fig. A4.3 the moment $T_o$ of the inertia forces about the center of rotation (o) equals $\mathbf{Mr}\omega = \mathbf{M}\omega^2$. The term $\mathbf{M}\omega^2$ is the mass moment of inertia of the airplane about point (o). Since an airplane has considerable pitching moment of inertia about its own center of gravity axis, it should be included. Thus by the parallel axis:

9. $T_o = I_o + I_{c.g.}\omega$

where $I_o = \frac{1}{2}M^2$ and $I_{c.g.}$ = moment of inertia of airplane about $Y$ axis through c.g. of airplane.

Inertia Forces For Pitching Rotation of Airplane about $Y$ Axis Through c.g. Airplane.

In flight, an air gust may strike the horizontal tail producing a tail force which has a moment about the airplane c.g. In some landing conditions the ground or water forces do not pass through the airplane c.g., thus producing a moment about the airplane c.g. These moments cause the airplane to rotate about the $Y$ axis through the c.g.

Therefore for this effect alone the center of rotation in Fig. A4.3 is not at (o) but at...
the c.g. of airplane, or \( F = 0 \). Thus \( F_n \) and \( F_t \) equal zero and thus the only inertia force for the pure rotation is \( I \cdot \omega \), (a couple) and thus the moment of this inertia couple about the c.g. is \( T \cdot c.g. = I \cdot c.g. \).

As explained before if the inertia forces are included with all other applied forces on the airplane, then the airplane is in static equilibrium and the problem is handled by the static equations for equilibrium.

A4.5 Air Forces on Wing.

The wing of an airplane carries the major portion of the air forces. In level steady flight the vertical upward force of the air on the wing, practically equals the weight of the airplane. The term airfoil is used when referring to the shape of the cross-section of a wing. Figs. A4.4 and A4.5 illustrate the air pressure intensity diagram due to an air-stream flowing around an airfoil shape for both a positive and negative angle of attack. The shape and intensity of this diagram is influenced by many factors, such as the shape of the airfoil itself, as the thickness to chord ratio, the number of the top and bottom surfaces etc. A normal wing is attached to a fuselage and it may support external power plants, wing tip tanks etc. Furthermore the normal wing is usually tapered in planform and thickness and may possess leading and trailing slots and flaps to produce high lift or control effects. The airflow around the wing is affected by such factors as listed above and thus wind tunnel tests are usually necessary to obtain a true picture of the air forces on a wing relative to their chordwise spanwise distribution.

Resultant Air Force, Center of Pressure.

It is convenient when dealing with the balancing or equilibrium of the airplane as a whole, to deal with the resultant of the total air forces on the wing. For example, consider the two air pressure intensity diagrams in Figs. A4.6 and A4.7. These distributed force systems can be replaced by their resultant \( R \), which of course must be known in magnitude, direction and location. The location is specified by a term called the center of pressure which is the point where the resultant \( R \) intersects the airfoil chord line. As the angle of attack is changed the resultant air force changes in magnitude, direction and center of pressure location.

A4.6 Forces on Airplane in Flight.

Fig. A4.10 illustrates in general the main forces on the airplane in an accelerated flight condition.
T = engine thrust.
L = total wing lift plus fuselage lift.
D = total airplane drag.
\( M_a \) = moment of L and D with reference to wing a.c. (aerodynamic center)
W = weight of airplane.
\( I_L \) = inertia force normal to flight path.
\( I_D \) = inertia force parallel to flight path.
\( I_m \) = rotation inertia moment.
E = tail load normal to flight path.

For a horizontal constant velocity flight condition, the inertia forces \( I_L \), \( I_D \), and \( I_m \) would be zero. For an accelerated flight condition involving translation but not angular acceleration about its own c.g. axis, the inertia moment \( I_m \) would be zero, but \( I_L \) and \( I_D \) would have values.

Equations of Equilibrium For Steady Flight.
From Fig. A4.10 we can write:

\[
\begin{align*}
\Sigma F_x &= 0, \quad D + W \sin \phi - T \cos \beta = 0 \\
\Sigma F_y &= 0, \quad L - W \cos \phi + T \sin \beta - E = 0 \\
\Sigma M_y &= 0, \quad -M_a - L_a - D_b + T_c \cos \beta + E_a = 0
\end{align*}
\]

Equations of Equilibrium in Accelerated Flight.

\[
\begin{align*}
\Sigma F_x &= 0, \quad D + W \sin \phi - T \cos \beta - I_D = 0 \\
\Sigma F_y &= 0, \quad L - W \cos \phi + T \sin \beta - I_L - E = 0 \\
\Sigma M_y &= 0, \quad -M_a - L_a - D_b + T_c \cos \beta + E_a + I_m = 0
\end{align*}
\]

Forces - Plus is up and toward tail.
Moment - Clockwise is positive.
Distances from c.g. to force - Plus is up and toward tail.

A4.7 Load Factors.
The term load factor normally given the symbol \( n \) can be defined as the numerical mul-

* The bar through letter \( Z \) has no significance. Same meaning without bar.

forces in steady horizontal flight. \( L \) represents the total airplane lift (wing plus tail).
Therefore \( L = W \). Now assume the airplane is accelerated upward along the \( Z \) axis. Fig. A4.11 shows the additional inertia force \( M_a/g \) acting downward, or opposite to the direction of acceleration. The total airplane lift \( L \) for the unaccelerated condition in Fig. A4.11 must be multiplied by a load factor \( n_L \) to produce static equilibrium in the \( Z \) direction.

Thus, \( n_L = \frac{L - W}{g} \)

Since \( L = W \)
Hence \( n_L = 1 + \frac{a}{g} \)

An airplane can of course be accelerated along the \( X \) axis as well as the \( Z \) axis. Thus in Fig. A4.13 the magnitude of the engine thrust \( T \) is greater than the airplane Drag \( D \), which causes the airplane to accelerate forward. It is convenient to express the inertia force in the \( X \) direction in terms of the load factor \( n_X \) and the
weight \( W \) of the airplane, hence

\[
\frac{\mathbf{n}_x}{\mathbf{W}} = \frac{\dot{\mathbf{a}}}{g} \quad \text{(see Fig. A4.13)}
\]

\( \Sigma F_x = 0 \), whence \( T-D-n_x W = 0 \)

Hence \( n_x = \frac{T-D}{W} \)

Therefore the loads on the airplane can be discussed in terms of load factors. The applied or limit load factors are the maximum load factors that might occur during the service of the particular airplane. These loads as discussed in Art. A4.2 must be taken by the airplane structure without appreciable permanent deformation. The design load factors are equal to the limit load factors multiplied by the factor of safety, and these design loads must be carried by the structure without rupture or collapse, or in other words, complete failure.

A4.8 Design Flight Requirements for Airplane.

The Civil and Military Aeronautics Authorities issue requirements which specify the design conditions for the various classification of airplanes. Generally speaking, any airplane flight altitude can be defined by stating the existing values of load factors (acceleration) and the airspeed (or more properly the dynamic pressure).

The accelerations on the airplane are produced from two causes, namely, maneuvers and air gusts. The accelerations due to maneuvers are subject to the control of the pilot who can manipulate the controls so as not to exceed a certain acceleration. In highly maneuverable military airplanes, an accelerometer is included in the cockpit instruments as a guide to limit the acceleration factor. For commercial airplanes the maneuver factors are made high enough to safely take care of any maneuvers that would be required in the necessary flight operations of the particular type of airplane. These limiting maneuver factors are based on years of operating experience and have given satisfactory results from a safety standpoint without penalizing the airplane from a weight design consideration.

The accelerations due to the airplane striking an air gust are not under the control of the pilot since it depends on the direction and velocity of the air gust. From much accumulated data obtained by installing accelerometers in commercial and military aircraft and flying them in all types of weather and locations, it has been found that a gust velocity of 30 ft. per second appears sufficient.

The speed or velocity of the airplane likewise affects the loads on the airplane. The higher the velocity the higher the aerodynamic wing moment. Furthermore the gust accelerations increase with airplane velocity, thus it is customary to limit the particular airplane to a definite maximum flight velocity. For commercial airplanes the velocity is limited to a reasonable glide speed which is sufficient to take care of reasonable flight operations.

A4.9 Gust Load Factors.

When a sharp edge must strike the airplane in a direction normal to the thrust line (X axis), a sudden change takes place in the wing angle of attack with no sudden change in airplane speed. The normal force coefficient \( C_{nA} \) can be assumed to vary linearly with the angle of attack. Thus in Fig. (a), let point (B) represent the normal airplane force coefficient \( C_{nA} \) necessary to maintain level flight with a velocity \( V \) and point (C) the value of \( C_{nA} \) after a sharp edged gust \( KU \) has caused a sudden change \( \Delta n \) in the angle of attack without change in \( V \). The total increase in the airplane load in the Z direction can therefore be expressed by the ratio \( C_{nA} \) at B.

From Fig. (b) for small angles, \( \Delta n = KU/V \) and from Fig. (a), \( \Delta C_{nA} = m \Delta n \), where \( m \) is the slope of the airplane normal force curve (\( C_{nA} \) per radian).

\[
\frac{KU}{V} \quad \text{Fig. b}
\]

\[
\Delta C_{nA}
\]

\[
\Delta n = \frac{\Delta C_{nA}}{C_{nA}} = \frac{(KU_m)}{V} \quad \text{KUVS} = \frac{KUVS}{W} \quad \text{per radian}.
\]

where

\[ U = \text{gust velocity in ft./sec.} \]

\[ K' = \text{gust correction factor depending on wing loading} \]

\[ Y = \text{indicated air speed in miles per hour} \]

\[ S = \text{wing area in sq. ft.} \]

\[ W = \text{gross weight of airplane.} \]

* NACA Technical Note 2964 (June 1953), proposes that the alleviation factor \( K \) should be replaced by a gust factor, \( K' = 0.88 \text{Mg}/(S + Mg) \). In this expression \( Mg \) is the airplane mass ratio or mass parameter, 2 W/\( \text{pdeg} \), in which \( c \) is the mean geometric chord in feet and \( g \) the acceleration due to gravity.
If \( U \) is taken as 30 ft./sec. and \( m \) as the change in \( C_{L} \) with respect to angle of attack in absolute units per degree, equation (A) reduces to the following

\[
\Delta n = \frac{3KnV}{W/S} \quad \text{(B)}
\]

Therefore the gust load factor \( n \) when airplane is flying in horizontal altitude equals

\[
n = 1 + \frac{3KnV}{W/S} \quad \text{(C)}
\]

and when airplane is in a vertical altitude

\[
n = \frac{3KnV}{W/S} \quad \text{(D)}
\]

A4.10 Illustration of Main Flight Conditions.

Velocity-Load Factor Diagram.

As indicated before the main design flight conditions for an airplane can be given by stating the limiting values of the acceleration and speed and in addition the maximum value of the applied gust velocity. As an illustration, the design loading requirements for a certain airplane could be stated as follows: "The proposed airplane shall be designed for applied positive and negative accelerations of \( \pm 6.0g \) and \( \pm 3.0g \) respectively at all speeds from that corresponding to \( CL_{\text{max}} \) up to 1.4 times the maximum level flight speed. Furthermore, the airplane shall withstand any applied loads due to a 30 ft./sec. gust acting in any direction up to the restricted speed of 1.4 times the maximum level flight speed. A design factor of safety of 1.5 shall be used on these applied loads".

In graphical form these design requirements can be represented by plotting load factor and velocity to obtain a diagram which is generally referred to as the Velocity-acceleration diagram. The results of the above specification would be similar to that of Fig. A4.14. Thus, the lines \( AB \) and \( CD \) represent the restricted positive and negative maneuver load factors which are limited to speeds inside line \( BD \) which is taken as 1.4 times the maximum level flight speed in this illustration. These restricted maneuver lines are terminated at points \( A \) and \( C \) by their intersection with the maximum \( CL \) values of the airplane. At speeds between \( A \) and \( B \), the pilot must be careful not to exceed the maneuver accelerations, since in general, it would be possible for him to manipulate the controls to exceed these values. At speeds below \( A \) and \( C \), there need be no care of the pilot as far as loads on the airplane are concerned since a maneuver producing \( CL_{\text{max}} \) would give an acceleration less than the limited values given by lines \( AB \) and \( CD \).

The positive and negative gust accelerations due to a 30 ft./sec. gust normal to flight path are shown on Fig. A4.14. In this example diagram, a positive gust is not critical within the restricted velocity of the airplane since the gust lines intersect the line \( BD \) below the line \( AB \). For a negative gust, the gust load factor becomes critical at velocities between \( F \) and \( D \) with a maximum acceleration as given by point \( E \).

For airplanes which have a relatively low required maneuver factor the gust accelerations may be critical for both positive and negative accelerations. Examination of the gust equation indicates that the most lightly loaded condition (smallest gross weight) produces the highest gust load factor, thus involving only partial pay load, fuel, etc.

On the diagram, the points \( A \) and \( B \) correspond in general to what is referred to as high angle of attack (H.A.A.) and low angle of attack (L.A.A.) respectively, and points \( C \) and \( D \) the inverted (H.A.A.) and (L.A.A.*) conditions respectively.

Generally speaking, if the airplane is designed for the air loads produced by the velocity and acceleration conditions at points \( A, B, E, F, \) and \( C \) it should be safe from a structural strength standpoint if flown within the specified limits regarding velocity and acceleration.

Basically, the flight condition requirements of the Civil Aeronautics Authority, Army, and Navy are based on consideration of specified velocities and accelerations and a consideration of gusts, thus a student understanding the basic discussion above should have no difficulty understanding the design requirements of these three government agencies.

For stress analysis purposes, all speeds are expressed as indicated air speeds. The "indicated" air speed is defined as the speed which would be indicated by a perfect air-speed indicator, that is, one that would indicate true air speed at sea level under standard atmospheric conditions. The relation between the actual air speed \( V_a \) and the indicated air speed \( V_i \) is given by the equation

\[
V_i = \sqrt{\frac{\rho_a}{\rho_0}} V_a
\]

where

\[
\begin{align*}
V_i & = \text{indicated airspeed} \\
V_a & = \text{actual airspeed} \\
\rho_0 & = \text{standard air density at sea level} \\
\rho_a & = \text{density of air in which } V_a \text{ is attained}
\end{align*}
\]
A4.11 Special Flight Design Conditions.

There are many other flight conditions which may be critical for certain portions of the wing or fuselage structure. Most airplanes are equipped with flaps, to decrease the landing speed and such flaps are lowered at speeds at least twice that of the minimum landing speed. Since the flapped airfoil has different values for the magnitude and location of the airfoil characteristics, the wing structure must be checked for all possible flap conditions within the specified requirement relative to maximum speed at which the flaps may be operated. Generally speaking, the flap conditions will effect only the wing portion inboard of the flap and it is usually only critical for the rear beam web or shear wall and for the top and bottom walls of the torsion box. This is due to the fact that the deflection flap moves the center of pressure considerably aft thus producing more shear load on the rear shear wall as well as torsional moment on the conventional cantilever box metal beam.

The airplane must likewise be investigated for aileron conditions. Operation of the ailerons produces a different air load on each side of the airplane wing which produces an angular rolling acceleration of the airplane. Furthermore, the deflected ailerons change the magnitude and location of the airfoil characteristics, thus calculations must be carried out to determine whether the loads in the aileron conditions are more critical than those for the normal flight conditions.

For angular acceleration resulting from pitching moments due to air gusts on the tail, the loads on the wing should be checked for cases where the engines are attached to the wing and are located forward of the leading edge.

In cases where the landing gear is attached to wing or when the fuel and engines are carried in and on the wing, the loads produced by the wing structure in a landing condition may be critical for some portions of the wing structure inboard of landing gear and engine attachment points.

A4.12 Example Problems Involving Accelerated Motion of Rigid Airplane.

As previously explained, it is general practice to place the airplane under accelerated conditions of motion into a condition of static equilibrium by adding the inertia forces to the applied force system acting on the airplane. It is usually assumed that the airplane is a rigid body. Several example problems will be presented to illustrate this general procedure.

Example Problem 1

Fig. A4.15 illustrates an airplane landing on a Navy aircraft carrier and being arrested by a cable pull T on the airplane arresting hook. If the airplane weight is 12,000 lbs. and the airplane is given a constant acceleration of 3.5g (112.7 ft/sec²), find the hook pull T, the wheel reaction R, and the distance (d) between the line of action of the hook pull and the airplane c.g. If the landing velocity is 60 M.P.H. what is the stopping distance.

![Diagram of airplane landing](image)

Solution:

On contact of the airplane with the arresting cable, the airplane is decelerated to the right relative to Fig. A4.15. The motion is pure translation horizontally. The inertia force is

\[ M_a = \frac{W a}{g} = \frac{12000}{3.5g} = 42000 \text{ lb.} \]

The inertia force acts opposite to the direction of acceleration, hence to the left as shown in Fig. A4.15. The unknown forces T and R can now be solved for by using the static equations of equilibrium.

\[ \Sigma F_x = -42000 + T \cos 10^\circ = 0 \]

hence,

\[ T = 42700 \text{ lb.} \]

\[ \Sigma F_y = -12000 + R - 42700 \times \sin 10^\circ = 0 \]
hence,  \[ R = 19420 \text{ lb.} \]

To find the distance \( d \) take moments about c.g. of airplane,

\[ \Sigma M_c = 19420 \times 24 - 42700 \times d = 0 \]

hence,  \[ d = 10.9 \text{ in.} \]

Landing velocity \( V_o = 60 \text{ M.P.H.} = 88 \text{ ft/sec.} \)

\[ V^2 - V_o^2 = 2as \]

Subt:  \[ a^2 = 88^2 = 2(-112.7) \text{ s} \]

hence stopping distance \( s = 34.4 \text{ ft.} \)

**Example Problem 2**

An airplane equipped with float is catapulted into the air from a Navy Cruiser as illustrated in Fig. A4.16. The catapulting force \( P \) gives the airplane a constant horizontal acceleration of \( 3g(96.6 \text{ ft/sec}^2) \). The gross weight of airplane \( 9000 \text{ lb} \) and the catapult track is \( 35 \text{ ft.} \) long. Find the catapulting force \( P \) and the reactions \( R_a \) and \( R_g \) from the catapult car. The engine thrust is \( 900 \text{ lb.} \) What is airplane velocity at end of track run?

![Fig. A4.16](image_url)

**Solution:**

The forces will be determined just after the beginning of the catapult run, where the car velocity is small, and thus the lift on the airplane wing and the airplane drag can be neglected.

Horizontal inertia force acting toward the airplane tail equals,

\[ M_a = \left(\frac{9000}{g}\right) 3.0g = 27000 \text{ lb.} \]

From statics:

\[ \Sigma F_x = -900 - P + 27000 = 0, \text{ hence } P = 26100 \text{ lb.} \]

To find \( R_a \) take moments about point \( A, \)

\[ \Sigma M_A = 9000 \times 55 + 27000 \times 78 - 900 \times 83 - 85R_a = 0 \]

**Example Problem 3**

Assume that the transport airplane as illustrated in Fig. A4.17 has just touched down in landing and that a braking force of \( 35000 \text{ lb.} \) on the rear wheels is being applied to bring the airplane to rest. The landing horizontal velocity is \( 86 \text{ M.P.H.} \) \( (125 \text{ ft/sec.}) \). Neglecting air forces on the airplane and assuming the propeller forces are zero, what are the ground reactions \( R_a \) and \( R_g \). What is the landing run distance with the constant braking force?

![Fig. A4.17](image_url)

**Solution:**

The airplane is being decelerated horizontally hence the inertia force through the airplane c.g. acts toward the front of the airplane. Since the braking force is given we can solve for the deceleration factor by the equilibrium equation,

\[ \Sigma F_x = 35000 - M_a = 0 \]

hence,

\[ M_a = 35000 \]

or

\[ \frac{(2)}{9} a_x = 35000 \]

whence

\[ a_x = \left(\frac{35000}{100000}\right) 32.2 = 11.27 \text{ ft/sec}^2 \]

To find landing run \( s \),

\[ V^2 - V_o^2 = 2 a_s \]

\[ 0 - 88^2 = 2(-11.27) \text{ s} \]

hence,

\[ s = 635 \text{ ft.} \]
To find $R_x$ take moments about point (A)

\[ \Sigma M_x = 100,000 \times 21 - 35,000 \times 9 + 36 \times R_x = 0 \]

\[ R_x = 47,000 \text{ lb. (2 wheels)} \]

\[ \Sigma F_y = 47,000 - 100,000 + R_1 = 0 \]

\[ R_1 = 53,000 \text{ lb.} \]

**Example Problem 4**

The airplane in Fig. A4.18 weighs 14,000 lb. It is flying horizontally at a velocity of 500 M.P.H. (732 ft/sec) when the pilot pulls it upward into a curved path with a radius of curvature of 2500 ft. Assume the engine thrust and airplane drag equal, opposite and collinear with each other (not shown on Fig. A4.18).

Find:

(a) Acceleration of airplane in Z direction

(b) Wing Lift (L) and Tail (T) forces

(c) Airplane Load factor.

**Solution:**

Acceleration \( a_z = \frac{V^2}{r} = \frac{732^2}{2500} = 214.5 \text{ ft/sec}^2 \)

or \( 214.5/32.2 = 6.67 \text{g} \) (upward).

The inertia force normal to the flight path and acting down equals

\[ \Sigma M_z = \left( \frac{14,000}{g} \right) 6.67g = 93,700 \text{ lb.} \]

Placing this force on the airplane through the c.g. promotes static equilibrium, hence to find tail load \( T \) takes moments about wing aerodynamic center (c.p.)

\[ \Sigma M_{c.g.} = -(14,000 + 93,700) \times 8 + 210 \times T = 0 \]

hence

\[ T = 4100 \text{ lb. (down)} \]

To find Wing Lift (L) use

\[ \Sigma F_z = -4100 - 14,000 - 93,700 + L = 0 \]

\[ L = 111,800 \text{ lb.} \]

**Airplane Load Factor = \( \frac{\text{Airplane Lift}}{W} \)**

\[ = \frac{111,800 - 4100}{14,000} = 7.7 \]

**Example Problem 5**

Assume the airplane as used in example problem 4 is in the same attitude as used in that example problem. Now the airplane is further maneuvered by the pilot suddenly pushing the control stick forward so as to give the airplane a pitching acceleration of 4 rad/sec².

(a) Find the inertia forces and the tail load \( T \), assuming the lift force on the wing does not change.

(b) Find the forces on the jet engine which weighs 1500 lb. and whose c.g. location is shown in Fig. A4.19.

Assume moment of inertia \( I_y \) (pitching) of the airplane equals 300,000 lb. sec², in.

**Solution:**

Fig. A4.19 shows a free body of the airplane with the lift and inertia forces as found in Problem 4.

The additional inertia force due to the angular acceleration \( a = 4 \text{ rad/sec}^2 \) equals,

\[ I_y = 300,000 \times 4 = 1,200,000 \text{ in. lb.} \]

which acts clockwise or counter to the direction of angular acceleration.

The airplane is now in static equilibrium and to find the tail load \( T \) take moments about airplane c.g.

\[ \Sigma M_{c.g.} = 1,200,000 - 111,800 \times 8 - 218 \times T = 0 \]

\[ T = 1400 \text{ lb.} \]

To find \( M_z \) take,

\[ \Sigma F_z = 111,800 - 14,000 - 1400 - \frac{M_z}{g} = 0 \]

\[ M_z = 99,200 \text{ lb.} \]
hence, \( a_g = \left( \frac{38200}{14800} \right) g = 7.1 \frac{g}{ft/sec^2} \).

The c.g. of the engine is 50 inches aft of the airplane c.g. as shown in Fig. A4.19. The force on the engine will be its own weight of 1500 lb., and the inertia forces due to \( a_g \) and \( g \).

Inertia force due to \( a_g \) equals,

\[
M_a = \left( \frac{1500}{g} \right) 7.1g = 10630 \text{ lb.}
\]

Inertia force due to angular acceleration \( c \) equals,

\[
M_c = \frac{1500}{32.2 \times 12} \times 80 \times 4 = 778 \text{ lb. (down)}
\]

Then the resultant force on the engine equals

\[
1500 + 10630 + 778 = 12908 \text{ lb. (down)}
\]

Note if the engine had been forward of the airplane c.g., the inertia force of 778 lb. would act upward instead of downward.

In calculating the inertia force on a certain airplane item due to angular acceleration, the equation \( F = M \cdot a \) assumes that the particular item had negligible mass moment of inertia about its own centroidal \( y \) axis. In the case of a large item this centroidal mass moment of inertia may be appreciable and should be included in the \( I_{c.g.} \) of airplane.

Then to find the inertia force for such an item the equation \( F = M \cdot a \) should be modified to be

\[ F = \left( I_{c.g.} \cdot a \right) / r \]

where \( r \) = distance or arm from airplane c.g. to c.g. of item.

\[ I_{c.g.} = \text{mass moment of inertia of item about airplane c.g. equals } I_y + M \cdot a^2 \]

where \( I_y \) is mass moment of inertia of item about its own centroidal \( y \) axis.

\( F \) = inertia force in lbs. normal to radius \( r \).

Example Problem 6

Fig. A4.20 shows a large transport airplane whose gross weight is 100,000 lb. The airplane pitching moment of inertia

\[ I_y = 40,000,000 \text{ lb. sec}^2 \text{ in.} \]

The airplane is making a level landing with nose wheel slightly off ground. The reaction on the rear wheels is 319,000 lb. inclined at such an angle to give a drag component of 100,000 lb. and a vertical component of 300,000 lb.

Find:

(a) The inertia forces on the airplane.

(b) The resultant load on the pilot whose weight is 180 lb. and whose location is shown in Fig. A4.20.

Solution:

The wing lift will be neglected in this example problem.

The inertia forces on the airplane are forces \( M_{a_x} \) and \( M_{a_y} \) and the couple \( I_{c.g.} \).

To find \( M_{a_x} \) take,

\[ E_{F_x} = 100,000 - M_{a_x} = 0 \]

or

\[ M_{a_x} = 100,000 \text{ lb.} \]

hence,

\[ a_x = \frac{100,000}{M} = \frac{100,000}{100,000} g = g \rightarrow \]

To find \( M_{a_y} \) take,

\[ E_{F_y} = 300,000 - 100,000 - M_{a_y} = 0 \]

hence

\[ M_{a_y} = 200,000 \text{ lb.} \]

To find the inertia couple \( I_{c.g.} \), take moments about airplane c.g.,

\[ E_{M_{c.g.}} = -100,000 \times 120 - 300,000 \times 84 \]

\[ + I_{c.g.} \cdot a = 0 \]

\[ I_{c.g., a} = 37,200,000 \text{ lb.} \]

hence angular acceleration \( a \) = 37,200,000

\[ 40,000,000 = 0.33 \text{ rad/sec}^2 \]
Calculations of resultant load on pilot:

\[
\begin{align*}
\vec{M}_a &= \text{c.g. of pilot} \\
\vec{N}_a &= \text{c.g. airplane} \\
\vec{W} &= 180^\circ \\
\end{align*}
\]

Fig. A4.21

Fig. A4.21 shows the airplane c.g. accelerations.

The forces on the pilot consist of the pilot's weight of 150 lb. and the various inertia forces as indicated in the figure.

\[
\begin{align*}
\vec{M}_a &= \left(\frac{180}{g}\right) \cdot 1g = 180^\circ \\
\vec{M}_b &= \left(\frac{180}{g}\right) \cdot 2.0g = 360^\circ \\
\end{align*}
\]

The inertia force due to the angular acceleration acts normal to the radius arm between the airplane c.g. and the pilot. For convenience this normal force will be replaced by its x and y components.

\[
\begin{align*}
F_x &= \vec{M}_a = \frac{180}{32.2 \times 12} \times 40 \times 0.33 = 17 \text{ lb.} \\
F_y &= \vec{M}_a = \frac{180}{32.2 \times 12} \times 372 \times 0.33 = 161 \text{ lb.} \\
\end{align*}
\]

Total force in x direction on pilot equals

\[
180 - 17 = 163 \text{ lb.}
\]

Total force in y direction:

\[
360 + 180 - 161 = 379 \text{ lb.}
\]

Hence Resultant force R, equals

\[
\sqrt{379^2 + 163^2} = 410^\circ
\]


The example problems of Art. A4.12 assume that the airplane is a rigid body (suffers no structural deformation). On the basis of this assumption the applied loads on the airplane in either flight or landing conditions are placed in equilibrium with the inertia forces which occur due to the acceleration of the airplane. It is obvious that an airplane structure like any other structure is not a rigid body, particularly a cantilever wing which undergoes rather large bending deflections in both flight and landing conditions. Figure A4.21 shows a composite photograph taken of a test wing for the Boeing B-47 airplane. The maximum upward and downward deflections shown are for design loads, which in general are 1.5 times the applied loads. It would not be correct to say that the wing deflections under the applied loads for these two high angle of attack conditions would be 2/3 the deflections shown in the photograph since under the design loads a considerable portion of the wing would be stressed beyond the elastic limit of the material or into the plastic range where the stiffness modulus is considerably less than the modulus of elasticity, hence the deflections under the applied loads would be somewhat less than 2/3 those shown in the photograph. This photograph thus indicates very strikingly that a wing structure is far from being a rigid body.

Static loads are loads which are gradually applied and cause no appreciable shock or vibration of structure. On high speed aircraft, air gusts, flight maneuvers and landing reactions are applied quite rapidly and thus can be classed as dynamic loads. Therefore when these dynamic loads strike a flexible (non-rigid) airplane cantilever wing, a rather large wing deflection is produced and the wing tends to vibrate. This vibration therefore causes additional accelerations of the mass units of the wing which means additional inertia forces on the wing. Furthermore if the time rate of application of the external applied forces approaches the natural bending frequencies of the wing, the vibration excited can produce large additional wing stresses.
Up until World War II practically all airplanes were assumed as rigid bodies for structural design purposes. During the war failure of aircraft occurred under load conditions which the conventional design procedure based on rigid body analysis, indicated satisfactory or safe stresses. The failures were no doubt due to dynamic overstress because the airplane is not a rigid body.

Furthermore, airplane design progress has resulted in thin wings and relatively large wing spans, and in many cases these wings carry concentrated masses, such as, power plants, bomb, wing tip fuel tanks etc. Thus the flexibility of wings have increased which means the natural bending frequencies have decreased. This fact together with the fact that airplane speeds have greatly increased and thus cause air gust loads to be applied more rapidly, or the loading is becoming more dynamic in character and thus the overall load effect on the wing structure is appreciable and cannot be neglected in the strength design of the wing.

General Dynamic Effect of Air Forces on Wing Loads.

The critical airloads on an airplane are caused by maneuvering the airplane by the pilot or in striking a transverse air gust. A transport airplane does not have to be designed for sharp maneuvers producing high airplane accelerations in its job of transporting passengers, thus the time of applying the maneuver loads is considerably more than a fighter type airplane pulling up sharply from high speeds.

Fig. A4.22 shows the result of a pull-up maneuver on the Douglas D.C.3 airplane at 180 M.P.H. relative to load factor versus time of application of load. As indicated the peak load of load factor 3.25 was obtained at the end of one second of time.

The author estimates the natural frequency of the D.C.3 wing to be around 10 to 15 cycles per second, thus a loading time of 1 second against a time of 1/10 or 1/15 for half a wing deflection cycle indicates that dynamic over stress should not be appreciable. In general, it can be said that dynamic over stress under maneuvering loads on transport airplanes is not as great as from other conditions such as air gusts or landing.

Dynamic Effect of Air Gusts.

The higher the air gust velocity and the higher the airplane velocity, the less the time for applying the load on the wing when striking the air gust.

NACA Technical Note 2424 reports the flight test results on a twin-engine Martin transport airplane. Strain gages were placed at various points on the wing structure, and strains were read for various gust conditions for which the normal airplane accelerations were also recorded. Then slow pull-up maneuvers were run to give similar airplane normal accelerations. The wing had a natural frequency of 3.5 cps and the airplane speed was 250 M.P.H. Two of the conclusions given in this report are: (1) The bending strains per unit normal acceleration under air gusts were approximately 20 percent higher than those of slow pull-ups for all measuring positions and flight conditions of the tests, and (2) The dynamic component of the wing bending strains appeared to be due primarily to excitation of the fundamental wing bending mode.

These results thus indicate that air gusts apply a load more rapidly to a wing than a maneuver load giving the same airplane normal acceleration for a commercial transport type of airplane, and thus the dynamic strain effect on the wing is more pronounced for gust conditions.

Fig. A4.23 24 and 25 show results of dynamic effect of air gusts on a large wing as determined by Bisplinghoff*. The results in these figures show that dynamic effects tend to considerably increase wing forces on some portions of the wing and decrease it on other portions.

---

It has also been found that landing loads applied through the conventional landing gear or by water pressure on a flying boat are applied rapid enough to be classed as dynamic loads and such loads applied to wings of large span produce dynamic stresses which cannot be neglected in the safe design of such structures.

**A4.14 General Conclusions on Influence of Dynamic Loading on Structural Design of Airplane.**

The advent of the turbo-jet and the rocket type engines has opened up a range of possible airplane airspeeds hardly dreamed of only a few years ago, and already trans-sonic and supersonic speed airplanes are a common development. From an aerodynamic standpoint such speeds have dictated a thin airfoil section which has thus promoted a high density wing. Thus for airplanes with appreciable wing spans like Military bombers and near future jet commercial transports, which usually carry large concentrated masses on the wing such as engines, fuel tanks etc., the assumption that the airplane is a rigid body is not sufficiently accurate enough because the dynamic stresses are appreciable.

The calculation of the dynamic loading on the wing requires that the mass and stiffness distribution of the wing structure be known. Since these factors are not known when the structural design of a wing is started, the general procedure in design would be to first base the design on the assumption that the wing is a rigid body plus correction factors based on past design experience or available research information to approximately take care of the influence of the elastic wing on the airplane aerodynamic characteristics and the build-up dynamic inertia forces. With the wing thus initially designed by this procedure, it then can be checked by a complete dynamic analysis and modified as the results dictate and then recalculated for the modified elastic wing. This procedure is now practical because of the availability of high speed computers.

**A4.15 PROBLEMS.**

1. The airplane in Fig. A4.26 is being launched from the deck of an aircraft carrier by the cable pull T which gives the airplane a forward acceleration of 3.25g. The gross weight of the airplane is 15,000 lb.

   (a) Find the tension load T in the launching cable, and the wheel reactions R_a and R_b.

   (b) If the flying speed is 75 M.P.H., what launching distance is required and the launching time t?

2. Assume the airplane of Fig. A4.26 is landing at 75 M.P.H. on a runway and brakes are applied to the rear wheels equal to 4 of the vertical rear wheel reaction. What is the hori-

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**Fig. A4.26**

**Fig. A4.27**

**Fig. A4.28**

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3. The flying patrol boat in Fig. A4.27 makes a water landing with the resultant bottom water pressure of 250,000 lb, as shown in the figure. Assume lift and tail loads as shown. The pitching moment of inertia of the airplane is 10 million lb. sec.² in. Determine the airplane pitching acceleration. What is the total load on the crew member who weighs 200 lb, and is located in a seat at the rear end of the hull?

4. The jet-plane in Fig. A4.28 is diving at a speed of 600 M.P.H. when pilot starts a 8g pull-out. Weight of airplane is 18,000 lb. Assume that engine thrust and total airplane drag are equal, opposite and collinear.

   (a) Find radius of flight path at start of pull-out.

   (b) Find inertia force in Z direction.

   (c) Find lift L and tail load T.
CHAPTER A5
BEAMS - SHEAR AND MOMENTS

A5.1 Introduction.

In general, a structural member that supports loads perpendicular to its longitudinal axis is referred to as a beam. The structure of aircraft provides excellent examples of beam units, such as the wing and fuselage. Very seldom do bending forces act alone on a major aircraft structural unit, but are accompanied by axial and torsional forces. However, the bending forces and the resulting beam stresses due to bending of the beam are usually of primary importance in the design of the beam structure.

A5.2 Statically Determinate and Statically Indeterminate Beams.

A beam can be considered as subjected to known applied loads and unknown supporting reactions. If the distribution of the applied known loads to the supporting reactions can be determined from the conditions of static equilibrium alone, namely, the summation of forces and moments equal zero, then the beam is considered as a statically determinate beam. However, if the distribution of the known applied loads to the supporting beam reactions is influenced by the behavior of the beam material during the loading, then the supporting reactions cannot be found by the statical equilibrium equations alone, and the beam is classified as a statically indeterminate beam. To solve such a beam, other conditions of fact based on the beam deformations must be used in combination with the statical equilibrium equations.

A5.3 Shear and Bending Moment.

A given beam is subjected to a certain applied known loading. The beam reactions to hold the beam in statical equilibrium are then calculated by the necessary equations of statical equilibrium, namely:

\[ EV = 0, \text{ or the algebraic summation of all vertical forces equal zero.} \]
\[ EH = 0, \text{ or the algebraic summation of all horizontal forces equal zero.} \]
\[ EM = 0, \text{ or the algebraic summation of all the moments equal zero.} \]

With the entire beam in statical equilibrium, it follows that every portion of the beam must likewise be in statical equilibrium. Now consider the beam in Fig. A5.1. The known applied load of \( P = 100 \) lb. is held in equilibrium by the two reactions of 25 and 75 lbs. as shown and are calculated from simple statics. [Beam weight is neglected in this problem]. Now consider the beam as cut at section a-a and consider the right side portion as a free body in equilibrium as shown in Fig. A5.2. For statical equilibrium, \( EV, EH \) and \( EM \) must equal zero for all forces and moments acting on this beam portion. Considering \( EV = 0 \) in Fig. A5.3:

\[ EV = 75 - 100 = -25 \text{ lb.} \]

thus, under the forces shown, the force system is unbalanced in the V direction, and therefore an internal resisting force \( V_z \) equal to 25 lb. must have existed on section a-a to produce equilibrium of forces in the V direction. Fig. A5.4 shows the resisting shear force, \( V_z = 25 \) lb. which must exist for equilibrium.

Considering \( EM = 0 \) in Fig. A5.3, take moments about some point 0 on section a-a,

\[ EM_z = -75 \times 15 + 100 \times 5 = -625 \text{ in.lbs.} \]

or an unbalanced moment of -625 tends to rotate the portion of the beam about section a-a. A counteracting resisting moment \( M = 625 \) must exist on section a-a to provide equilibrium. Fig. A5.4 shows the free body with the \( V_z \) and \( M_z \) acting.

Now \( EH \) must equal zero. The external forces as well as the internal resisting shear \( V_z \) have no horizontal components. Therefore, the internal forces producing the resisting moment \( M_z \) must be such as to have no horizontal

A5.1
unbalanced force, which means that the resisting moment \( M_i \) in the form of a couple, as shown in Fig. A5.5, or \( M_i = Cd \) or \( Td \) and \( T \) must equal \( C \) to make \( ZH = 0 \).

The tendency of the loads and reactions acting on a beam to shear or move one portion of a beam up or down relative to the adjacent portion of the beam is called the External Vertical Shear, or commonly referred to as the beam Vertical Shear and is represented by the term \( V \).

From equation (1), the Vertical Shear at any section of a beam can be defined as the algebraic sum of all the forces and reactions acting to one side of the section at which the shear is desired. If the portion of the beam to the left of the section tends to move up relative to the right portion, the sign of the Vertical Shear is taken as positive shear and negative if the tendency is opposite. Or in other words, if the algebraic sum of the forces is up on the left or down on the right side, then the Vertical Shear is positive, and negative for down on the left and up on the right.

From equation (2), the Bending Moment at any section of a beam can be defined as the algebraic sum of the moments of all the forces acting to either side of the section about the section. If this bending moment tends to produce compression (shortening) of the upper fibers and tension (stretching) of the lower fibers of the beam, the bending moment is classed as a positive bending moment, and negative for the reverse condition.

**A5.4 Shear and Moment Diagrams.**

In aircraft design, a large proportion of the beams are tapered in depth and section, and also carry a variable distributed load. Thus, to design or check the various sections of such beams, it is necessary to have a complete picture as to the value of the vertical shear and bending moment at all sections along the beam. If these values are plotted as ordinates from a base line, the resulting curves are referred to as Shear and Moment diagrams. A few example Shear and Moment diagrams will be plotted, to refresh the students knowledge regarding these diagrams.

**Example Problem 1.**

Draw a shear and bending moment diagram for the beam shown in Fig. A5.6. Neglect the weight of the beam.

In general, the first step is to determine the reactions.

To find \( R_3 \), take moments about point A.

\[
ZM_A = -4 \times 500 + 1000 \times 5 + 500 \times 13 - 10R_3 = 0
\]

hence \( R_3 = 590 \text{ lb} \).

\[
ZV = -500 + R_A - 1000 - 300 + 690 = 0
\]

hence \( R_A = 1110 \text{ lb} \).

Calculations for Shear Diagram:

We start at the left end of the beam.

Considering a section just to the right of the 500 lb. load, or section 1-1, and considering the portion to the left of the section, the Vertical Shear at 1-1 = \( ZV = -500 \) (negative, down on left.)

\[
+610 \text{ lb} \quad 610 \text{ lb} \quad -300 \text{ lb} \quad -300 \text{ lb}
\]

**Fig. A5.7 (Shear Diagram)**

Next, consider section 2-2, just to left of reaction \( R_A \).

\[
ZV = -500 \text{, or same as at section 1-1.}
\]

Next, consider section 3-3, just to right of \( R_A \).

\[
ZV = -500 + 1110 = 610 \text{ (positive, up on left side of section).}
\]

Next, consider section 4-4, just to left of 1000 load.

\[
ZV = -500 + 1110 = 610 \text{ (same as at section 3-3).}
\]

Section 5-5, to right of 1000 load:

\[
ZV = -500 + 1110 - 1000 = -390 \text{ (down on left).}
\]

Check this shear at section 5-5 by using the portion of the beam to the right of 5-5 as a free body.

\[
ZV = -300 + 690 = 390, \text{ which checks (sign of shear is minus, because } ZV \text{ is up on right). Section 6-6, use the portion to right as a free body:}
\]

\[
ZV = -300 + 690 = 390 \text{ (minus shear).}
\]

Section 7-7:

\[
ZV = -300 \text{ (positive shear, down on right)}
\]
Section 8-8:

\[ ZV = -300 \text{ (positive shear)} \]

Fig. A5.7 shows the plotted values on the shear diagram.

Calculation of the Moment Diagram.

Start at section 1-1, and consider the forces to the left only:

\[ EM = -500 \times 0 = 0 \]

Since sections 2-2 and 3-3 are only a differential distance apart, assume a section just above \( R_A \) and consider the forces on the left side only:

\[ EM = -500 \times 4 = -2000 \text{ in. lb. (Negative moment, because of tension in the top fibers).} \]

Consider the section under the 1000 in. lb. load:

\[ EM \text{ to left} = -500 \times 9 + 1110 \times 5 = 1050 \text{ in. lb. (positive moment, compressing the top fibers).} \]

Check by considering the forces to the right:

\[ EM \text{ right} = 300 \times 3 - 690 \times 5 = -1050 \text{ in. lb.} \]

Next, consider a section over \( R_B \):

\[ EM \text{ right} = 300 \times 3 = 900 \text{ in. lb. (Negative moment, tension in top fibers).} \]

At Section 8-8:

\[ EM \text{ right} = 300 \times 0 = 0 \]

Fig. A5.8 shows the plotted values.

From the above results it may be noticed that when the bending moment is obtained from the forces that lie to the left of any section, the bending moment is positive when it is clockwise. If obtained from the forces to the right, it is positive when counter-clockwise. The student should sketch in the approximate shape of the deflected structure and determine the signs from whether tension or compression exists in the upper and lower fibers.

Example Problem 2

Calculate and draw the shear and moment diagrams for the beam and loading shown in Fig. A5.9.

First, determine the reactions, \( R_A \) and \( R_B \):

\[ EM_A = 36 \times 10 \times 18 + 120 \times 9 - 36R_B = 0 \]

\[ R_B = 210 \text{ lb.} \]

\[ ZV = -120 - 36 \times 10 + 210 + R_A = 0 \]

\[ R_A = 270 \text{ lb.} \]

Shear Diagram:

The vertical shear just to the right of the reaction at \( A \) is equal to 270 up, or positive. This is plotted as line \( ZE \) in Fig. A5.10. The vertical shear at section \( C \) just to the left of the load and considering the forces to the left of the section = 270 - 9 \times 10 = 180 \text{ lb. up, or positive.} \ The vertical shear for any section between \( A \) and \( C \) at a distance \( x \) from \( A \) is:

\[ V_x = 270 - 10x \text{, and hence, the shear decreases at a constant rate of 10 lb./in. from 270 at } A \text{ to 180 at } C \text{.} \]

The vertical shear at section \( D \), just to the right of load is,

\[ \theta_b = ZV \text{ at } E = 270 - 10 \times 9 - 120 = 60 \text{ up, or positive.} \]

The vertical shear between points \( D \) and \( B \), when \( x \) is the distance of any section between \( D \) and \( B \) from \( A \):

\[ V_{pg} = 270 - 120 - 10x \]

At point \( B \), \( x = 36 \):

\[ V_B = 270 - 120 - 10 \times 36 = -210 \text{ lb.} \]

which checks the reaction \( R_B \).

Since the shear decreases at a rate of 10 lb/in. from \( D \) to \( B \), it will be 6\(^\circ\) from \( D \) to a point where the shear is zero, since the shear at \( D \) is 60 lb.

This point could also be located by equating equation (1) to zero and solving for \( x \) as follows:

\[ 0 = 270 - 120 - 10x \text{, or } x = \frac{150}{10} = 15' \text{ from } A \text{.} \]

If the shear diagram has passed through zero under the concentrated load, then the method of equating the shear equation to zero and solving for \( x \) could not be used, thus in general, it is best to draw a shear diagram to find when shear is zero. Fig. A5.10 shows the plotted shear diagram.

Moment Diagram:

At section \( A \) just to the right of reaction \( R_A \), the bending moment, considering the forces to the left, is zero, since the arm of \( R_A \) is zero.

The bending moment at any section between \( A \) and \( C \), at a distance \( x \) from the left reaction \( R_A \), is,

\[ M_x = R_A \times x - \frac{w \times x^2}{2} \]

In equation (2), \( x \) cannot be greater than 9.
The equation for the bending moment between D and B (x greater than 9) is:
\[ M_x = R_A x - P (x - 9) - \frac{wx^2}{2} \]  
\[ = 270 x - 120 (x - 9) - \frac{10x^2}{2} \]  
\[ = 1080 + 150 x - 5x^2 \]  \( (3) \)

At section C, \( x = 9 \), substitute in equation (4):
\[ M_x = 1080 + 150 \times 9 - 5 \times 9^2 = 2025 \text{ in.lb.} \]  (positive, compression in top fibers).

At the point of zero shear, \( x = 15 \).
\[ M = 1080 + 150 \times 15 - 5 \times 15^2 = 2205 \]

Thus, by substituting in equation (2) and (4) the moment diagram as plotted in Fig. A5.11 is obtained.

A5.5 Section of Maximum Bending Moment.

The general expression for the bending moment on the beam of example problem 2 is from equation (3):
\[ M_x = R_A x - P (x - 9) - \frac{wx^2}{2} \]

Now, the value of \( x \) that will make \( M_x \) a maximum or minimum is the value that will make the first derivative of \( M_x \) with respect to \( x \) equal to zero, or
\[ \frac{dM}{dx} = R_A - P - wx = 0 \]  \( (5) \)

Therefore, the value of \( x \) that will make \( M_x \) a maximum or minimum may be found from the equation
\[ R_A - P - wx = 0 \]

But, observation of this equation indicates that the term \( R_A - P - wx \) is the shear for the section at a distance \( x \) from the left reaction. Therefore, where the shear is zero, the bending moment is maximum. Thus, the shear diagram which shows where the shear is zero is a convenient medium for locating the points of maximum bending moment.

A5.6 Relation Between Shear and Bending Moment.

Equation (5) can also be written \( \frac{dM}{dx} = V \), since the right hand portion of equation (5) is equal to the shear.

Hence, \( dM = Vdx \)  \( (6) \)

Which means that the difference \( dM \) between the bending moments at two sections that are a distance \( dx \) apart, is equal to the area \( Vdx \) under the shear curve between the two sections. Thus, for two sections \( x_1 \) and \( x_2 \),

\[ \begin{bmatrix} M_x \\ dM \end{bmatrix} = \begin{bmatrix} x_2 \\ Vdx \end{bmatrix} \]

Thus, the area of the shear diagram between any two points equals the change in bending moment between these two points.

To illustrate this relationship, consider the shear diagram in example problem 2 (Fig. A5.10). The change in bending moment between the left reaction \( R_A \) and the load is equal to the area of the shear diagram between these two points, or
\[ \frac{270 + 180}{2} \times 9 = 2025 \text{ in.lb.} \]
Since the bending moment at the left support is zero, this change therefore equals the true moment at a section under the load \( P \).

Adding to this the area of the small triangle between point D and the point of zero shear, or \( \frac{80}{2} \times 6 = 180 \), we obtain 2205 in.lb. as the maximum moment. This can be checked by taking the area of the shear diagram between the point of zero shear and point B \( = \frac{210}{2} \times 21 = 2205 \text{ in.} \text{lb.} \)


Example Problem 3.

Fig. A5.12 Illustrates a landing gear oleo strut ADEO braced by struts BD and CE. A landing ground load of 15000 lb. is applied through the wheel axle as shown. Let it be required to find the axial load in all members and the shear and bending moment diagram for the oleo strut.

\[ V_c \rightarrow 15.0 \quad 5.77 \quad V_B \]

\[ H_C \rightarrow O \rightarrow C \rightarrow H_A \rightarrow A \rightarrow B \rightarrow H_B \]

\[ 10'' \quad 16'' \]

\[ Pins \ at \ Points \ B, \ C, \ E. \ No \ Vertical \ Resistance \ at \ Point \ A. \]

\[ 15000 \ lb. \]

\[ 16'' \]

**SOLUTION:**

To find \( V_c \) take moments about point B,
\[ \begin{align*}
\sum M = & -15000 \times 0.5 \times 42 + 15000 \times 0.968 \times 8.77 + 20.77 \times V_c - 0, \\
V_c = & 11550 \text{ lb.}
\end{align*} \]

The axial load in member CE therefore equals...
11550/cos 30° or 13330 lb. . \therefore H_C = 11550 \times 15/26 = 6660

To find \( H_A \) take moments about point D,
\[
\Sigma M_D = 11550 \times 15 - 6660 \times 10 = 15000 \times 0.5 \times 32 + 10 \times H_A = 0.
\]
hence, \( H_A = 13340 \) lb.

To find \( V_B \) take \( \Sigma F_y = 0, \)
\[
\Sigma F_y = 15000 \cos 30° + 11550 - V_B = 0.
\]
hence, \( V_B = 24550 \) lb.

The axial load in member BD therefore equals 24550/cos 30° = 28360 lb. (compression). The reaction \( H_B \) therefore equals 28360 x sin 30° = 14180 lb.

Fig. A5.13 shows the I-steel strut as a free body with the reactions at A, D, and E as calculated. Fig. A5.13 also shows the axial load, vertical shear, and bending moment diagrams.

The bending moments due to applied loads without regard to bending deformation of the beam are usually referred to as the primary bending moments. If a member carries axial loads additional bending moments will be produced due to the axial loads times the lateral deflection of the beam, and these bending moments are usually referred to as secondary bending moments. (Arts. 225-30 cover the calculation of secondary moments).

Example Problem 4.

Fig. A5.14 shows a beam loaded with both transverse and longitudinal loads. This beam loading is typical of interior beams in the airplane fuselage which support all kinds of fixed equipment. The reactions for the beam are at points A and B. Required: - Shear and bending moment diagrams.

SOLUTION:

Calculations of reactions at A and B:

To find \( V_B \) take moments about point A,
\[
\Sigma M_A = -500 \times 7 - 500 \times 6 + 1000 \times 20 + 1000 \times 
\sin 45° \times 10 + 1000 \cos 45° \times 2 - 22 \times V_B = 0.
\]
hence, \( V_B = 999.3 \) lb. (up).

To find \( V_A \) take \( \Sigma V = 0, \)
\[
\Sigma V = 999.3 - 1000 - 1000 \sin 45° - 500 + V_A = 0.
\]
hence, \( V_A = 1207.3 \) lb. (up).

To find \( H_B \) take \( \Sigma H = 0, \)
\[
\Sigma H = -500 + 1000 \cos 45° - H_B = 0,
\]
hence \( H_B = 207.1 \) lb.

With the exception of the 1000 lb. load at 45°, all loads are applied to brackets which in turn are fastened to the beam. Therefore the next step is to find the reaction of the loaded brackets at the beam centerline support points. The load at E and the reaction \( H_B \) at B will be also referred to beam centerline.

Fig. A5.15 (a,b,c,d) show the cantilever brackets as free bodies. The reactions at the base of these cantilevers will be determined. These reactions reversed will then be the applied loads to the beam at points C, D, F, and E.

For bracket at C, to find \( H_C \) take \( \Sigma H = 0, \) or obviously \( H_C = 500 \) lb. In like manner use \( \Sigma V = 0 \) to find \( V_C = 500 \) lb. To find \( H_B \) take moments about point C,
\[
\Sigma M_C = -500 \times 2 + 500 \times 8 - M_B = 0,
\]
hence \( M_B = 3000 \) in. lb. The student should check the reactions at the base of the cantilever brackets at D and F (See Fig. b.c).
The load of 1000 at 45° and applied at point E' will be referred to point E' the centerline of beam. Fig. 4 shows the reaction at E due to the load at E'. The reaction at B should also be referred to the beam centerline. Fig. 5.16 shows the beam with the applied loads at points C D E' F and E'. Figs. 5.17, 18 and 19 show the axial load, vertical shear and bending moment diagrams under the beam loading of Fig. 5.16.

![Diagram](image)

The shear diagram is determined in the same manner as explained before. The applied external couples do not enter into the vertical shear calculations. The bending moment diagram can be calculated by taking the algebraic sum of all couples and moments of all forces lying to the one side of a particular section. If it is desired to use the area of the shear diagram to obtain the bending moments, it is necessary to add the couple moments to the shear areas to obtain the true bending moment. For example, the bending moment just to the left of point E will be equal in magnitude to the area of the shear diagram between C and E plus the sum of all applied couple moments between C and E but not including that at E.

To illustrate the calculations: 

\[ (-500 \times 5) + (707.8 \times 10) = 4578 \text{ in. lb. (from area of shear diagram).} \]

\[ (3000 - 4000) = -1000 \text{ in. lb. (from sum of couple moments).} \]

Thus bending moment at E = 4578 - 1000 = 3578 in. lb.

The bending moment at E' right will equal that at E' left plus the couple moment at E or 3579 + 707 = 4285 in. lb.

The student should realize that when couple moments are applied to a beam it is possible to have maximum peak moments without the Vertical Shear passing through zero. To illustrate this fact, consider the beam of Fig. 5.20, namely, a simple supported beam with an externally applied couple moment of 10 in. lb. magnitude at point C the center point of the beam. The shear and bending moment diagrams are as indicated and a maximum bending moment occurs at C but the shear diagram does not pass through zero.

![Diagram](image)

A couple is two equal and opposite forces not in the same straight line. Let it be assumed that the 10 in. lb. couple is made up of forces equal to 100 lb. each and an arm between them of 0.1 inch as illustrated in Fig. 5.21.

![Diagram](image)

The shear diagram is as shown in Fig. 5.21 and now passes through zero under each of the couple forces. Thus, if we assume the couple moment has a dx arm the shear to the right of C is one lb. and then changes to some unknown negative value and then back to one lb. positive as the distance dx is covered in going to the left. Thus the shear goes to zero twice in the region of point C.

### A5.7 Moment Diagrams as Made up of Parts

In calculating the deflection of statically determinate beams (See Chapter A7) and solving statically indeterminate structures (See Chapter A8), the area under the bending moment curve is required, thus it is often convenient to treat each load and reaction as a separate acting force and draw the moment diagram for each force. The true bending moment at a particular point will then equal the algebraic summation of the ordinates of all the various moment curves at this particular point or adding the various separate moment diagrams will give the true bending moment diagram. Figs. 5.22 and 5.23 illustrate the drawing of the bending moment diagram in parts. In these examples, we start from the left end and proceed to the right end and draw the moment curve for each force as though the beam was a cantilever with the fixed
at B and thus leaving only 3 unknown elements of the reaction at B. Fig. A5.26 shows the bending moment curves for each load acting separately on this cantilever frame. Fig. A5.26 shows the true bending moment as the summation of the various moment curves of Fig. A5.25.

As another solution of this fixed-ended frame, one could assume the statically-determinate modification as a frame pinned at A and pinned with rollers at B as illustrated in Fig. A5.27. This assumed structure is statically-determinate because there are only 3 unknown elements, namely the magnitude and direction of the reaction at A and the magnitude of the reaction at B. For convenience the reaction at A is resolved into two magnitudes as H and V components. The reactions V_A, H_A, and V_B can then be found by statics and the results are shown on Fig. A5.27. Fig. A5.28 shows the bending moment diagram on this frame due to each load or reaction acting separately, starting at A and proceeding clockwise to B. Fig. A5.29 shows the true bending moment diagram as the summation of the separate diagrams.

A5.8 Forces at a Section in Terms of Forces at a Previous Station.

STRAIGHT BEAMS

Aircraft structures present many beams which carry a varying distributed load. Minimum structural weight is of paramount importance in aircraft structural design thus it is desirable to have the complete bending moment diagram for the structure so that each portion of the structure can be proportioned efficiently. To decrease the amount of numerical work required in obtaining the complete shear
and bending moment diagrams it usually saves
time to express the shear and moment at a given
station in terms of the shear and moment at a
previous station plus the effect of any loads
lying between these two stations. To illustrate,
Fig. A5.30 shows a cantilever beam carrying a
considerable number of transverse loads \( F \) of dif-
ferent magnitudes. Fig. A5.31 shows a free body
of the beam portion between stations 1 and 2.
The vertical shear \( V \) at station 1 equals the
summation of the forces to the left of station 1
and \( M \), the bending moment at station 1 equals
the algebraic sum of the moments of all forces
lying to left of station 1 about station 1.
Now considering station 2: The vertical
shear \( V \) equals the shear at the previous sta-
tion 1 plus the algebraic sum of all forces \( F \)
lying between stations 1 and 2. Again consider-
ing Fig. A5.31, the bending moment \( M \) at station
2 can be written, \( M = M + V \cdot d + F \cdot a \), or
stated in words, the bending moment \( M \) at sta-
tion 2 is equal to the bending moment \( M \) at a
previous station 1, plus the shear \( V \) at the
previous station 1 times the arm \( d \), the dis-
tance between stations 1 and 2 plus the moments
of all forces lying between stations 1 and 2
about station 2.

A5.9 Equations for Curved Beams.
Many structural beams carry both longitudi-
nal and transverse loads and also the beams
may be made of straight elements to form a frame
or all beam elements may be curved to form a
curved frame or ring. For example the airplane
fuselage ring is a curved beam subjected to
forces of varying magnitude and direction along
its boundary due to the action of the fuselage
skin forces on the frame. Since the complete
bending moment diagram is usually desirable, it
is desirable to minimize the amount of numerical
work in obtaining the complete shear and bending
moment values. Fig. A5.33 shows a curved beam
loaded with a number of different vertical loads
\( F \) and horizontal loads \( Q \). Fig. A5.33 shows the
beam portion 1-2 cut out as a free body. \( H \)
represents the resultant horizontal force at
station 1 and equals the algebraic summation of
all the \( Q \) forces to the left of station 1. \( V \)
represents the resultant vertical force at
station 1 and equals the sum of all \( F \) forces to
left of station 1, and \( M \) equals the bending
moment about point (1) on station 1 due to the
moments of all forces lying to the left of
point 1.

Then from Fig. A5.33 we can write for the
resultant forces and moment at point (2) at
station 2:

\[
\begin{align*}
V &= V_1 + F_{1-x} \\
H &= H_1 + Q_{1-y} \\
M &= M_1 + V_1 \cdot d + H_1 \cdot h + F_{1-x} \cdot a - Q_{1-y} \cdot b
\end{align*}
\]
Having the resultant forces and moments for a
given point on a given station, it is usually
necessary in finding beam stresses to resolve
the forces into components normal and parallel
to the beam cross-section and also transfer
their location to a point on the neutral axis
of the beam cross-section.

For example Fig. A5.34 shows the resultant
forces and moment at point 1 of a beam cross-
section. They can be resolved into a normal
force \( N \) and a shear for \( S \) plus a moment \( M \) as
shown in Fig. A5.35 where,

\[
\begin{align*}
N &= H \cos \alpha + V \cos \beta \\
S &= V \sin \alpha - H \sin \beta
\end{align*}
\]
Later on when the beam section is being de-
signed it may be found that the neutral axis
lies at point 0 instead of point 1. Fig.
A5.36 shows the forces and moments referred to
point 0, with \( M_0 \) being equal to \( M - Ne \).
A5.10 Torsional Moments.

The loads which cause only bending of a beam are located so that their line of action passes through the flexural axis of the beam. Quite often, the loading on a beam does not act through the flexural axis of the beam and thus the beam undergoes both bending and twisting. The moments which cause the twisting action are usually referred to as torsional moments. The airplane wing is an excellent example of a beam structure that is subjected to combined bending and torsion. Since the center of pressure of the airfoil forces changes with angle of attack, and since there are many flight conditions it is impossible to eliminate torsional moments under all conditions of flight and landing. For the fuselage, the vertical tail surfaces is normally located above the fuselage and thus a load on this tail unit causes combined bending and twisting of the fuselage.

Fig. A5.34 illustrates a cantilever tube being subjected to a load P acting at point A on a fitting attached to the tube end. The flexural axis coincides with the tube centerline, or axis 1-1. Fig. A5.35 shows the load P being moved to the point (O) on the tube axis 1-1, however the original force P had a moment about (O) equal to Pr, thus the moment Pr must be added to the load P acting at (O) if the force system at point (O) is to be equivalent to the original force P at point A. The force P acting through (O) causes bending without twist and the moment Pr causes twisting only.

For the resolution of moments into various resultant planes of action, the student should refer to any textbook on statics.

A5.11 Shears and Moments on Wing.

Arts. A4.5 and A4.6 of Chapter A4 discusses the airloads on the wing and the equilibrium of the airplane as a whole in flight. As explained, it is customary to replace the distributed air forces on an airfoil by two resultant forces, namely, lift and drag forces acting through the aerodynamic center of the airfoil plus a wing moment. The airflow around a wing is not uniform in the spanwise direction, thus the airfoil force coefficients \( C_L \), \( C_D \) and \( C_M \) vary spanwise along the wing. Fig. A5.35 shows a typical spanwise variation of the \( C_L \) and \( C_D \) force coefficients in terms of a uniform spanwise variation \( C_l \) and \( C_d \).

Any particular type of airplane is designed to carry out a certain job or duty and to do that job requires a certain maximum airplane velocity with the maneuvering limited to certain maximum accelerations. These limiting accelerations are usually specified with reference to the X Y Z axes of the airplane. Since the directions of the lift and drag forces change with angle of attack it is simpler and convenient in stress analysis to resolve all forces with reference to the X Y Z axes which remain fixed in direction relative to the airplane.

As a time saving element in wing stress analysis, it is customary to make unit load analysis for wing shears and moments. The wing shears and moments for any design condition then follows as a matter of simple proportion and addition. For example it is customary:

(1) To assume a total arbitrary unit load acting on the wing in the Z direction through the aerodynamic section of the airfoil section and distributed spanwise according to that of the \( C_L \) or lift coefficient.

(2) A similar total load as in (1) but acting in the X direction.

(3) To assume a total unit wing load acting in the Z direction through the aerodynamic center and distributed spanwise according to that of the \( C_D \) or drag coefficient.

(4) Same as (3) but acting in the X direction.

(5) To assume a unit total wing moment and distributed spanwise according to that of the \( C_M \) or moment coefficient.

The above unit load conditions are for conditions of acceleration in translation of the
airplane as a rigid body. Unit load analyses are also made for angular accelerations of the airplane which can also occur in flight and landing maneuvers.

The subject of the calculation of loads on the airplane is far too large to cover in a structures book. This subject is usually covered in a separate course in most aeronautical curricula after a student has had initial courses in aerodynamics and structures. To illustrate the type of problem that is encountered in the calculation of the applied loads on the airplane, simplified problems concerning the wing and fuselage will be given.

A5.12 Example Problem of Calculating Wing Shears and Moments for One Unit Load Condition.

Fig. A5.37 shows the half wing planform of a cantilever wing. Fig. A5.38 shows a wing section at station 0. The reference Y axis has been taken as the 40 percent chord line which happens to be a straight line in this particular wing layout.

![Wing Section Diagram](image)

The total wing area is 17760 sq. in. For convenience a total unit distributed load of 17760 lbs. will be assumed acting on the half wing and acting upward in the Z direction and through the airfoil aerodynamic center. The spanwise distribution of this load will be according to the \( CL \) lift coefficient spanwise distribution. For simplicity in this example it will be assumed constant.

Table A5.1 shows the calculations in table form for determining the \( V_2 \) the wing shear in the Z direction, the bending moment \( M_X \) or moment about the X axis and \( M_Y \) the moment about the Y axis for a number of stations between the wing tip station 240 and the centerline station 0.

Column 1 of the table shows the number of stations selected. Column 2 shows the \( CL/CL \) ratio or the spanwise variation of the lift coefficient \( C_L \) in terms of a uniform distribution \( C_L^* \). In this example we have taken this ratio as unity since we have no wind tunnel or aerodynamic calculations for this wing relative to the spanwise distribution of the lift force coefficient. In an actual problem involving an airplane a curve such as that given in Fig. A5.38 would be available and the values to be used in Column 3 of Table A5.1 would be read from such a curve. Column 2 gives the wing chord length at each station. Column 4 gives the wing running load per inch of span at each station point. Since a total unit load of 17760 lb. was assumed acting on the half wing and since the wing area is 17760 sq. in., the running load per inch at any station equals the wing chord length at that station. In order to find shears and moments at the various station points, the distributed load is now broken down into concentrated loads which are equal to the distributed load on a strip and this concentrated strip load is taken as acting through the center of gravity of this distributed strip load. Columns 5, 6, and 7 show the calculations for determining the \( \Delta P_z \) strip loads. Column 8 shows the location of the \( \Delta P_z \) load which is at the centroid of a trapezoidal distributed load whose end values are given in Column 4. In determining these centroid locations it is convenient to use Table A3.4 of Chapter A3.

The values of the shear \( V_2 \) and the moment \( M_X \) at each station are calculated by the method explained in Art. A5.8. Columns 9, 10, 11 and 12 of Table A5.1 give the calculations. For example, the value of \( M_X = 9884 \) in Column (12) for station 220 equals 2436, the \( M_Y \) moment at the previous station in Column (12) plus 4908 in Column (10) which is the shear at the previous station (220) times the distance 10 inches plus the moment 2540 in Column (9) due to the strip load between stations 230 and 220, which gives a total of 9884 the value in Column (12).

The strip loads \( \Delta P_z \) act through the aerodynamic center (a.c.) of each airfoil strip. Column (13) and (14) give the x arms which is the distance from the a.c. to the reference Y axis. (See Fig. A5.38). Column 15 gives the \( M_Y \) moment for each strip load and Column 16 the \( M_Y \) moment at the various stations which equals the summation of the strip moments as one progresses from station 240 to zero.

Fig. A5.39 shows the results at station (0) as taken from Table A5.1.
**TABLE A5.1**

**CALCULATION OF WING SHEAR Vz AND WING MOMENTS Mz and Mw DUE TO TOTAL UNIT DISTRIBUTED HALFWING LOAD OF 17760 LBS. ACTING UPWARD IN Z DIRECTION AND APPLIED AT AERODYNAMIC CENTERS OF WING SECTIONS. (See Figs. A5.37, 38 for Wing Layout)**

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*Sum = 17760 Checks Total Load Limit Assumed on Half Wing.*

When the time comes to design the structural make-up of a cross-section to withstand these applied shears and moments, the structural designer may wish to refer the forces to another Y axis as for example one that passes through the shear center of the given section. This transfer of a force system with reference to another set of axes presents no difficulty.

**SHEARS AND MOMENTS ON AIRPLANE BODY**

**A5.13 Introduction.**

The body of an airplane acts essentially as a beam and in some conditions of flight or landing as a beam column which may also be subjected to twisting or torsional forces. Thus to design an airplane body requires a complete picture of the shearing, bending, twisting and axial forces which may be encountered in flight or landing. In the load analysis for wings, the direct air forces are the major forces. For the body load analysis the direct air pressures are secondary, the major forces being of a concentrated nature in the form of loads or reactions from units attached to the body, as the power plant, wing, landing gear, tail, etc. In addition, since the body usually serves as the load carrying medium, important forces are produced on the body in resisting the inertia forces of the weight of the interior equipment, installations, pay load etc.

As in the case of the wing, a large part of the load analysis can be made without much consideration as to the structural analysis of the body. The load analysis of an airplane body involves a large amount of calculation, and thus the treatment in this chapter must be of a simplified nature, and is presented chiefly for the purpose of showing the student in general how the problem of load analysis for an airplane body is approached.
A5.14 Design Conditions and Design Weights.

The airplane body must be designed to withstand all loads from specified flight conditions for both maneuver and gust conditions. Since accelerations due to air gusts vary inversely as the airplane weight, it is customary to analyze or check the body for a light load condition for flight conditions. In general, the design weights are specified by the government agencies. For landing conditions, however, the normal gross weight is used since it would be more critical than a lightly loaded condition. The general design conditions which are usually investigated in the design of the body are as follows:

**Flight Conditions:**

- **H.A.A.** (High angle of attack)
- **L.A.A.** (Low angle of attack)
- **I.L.A.A.** (Inverted low angle of attack)
- **I.H.A.A.** (Inverted high angle of attack)

The above conditions generally assume only translational acceleration. In addition, it is sometimes specified that the forces due to a certain angular acceleration of the airplane about the airplane c.g. must be considered. The body is usually required to withstand special tail loads both symmetrical and unsymmetrical which may be produced by air gusts, engine forces, etc. Also, the body should be checked for forces due to unsymmetrical air loads on the wing.

**Landing Conditions:**

In general, the body is investigated for the following landing conditions. The detailed requirements for each condition are given in the government specifications for both military and commercial airplanes.

- **Landplanes:** Level landing.
- Level landing with side load.
- Three point landing.
- Three point landing with ground loop.
- Nose over or turn over condition.
- Arresting. (Usually for only Navy carrier based airplanes).

- **Seaplanes or Boats:**
  - Step landing with and without angular acceleration.
  - Bow landing.
  - Stern landing.
  - Two wave landing.
  - Reaching conditions.
  - Catapulting conditions (Navy airplanes).

**Special Conditions or Forces:**

- Towing of airplane.
- Body supercharging.

A5.15 Body Weight and Balance Distribution.

The resisting inertia forces due to the dead weight of the body and its content plays an important part in the load analysis for the airplane body. When the initial aerodynamic and general layout and arrangement of the airplane is made, it is necessary that a complete weight and balance estimate of the airplane be made. This estimate is usually made by an engineer from the weight control section of the engineering department who has had experience in estimating the weight and distribution of airplane units. This estimate which is presented in report form gives the weights and (c.g.) locations of all major airplane units or installations as well as for many of the minor units which make up these major airplane assemblies or installations. This weight and balance report forms the basis for the dead weight inertia load analysis which forms an important part in the load analysis of the airplane body. The use of this weight and balance estimate will be illustrated in the example problem to follow later.

A5.16 Load Analysis. Unit Analysis.

Due to the many design conditions such as those listed in Art. A5.14, the general procedure in the load analysis of an airplane body is to base it on a series of unit analyses. The loads for any particular design condition then follows as a certain combination of the unit results with the proper multiplying factors. A simplified example problem follows which illustrates this unit method of approach.

A5.17 Example Problem Illustrating the Calculation of Shears and Moments on FuselageDue to Unit Load Conditions.

Fig. A5.40 and A5.41 shows a layout of the airplane body to be used in this example problem. It happens to be the body of an actual airplane and the wing used in the previous example problem was the wing that went with the airplane.

![Fig. A5.40](image1)

![Fig. A5.41](image2)
Table A5.2 gives the Weight and Balance estimate for the total airplane. This table is usually formulated by the Weight and Balance Section of the engineering department and it is necessary to have this information before the airplane load analysis can be made.

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
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<td>12</td>
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<td>200</td>
<td>99</td>
<td>18500</td>
<td>6</td>
<td>600</td>
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<td>13</td>
<td>Fuel System</td>
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<td>69</td>
<td>67800</td>
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<td>-20500</td>
</tr>
<tr>
<td>Gross weight</td>
<td>4300</td>
<td>375800</td>
<td>375800</td>
<td>-20500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Calculation of C.G. locations:
Gross wt. X = 375800/4300 = 88.3 " at Ref. Axis
Y = -40328/4300 = 9.4 " below thrust line

SOLUTION:

**WEIGHT AND BALANCE OF BODY ITEMS.
WEIGHT DISTRIBUTION.**

Table A5.3 gives the weight and balance calculations for all items attached to fuselage or carried in the fuselage, except the wing and items attached to the wing as the front landing gear and the fuel.

In order to obtain a close approximation to the true shears and moments on the fuselage due to the dead weight inertia loads, it is necessary to distribute the weights of the various items as given in Table A5.3. Fig. A5.42 shows a side view of the airplane with the center of gravity locations of the weight items of Table A5.3 indicated by the (+) signs. In the various design conditions, the direction of the weight inertia forces changes, thus it is convenient and customary to resolve the inertia forces into X and Z components. Thus, in Fig. A5.43, the weights as given in Table A5.3 are assumed acting in the Z direction through their (c.g.) locations. The loads shown would not give a true picture as to the shears and moments along the fuselage, thus these loads should be distributed in a manner which should simulate the actual weight distribution. In most weight and balance reports, the weight items are broken down into considerable more detail than that shown in Table A5.3, which makes the weight distribution more evident. The person making the
WEIGHT DISTRIBUTION TO FUSELAGE STATIONS

Fig. A5.42. Location of weight items of Table A5.3.

Fig. A5.43. Weight items of Table A5.3 acting in Z direction.

Fig. A5.44. Results of fuselage weight distribution to stations.

Fig. A5.45. Final weight distribution to station points.

Fig. A5.46. Weight items from Table A5.3 acting in X direction.

Fig. A5.47. Vertical distribution of fuselage dead weight.

Fig. A5.48. Fuselage weight referred to X axis plus couples.

Fig. A5.49. Final weight distribution in X direction referred to X axis plus proper couples.
shows how the dead weight of 350# is distributed to the various station points considering the weights to be acting in the Z direction.

Table A5.4 shows the results of this station point weight distribution for the weight items of Table A5.3. The values in the horizontal rows opposite each weight item shows the distribution to the various fuselage stations. The summation of the weights in each vertical column at each station point as given in the third horizontal row from the bottom of the table gives the final station point weight. These weights are shown in Fig. A5.46 for weights acting in the Z direction. The moment of each total-station load about the Z axis is given in the second horizontal row from the bottom of Table A5.4. The summation of the moments in this row must equal the total weight moments of Table A5.3 or 219,700"#. This check is shown in the last vertical column of Table A5.4.

The distributed system must also be distributed in the Z or vertical direction in such a manner as to have the same resultant c.g. location as the original weight system which is illustrated in Fig. A5.46. Fig. A5.47 illustrates how the fuselage weight distributed system as shown in Fig. A5.44 is distributed in the vertical direction at the various station points so that the moment of this system about the X axis is equal to that of the original fuselage weight of 350#. For convenience, these distributed fuselage weights can be transferred to the X axis plus a moment as shown in Fig. A5.48.

Table A5.4 shows the vertical distribution of the various items at the various station points. The bottom horizontal row gives the moment about the Y axes of the loads at each station point, which equals the individual loads times their Z distances. The summation of the values in this horizontal row must equal the total weight moment of Table A5.3. This check is shown at the bottom of the last vertical column. Fig. A5.49 shows the results as given in Table A5.4 for the weight distribution in the X direction.

### TABLE A5.4

<table>
<thead>
<tr>
<th>Station No.</th>
<th>Distance x from Ref. X Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item No.</td>
<td>Name</td>
</tr>
<tr>
<td>1</td>
<td>Powerplant Group</td>
</tr>
<tr>
<td>2</td>
<td>Fuselage Group</td>
</tr>
<tr>
<td>3</td>
<td>Tail Group</td>
</tr>
<tr>
<td>4</td>
<td>Pressure Systems</td>
</tr>
<tr>
<td>5</td>
<td>Electrical System</td>
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<td>6</td>
<td>Tail Shear Group</td>
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<td>Fuselage Shear</td>
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<td>Landing Gear</td>
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<td>Fittings</td>
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<td>Plug</td>
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<td>11</td>
<td>Nacelle</td>
</tr>
<tr>
<td>12</td>
<td>Total</td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
<tr>
<td>Horizontal</td>
<td></td>
</tr>
<tr>
<td>Vertical</td>
<td></td>
</tr>
</tbody>
</table>

A5.18 Unit Analysis for Fuselage Shears and Moments.

Since there are many flight and landing conditions, considerable time can be saved if a unit analysis is made for the fuselage shears, axial and bending forces. The design values in general then follow as a summation of the values in the unit analysis times a proper multiplication factor.

The loads on the fuselage in general consists of tail loads, engine loads, wing reactions, landing gear reactions if attached to fuselage and inertia forces due to the airplane acceleration which may be due to both translational and angular acceleration of the airplane. For simplicity, these loads can be resolved into components parallel to the Z and X axes.

To illustrate the unit analysis procedure, a unit analysis for our example problem will be carried out for the following unit conditions:

1. Unit acceleration or load factor in Z direction and acting up.
2. Unit acceleration or load factor in X direction and acting forward.
3. Unit tail load normal to X axis acting down.

Unit analyses are also usually carried out for engine thrust and engine torque, side load on tail and angular acceleration, but to keep the example calculations from becoming too lengthy only the above 3 unit conditions will be carried out in detail. The others will be discussed in detail in later paragraphs.

Solution for Unit Load Factor in Z Direction.

Fig. A5.50 shows the dead weight loads.
acting in the Z direction as taken from Table A5.4 or Fig. A5.45. The wing is attached to the fuselage at stations 73 and 116 as shown on Fig. A5.50. The fittings at these points are assumed as designed to cause all the drag or reaction in the X direction to be taken off entirely at the front fitting on station 73.

To place the fuselage in equilibrium, the wing reaction will be calculated:

\[ \Sigma F_X = 0, \; R_H + 0 = 0, \; \text{hence} \; R_H = 0 \]

\[ \Sigma M \text{ at station } 0 = 219700 - 115 R_R - 73 R_P = 0 \]  
(Note: 219700 from Table A5.3)

\[ \Sigma F_Z = -2555 + R_P + R_R = 0 \]  
(Solving equations (A) and (B) for \( R_P \) and \( R_R \):

\[ R_P = 1780 \text{ lb}, \; R_R = 775 \text{ lb.} \]

Table A5.5 gives the calculations for the fuselage shear and bending moments at the various station points.

### Table A5.5

<table>
<thead>
<tr>
<th>Sta. No.</th>
<th>Load or Reaction</th>
<th>( V = \text{shear} )</th>
<th>( \Delta x = \text{Dist. between stations} )</th>
<th>( \Delta M = V \Delta x ) (in. lbs.)</th>
<th>Moment ( M^* ) (in. lbs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>315</td>
<td>0</td>
<td>22</td>
<td>0</td>
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<td>290</td>
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<td>-4750</td>
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<tr>
<td>230</td>
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<td>73</td>
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<td>11</td>
<td>0</td>
<td>-893</td>
<td>-34827</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

(3) Up on left and down on right side of a section is positive shear.
(6) Tension in upper fuselage portion is negative bending moment.
(1) * refers to aft side of station.
(2) * refers to forward side of station.
(3) \( M = M \) at previous station in col. 6 plus \( \Delta M \) in col. 5.

**Solution for Unit Load Factor in X Direction.**

Fig. A5.51 shows the panel point dead weight distribution for loads acting in the X direction and aft, as taken from Table A5.4 or Fig. A5.40. To place the fuselage in equilibrium the wing reactions at points (A) and (B) will be calculated.

\[ \Sigma F_X = 2555 - R_H = 0, \; \text{hence} \; R_H = 2555 \text{ lb. (forward)} \]

Take moments about point (A)

\[ \Sigma M_A = 2555 \times 17 + 5920 - 43 R_R = 0, \]  
(hence \( R_R = 1147.8 \) (up)

\[ \Sigma F_Z = 1147.8 - R_P = 0, \]  
(hence \( R_F = 1147.8 \) lb. (down)

Table A5.6 gives the calculations for the shears, moments and axial loads for the loading of Fig. A5.51.
### Table A5.6

<table>
<thead>
<tr>
<th>Sta. No.</th>
<th>W_a \ (\text{in. lbs})</th>
<th>W_x \ (\text{in. lbs})</th>
<th>P_x \ (\text{in. lbs})</th>
<th>X \ (\text{in.})</th>
<th>M_x \ (\text{in. lbs})</th>
<th>M_y \ (\text{in. lbs})</th>
<th>M_z \ (\text{in. lbs})</th>
<th>Load Factors</th>
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<tr>
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</table>

Solution for Unit Horizontal Tail Load Acting Down.

The fuselage shears and moments will be computed for a unit tail load of 100 lb on the tail in the direction of the 2 direction, with balancing reactions at the wing attachment points. The center of pressure on the horizontal tail is at station 277.5. Fig. A5.52 shows the fuselage loading. To find wing reactions at (A) and (B):

\[ \Sigma M_A = 100 \times (277.5 - 73) - 45 R_R = 0, \]

hence \( R_R = 475.6 \text{ (up)} \)

\[ \Sigma F_x = -100 + 475.6 - R_F = 0, \]

hence \( R_F = 375.6 \text{ (down)} \)

Table A5.7 gives the detailed calculations for the loads and moments at the various stations.

### Table A5.7

<table>
<thead>
<tr>
<th>Sta. No.</th>
<th>Load or Reaction w lbs</th>
<th>X \ (\text{in.})</th>
<th>M_x \ (\text{in. lbs})</th>
<th>M_y \ (\text{in. lbs})</th>
<th>M_z \ (\text{in. lbs})</th>
<th>Moment \ (\text{in. lbs})</th>
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### Calculation of Applied Fuselage Shears, Moments and Axial Loads for a Specific Flight Condition.

Using the results in Tables A5.5, A5.6, and A5.7, the applied shears and moments for a given flight condition follow as a matter of proportion and addition. To illustrate, the applied values for one flight condition will be given. It will be assumed that the aerodynamic calculations for this airplane for the (H.A.A.) high angle of attack condition gave the following results, which the student will have to accept without knowledge of how they were obtained.
Applied load factor in Z direction = -6.0
down
Applied load factor in X direction = 1.333 up.
Applied tail load = 110 lb. up.

Thus with the load factors in the Z and X directions and the tail load known, Table A5.6 can be filled in as illustrated. In a similar manner the values for other flight conditions can be found, the only difference being a new set of multiplying factors since the applied loads would be different.

<p>| TABLE A5.6 |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|</p>
<table>
<thead>
<tr>
<th>Ra</th>
<th>Vf for Load Factor</th>
<th>Vf for Tail Load</th>
<th>Vf for Tail Load</th>
<th>M for Z Load Factor</th>
<th>M for X Load Factor</th>
<th>M for Tail Load Factor</th>
<th>Axial Load Factor</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**NOTES:** (Moments & axial loads are referred to the thrust line.)

Col. 1 & 2: x values in column 3 of Table A5.5.  
Col. 3: 1.333 x values in column 5 of Table A5.5.  
Col. 4: 1.333 x values in column 3 of Table A5.5.  
Col. 5: columns (4) + columns (3) - columns (2).  
Col. 6: 1.333 x values in column 5 of Table A5.5.  
Col. 7: columns (6) + columns (4) - columns (2).  
Col. 8: 1.333 x values in column 5 of Table A5.5.  
Col. 9: columns (4) + columns (3) - columns (2).  
Col. 10: 1.333 x values in column 5 of Table A5.5.  

A5.19 Example of Fuselage Shears and Moments for Landing Conditions.

Fig. A5.53 illustrates the airplane in a level landing condition. The ground reaction is assumed to pass the center of landing gear wheel and c.g. of airplane. The fuselage shears, moments and axial loads are required when the vertical ultimate load factor is 7. (Gross weight = 4300#).

**SOLUTION:**

The vertical or Z component of the ground reaction R is specified as 7 load factors which equals 7 x 4500 = 31000#. One half of this is acting on each wheel.

The horizontal or X component of R is 30100 tan 23° = 426 x 30100 = 12800# and acting aft.

The horizontal load factor on airplane equals 12800/4500 = 2.86.

A5.20 Inertia Loads Due to Angular Acceleration.

In some of the flying conditions, it is sometimes specified that the airplane must be subjected to an angular acceleration as well as translational acceleration. This angular acceleration of the airplane produces inertia forces which must be calculated if the airplane is to be treated as a body in static.
equilibrium. In some cases, a tail load due to a gust on the tail is specified which produces a moment about the airplane c.g. which produces angular acceleration of the airplane. In certain landing conditions, the ground forces do not pass through the airplane c.g. thus producing a moment about the c.g. which for stress analysis purposes is balanced by inertia forces.

Moment of Inertia of Airplane

The calculation of the moment of inertia of an airplane about the center of gravity axes was explained on page A5.8 of Chapter A5. A detailed example solution was given in detail in Table 6A of Chapter A5. The general equations for the moments of inertia of the airplane about the reference axes are:

\[
I_y = 2\, \text{wx} + 2\, \text{wx} + 2\, \Delta\, I_y
\]

\[
I_x = 2\, \text{wy} + 2\, \text{wy} + 2\, \Delta\, I_x
\]

\[
I_z = 2\, \text{wx} + 2\, \text{wx} + 2\, \Delta\, I_z
\]

The last term in each of the above equations represents the moment of inertia of each weight item about its own centroidal axes parallel to the reference axes.

A5.21 Solution for Inertia Loads Due to Unit 100,000 Lbs. Pitching Moment.

To illustrate the general procedure of determining the balancing inertia loads when the airplane is subjected to an unbalanced moment about the c.g., an analysis will be made for a unit 100,000 lb. moment. Table A5.9 gives the necessary calculations.

From kinetics:

\[
\text{Pitching angular acceleration } a = \frac{M_y}{I_y}\] (rad/sec.²)

where

\[
M_y = \text{unbalanced external pitching moment about c.g. of airplane.}
\]

\[
I_y = \text{pitching moment of inertia of airplane about c.g.} = Z\omega^2
\]

The tangential inertia force \( F \) for a mass \( w/g \) due to an angular acceleration \( a \) equals,

\[
F = \frac{w\, \omega}{g}, \text{ but } a = \frac{M_y}{I_y}\]

hence

\[
F = \frac{M_y}{I_y}\, w\, r, \text{ where } r \text{ is the distance from the weight } w \text{ to the airplane c.g.}
\]

It is convenient to treat the inertia force \( F \) as resolved into two components \( F_x \) and \( F_z \).

\[
F_x = \frac{M_y}{I_y}\, w\, z_c \quad \text{(1)}
\]

\[
F_z = \frac{M_y}{I_y}\, w\, x_c \quad \text{(2)}
\]

From Table A5.9, \( I_y = 16097800 \)

\[
M_y \text{ was assumed as } 100,000
\]

hence

\[
F_x = \frac{100000\, w\, x_c}{16097800} = 0.00621\, w\, x_c
\]

\[
F_z = 0.00621\, w\, z_c
\]

where \( z_c \) and \( x_c \) are the \( z \) and \( x \) distances of the weight \( w \) to the airplane c.g.

Columns No. 9 and 10 of Table A5.9 gives the values of these inertia components. Fig. A5.54 shows these inertia loads applied to the fuselage. The reactions at wing attachment points should be computed and then a table of fuselage shear, moments and axial loads should be made up. This table could then be used for all conditions involving angular acceleration of the airplane.

It should be realized that the inertia resisting loads in Table A5.9 are only approximately, since the moment of inertia neglects the centroidal moment of inertia of the big items, such as the power plant, wing, etc. The example is only for the purpose of illustrating the general procedure of determining the inertia resisting loads due to angular acceleration. The same general procedure can be followed in considering unbalanced external moments about the \( z \) and \( x \) axes, commonly referred to as yawing and rolling moments.

![Table A5.9](image)

### Table A5.9

<table>
<thead>
<tr>
<th>Plane No.</th>
<th>Arm 2</th>
<th>Arm 3</th>
<th>Moment about 2</th>
<th>Inert. Force Fx</th>
<th>Inert. Force Fz</th>
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<tr>
<td>11</td>
<td>102</td>
<td>11</td>
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<td>15.4</td>
<td>15.4</td>
<td>200000</td>
<td>200000</td>
</tr>
</tbody>
</table>

\[I_y = \frac{16097800}{100000} = 160.978\]

Columns (3) and (4) taken from Table A5.2.

Column (5) \( z_c = 2 - 0.5 \), see Table A5.2 for c.g.

Column (6) \( x_c = 0.75 \)

Column (7) \( F_x = 0.00621\, w\, x_c \)

Column (8) \( F_z = 0.00621\, w\, z_c \)

Column (9) \( = \frac{100000}{16097800} \)

\[I_y = \frac{16097800}{100000} = 160.978\]
A5.22 Problems

(I) Draw the shear, bending moment and axial load diagrams for loaded structures in Figs. 55 to 60c.

Fig. A5.54

[Diagram showing shear, bending moment, and axial load diagrams]

(RF) 50 80 120 170 200 230 260 290 315

[Table with load values]

11

22.6

28.2

28.4

29.3

159

53.2

8.9

2.9

20.5

0.1

31.2

[Diagram with figure A5.54 annotations]

III. Fig. A5.62 shows the plan form of a cantilever wing. Assume a constant normal distributed load on the surface equal to 50 lb./sq.ft. Write expressions for shear and bending moment on wing and find values at 25, 100, 150 and 200 inches from root.

Fig. A5.62

[Diagram showing plan form of cantilever wing]

100° Root

Planform

30° Fig. A5.62

[Diagram showing plan form with annotations]

100° Root

Planform

30° Fig. A5.62

90°

Leading Edge

Planform

Fig. A5.63

y ref. axis

150°

Relative Spanwise Distribution

Fig. A5.63 shows plan form of a cantilever wing. The total distributed air load normal to surface is 10000 lb. The relative spanwise distribution is shown. Take center of pressure at 24 percent of chord from leading edge. Divide wing into 10 inch width strips and calculate $V_2$, $M_x$ and $M_y$, and plot curves for same.

IV. Fig. A5.64 shows an externally braced monoplane wing. Take an average wing lift load of 90 lb./sq.ft. normal to wing with center of pressure at 27 percent of the chord from leading edge of wing and calculate and draw the front and rear beam primary shears and bending moment diagrams.

Fig. A5.64

[Diagram showing externally braced monoplane wing]

44

Hinge Rear Beam

PLAN VIEW

Hinge Front Beam

[Diagram with figure A5.64 annotations]
BENDING MOMENTS - BEAM - COLUMN ACTION

A5.23 Introduction

A beam-column is a member subjected to transverse loads or end moments plus axial loads. The transverse loading, or end moments, produces bending moments which, in turn, produce lateral bending deflection of the member. The axial loads produce secondary bending moments due to the axial load times this lateral deflection. Compressive axial loads tend to increase the primary transverse bending moments, whereas tensile axial loads tend to decrease them.

Beam-column members are quite common in airplane structures. For example, the beams of externally braced wing and tail surfaces are typical examples, the air loads producing transverse beam loads and the struts introducing axial beam loads. In landing gears, one member is usually subjected to large bending and axial loads. In tubular fuselage trusses, lateral loads due to installations supported on members between truss joints produce beam-column action.

In general, beam column members in airplane structures are comparatively long and slender compared to those in buildings and bridges; thus, the secondary bending moments due to the axial loads are frequently of considerable proportion and need to be considered in the design of the members.

This chapter deals briefly on the theory of single span beam-column members. A summary of equations and design tables is included together with examples of their use. The information in this chapter is used frequently in other chapters where practical analysis and design of beam-column members is considered. For a completed and comprehensive treatment of beam-column theory and derivation of equations, see Miles and Newell -"Airplane Structures".

A5.24 General Action of a Member Subjected to Combined Axial and Transverse Loads

Sub-figure a of Fig. A5.65 shows a member subjected to transverse loads \( W \) and axial compressive loads \( P \). The transverse loads \( W \) produce a primary bending distribution on the member as shown in Fig. b. This bending will produce a transverse deflection curve as illustrated in Fig. e. The end loads \( P \) now produce an additional secondary bending moment due to the end load \( P \) times the deflection \( f \), or the bending moment diagram of Fig. d. This first secondary moment distribution produces the additional lateral deflection curve of Fig. e and the end load \( P \) will again produce further bending moments due to this deflection. If the axial load is not too large, these successive deflections will gradually converge and the members will reach a state of equilibrium. These secondary bending moments could be found by successive steps by the various deflection principles given in Chapter A.7. However, for prismatic beams this convergence can be expressed as a mathematical series and thus save much time over the above successive step method. For members of variable moment of inertia, the secondary moments will usually have to be found by successive steps.

If the end loads \( P \) are tension, they will tend to decrease the primary moments; thus, in general, the case of axial compression is more important in practical design, since buckling and instability enter into the problem.

A5.25 Equations for a Compressive Axially Loaded Strut with Uniformly Distributed Side Load

Fig. A5.66 shows a prismatic beam of length \( L \) subjected to a concentric compressive load \( P \) and a uniformly transverse distributed load \( W \), with the beam supported laterally at each end, and with end restraining moments \( M_1 \) and \( M_2 \). It is assumed that the general conditions for the beam theory hold, namely; that plane sections remain plane after bending; that stress is proportional to strain in both tension and compression.

At any point a distance \( x \) from the beam end, the moment expression is,

\[
M = M_1 + \frac{(M_2 - M_1)}{L} \times \frac{WL}{2} + \frac{Wx^2}{2} - Py = - - - - - - - - (A5.1)
\]

From applied mechanics, we know that

\[
M = EI \frac{d^2y}{dx^2}
\]

therefore, differentiating equation (A5.1) twice with respect to \( x \) gives

\[
\frac{d^2M}{dx^2} + \frac{P}{EI} M = W - - - - - - - - - - (A5.2)
\]

For simplification, let \( J = \frac{EI}{P} \); hence \( \frac{d^2M}{dx^2} = \frac{1}{J} W \).

For simplicity, after integrating, let \( \frac{dM}{dx} = \frac{1}{J} W \), which gives

\[
\frac{d^2M}{dx^2} + \frac{1}{J} M = W
\]
The solution of this differential equation gives:

\[ M = C_1 \sin \frac{x}{L} + C_2 \cos \frac{x}{L} + \frac{wL^2}{E} + \omega_j^2 \]  

where \( C_1 \) and \( C_2 \) are constants of integration and \( \sin \frac{x}{L} \) and \( \cos \frac{x}{L} \) are the limits of an infinite series of variable \( x \). When \( x = 0 \), \( M = M_1 \) and when \( x = L \), \( M = M_2 \), therefore:

\[ C_1 = \frac{M_2 - \omega_j^2}{\sin \frac{L}{j}} - \frac{M_1 - \omega_j^2}{\tan \frac{L}{j}} \]

\[ C_2 = \frac{M_2 - \omega_j^2}{\sin \frac{L}{j}} - \frac{M_1 - \omega_j^2}{\cos \frac{L}{j}} \]

and \( C_3 = M_1 - \omega_j^2 \)

Let \( D_1 = M_1 - \omega_j^2 \) and \( D_2 = M_2 - \omega_j^2 \). Then, substituting in equation (A5.3),

\[ M = \frac{D_1 - D_2 \cos \frac{x}{L}}{\sin \frac{x}{L}} + \frac{D_2}{\cos \frac{x}{L}} + \omega_j^2 \]  

(A5.4)

To find the location of the maximum moment, differentiate equation (A5.3) and equate to zero.

\[ \frac{dM}{dx} = 0 = \frac{C_1}{j} \cos \frac{x}{L} - \frac{C_2}{j} \sin \frac{x}{L} \]

whence

\[ \tan \frac{x}{L} = \frac{C_2}{j} \frac{C_2}{D_1 - D_2 \cos \frac{x}{L}} \]

(A5.5)

The value of \( x \) must fall within \( x = 0 \) to \( x = L \), otherwise \( M_1 \) or \( M_2 \) is the maximum value.

The value of the maximum span moment can be found by substituting the value of equation (A5.5) in (A5.4), which gives

\[ M_{\text{max}} = \frac{D_1 - D_2 \cos \frac{x}{L}}{\sin \frac{x}{L}} \]

(A5.6)

The moment \( M \) at any point \( x \) along the span can also be written:

\[ M = D_1 \left( \tan \frac{x}{L} \sin \frac{x}{L} \cos \frac{x}{L} \right) + \omega_j^2 \]

(A5.7)

where \( x_0 \) refers to the value of \( x \) where the span moment is maximum, or equation (A5.6). Since it is customary to locate the point of maximum span bending moment and its value before investigating other span points, the value of \( x_0 \) is known from equation (A5.6) and thus is available to use in equation (A5.7) for finding moments at other points along the span.

If the equation for the beam deflection is desired, it can be found by substituting the value of \( M \) from equation (A5.3) in equation (A5.1), which gives:

\[ y = \frac{1}{E} \left( M_1 + \frac{M_2 - M_1}{L} x - \frac{wL^2}{2} + \omega_j^2 \right) \]

(A5.8)

\[ + \frac{D_1 - D_2 \cos \frac{x}{L}}{\sin \frac{x}{L}} \sin \frac{x}{L} - D_2 \cos \frac{x}{L} - w_j^2 \]

(A5.9)

The slope of the elastic curve at any point is given by the first derivative of equation (A5.7a)

\[ 1 = \frac{1}{E} \left( \frac{M_1 - M_2}{L} x \frac{wL}{2} - \frac{C_1 \cos \frac{x}{L} + C_2 \sin \frac{x}{L}}{j} \right) \]

(A5.10)

A5.26 Formulas for Other Single Span Loadings

In investigating other transverse loadings for a single span carrying axial compression, it is found that the expression for bending moment in the span always takes the form:

\[ M = C_1 \sin \frac{x}{L} + C_2 \cos \frac{x}{L} + f(w) \]

where \( f(w) \) is a term which does not include the axial load \( P \) or the end moments \( M_1 \) and \( M_2 \). The expressions for \( f(w) \), \( C_1 \) and \( C_2 \) depend on the type of the transverse load.

Table A5.1 gives the value of these 3 terms for types of transverse loading on a single span which are frequently encountered in airplane structures. The Table also gives equations for the point of maximum bending moment and its magnitude.

Table A5.11 is a table of sines, cosines, and tangents for \( L/j \) in radians which is more convenient to use than the usual type of trigonomometric tables. This table is based on values given in Appendix I of Air Corps Information Circular #493. The 3 difference have been added to facilitate rapid use of the tables.

For single span beams, the critical value of \( L/j \) is \( n \); that is, if the axial compressive load is such that the term \( L/j = n \), the center region of the beam will tend to deflect until the combined stresses equal the failing stress of the material.

A5.27 Moments for Combinations of the Various Load Systems as Given in Table A5.1, Margins of Safety. Accuracy of Calculations.

The principle of superposition does not apply to a beam-column, because the sum of the bending moments due to the transverse loads and the axial loads acting separately are not the same as the moments when they act simultaneously. In combining several transverse load systems with their accompanying axial loads, the principle of superposition can be said to apply if each transverse loading is used with the total axial load for the systems which are being combined. Thus, in Table A5.1, to find the moments for several combined loadings, add the values of \( C_1 \), \( C_2 \), and \( f(w) \) for the several loadings and use
Table A5.1
Values of Terms $C_1$, $C_2$, and $f(w)$ in Equation

$$M = C_1 \sin x + C_2 \cos x + f(w)$$

Single Span - Axial Compression - Uniform Section

<table>
<thead>
<tr>
<th>Loading</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$f(w)$</th>
<th>Eq. for Point of Max. Bending Moment</th>
<th>Eq. for Max. Span Bending Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal End Moments, No Side Load</td>
<td>$M_1 \tan \frac{L}{2}$</td>
<td>$M_1$</td>
<td>$0$</td>
<td>$x = \frac{L}{2}$</td>
<td>$M_{max} = \frac{M_1}{\cos \frac{L}{2}}$</td>
</tr>
<tr>
<td>Unequal End Moments, No Side Load</td>
<td>$M_2 - M_1 \cos \frac{L}{J}$</td>
<td>$M_1$</td>
<td>$0$</td>
<td>$\tan x = \frac{M_2 - M_1 \cos \frac{L}{j}}{M_1 \sin \frac{L}{j}}$</td>
<td>$M_{max} = \frac{M_1}{\cos \frac{L}{j}}$</td>
</tr>
<tr>
<td>Uniform Side Load, No End Moments</td>
<td>$w_j^2 (\cos \frac{L}{1}) \sin \frac{L}{j}$</td>
<td>$-w_j^2$</td>
<td>$w_j^2$</td>
<td>$x = 0.5L$</td>
<td>$M_{max} = w_j^2 (1 - \sec \frac{L}{j})$</td>
</tr>
<tr>
<td>Uniform Side Load Plus End Moments</td>
<td>$D_2 - D_1 \cos \frac{L}{j}$</td>
<td>$-w_j^2$</td>
<td>$w_j^2$</td>
<td>$\tan \frac{x}{\cos \frac{L}{j}} = \frac{D_2 - D_1 \cos \frac{L}{j}}{D_1 \sin \frac{L}{j}}$</td>
<td>$M_{max} = \frac{D_1 + w_j^2}{\cos \frac{L}{j}}$</td>
</tr>
<tr>
<td>Concentrated Side Load, No End Moments</td>
<td>$x &lt; a$, $-w_j \sin \frac{L}{j}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\tan \frac{x}{\cos \frac{L}{j}} = \frac{C_1}{C_2}$</td>
<td>$M_{max} = (C_1^2 + C_2^2)^{1/2}$</td>
</tr>
<tr>
<td>Triangular Loading, No End Moments</td>
<td>$-\frac{w_j^2 \sin \frac{L}{j}}{2}$</td>
<td>$0$</td>
<td>$\frac{w_j^2}{L}$</td>
<td>(Note A) To obtain Maximum Moment, compute moment at 3 or 4 points in span. Draw a smooth curve thru plotted results.</td>
<td></td>
</tr>
<tr>
<td>Triangular Loading, No End Moments</td>
<td>$\frac{w_j^2 \tan \frac{L}{j}}{2}$</td>
<td>$-\frac{w_j^2}{L}$</td>
<td>$\frac{w_j^2 (1-x)}{L}$</td>
<td>(See Note A)</td>
<td></td>
</tr>
</tbody>
</table>

$w$ or $W$ is positive when upward.
$M$ is positive when it tends to cause compression on the upper fibers of the beam at the section being considered.

Reference: ACIC 493; Niles, Airplane Design; Newell and Niles Airplane Structures

For Table of many other loadings, see NACA T.M. 983.

52
<table>
<thead>
<tr>
<th>L in Radians</th>
<th>( \sin L )</th>
<th>( \cos L )</th>
<th>( \tan L )</th>
<th>( \cot L )</th>
<th>( \csc L )</th>
<th>( \sec L )</th>
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</thead>
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<td>1.00000</td>
<td>0.0000</td>
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Note: The table continues with similar entries for all radians up to 6.28318 (\(2\pi\)) radians.
| $L_j$ in Radians | Sin $L_j$ | $\Delta$ Sin $L_j$ | Com $L_j$ | $\Delta$ Com $L_j$ | Tan $L_j$ | $\Delta$ Tan $L_j$ |
|------------------|----------|-------------------|----------| ....................| ---------| ....................|
| 3.00             | 0.14112  | -0.99999         | 0.00136  | -0.14254           | 0.01019  |
| 3.01             | 0.13121  | -0.99135         | 0.00127  | -0.13235           | 0.01016  |
| 3.02             | 0.12129  | -0.98262         | 0.00116  | -0.12219           | 0.01013  |
| 3.03             | 0.11138  | -0.99378         | 0.00106  | -0.11206           | 0.01011  |
| 3.04             | 0.10142  | -0.98484         | 0.00087  | -0.10195           | 0.01010  |
| 3.05             | 0.09146  | -0.99581         | 0.00086  | -0.09185           | 0.01008  |
| 3.06             | 0.08150  | -0.99687         | 0.00077  | -0.08177           | 0.01006  |
| 3.07             | 0.07153  | -0.99744         | 0.00066  | -0.07171           | 0.01004  |
| 3.08             | 0.06155  | -0.99810         | 0.00057  | -0.06167           | 0.01003  |
| 3.09             | 0.05156  | -0.99867         | 0.00046  | -0.05184           | 0.01002  |
| 3.10             | 0.04159  | -0.99913         | 0.00037  | -0.04162           | 0.01001  |
| 3.11             | 0.03159  | -0.99950         | 0.00027  | -0.03161           | 0.01001  |
| 3.12             | 0.02160  | -0.99977         | 0.00016  | -0.02160           | 0.01000  |
| 3.13             | 0.01160  | -0.99993         | 0.00007  | -0.01180           | 0.01000  |
| 3.14             | 0.00160  | -1.00000         | 0.00003  | -0.00180           | 0.01000  |
| 3.15             | -0.00841 | -0.9997         | 0.00014  | 0.0048           | 0.01000  |
| 3.16             | -0.01841 | -0.99984         | 0.00023  | 0.01841           | 0.01000  |
| 3.17             | -0.02840 | -0.99960         | 0.00034  | 0.02841           | 0.01002  |
| 3.18             | -0.03840 | -0.99926         | 0.00043  | 0.03843           | 0.01002  |
| 3.19             | -0.04839 | -0.99883         | 0.00083  | 0.04845           | 0.01003  |
| 3.20             | -0.05838 | -0.99830         | 0.00064  | 0.05848           | 0.01004  |
| 3.21             | -0.06836 | -0.99766         | 0.00073  | 0.06852           | 0.01005  |
| 3.22             | -0.07833 | -0.99693         | 0.00084  | 0.07857           | 0.01007  |
| 3.23             | -0.08829 | -0.99609         | 0.00093  | 0.08864           | 0.01009  |
| 3.24             | -0.09825 | -0.99516         | 0.00103  | 0.09873           | 0.01010  |
| 3.25             | -0.10820 | -0.99413         | 0.00108  | 0.10883           | 0.01010  |
these values in the general expression for $M$ as given at the top of the Table.

In a beam-column member, the bending moments do not vary directly as the load is increased. Thus, the student should realize that margins of safety based on direct proportion of moments to loads are incorrect and lie on the unsafe side.

It is recommended that four significant figures be used in computations, making use of the so-called precise equations, since the results in many cases involve small differences between large numbers.

**A5.28 Example Problems**

**Example Problem #1**

Fig. A5.67 illustrates a typical upper, outer panel wing beam of a biplane. Let it be required to determine the maximum negative bending moment between points (1) and (2), generally referred to as the maximum span moment. To obtain the true bending moments on the beam, the axial beam load as well as the end moments at (1) and (2) are necessary since they influence the deflection of the beam.

**Solution:**

To obtain the horizontal component $T_h$ of the lift strut load, we take moments about the hinge point at the left end.

$$\Sigma M = -2000 \times 50 - 540 \times 115$$

$$-1500 \times 100 - 70 \times 75 \quad T_h = 0$$

hence

$$T_h = 4420\#$$

The axial compressive load induced by the lift strut at point (2) then equals $-4420\#$.

Taking $E = 0$ for the load system of Fig. A5.67 gives $P = -T_h = 4420\#$. The end moment on the beam at (1) equals the end load times the eccentricity of the hinge from the neutral axis of the beam, or $4420 \times 0.75 = 3315\#$ positive because it produces compression in the top fibers. The moment at (2) due to the cantilever overhang equals $(20 \times 10) 58 \times 10 = 8640\#$. Fig. A5.58 shows the beam portion between points (1) and (2) as a free body.

From Art. A5.25, we have the following precise equations for a beam carrying a transverse uniform distributed load with end compressive loads.

$$\tan \frac{X}{j} = \frac{D_s - D_i \cos \frac{L}{j}}{D_i \sin \frac{L}{j}} \quad \text{(A)}$$

and

$$M_{\text{max}} = \frac{D_i}{\cos \frac{X}{j}} + w_j \quad \text{(B)}$$

Evaluating terms for substitution in these equations, we obtain,

$$M_1 = 3315\#$$

$$M_a = 8640\#$$

$$P = 4420\# \text{ compression}$$

$$I = 10 \text{ in.}^4 \text{ given and assumed constant throughout the span.}$$

$$j = \sqrt{\frac{1.3 \times 10^6 \times 10}{4420}} = 2541 \approx 54.23$$

$$w_j = 20 \times 2941 = 58820$$

$$D_s = M_1 - w_j = 3315 - 58820 = -55505$$

$$D_a = M_a - w_j = 8640 - 58820 = -50180$$

$$L = 100$$

$$\frac{L}{j} = \frac{54.23}{1.844} = 29.700\#$$

From Table A5.II $\sin \frac{L}{j} = 0.86290$ and $\cos \frac{L}{j} = -0.50190$

Substituting in equation (A)

$$\tan \frac{X}{j} = \frac{D_s - D_i \cos \frac{L}{j}}{D_i \sin \frac{L}{j}}$$

$$= \frac{-55505 - (-55505 \times -0.50190)}{-55505 \times 0.86290} = -1.2192$$

$$\frac{X}{j} = \tan^{-1} 1.2192 = 53.419\text{ from Table A5.II}$$

Hence, $x = 53.419 \times 54.23 = 29.700\#$, which equals the distance from the left end of the beam to the point of maximum span moment.

$$M_{\text{max}} = \frac{D_i}{\cos \frac{X}{j}} + w_j \quad \cos \frac{X}{j} = 0.86290$$

Hence

$$M_{\text{max}} = -55505 + 58820 = 32,700\#$$

To obtain an idea as to the magnitude of the secondary bending moment, that is, the moment due to the axial load times the lateral beam deflection, the primary bending moment at a point $48\text{"}$
from the left end will be computed.

\[ M_2 = 3315 + 48 \times 20 \times 24 - 940 \times 48 = -18765'\]

Thus the secondary bending moment equals -28700 + 18765 = -9935' which is a large percentage of the primary moment. The transverse deflection of the beam at the point of max. span moment then equals -9935 = 2.26 inches upward.

- 4426

Bending Moment at any Point Along Span

Let the moment at a point 10 feet from point (2) be required. In this case, \( x = 100 - 10 = 90 \)

\[ M = D_o \left( \frac{x}{J} \sin \frac{x}{J} \cos \frac{x}{J} \right) + wJ^2 \quad \text{(Ref. Eq. A5.7)} \]

\[ \frac{x}{J} = \frac{90}{54.23} = 1.6696, \sin \frac{x}{J} = 0.99605 \]

\[ \cos \frac{x}{J} = -0.08667 \]

\[ \tan \frac{x}{J} = 1.2192 = \text{value for } x \text{ at the point of maximum bending moment} \]

Hence,

\[ M = -55505 \left( \frac{1.2192 \times 0.99605}{-0.08667} \right) + 58620 = -3654'\]

Example Problem #2

Fig. A5.69 shows a simplified landing gear structure carrying a vertical load of 12000# on the axle. Member ABC is continuous thru B and pinned at C. Let it be required to determine the bending moment at the midpoint of member BC and its lateral deflection due to the 12000# vertical design load.

![Fig. A5.69](image)

Line of action of DB goes through E

Solution:

Solving for reactions at C by statics, we obtain the axial load in BC = -20000. The bending moment at B due to 3' eccentricity of the wheel load = 3 \times 12000 = 36000'.

Fig. A5.70 shows a free body of portion BC of member ABC. From Table A5.1

\[ M = C_1 \sin \frac{x}{J} + C_2 \cos \frac{x}{J} + f(w) \]

Substituting values of \( C_1 \) and \( C_2 \) and \( f(w) \) from Table A5.1 in the above equations:

\[ M = \left( \frac{M_1}{M} \cos \frac{L}{J} \right) \sin \frac{x}{J} + M \cos \frac{x}{J} \]

20,000#

-21/2 - 083 Steel Tube

L = 41.762

20,000#

Fig. A5.70

But, \( M_1 = 0 \) in our problem, hence,

\[ M = \frac{M_1}{M} \sin \frac{x}{J} \]

\[ J = \sqrt{J^2 \sin \frac{x}{J}} \]

\[ \frac{L}{J} = \frac{41.762}{25.825} = 1.617 - - - \sin \frac{L}{J} = 0.99622 \]

\[ x = \frac{L}{2} = 20.381 \]

\[ \frac{x}{J} = \frac{20.381}{25.825} = 0.7865 - - - \sin \frac{x}{J} = 0.72327 \]

Substituting in the above equations for M

\[ M = \frac{36000 \times 0.72327}{0.99622} = 26066'\]

This compares with a primary moment of 36000/2 = 18000'. The deflection at the midpoint of BC = 26066 - 18000 = .403 in.

20000

The maximum moment is given by the equation:

\[ M_{max} = \frac{M_1}{M} \sin \frac{L}{J} \]

and it occurs at \( x = \frac{L}{2} \) (See Table A5.1)

A5.29 Stresses Above Proportional Limit Stress of Material.

The equations as presented in this chapter assume that E is constant or in other words the stresses are within the elastic range. In aircraft structural design the applied or limit loads must be taken without suffering permanent deformation, hence E is constant under such loads. However the aircraft structure must take the design loads which equal the limit load times a factor of safety (usually 1.5) without failure. In many cases structural failure will occur under stresses in the plastic range where the material stiffness is less and not constant.

A good approximation for an effective modulus E' is obtained as follows:-

(1) Compute \( F_C = P/A \) for the given number.

(2) With this value of \( F_C \) enter the basic column curve diagram for the given material (for end fixity C = 1) and find value of L'/\rho corresponding to the stress \( F_C \).
(3) Using these values of $L'/p$ and $F_c$, compute

$$E' = \frac{F_c (L'/p)}{\pi^2}$$

(4) Then $j = \left(\frac{E'I}{p}\right)^{1/2}$

Basic column curves for various materials are given in another chapter of this book.

A5.30 Problems

![Diagram](image1.png)

Fig. A5.71

(1) Fig. A5.71 shows a 1-1/2 = .065 steel tube subjected to both end and lateral loads. Determine the maximum bending moment on the tube. Compare the result with the bending moment due to the side load only. $E = 29 \times 10^6$ psi. I of tube = .075 in. Compute lateral deflection at point of maximum bending moment.

(2) The beam-column member in Fig. A5.72 is made of 24ST aluminum alloy. Calculate and plot a curve of the bending moments on the member. Also plot bending moment due to lateral loads only. $E = 10.3 \times 10^6$ psi. I = 6.0 in. ^2

(3) Determine the maximum bending moment for the wood wing beam and loading of Fig. A5.73. I of beam section = 17 in. ^2 $E = 1.3 \times 10^6$.

![Diagram](image2.png)

Fig. A5.72

![Diagram](image3.png)

Fig. A5.73

(4) Determine the bending moment at the centerline of the beam-columns shown in Fig. A5.74. Assume $E = 64,000,000$ lb. in. sq.

![Diagram](image4.png)

Fig. A5.74

(5) For the beam-column in Fig. A5.75 calculate the bending moment at the centerline of the member. Assume $E = 1,300,300$ psi. and $I = 10$ in.

![Diagram](image5.png)

Fig. A5.75

(6) For the beam-column loading in Fig. A5.76, calculate bending moment at center point of beam. Take $E = 1,200,000$ psi and $I = 10$ in.

![Diagram](image6.png)

Fig. A5.76

A5.31 Beam-Column in Continuous Structures.

The secondary moments in a particular member due to beam-column-action also effect or influence the deflections in adjacent members of a continuous structure. This rather involved problem can be handled quite simply and rapidly by the moment distribution method as explained and illustrated in Arts. All.12 to 15 of Chapter A11.
CHAPTER A6
TORSION. - STRESSES AND DEFLECTIONS

A6.1 Introduction.
Problems involving torsion are common in aircraft structures. The metal covered airplane wing and fuselage are basically thin-walled tubular structures and are subjected to large torsional moments in certain flight and landing conditions. The various mechanical control systems in an airplane often contain units of various cross-sectional shapes which are subjected to torsional forces under operating conditions, hence a knowledge of torsional stresses and distortions of members is necessary in aircraft structural design.

A6.2. Torsion of Members with Circular Cross Sections.
The following conditions are assumed in the derivation of the equations for torsional stresses and distortions:

1. The member is a circular, solid or hollow round cylinder.
2. Sections remain circular after application of torque.
3. Diameters remain straight after twisting of section.
4. Material is homogeneous, isotropic and elastic.
5. The applied loads lie in a plane or planes perpendicular to the axis of the shaft or cylinder.

Fig. A6.1 shows a straight cylindrical bar subjected to two equal but opposite torsional couples. The bar twists and each section is subjected to a shearing stress. Assuming the left end as stationary relative to the rest of the bar a line AB on the surface will move to AB' under these shearing stresses and this rotation at any section will be proportional to the distance from the fixed support. It is assumed that any radial line undergoes angular displacement only, or OB remains straight when moving to OB'.

The unit shearing strain in a distance L equals,

\[ \tau = \frac{OB'}{AB} = \frac{r \theta}{L} \]

Let G equal modulus of rigidity of the material and let \( \tau \) equal the unit shearing stress at the extreme fiber on the cross-section.

Hence, \( \tau = \frac{G \theta}{L} \) \hspace{1cm} (1)

In Fig. A6.2 let \( \tau_0 \) equal the unit shearing stress on a circular strip \( dA \) at a distance \( r \) from 0. Then

\[ \tau_0 = \frac{d \tau}{d r} = \frac{d r \theta G}{L} \]

The moment of the shearing stress on the circular strip \( dA \) about 0 the axis of the bar is equal to,

\[ dM = \tau_0 \rho dA = \frac{\rho \theta G dA}{L} \]

and thus the total internal torsional resisting moment is,

\[ M_{\text{int.}} = \int \frac{G \theta dA}{L} \]

For equilibrium, the internal resisting moment equals the external torsional moment \( T \), and since \( G \theta / L \) is a constant, we can write,

\[ T = M_{\text{int.}} = \frac{G \theta}{L} \int \rho dA = \frac{G \theta J}{L} \] \hspace{1cm} (2)

where \( J \) is polar moment of inertia of the shaft cross section and equals twice the moment of inertia about a diameter.

From equation (1) \( \frac{G \theta}{L} = \frac{T}{r} \)

Hence, \( T = \frac{r J}{L} \) \hspace{1cm} (3)

or \( \tau = \frac{T r}{\rho} \) \hspace{1cm} (4)

also from equation (2), solving for the twist \( \theta \),

\[ \theta = \frac{T}{G J} \] \hspace{1cm} (5)

(\( \theta \) is measured in radians).
A6.3 Transmission of Power by a Cylindrical Shaft.

The work done by a twisting couple \( T \) in moving through an angular displacement is equal to the product of the magnitude of the couple and the angular displacement in radians. If the angular displacement is one revolution, the work done equals \( 2\pi \) T. If \( T \) is expressed in inch-pounds and \( N \) is the angular velocity in revolutions per minute, then the horsepower transmitted by a rotating shaft may be written,

\[
\text{H.P.} = \frac{2\pi NT}{33,000} \quad \text{------------------------- (6)}
\]

where 33,000 represents inch pounds of work of one horsepower for one minute. Equation (6) may be written:

\[
T = \frac{\text{H.P.} \times 33,000}{2\pi N} \quad \text{------------------------- (7)}
\]

**EXAMPLE PROBLEMS.**

**Problem 1.**

Fig. A6.3 shows a conventional control stick-torque tube operating unit. For a side load of 150 lbs. on stick grip, determine the shearing stress on aileron torque tube and the angle of twist between points A and B.

**SOLUTION:**

Torsional moment on tube AB due to side stick force of 150 lbs. is 150 \( \times \) 26 = 3900 in. lb. The resistance to this torque is provided by the aileron operating system attached to aileron horn and the horn pull equals 3900/11 = 355 lb.

The polar moment of inertia of a \( \frac{1}{4} \) - 0.068 round tube equals 0.1368 in\(^4\).

Maximum shearing stress is \( \tau = \frac{T}{J} \) = \( \frac{3900 \times 0.75}{0.1368} \) = 21400 psi.

The angular twist of the tube between points A and B equals

\[
\theta = \frac{T r}{0.1368} = \frac{3900 \times 26}{3,800,000 \times 0.1368} = 0.21 \text{ radians}
\]
or 12 degrees.

**Problem 2.**

Fig. A6.4 illustrates an aileron control surface, consisting of a circular torque tube (1-1/4 - 0.049 in size) supported on three hinge brackets and with the control rod fitting attached to the torque tube above the center support bracket. Find the maximum torsional shearing stress in the tube if the air load on the aileron is as indicated in Fig. A6.4, and also compute the angle of twist of tube between horn section and end of aileron.

**SOLUTION:**

The airload on the surface tends to rotate the aileron around the torque tube, but movement is prevented or created by a control rod attached to the torque tube over the center support bracket.

The total load on a strip of aileron one inch wide = 40(15 x 1/44) = 4.16 lb.

Let \( w \) equal intensity of loading per inch of aileron span at the leading edge point of the aileron surface, (see pressure diagram in Fig. A6.4).

Then \( S = (0.5w)12 = 4.16 \)

hence \( w = 0.463 \) lb.

The total load \( P_1 \) forward of the centerline of torque tube = 0.463 x 3 = 1.389 lb, and \( P_2 \) the load on aileron portion aft of hinge line = 0.463 x 0.5 x 12 = 2.778 lb.

The torsional moment per running inch of torque tube: \( M = -1.389 \times 1.5 + 2.778 \times 4 = 9.0 \) in. lb. Hence, the maximum torque, which occurs at the center of the aileron, equals 9.0 x 2\pi = 261 in. lb.

\[
\tau(\text{max}) = \frac{T}{J} = \frac{261 \times 0.625}{0.06678} = 2450 \text{ psi.}
\]

\( J = 0.06678 \text{ in}^4. \)

Since the tube section is constant and the torque...
varies directly as the distance from the end of the ailerons, the angle of twist \( \theta \) can be computed by using the average torque as acting on entire length of the tube to one side of horn or a distant \( L = 25'' \), hence

\[
\theta = \frac{T_l}{GJ} = \frac{251 \times 29}{2 \times 3800000 \times 0.06678} = 57.3 \text{ degrees}
\]

A6.4 Torsion of Members with Non-Circular Cross-Sections.

The formulas derived in Art. A6.2 cannot be used for non-circular shapes since the assumptions made do not hold. In a circular shaft subjected to pure torsion, the shearing stress distribution is as indicated in Fig. A6.5, namely, The maximum shearing stress is located at the most remote fiber from the centerline axis of the bar and is perpendicular to the radius to the stressed point. At a given distance from the axis of rotation the shear stress is constant in both directions as illustrated in Fig. A6.5, which means that ends of segments of the bar as it twists remain parallel to each other or in other words the bar sections do not warp out of their plane when the bar twists.

If the conditions of Fig. A6.5 are applied to the rectangular bar of Fig. A6.6, the most stressed fibers will be at the corners and the stress will be directed as shown. The stress would then have a component normal to the surface as well as along the surface and this is not true. The theory of elasticity shows that the maximum shear stress occurs at the centerline of the long sides as illustrated in Fig. A6.6 and that the stress at the corners is zero. Thus when a rectangular bar twists, the shear stresses are not constant at the same distances from the axis of rotation and thus the ends of segments cut through the bar would not remain parallel to each other when the bar twists or in other words, warping of the section out of its plane takes place. Fig. A6.7 illustrates this action in a twisted rectangular bar. The ends of the bar are warped or suffer distortion normal to the original unstressed plane of the bar ends.

Further discussion and a summary of equations for determining the shear stresses and twists of non-circular cross-sections is given in Art. A6.6.

A6.5 Elastic Membrane Analogy.

The shape of a warped cross-section of a non-circular cross-section in torsion is needed in the analysis by the theory of elasticity, and as a result only a few shapes such as rectangles, ellipses, triangles, etc., have been solved by the theoretical approach. However, a close approximation can be made experimentally for almost any shape of cross-section by the use of the membrane analogy.

It was pointed out by Prandtl that the equation of torsion of a bar and the equation for the deflection of a membrane subjected to uniform pressure have the same form. Thus if an elastic membrane is stretched over an opening which has the same shape as the cross-section of the bar being considered and then if the membrane is deflected by subjecting it to a slight difference of pressure on the two sides, the resulting deflected shape of the membrane provides certain quantities which can be measured experimentally and then used in the theoretical equations. However, possibly the main advantage of the membrane theory is, that it provides a method of visualizing to a considerable degree of accuracy how the stress conditions vary over a complicated cross-section of a bar in torsion.

The membrane analogy provides the following relationships between the deflected membrane and the twisted bar.

(1) Lines of equal deflection on the membrane (contour lines) correspond to shearing stress lines of the twisted bar.

(2) The tangent to a contour line at any point on the membrane surface gives the direction of the resultant shear stress at the corresponding point on the cross-section of the bar being twisted.

(3) The maximum slope of the deflected membrane at any point, with respect to the edge support plane is equal in magnitude to the shear stress at the corresponding point on the cross-section of the twisted bar.

(4) The applied torsion on the twisted bar is proportional to twice the volume included between the deflected membrane and a plane through the supporting edges.

To illustrate, consider a bar with a rectangular cross-section as indicated in Fig. A6.8. Over an opening of the same shape we stretch a thin membrane and deflect it normal to the cross-section by a small uniform pressure. Equal deflection contour lines for this deflected membrane will take the shape as illustrated in Fig. A6.9. These contour lines which correspond to direction of shearing stress in the twisted bar are nearly circular near the center region of the bar, but tend
to take the shape of the bar boundary as the boundary is approached. Fig. A6.9a shows a section through the contour lines of the deflected membrane along the lines 1-1, 2-2 and 3-3 of Fig. A6.9. It is obvious that the slopes of the deflected surface along line 1-1 will be greater than along lines 2-2 or 3-3. From this we can conclude that the shear stress at any point on line 1-1 will be greater than the shear stress for corresponding points on lines 2-2 and 3-3. The maximum slope and therefore the maximum stress will occur at the ends of line 1-1. The slope of the deflected membrane will be zero at the center of the membrane and at the four corners, and thus the shear stress at these points will be zero.

### A6.6 Torsion of Open Sections Composed of Thin Plates

Members having cross-sections made up of narrow or thin rectangular elements are sometimes used in aircraft structures to carry torsional loads such as the angle, channel, and tee shapes.

For a bar of rectangular cross-section of width b and thickness t a mathematical elasticity analysis gives the following equations for maximum shearing stress and the angle of twist per unit length.

\[ \tau_{\text{max}} = \frac{T}{a \ b \ t} \]  
\[ \theta = \frac{T}{G \ b \ t^3} \]  

Values of a and \( \theta \) are given in Table A6.1.

### Table A6.1

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<td>0.214</td>
<td>0.239</td>
<td>0.264</td>
<td>0.287</td>
<td>0.313</td>
<td>0.333</td>
<td>0.359</td>
<td>0.439</td>
<td>0.500</td>
<td></td>
</tr>
</tbody>
</table>

From Table A6.1 it is noticed that for large values of \( b/t \), the values of the constants is 1/3, and thus for such narrow rectangles, equations (6) and (7) reduce to:

\[ \tau_{\text{max}} = \frac{3 \ T}{b \ t^3} \]  
\[ \theta = \frac{3 \ T}{G \ b \ t^3} \]

Although equations (6) and (7) have been derived for a narrow rectangular shape, they can be applied to an approximate analysis of shapes made up of thin rectangular members such as illustrated in Fig. A6.10. The more generous the fillet or corner radius, the smaller the stress concentration at these junctions and therefore the more accuracy of these approximate formulas. Thus for a section made up of a continuous plate such as illustrated in Fig. a, the width b can be taken as centerline length for above type of sections.

For the tee section of Fig. A6.10:

\[ \theta = \frac{T}{G \ t^3} = \frac{3 \ T}{G \ 2 \ b t^3} \]

For the maximum shearing stress on leg \( b_1 \):

\[ \tau_{b_1} = \frac{T \ t_1}{J} = \frac{3 \ T \ t_1}{b_1 t_1 + b_1 t_2} \]  

and for the plate \( b_2 \):

\[ \tau_{b_2} = \frac{T \ t_2}{J} = \frac{3 \ T \ t_2}{b_2 t_1 + b_2 t_2} \]

If \( t_1 = t_2 = t \), then

\[ \theta = \frac{3 \ T}{G \ t^3 (b_1 + b_2)} \]

\[ \tau = \frac{3 \ T}{t^3 (b_1 + b_2)} \]
EXAMPLE PROBLEM SHOWING TORSIONAL STIFFNESS OF
CLOSED THIN WALLED TUBE COMPARED
TO OPEN OR SLOTTED TUBE.

Fig. A6.11a shows a 1 inch diameter tube
with .035 wall thickness, and Fig. A6.11b shows
the same tube but with a cut in the wall making
it an open section.

For the round tube \( J_1 = 0.02474 \text{ in}^4 \).

For open tube \( J_2 = \frac{1}{3} \times 3.14 \times .035^2 = 0.000045 \)

Fig. A6.11a  Fig. A6.11b

Let \( \Theta_1 \) equal twist of closed tube and \( \Theta_2 \)
equal twist of open tube. The twist, will then
be inversely proportional to \( J \) since \( \Theta = \frac{T}{KJ} \).

Therefore the closed tube is \( J_1/J_2 = 0.02474/\)
0.000045 = 550 times as stiff as the open tube.
This result shows why open sections are not ef-
ficient torsional members relative to
torsional deflection.

<table>
<thead>
<tr>
<th>SECTION</th>
<th>K</th>
<th>FORMULA FOR SHEAR STRESS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOLID ELLIPTICAL SECTION</td>
<td>( K = \frac{\pi^2 a^4}{24} )</td>
<td>( T = \text{TMB} ) at ends of minor axis.</td>
</tr>
<tr>
<td>SOLID SQUARE</td>
<td>( K = 0.141a^4 )</td>
<td>( T = \text{TMB} ) at midpoint of side.</td>
</tr>
<tr>
<td>SOLID RECTANGLE</td>
<td>( K = \frac{2b}{2a} \left( \frac{18}{16} - 3.36 \right) )</td>
<td>( T = \text{TMB} ) at midpoint of long side.</td>
</tr>
<tr>
<td>SOLID TRIANGLE</td>
<td>( K = \frac{1.73a^4}{5b} )</td>
<td>( T = \text{TMB} ) at midpoint of side.</td>
</tr>
</tbody>
</table>

For an extensive list of formulas for many shapes both solid and hollow, refer to book, "Formulas For Stress and Strain" by Roark, 1984 Edition.

A6.7 Torsion of Solid Non-Circular Shapes and Thick-Walled Tubular Shapes.

Table A6.3 summarizes the formulas for
torsional deflection and stress for a few
shapes. These formulas are based on the as-
sumption that the cross-sections are free to
warp (no end restraints). Material is homo-
geneous and stresses are within the elastic
range.

A6.8 Torsion of Thin-Walled Closed Sections.

The structure of aircraft wings, fuselages and
control surfaces are essentially thin-walled
of tubes or more cells. Flight and landing
loads often produce torsional forces on these
major structural units, thus the determination
of the torsional stress and deformation of such
structures plays an important part in aircraft
structural analysis and design.

Fig. A6.12 shows a portion of a thin-

Fig. A6.12  Fig. A6.13

Let \( q_0 \) be the shear force intensity at point
(a) on the cross-section and \( q_0 \) that at point
(b).

Now consider the segment a Y b of the tube
wall as shown in Fig. A6.12 as a free body. The
applied shear force intensity along the segment
edges parallel to the y axis will be given the
values \( q_{ay} \) and \( q_{by} \) as shown in Fig. A6.12. For
a plate in pure shear the shearing stress at a
point in one plane equals the stress in a plane
at right angles to the first plane, hence
\( q_{ay} = q_{ay} \) and \( q_{by} = q_{by} \).

Since the tube sections are free to warp
there can be no longitudinal stresses on the
tube wall. Considering the equilibrium of the
segment in the y direction,
\[ \Sigma y = 0 = q_{ay} \frac{a}{2} - q_{by} \frac{a}{2} \]
and therefore \( q_{ay} = q_{by} \) or in other words the shear
force intensity around the tube wall is con-
stant. The shear stress at any point \( \tau = q/a \).
If the wall thickness \( t \) changes the shear stress.
changes but the shear force \( q \) does not change, or

\[ \tau_a t_a = \tau_b t_b = \text{constant}. \]

The product \( \tau t \) is generally referred to as the shear flow and is given the symbol \( q \). The name shear flow possibly came from the fact that the equation \( \tau t = \text{constant} \), resembles the equation of continuity of fluid flow \( qS = \text{constant} \) where \( q \) is the flow velocity and \( S \) the tube cross-sectional area.

We will now take moments of the shear flow \( q \) on the tube cross-section about some point \( (o) \). In Fig. A6.13 the force \( df \) on the wall element \( ds = qds \). Its arm from the assumed moment center \( (o) \) is \( h \). Thus the moment of \( df \) about \( (o) \) is \( qdh \). However, \( ds \) times \( h \) is twice the area of the shaded triangle in Fig. A6.13.

Hence the torsional moment \( dt \) of the force on the element \( ds \) equals,

\[ dt = qdh = 2qda \]

and thus for the total torque for the entire shear flow around the tube wall equals,

\[ T = \int q2a \text{ and since } q \text{ is constant} \]

\[ T = q2a \hspace{1cm} (14) \]

or

\[ q = \frac{T}{2a} \hspace{1cm} (15) \]

where \( A \) is the enclosed area of the mean periphery of the tube wall.

The shear stress \( \tau \) at any point on the tube wall is equal to \( q \), the shear force per inch of wall divided by the area of this one inch length or \( 1 \times t \) or

\[ \tau = \frac{q}{t} = \frac{T}{2at} \hspace{1cm} (16) \]

**TUBE TWIST**

Consider a small element cut from the tube wall and treated as a free body in Fig. A6.14, with \( ds \) in the plane of the tube cross-section and a unit length parallel to the tube axis. Under the shearing strains the plate element

![Fig. A6.14](image)

deforms as illustrated in Fig. A6.15, that is, the face \( a-a \) moves with respect to face \( 2-2 \) a distance \( \delta \). The force on edge \( a-a \) equals \( qds \) and it moves through a distance \( \delta \).

![Fig. A6.15](image)

The elastic strain energy \( dU \) stored in this element therefore equals,

\[ dU = \frac{qds}{2} \cdot \delta \]

However the shear strain \( \delta \) can be written,

\[ \delta = \frac{T}{G} = \frac{2a}{G} \text{ but } q = \frac{T}{2a} \]

hence

\[ dU = \frac{T^2}{2a^2G}ds \]

or

\[ U = \int \frac{T^2}{2a^2G}ds, \text{ the integral of } \]

is the line integral around the periphery of the tube. From Chapter A7 from Castigliano’s theorem,

\[ U = \frac{2U}{T} = \int \frac{T}{4a^2G}ds \hspace{1cm} (17) \]

since all values except \( t \) are constant, equation (17) can be written,

\[ \phi = \frac{T}{4a^2G} \int ds \hspace{1cm} (18) \]

and since \( T = 2qa \), then also,

\[ \phi = \frac{q}{2aG} \int ds \hspace{1cm} (19) \]

where \( \phi \) is angle of twist in radians per unit length of one inch of tube. For a tube length of \( L \),

\[ \phi = \frac{q}{2aG} \int \frac{ds}{t} \hspace{1cm} (20) \]

**A6.9 Expression for Torsional Moment in Terms of Internal Shear Flow Systems for Multiple Cell Closed Sections.**

Fig. A6.16 shows the internal shear flow pattern for a 2-cell thin-walled tube, when the tube is subjected to an external torque. \( q_a, q_b \) and \( q_3 \) represent the shear load per inch on the three different portions of the cell walls.

For equilibrium of shear forces at the junction point of the interior web with the outside wall, we know that

\[ q_1 = q_2 + q_3 \hspace{1cm} (21) \]

![Fig. A6.16](image)

Choose any moment axis such as point \( (o) \). Referring back to Fig. A6.13, we found that the moment of a constant shear force \( q \) acting along a wall length \( ds \) about a point \( (o) \) was equal in
ANSALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

magnitude to twice the area of the geometrical shape formed by radii from the moment center to the ends of the wall element ds times the shear flow q.

Let $T_0$ = moment of shear flow about point (o). Then from Fig. A6.16,

$$T_0 = 2q_1(A_1 + A_2) + 2q_2A_{a2} - 2q_2A_{a2}$$

$$= 2q_1A_1 + 2q_2A_{ab} + 2q_2A_{a2} - 2q_2A_{a2}$$

$$= 2q_1A_1 + 2q_2A_{ab}$$

But from equation (21), $q_3 = q_1 - q_2$.

Substituting the value of $q_3$ in (22)

$$T_0 = 2q_1A_1 + 2q_2A_{ab} + 2q_2A_{a2} - 2q_2A_{a2} + 2q_2A_{ab}$$

But $A_{a2} = A_{ab} + A_{a2}$

Hence, $T_0 = 2q_1A_1 + 2q_2A_{ab} - \ldots$ (23)

where $A_1$ = area of cell (1) and $A_2 = area of cell (2)$. Therefore, the moment of the internal shear system of a multiple cell tube carrying pure torsional shear stresses is equal to the sum of twice the enclosed area of each cell times the shear load per inch which exists on the outside wall of that cell. (Note: The cell is referred to as an inside wall of either cell).

A6.10 Distribution of Torsional Shear Stresses in a Multiple-Cell Thin-Walled Closed Section. Angle of Twist.

Fig. A6.17 shows in general the internal shear flow pattern on a 3-cell tube produced by a pure torque load on the tube. The cells are numbered (1), (2), and (3), and the area outside the tube is designated as cell (o). Thus, to designate the outside wall of cell (1), we refer to it as lying between cells 1-2; for the outside wall of cell 2, as 2-3; and for the web between cells (1) and (2) as 1-2, etc.

$q_1$ = shear load per inch = $\tau_1t_1$ in the outside wall of cell (1), where $\tau_1$ equals the unit stress and $t_1$ = wall thickness. Likewise, $q_2 = \tau_2t_2$ and $q_3 = \tau_3t_3$ = shear load per inch in outside walls of cells (2) and (3) respectively. For equilibrium of shear forces at the junction points of interior webs with the outside walls, we have $(q_1 - q_2)$ equal to the shear load per inch in the web 1-2 and $(q_2 - q_3)$ for web 2-3.

For equilibrium, the torsional moment of the internal shear system must equal the external torque on the tube at this particular section. Thus, from the conclusions of article A5.9, we can write:

$$T = 2q_1A_1 + 2q_2A_{ab} + 2q_3A_{a2}$$

For elastic continuity, the twist of each cell must be equal, or $q_1 = q_2 = q_3$.

From equation (19), the angular twist of a cell is

$$\theta = \frac{q}{2\pi 0} \int ds$$

Thus, for each cell of a multiple cell structure an expression $\frac{q}{A} \int ds$ can be written and equated to the constant value $\Theta$. Let $a_1$, represent a line integral $\int ds$ for cell wall 1-0, and $a_{a1}$, $a_{a2}$, $a_{a3}$ and $a_{a0}$ the line integrals $\int ds$ for the other outside wall and interior web portions of the 3-cell tube. Let clockwise direction of wall shear stresses in any cell be positive in sign. Now, substituting in Equation (25), we have:

$$\begin{align*}
cell (1) & \frac{1}{A_1} \left[ q_1 a_{a1} + (q_1 - q_2) a_{a2} \right] = \Theta \\
cell (2) & \frac{1}{A_2} \left[ (q_2 - q_1) a_{a2} + q_3 a_{a3} + (q_3 - q_2) a_{a3} \right] = \Theta \\
cell (3) & \frac{1}{A_3} \left[ (q_3 - q_2) a_{a3} + q_3 a_{a0} \right] = \Theta
\end{align*}$$

Equations (24, 25, 27 and 28) are sufficient to determine the true values of $q_1$, $q_2$, $q_3$ and $\Theta$.

Thus, to determine the torsional stress distribution in a multiple cell structure, we write equation (25) for each cell and these equations together with the general torque equation, similar to equation (24), provides sufficient conditions for the solution of the shear stresses and the angle of twist.

A6.11 Stress Distribution and Angle of Twist for 2-Cell Thin-Wall Closed Section.

For a two cell tube, the equations can be simplified to give the values of $q_1$, $q_2$, and $\Theta$ directly. For tubes with more than two cells, the equations become too complicated, and thus the equations should be solved simultaneously. Equations for two-cell tube (Fig. A6.18):
q_1 = \frac{1}{2} \left[ \frac{(a_{so}A_1 + a_{14}A_4)}{a_{so}A_1^2 + a_{14}A_4^2 + a_{so}A_4^2} \right] T \quad \ldots \quad (29)

q_2 = \frac{1}{2} \left[ \frac{a_{14}A_4 + a_{14}A_4}{a_{so}A_1^2 + a_{14}A_4^2 + a_{so}A_4^2} \right] T \quad \ldots \quad (30)

J = 4 \left[ \frac{a_{so}A_1^2 + a_{14}A_4^2 + a_{so}A_4^2}{a_{so}A_1^2 + a_{14}A_4^2 + a_{so}A_4^2} \right] \quad \ldots \quad (31)

Q = \frac{T}{G} \quad \ldots \quad (32)

where \( A = A_1 + A_4 \).

A6.12 Example Problems of Torsional Stresses in Multiple-Cell-Thin-Walled Tubes.

Example 1 - Torsional Stresses in Un-symmetrical Two-Cell-Tube.

Fig. A6.19 shows a typical 2-cell tubular section as formed by a conventional airfoil shape, and having one interior web. An external applied torque \( T \) of 83450 in. lb. is assumed acting as shown. The internal shear resisting pattern is required.

Calculation of Cell Constants

Cell areas: \( A_1 = 105.8 \text{ sq. in.} \)
\( a_{so} = 397.4 \text{ sq. in.} \)
\( A = 493.2 \text{ sq. in.} \)

Line integrals \( a = \oint \frac{ds}{t} \): 
\( a_{14} = 26.9 \times 1075; \ a_{14} = 13.4 \times 0.04 = 335 \)
\( a_{so} = 25.25 \times 0.04 + 16.1 \times 0.05 + 26.3 \times 0.032 = 1735 \)

Solution by equating angular twist of each cell.

General equation \( 2G = \frac{q_1}{A} \oint \frac{ds}{t} \). Clockwise flow of \( q \) is positive.

Cell 1 Subt. in general equation

\( 2G = \frac{1}{105.8} \left[ -q_1 \times 1075 + (-q_1 + q_4) \times 335 \right] = -13.33 q_1 + 3.165 q_4 \) \quad \ldots \quad (33)

Cell 2

\( 2G = \frac{1}{397.4} \left[ -q_4 - q_1 \right] \times 335 - 1735 q_4 = 0.865 q_4 - 5.34 q_4 \) \quad \ldots \quad (34)

Equating (33) and (34)

\( -14.195 q_1 + 8.605 q_4 = 0 \) \quad \ldots \quad (35)

The summation of the external and internal resisting torque must equal zero.

Solving equations (33) and (36), \( q_1 = 55.8/\text{in.} \) and \( q_4 = 32.5/\text{in.} \). Since results come out positive, the assumed direction of counter-clockwise was correct for \( q_1 \) and \( q_4 \) or true signs are \( q_1 = -55.8 \) and \( q_4 = -32.5 \).

\( q_4 = -55.8 + 32.5 = 35.9/\text{in.} \) (as viewed from Cell 1).

Fig. A6.20 shows the resulting shear pattern. The angular twist of the complete cell can be found by substituting values of \( q_1 \) and \( q_4 \) in either equations (33) or (34), since twist of each cell must be the same and equal to twist of tube as a whole.
Example Problem 2.

Determine the torsional shear stresses in the symmetrical 2 cell section of Fig. A6.21 when subjected to a torque \( T \). Neglect any resistance of stringers in resisting torsional moment.

**SOLUTION:**

**Calculation of Terms:**

**Area of cells:**

\[ A_1 = 100 \quad A_2 = 100 \]

\[ A = A_1 + A_2 = 200 \]

**Line integrals**

\[ \int_{A_1} \frac{dA}{r} = \frac{10}{.05} + \frac{20}{.03} = 866.7 \]

\[ a_{1o} = \frac{10}{.05} = 200 \]

\[ a_{1s} = \frac{10}{.03} = 333.3 \]

\[ a_{2o} = \frac{20}{.03} + \frac{10}{.04} = 916.7 \]

Solution of Equations from Article A6.11:

\[ q_1 = \frac{1}{2} \left[ \frac{3a_{0A_1} + 3a_{1A}}{2a_{0A_1}^{2} + 3a_{1A}^{2} + 3a_{0A_2}^{2}} \right] T \]

\[ q_1 = \frac{1}{2} \left[ \frac{916.7 \times 100 + 333.3 \times 200}{916.7 \times 100^2 + 333.3 \times 200^2 + 866.7 \times 100^2} \right] T \]

\[ = .002546 T \]

\[ J = 4 \left[ \frac{a_{0aA_1^3} + a_{1aA}^3 + a_{10aA_2^2}}{a_{10aA_1} + a_{1aA_2} + a_{0aA_2}} \right] \]

\[ = 4 \left[ \frac{916.7 \times 100 \times 333.3 \times 200^2 + 866.7 \times 100^2}{916.7 \times 333.3 \times 333.3 \times 916.7 \times 916.7 \times 866.7} \right] \]

\[ = 89.76 \]

\[ q = \frac{T}{JG} = \frac{T}{G \times 89.76} = .01115 T \] (rad. per unit length of cell).

A6.13 Example 3 - Three-Cell-Tube.

Fig. A6.22 shows a thin-walled tubular section composed of three cells. The internal shear flow pattern will be determined in resisting the external torque of 100,000 in as shown.

**SOLUTION:**

**Calculation of cell constants**

**Cell areas:**

\[ A_1 = 39.3 \quad A_2 = 100 \quad A_3 = 100 \]

**Line integrals**

\[ \int_{A_1} \frac{dA}{r} = \frac{\pi \times 10}{2 \times .035} = 629 \]

\[ a_{1o} = \frac{\pi \times 10}{2 \times .035} = 200 \]

\[ a_{1s} = \frac{20}{.03} = 667 \]

\[ a_{2o} = \frac{20}{.03} + \frac{10}{.04} = 917 \]

Equating the external torque to the internal resisting torque:

\[ 2qa_1 + 2qa_2 + 2qa_3 + T = 0 \]

Substituting:

\[ 78.6 q_1 + 200 q_2 + 200 q_3 - 100,000 = 0 \] - (37)

Writing the expression for the angular twist of each cell:

**Cell (1)**

\[ 200 = \frac{1}{A_1} \left[ q_1 a_{1o} + (q_1 - q_1) a_{1s} \right] \]
Substituting:
\[ 2\Theta = \frac{1}{33.3} \left[ 629 \, q_1 + 200 \, q_3 - 200 \, q_a \right] - - - - (38) \]

Cell (2)
\[ 2\Theta = \frac{1}{A_1} \left[ (q_2 - q_1) \, a_{a1} + q_2 a_{a2} + (q_a - q_3) \, a_{a3} \right] \]

Substituting:
\[ 2\Theta = \frac{1}{100} \left[ 200 \, q_1 - 200 \, q_3 + 333 \, q_a - 333 \, q_3 \right] - - - - - - - - - - - - (39) \]

Cell (3)
\[ 2\Theta = \frac{1}{A_3} \left[ (q_3 - q_2) \, a_{a3} + q_3 a_{a4} \right] \]

Substituting:
\[ 2\Theta = \frac{1}{100} \left[ 333 \, q_3 - 333 \, q_a + 917 \, q_3 \right] - - - - (40) \]

Solving equations (37) to (40), we obtain,
- \( q_1 = 143.4 \# / \text{in.} \)
- \( q_2 = 234.1 \# / \text{in.} \)
- \( q_3 = 208.8 \# / \text{in.} \)
- \( q_a - q_3 = 90.7 \# / \text{in.} \)
- \( q_a - q_3 = 25.3 \# / \text{in.} \)

Fig. A6.23 shows the resulting internal shear flow pattern. The angle of twist, if desired, can be found by substituting values of shear flows in any of the equations (38) to (40),

A6.14 Torsional Shear Flow in Multiple Cell Beams by Method of Successive Corrections.
The trend in airplane wing structural design particularly in high speed airplanes is toward multiple cell arrangement as illustrated in Fig. A6.24, namely a wing cross-section made up of a relatively large number of cells.

![Fig. A6.24](image)

With one unknown shear flow \( q \) for each cell, the solution by the previous equations becomes quite laborious.

The method of successive approximations provides a simple, rapid method for finding the shear flow in multiple cells under pure torsion.

**EXPLANATION OF SUCCESSIVE CORRECTION METHOD.**

Consider a two cell tube as shown in Fig. a. To begin with assume each cell as acting independently, and subject cell (1) to such a shear flow \( q_1 \) as to make \( \Theta_1 = 1 \).

From equation (19) we can write,
\[ q = \frac{G}{2A} \int \frac{d\Theta}{t} \]

Now assume \( \Theta_1 = 1 \), then
\[ q = \frac{G}{2A} \int \frac{d\Theta}{t} \]

Since practically all cellular aircraft beams have wall and web panels of constant thickness for each particular unit, the term \( \int \frac{d\Theta}{t} \) for simplicity will be written \( 2 \frac{L}{t} \) where \( L \) equals the length of a wall or web panel and \( t \) its thickness. Thus we can write,
\[ q = \frac{2A}{E_{\text{cell}}} \frac{L}{t} \]

Therefore assuming \( \Theta_1 = 1 \) for cell (1) of Fig. a, we can write from equation (41):

\[ q_1 = \frac{2A_s}{L} \times \frac{2 \times 89.3}{.04} + \frac{10}{.05} = \frac{173.6}{842} = 0.212 \text{ lb./in.} \]

Fig. b. shows the results.

\[ q = .212 \]

(1)

\[ q = .109 \]

(2)

Fig. c

In a similar manner assume cell (2) subjected to a shear flow \( q_a \) to make \( \theta_a = 1 \). Then

\[ q_a = \frac{2A_s}{L} \times \frac{2 \times 89.3}{.05} + \frac{15.7}{.06} = \frac{78.6}{723} = 0.108\text{ lb./in.} \]

Fig. c shows the results.

Now assume the two cells are joined together with the interior web (1-2) as a common part of both cells.

See Fig. d. The interior web is now subjected to a resultant shear flow of \( q_1 - q_a = (.212 - .109) = .103\text{ lb./in.} \).

Obviously this change of shear flow on the interior web will cause the cell twist to be different for each cell instead of the same when the cells were considered acting separately. To verify this conclusion the twist measured by the term \( \theta_a \) will be computed for each cell.

Cell (1), \( \theta_a = \frac{1}{2A_s} \times \frac{10}{.05} \times \frac{25.7}{.04} + \frac{212 - 109}{.05} \]

\[ = 0.875 \]

Cell (2), \( \theta_a = \frac{1}{2A_s} \times \frac{10}{.05} \times \frac{25.7}{.04} + \frac{109 - 212}{.05} \]

\[ = 0.4375 \]

Since \( \theta_a \) must equal \( \theta_a \) if the cross-section is not to distort from its original shape, it is evident that the above shear flows are not the true ones when the two cells act together as a unit.

Now consider cell (1) in Fig. d. In bringing up and attaching cell (2) the common web

\[ (1-2) \] is subjected to a shear flow \( q_a = .109\text{ lb./in.} \) (counterclockwise with respect to cell (1) and therefore negative). In addition to the shear flow \( q_1 = .212 \) of cell (1). The negative shear flow \( q_a = .109 \) on web 1-2 decreases the twist of cell (1) as calculated above with the resulting value for \( \theta_a = .37 \) instead of 1.0 as started with.

Thus in order to make \( \theta_a = 1 \) again, we will have to add a constant shear flow \( q_1 \) to cell (1) which will cancel the negative twist due to \( q_a \) acting on web (1-2). Since we are considering only cell (1) we can compare cell wall strains instead of cell twist since in equation (41), the term \( 2A_s \) is constant.

Thus adding a constant shear flow \( q_1 \) to cancel influences of \( q_a \) on web 1-2, we can write:

\[ q_1 \left( \frac{L}{L} \right) \text{ cell (1)} - q_a \left( \frac{L}{L} \right) \text{ web 1-2} = 0 \]

hence

\[ q_1 = q_a \left[ \left( \frac{L}{L} \right) \text{ web 1-2} \right] - \left( \frac{L}{L} \right) \text{ cell (1)} \]

Substituting values in equation (42)

\[ q_1 = q_a \left[ \frac{10}{25.7 + 10} \times \frac{.04}{.05} \right] = \frac{200}{842} q_a = .237 q_a \]

Thus to make \( \theta_a \) equal to 1 we must correct the shear flow in cell (1) by adding a constant shear flow equal to .237 times the shear flow \( q_a \) in cell (2) which equals .237 x .109 = .0265\text{ lb./in.} Since this shear flow is in terms of the shear flow \( q_a \) of the adjacent cell it will be referred to as a correction carry over shear flow, and will consist of a carry over correction factor times \( q_a \).

Thus the carry over factor from cell (2) to cell (1) may be written as

\[ \text{C.O.F. (2 to 1)} = \left( \frac{L}{L} \right) \text{ web (1-2)} \text{ cell (1)} \]

.237 as found above in substitution in equation (42).

Now consider cell (2) in Fig. d. In bringing up and attaching cell (1), the common web (2-1) is subjected to a shear flow of \( q_1 = -0.212\text{ lb./in.} \) (counterclockwise as viewed from cell 2 and therefore negative). This additional shear flow changes \( \theta_a \) twist of cell (2) to a relative value of 0.4375 instead of 1.0 (see previous \( \theta_a \) calculations). Therefore to make \( \theta_a \) equal to 1.0 again, a corrective constant shear flow \( q_1 \) must be added to cell (2) to
cancel the twist effect of \( q_1 = 0.212 \) on web (2-1). Therefore we can write,

\[
q'_1 \left( \frac{1}{z} \right) \text{ cell (2)} - q_1 \left( \frac{1}{z} \right) \text{ web (2-1)} = 0
\]

hence

\[
q'_1 = q_1 \left( \frac{1}{z} \right) \text{ web (2-1)} \frac{1}{z} \text{ cell (2)}
\]  

Substituting in equation (43):

\[
q'_1 = q_1 \left( \frac{200}{723} \right) = 0.277 q_1 = .277 \times .212 = 0.0587\#/\text{in}.
\]

Thus the carry over factor from cell (1) to cell (2) in terms of \( q_1 \) to make \( \Phi_2 = 1 \) again can be written

\[\text{C.O.F. (1 to 2)} = \frac{\left( \frac{1}{z} \right)}{z} \text{ web (2-1) } = \frac{200}{723} = 0.277
\]

Fig. e shows the constant shear flow \( q'_1 \) and \( q_1 \) that were added to make \( \Phi_1 = 1 \) for each cell. However these corrective shear flows were added assuming the cells were again independent of each other or did not have the common web (1-2). Thus in bringing the cells together again the interior web is subjected to be resultant shear flow of \( q'_1 - q_1 \). In other words if we were to add the shear flows of Fig. e to those of Fig. d, we would not have \( \Phi_1 \) and \( \Phi_2 \) equal to 1. The resulting values would be closer to 1.0 than were found for the shear flow system of Fig. d.

Considering Fig. e, we will now add another set of corrective shear flows \( q''_1 \) and \( q'_1 \) to cells (1) and (2) respectively to make \( \Phi_1 \) and \( \Phi_2 \) = 1 for cells acting independently.

Considering cell (1), and proceeding with same reasoning as before,

\[
q'_1 \left( \frac{1}{z} \right) \text{ cell (1)} - q_1 \left( \frac{1}{z} \right) \text{ web (1-2)} = 0
\]

Hence

\[
q''_1 \left( \frac{1}{z} \right) \text{ cell (1)} - q_1 \left( \frac{1}{z} \right) \text{ web (1-2)} = 0.0587 \times 0.237 = 0.0139\#/\text{in}.
\]

or

\[
q''_1 = (\text{C.O.F.}) \text{ 2 to 1 times } q'_1 = 0.237 \times 0.0587 = 0.0139
\]

Considering Cell (2)

\[
q''_1 = q_1 \left( \text{C.O.F.} \right) 1 \text{ to } = 0.237 \times 0.227 = 0.0717\#/\text{in}.
\]

Fig. f shows the resulting second set of corrective constant shear flows for each cell. Since our corrective shear flows are rapidly getting smaller, the continuation of the process depends on the degree of accuracy we wish for the final results. Suppose we add one more set of corrective constant shear flows \( q'''_1 \) and \( q'_1 \). Using the carry over correction factors previously found we obtain,

\[
q'''_1 = 0.237 \times 0.0717 = 0.007\#/\text{in}
\]

\[
q''_1 = 0.277 \times 0.0139 = 0.00385\#/\text{in}
\]

Fig. g shows the results.

The final or resulting cell shear flows then equal the original shear flows plus all corrective cell shear flows, or

\[
q_1 (\text{final}) = q_1 + q''_1 + q''_1 + q''_1
\]

\[
q_1 (\text{final}) = q_1 + q''_1 + q''_1 + q''_1
\]

Fig. h shows the final results. To check the final twist of each cell the value \( \Phi_1 \) will be computed for each cell using the \( q \) values in Fig. h.

Cell (1)

\[
\Phi_1 = \frac{1}{2 x 29.3} \left[ 0.2534 \times 25.7 + 0.0747 \times 10 \times 0.05 \right] = 0.997
\]

\[
\Phi_2 = \frac{1}{2 x 29.3} \left[ 0.1787 \times 15.7 - 0.0747 \times 10 \times 0.05 \right] = 0.997
\]

A6.15 Use of Operations Table to Organize Solution by Successive Corrections.

Operations Table 1 arranges the calculations so that the steps dealing with the corrective shear flows can be carried out rapidly and with a minimum of thought.
**ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES**

### OPERATIONS TABLE 1

<table>
<thead>
<tr>
<th>Cell 1</th>
<th>Cell 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.237</td>
</tr>
<tr>
<td>2</td>
<td>.327</td>
</tr>
<tr>
<td>3</td>
<td>.0137</td>
</tr>
<tr>
<td>4</td>
<td>.0265</td>
</tr>
<tr>
<td>5</td>
<td>.017</td>
</tr>
<tr>
<td>6</td>
<td>.00592</td>
</tr>
<tr>
<td>7</td>
<td>2.444</td>
</tr>
<tr>
<td>8</td>
<td>45.2</td>
</tr>
<tr>
<td></td>
<td>58.3</td>
</tr>
</tbody>
</table>

#### Explanation of Table 1

Line 1 gives the carry over factors for each cell, computed as explained before. Line 2 gives the necessary constant shear flow q in each cell to give unit rate of twist to each cell acting independently. (Gθ = 1). Line 3 gives the first set of constant corrective shear flows to add to each cell. The corrective q referred to as the carry over q or C.O.q in the table consists of the q in the adjacent cell times the C.O. factor of that cell.

Thus .237 x .109 = .0265 is carry over from cell (2) to cell (1) and .237 x .212 = .0587 is carry over from cell (1) to cell (2).

Line 4 gives the second set of corrective carry over shear flows, namely .277 x .0265 = .00717 to cell (2) and .277 x .0587 = .0139 to cell (1). Line 5 repeats the corrective carry over process once more. Line 6 gives the final q values which equal the original q plus all carry over q values.

---

### Example Problem 1 (2 cells)

Determine the internal shear flow system for the two cell tube in Fig. A.6.25 when subjected to a torque of 20,000 in. lbs.

![Fig. A6.25](image)

### OPERATIONS TABLE 2

<table>
<thead>
<tr>
<th>Cell 1</th>
<th>Cell 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>25.8</td>
</tr>
<tr>
<td>3</td>
<td>10.4</td>
</tr>
<tr>
<td>4</td>
<td>2.1</td>
</tr>
<tr>
<td>5</td>
<td>10.4</td>
</tr>
<tr>
<td>6</td>
<td>5.4</td>
</tr>
<tr>
<td>7</td>
<td>1.6</td>
</tr>
<tr>
<td>8</td>
<td>0.1</td>
</tr>
<tr>
<td>9</td>
<td>0.7</td>
</tr>
<tr>
<td>10</td>
<td>7778</td>
</tr>
<tr>
<td>11</td>
<td>18876</td>
</tr>
<tr>
<td>12</td>
<td>52.2</td>
</tr>
</tbody>
</table>

#### Explanation of solution as given in Table (2):

Line 1 gives the values of the carry over factors. The C.O. factors are calculated as follows:

- C.O. factor cell (1) to cell (2):
  \[ q_1 = \frac{2A_1}{L} \]
  \[ q_2 = \frac{2A_2}{L} \]

For cell (1)

\[ q_1 = \frac{2\times25}{20} = \frac{50}{20} = 2.5 \]

For cell (2)

\[ q_2 = \frac{2\times10}{20} = \frac{20}{20} = 1 \]

\[ q_3 = 0.5 \]

\[ q_4 = 0.4 \]

\[ q_5 = 0.2 \]

\[ q_6 = 0.1 \]

\[ q_7 = 0.05 \]

\[ q_8 = 0.05 \]

\[ q_9 = 0.05 \]

\[ q_{10} = 0.05 \]

\[ q_{11} = 0.05 \]

\[ q_{12} = 0.05 \]

\[ T = 2Aq \]
For cell (1) \( T = 2 \times 100 \times 32.85 = 7770 \) in. lb.
For cell (2) \( T = 2 \times 100 \times 55.5 = 11100 \) in. lb.

Line 10 gives the sum of the above two values which equals 18870 in. lb.

The original requirement of the problem was the shear flow system for a torque of 20,000 lb.
Therefore the required \( q \) values follow by direct proportion, whence

\[
q_1 = \frac{20000}{18870} \times 32.85 = 41.2\, \text{lb/in.}
\]

\[
q_2 = \frac{20000}{18870} \times 55.5 = 58.34\, \text{lb/in.}
\]

These values are shown in line 11 of Table 2. Check on twist of cells under final \( q \) values.
The relative total strain around each cell boundary is given by the term \( \frac{4}{A} \int q \frac{L}{t} \) for the cell.

Thus for cell (1)
\[
\frac{1}{A_1} \int q \frac{L}{t} = \frac{1}{100} \left[ 41.2 \left( \frac{30}{0.05} \right) + 55.5 \right] \frac{10}{0.05} = 212
\]

For cell (2),
\[
\frac{1}{A_2} \int q \frac{L}{t} = \frac{1}{100} \left[ (55.5 - 41.2) \frac{10}{0.05} + 58.34 \times \frac{30}{0.1} \right]
\]

= 212

Thus both cells have the same twist.

In the above calculations \( q_1 \) and \( q_2 \) act clockwise in each cell, hence the shear flow on the interior common web is the difference of the two \( q \) values.

Example Problem 2. Three cells
The three cell structure in Fig. A6.26 is subjected to an external torque of \(-100,000\) in. lb. Determine the internal resisting shear flow pattern.

\[
\begin{array}{c}
12'' \\
.04 \\
.032 \\
12'' .04 \\
\end{array}
\begin{array}{c}
.05 \\
Cell 1 \\
Cell 2 \\
Cell 3 \\
.04 \\
.032 \\
36^\circ
\end{array}
\]

Fig. A6.26

Explanation of solution as given in Table 3:

<table>
<thead>
<tr>
<th>Cell 1</th>
<th>Cell 2</th>
<th>Cell 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>144</td>
<td>96</td>
<td>56.5</td>
</tr>
<tr>
<td>$\frac{L}{t} = 1140$</td>
<td>$\frac{L}{t} = 1040$</td>
<td>$\frac{L}{t} = 1065$</td>
</tr>
</tbody>
</table>

Shear flow \( q \) for \( GG = 1 \) for each cell acting independently:

<table>
<thead>
<tr>
<th>Cell 1</th>
<th>Cell 2</th>
<th>Cell 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = \frac{2A_1}{L/t} = 299 = 0.293$</td>
<td>$q = \frac{2A_2}{L/t} = 192 = 0.1845$</td>
<td>$q = \frac{2A_3}{L/t} = 113 = 0.107$</td>
</tr>
</tbody>
</table>

To avoid small numbers these values of \( q \) are multiplied by 100 and entered on line 2 of the table. Calculation of carry factors as given in line 1 of Table 3.

Cell (1) to (2)

\[
\text{C.O. (1-2) } = \frac{\left( \frac{L}{t} \right)_{\text{web 1-2}}}{\left( \frac{L}{t} \right)_{\text{cell 2}}} = \frac{240}{1040} = 0.23
\]

Cell (2) to (1)

\[
\text{C.O. (2-1) } = \frac{\left( \frac{L}{t} \right)_{\text{web 2-1}}}{\left( \frac{L}{t} \right)_{\text{cell 1}}} = \frac{240}{1140} = 0.21
\]

Cell (2) to (3)

\[
\text{C.O. (2-3) } = \frac{\left( \frac{L}{t} \right)_{\text{web 2-3}}}{\left( \frac{L}{t} \right)_{\text{cell 3}}} = \frac{300}{1065} = 0.284
\]
Cell (3) to (2)

\[
\text{C.O. (3-2)} = \left( \frac{L}{T} \right)_{\text{web 2-3}} \times \frac{300}{1040} = 0.288
\]

The balance of the solution or procedure in Table 3 is the same as explained for Problem 1 in Table 2. It should be noticed that cell (2) being between two cells receives carry over \( q \) values from both adjacent cells and these two values are added together before being distributed or carried over again to adjacent cells. For example consider line 3 in Table 3 and cell (2). The \( q \) value 5.82 representing 0.23 \( x \) 23.3 is brought over from cell (1) and the \( q \) value 3.08 = 0.228 \( x \) 10.70 from cell (3). These two values are added together to 5.82 + 3.08 = 8.90. The carry over \( q \) to cell (1) is then 0.21 \( x \) 8.90 = 1.87 and to cell (3) is 0.224 \( x \) 8.90 = 2.05.

In line 6, the final \( q \) in cell (2) equals the original \( q \) of 18.45 plus all carry over \( q \) values from each adjacent cell.

Line 10 in Table 3 shows the total torque developed by the resultant internal shear flow is 17435". Since the problem was to find the shear flow system for a torque of -100,000 in. lb., the values of \( q \) in line 8 must be multiplied by the factor 100,000/17435.

Line 11 shows the final \( q \) values.

Example Problem 3. Four cells.

Determine the internal shear flow system for the four cell structure in Fig. A6.27 when subjected to a torsional moment of -100,000 in. lb.

Cell 1: 8" 20 90 40 1000 20 90 40 1000
Cell 2: 16" (500) (625) (1600)
Cell 3: 300 (280) (250) 0.04 0.05 0.02 0.02
Cell 4: (500) (625) (1600)

Fig. A6.27

In Fig. A6.27 the values in the rectangles represent the cell areas. The values in ( ) represent the L/t values for the particular wall or web. After studying example problems 1 and 2 one should have no trouble checking the values as given in Table 4. Line 10 shows the correction of \( q \) values to develop a resisting torque of 100,000". The multiplying factor is 100,000/90630.

A6.16 Torsion of Thin-Walled Cylinder Having Closed Type Stiffeners.

The airplane thin-walled structure usually contains longitudinal stiffeners spaced around the outer walls as illustrated in Figs. A6.28 and A6.29.

For the open type stiffener as illustrated in Fig. A6.28, the torsional rigidity of the individual stiffeners as compared to the torsional rigidity of the thin-walled cell is so small to be negligible. However a closed type stiffener is essentially a small sized tube and its stiffness is much greater than an open section of similar size. Thus a cell with closed type stiffeners attached to its outer walls could be handled as a multiple cell structure, with each stiffener acting as a cell with a common wall with the outside surrounding cell. Since in general the stiffness provided by the stiffeners is comparatively small compared to the over-all cell, the approximate simplified procedure as given in NACA T.N. 542 by Kuhn can be used to usually give sufficient accuracy. In this approximate method, the thin-walled tube and closed stiffeners are converted or transformed into a single thin-walled tube by modifying the closed stiffeners by either one of the following procedures:

1. Replace each closed stiffener by a doubler plate having an effective thickness \( t_e = \frac{t_s K S}{d} \), and calculate \( S \) ds/t with these doubler plates in place. The enclosed area of the torsion tube still remains \( A \) or the same. See Fig. A6.30.

\[
\text{Transformation by Procedure (1)}
\]

\[
\text{Transformation by Procedure (2)}
\]

In Fig. A6.27 the values in the rectangles represent the cell areas. The values in ( ) represent the L/t values for the particular wall or web. After studying example problems 1 and 2 one should have no trouble checking the values as given in Table 4. Line 10 shows the correction of \( q \) values to develop a resisting torque of 100,000". The multiplying factor is 100,000/90630.

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1. Replace each closed stiffener by a doubler plate having an effective thickness \( t_e = \frac{t_s K S}{d} \), and calculate \( S \) ds/t with these doubler plates in place. The enclosed area of the torsion tube still remains \( A \) or the same. See Fig. A6.30.
(2) Replace the skin over each stiffener by a "liner" in the stiffener having a thickness \( t_s = \frac{t}{3} \). (See Fig. A6.31.) The enclosed area \( A \) of the cell now equals the original area less that area cross-latched in Fig. A6.31.

Procedure (1) slightly overestimates and procedure (2) slightly underestimates the stiffness effect of the stiffeners.

The corner members of a stiffened cell are usually open or solid sections and thus their torsional resistance can be simply added to the torsional stiffness of the thin-walled over-all cell.

A6.17 Effect of End Restraint on Members Carrying Torsion.

The equations derived in the previous part of this chapter assumed that cross-sections throughout the length of the torsion members were free to warp out of their plane and thus there could be no stresses normal to the cross-sections. In actual practical structures restraint against this free warping of sections is however often present. For example, the airplane cantilever wing from its attachment to a rather rigid fuselage structure is restrained against warping at the wing-fuselage attachment point. Another example of restraint is a heavy wing bulkhead such as those carrying a landing gear or power plant reaction. The flanges of these heavy bulkheads often possess considerable lateral bending stiffness, hence they tend to prevent warping of the wing cross-section. Since only torsional forces are being considered here as being applied to the member, the stresses produced normal to the cross-section of the member namely, tension and compression must add up zero for equilibrium. Thus the applied torque is carried by pure torsion action of the member and part by the longitudinal stresses normal to the member cross-sections. The percentage of the total torque carried by each action depends on the dimensions and shape of the cross-section and the length of the member.

Fig. A6.32 illustrates the distortion of an open section, namely, a channel section subjected to a pure torsional force \( T \) at its free end and fixed at the other or supporting end. Near the fixed and the applied torque is practically all resisted by the lateral bending of the top and bottom legs of the channel acting as short cantilever beams, thus forming the couple with \( H \) forces as illustrated in the Figure. Near the free end of the member, these top and bottom legs are now very long cantilever beams and thus their bending rigidity is small and thus the pure torsional rigidity of the section in this region is greater than the bending rigidity of the channel legs.

A6.18 Example Problem Illustrating Effect of End Restraint on a Member in Torsion.

Fig. A6.33 shows an I-beam subjected to a torsional moment \( T \) at its free end. The problem will be to determine what proportion of the torque \( T \) is taken by the flanges in bending and what proportion by pure shear, at two different sections, namely 10 inches and 40 inches from the fixed end of the I-beam.

**Fig. A6.33**

**Fig. A6.34**

**Fig. A6.35**

Fig. A6.34 shows the torque dividing into two parts, namely, the couple force \( F \) formed by bending of the flanges of the I-beam and the pure shearing stress system on the cross-section. Fig. A6.35 shows the twisting of the section through a distance \( s \).

The solution will consist in computing the angle of twist \( \theta \) under the two stress conditions and equating them.

Let \( T_b \) be the proportion of the total torque \( T \) carried by the flanges in bending forming the couple \( F \) in Fig. A6.34. From Fig. A6.35, the angle of twist can be written

\[
\theta_b = \frac{6}{0.5h} = \frac{F \cdot L^3}{3 \cdot 2 \cdot 21} = \frac{2 \cdot T_b \cdot L^3}{3 \cdot 4 \cdot 2 \cdot 21} \]

\[
\Rightarrow \frac{6}{0.5h} = \frac{2 \cdot T_b \cdot L^3}{3 \cdot 4 \cdot 2 \cdot 21}
\]
Note: The deflection of a cantilever beam with a load \( F \) at its end equals \( FL^3/3EI \), and the moment of inertia of a rectangle about its center axis equals \( bh^3/12 \).

hence \( \theta_b = \frac{3 FL^3}{2b^4} \)

Now let \( T_B \) be the portion of the total torque carried by the member in pure torsion. The approximate solution for open sections composed of rectangular elements as given in Art. A6.6, equation (12) will be used.

\[ \theta_t = \frac{3 T_B L}{Gt(2b + b_w)} \]

Equating \( \theta_b \) to \( \theta_t \), we can write

\[ \frac{T_B}{T_t} = \frac{3 E h^2 b^2}{8L^5 Gt^2(2b + b_w)} \]

Substituting values when \( L = 10 \) inches.

\[ \frac{T_B}{T_t} = \frac{3 \times 10.5 \times 10^4 \times 3.4 \times 1.7^2}{6 \times 10^2 \times 3.7 \times 10^4 \times 0.1^2(2 \times 1.73 + 3.3)} = 9.55 \]

Therefore, percent of total torsional \( T \) taken by bending of flanges equals,

\[ \frac{T_B - T_t}{T_B} (100) = \frac{T_B}{T_B + T_t} (100) = 90.5 \text{ percent.} \]

If we consider the section 40 inches from the fixed end, then \( L = 40 \) inches. Thus if 40\( ^* \) is placed in the above substitution instead of 10\( ^* \) the results for \( T_B/T_t \) would be 0.602 and the percent of the torque carried by the flanges in bending would be 36 as compared to 90.5 percent at \( L = 10 \) inches from support. Thus in general the effect of the end restraint decreases rapidly with increasing value of \( L \).

The effect of end restraint on thin walled tubes with longitudinal stiffeners is a more involved problem and cannot be handled in such a simplified manner. This problem is considered in other Chapters.

A6.18 Problems.

(1) In Fig. A6.35 pulley (1) is the driving pulley and (2) and (3) are the driven pulleys.

(2) A 1/2 HP, motor operating at 1000 RPM, rotates a 3/4"-0.035 aluminum alloy torque tube 30 inches long which drives the gear mechanism for operating a wing flap. Determine the maximum torsional stress in the 1-3/4 inch shaft between pulleys (1) and (2) and between (2) and (3).

(3) In the cellular section of Fig. A6.36 determine the torsional shear flow in resisting the external torque of 60000 in. lb. Wall and web thickness are given on the figure. Assume the tube is 100 in. long and find the torsional deflection. Material is aluminum alloy. (\( E = 3850000 \) psi.)

(4) In Fig. A6.36 remove the interior .035 web and compute torsional shear flow and deflections.

(5) In the 3-cell structure of Fig. A6.37 determine the internal resisting shear flow due to external torque of 100,000 in. lb. For a length of 100 inches calculate twist of cellular structure if \( G \) is assumed 2,500,000 psi.
(6) Remove the .05 interior web of Fig. A6.37 and calculate shear flow and twist.

(7) Remove both interior webs of Fig. A6.37 and calculate shear flow and twist.

(8) Each of the cellular structures in Fig. A6.38 is subjected to a torsional moment of 120,000 in.lbs. Using the method of successive approximation calculate resisting shear flow pattern.

All interior webs = .05 thickness
Top skin .064" thickness
Bottom skin .064" thickness

Fig. A6.38

The big helicopters of the future will be used in many important industrial and military operations. The helicopter presents many challenging problems for the structures engineers.

(Sketches from United Aircraft Corp. Publication "BEE-HIVE", Jan. 1958. Sikorsky Helicopters)
CHAPTER A7
DEFLECTIONS OF STRUCTURES
ALFRED F. SCHMITT

A7.1 Introduction.
Calculations of structural deflections are important for two reasons:

(1) A knowledge of the load-deformation characteristics of the airplane is of primary importance in studies of the influence of structural flexibility upon airplane performance.

(2) Calculations of deflections are necessary in solving for the internal load distributions of complex redundant structures.

The elastic deflection of a structure under load is the cumulative result of the strain deformation of the individual elements composing the structure. As such, one method of solution for the total deflection might involve a vectorial addition of these individual contributions. The involved geometry of most practical structures makes such an approach prohibitively difficult.

For complex structures the more popular techniques are analytical rather than vectorial. They deal directly with quantities which are not themselves deflections but from which deflections may be obtained by suitable operations. The methods employed herein for deflection calculations are analytical in nature.

A7.2 Work and Strain Energy.
Work as defined in mechanics is the product of force times distance. If the force varies over the distance then the work is computed by the integral calculus. Thus the work done by a varying force P in deforming a body an amount δ is

\[ \text{Work} = \int P \, \delta \]  

and is represented by the area under the load deformation (P-δ) curve as shown in Fig. A7.1.

If the deformed body is perfectly elastic the energy stored in the body may be completely recovered, the body unloading along the same P-δ curve followed for increasing load. This energy is called the elastic strain energy of deformation (hereafter the strain energy, for brevity) and is denoted by the symbol U. Thus

\[ U = \int P \, \delta \]  

Should the body be linearly elastic (as are most bodies dealt with in structural analysis) then the load-deformation curve is a straight line whose equation is

\[ P = k \delta \]  

and the strain energy is readily computed to be

\[ U = \frac{k \delta^2}{2} \text{ or, equally, } U = \frac{P^2}{2k}. \]

A7.3 Strain Energy Expressions for Various Loadings.

STRAIN ENERGY OF TENSION
A tensile load S acting at the end of a uniform bar of length L, cross sectional area A and elastic modulus E causes a deformation

\[ \delta = \frac{SL}{AE}. \]

Hence

\[ S = \frac{AE}{L} \delta \]  

and

\[ U = \int S \delta \, d\delta = \frac{AE}{L} \delta^2 \]  

or

\[ U = \frac{S^2L}{2AE}. \]  

(1)

Alternately,

Equations (1) and (2) are equivalent expressions for the strain energy in a uniform bar, the former expressing U in terms of the deformation and the latter expressing U in terms of load. The second form of expression is more convenient for general usage and succeeding strain energy formulas will be put in this form.

For a tensile bar having non-uniform properties (varying AE), or for which the axial load S varies, the strain energy is computed by the calculus. Thus the energy in a differential element of length dx is given by eq. (2) as

\[ dU = \frac{S^2 dx}{2AE} \]

where S and AE are average values over the length of element.

(1) Design Specialist, Convair-Astronautics
The total energy in the bar is obtained by summing over the length of the bar.

\[ U = \int \frac{S^2}{2AE} \, dx \]  

(3)

**Example Problem 1**

A linearly tapered aluminum bar is under an axial load of 1500 lbs, as shown in Fig. A7.2. Find \( U \).

\[ A(x) = A_0 \left(1 - \frac{x}{80}\right) \]

\[ A_0 = 2 \text{ in}^2 \]  

Fig. A7.2

**Solution:**

From statics, the internal load \( S = -1500 \) at every section.

\[ U = \int_0^L \frac{S^2}{2AE} \, dx = \int_0^{1500} \frac{(-1500)^2}{2AE} \, dx \]

\[ = \frac{2(-1500)^2}{10^6} \int_0^L \frac{dx}{20} = 4.5 \ln 2 \text{ inch lbs.} \]

Note that although \( P \) was a negative (compressive) load the strain energy remained positive.

**Example Problem 2**

Find \( U \) in a uniform bar under a running load

\[ q = q_0 \cos \frac{\pi x}{2L} \]  

(Fig. A7.3).

**Solution:**

The equation for \( S(x) \) was found by statics.

\[ S(x) = \frac{q_0}{\pi} \sin \frac{\pi x}{2L} dx \]

\[ = q_0 \frac{2L}{\pi} \sin \frac{\pi x}{2L} \]

Substitution into eq. (3) gave

\[ U = \frac{q_0^2}{2AE} \int_0^L \left(1 - \sin \frac{\pi x}{2L}\right) \, dx \]

\[ = \frac{2(65)^2}{40 \cdot (10 \times 10^3)} \cdot \frac{1}{2} \cdot (227)(40) \cdot 0.0820 \text{ in lbs.} \]

**STRAIN ENERGY OF FLEXURE**

A uniform beam of length \( L \) under the action of a pure couple undergoes an angular rotation proportional to the couple. Thus, from elementary strength of materials

\[ Q = \frac{L}{EI} M \]

where the constant EI, the product of elastic modulus by section moment of inertia, is called the "bending stiffness".

Since the rotation \( \theta \) builds up linearly with \( M \) the strain energy stored is

\[ U = \frac{1}{2} \frac{M^2}{EI} \]

where

\[ U = \frac{1}{2} \frac{M^2}{EI} \]

For a beam having non-uniform properties (varying EI) and/or for which \( M \) varies along the beam, the strain energy is computed by the calculus. From eq. (4) the strain energy in a beam element of differential length is

\[ dU = \frac{1}{2} \frac{M^2}{EI} \, dx \]

Hence summing over the complete beam to get the total strain energy one has

\[ U = \int \frac{1}{2} \frac{M^2}{EI} \, dx \]

(5)

**Example Problem 3**

For the beam of Fig. A7.4 derive the strain energy expression as a function of \( M_0 \).

**Solution:**

The bending moment diagram (Fig. A7-4a) was found and an analytic expression written for \( M \).
Then

\[ U = \frac{1}{2EI} \int_0^L M_s'(1 - \frac{X}{L})^2 \, dx \]

Example Problem 4
Determine the strain energy of flexure of the beam of Fig. A7.5. Neglect shear strain energy.

Solution:

The bending moment diagram was drawn first (Fig. A7.5a) and analytic expressions were written for \( M \).

Inspection of the diagram revealed that the energy of flexure in the right half of the beam must be identical with that of the left half. Hence

\[ U = 2 \cdot \frac{1}{2EI} \int_0^{L/2} (P_0 x)^2 \, dx \]

\[ = \frac{P_0^2 L^4}{24EI} \]

STRAIN ENERGY OF TORSION

A uniform circular shaft carrying a torque \( T \) experiences a total twist in a length \( L \) proportional to the torque. Thus from elementary strength of materials

\[ \Phi = \frac{T}{GJ} \]

(6)

where the constant \( GJ \), the product of the elastic modulus in shear by the cross section polar moment of inertia, is called the "torsional stiffness".

Since the twist \( \Phi \) builds up linearly with \( T \) the strain energy stored is

\[ U = \frac{1}{2} \frac{T^2}{GJ} \]

or

\[ U = \frac{1}{2} \frac{I^2}{GJ} \]

(7)

For a shaft of non-uniform properties and varying loading one has

\[ U = \frac{1}{2} \int \frac{T^2 \, dx}{GJ} \]

(8)

In passing it is worth remarking that one often encounters the group symbol "GJ" in use for the torsional stiffness of a non-circular shaft or beam such as an aircraft wing. In such a case the torsional stiffness has not been computed literally as \( G \times J \), but rather as defined by eq. (6), viz. the ratio of torque to rate of twist.

Example Problem 5
For a certain flight condition the torque on an airplane wing due to aerodynamic loading is given as shown graphically in Fig. A7.6. The variation of torsional stiffness \( GJ \) is given in like manner. Find the strain energy stored.

Solution:

A numerical integration of eq. (8) was performed using Simpson's rule. The work is shown in tabular form in table A7.1.

Values of \( T \) and \( GJ \) for selected wing stations were taken from the graphs provided and were entered in the table. For convenience in handling the numerical work all the variables were treated in non-dimensional form, eq. (6) being changed as follows

\[ U = \frac{1}{2} \int_0^l \frac{t^2}{GJ} \, dy = \frac{1}{2} \frac{L R^3}{GJR} \int_0^l \frac{\Phi^2}{GJ} \, dt \]
where
\[ \bar{T} = \frac{T}{T_R}, \quad \tau = \frac{y}{L} \]
\[ \frac{\bar{GJ}}{GJ} = \frac{GJ}{GJR} \]

The subscript \( R \) denotes "root" value \((y=0)\). The coefficients for Simpson's rule appearing in column (6) were taken from the expression for odd \( n \)
\[ I = \frac{4T}{3} \left( f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots \right) \]
\[ = + \frac{18}{4} f_{n-2} + 3f_{n-1} + \frac{5}{4} f_n \]

<table>
<thead>
<tr>
<th>Table A7.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>.2</td>
</tr>
<tr>
<td>.4</td>
</tr>
<tr>
<td>.6</td>
</tr>
<tr>
<td>1.0</td>
</tr>
</tbody>
</table>

Therefore the strain energy was
\[ U = \frac{LT_{Rs}}{GJR} \times 0.994 \]

STRAIN ENERGY OF SHEAR

A rectangular element "dx" by "dy" of thickness \( "t" \) into the page under the action of uniform shear stress \( \tau \) (psi) is shown in Fig. A7.6a.

From elementary strength of materials the angle of shear strain \( \gamma \) is proportional to the shear stress \( \tau \) as
\[ \gamma = \frac{\tau}{E} \]

where \( G \) is the material elastic modulus in shear. For the displacements as shown in the sketch only the downward load on the left hand side does any work. (In general all four sides move, but if the motion is referred to axes lying along two adjacent sides of the element, as was done here, the derivation is simplified). This load is equal to \( \tau x dy x t \) and moves an amount \( \delta x d \). Then
\[ dU = \frac{1}{2} \tau \delta t \cdot dx \cdot dy \]
\[ = \frac{1}{2} \gamma \frac{t^2}{G} \cdot dx \cdot dy \]
\[ = \frac{1}{2} \gamma \frac{t^2}{G} \cdot dx \cdot dy \]

Eq. (9) may be used to compute the shear strain energy in a beam under the action of a transverse shear loading \((V \mathrm{lbs})\). For this purpose \( V = \tau x dy x t \) and \( t x dy \) is called \( A \), the beam cross sectional area. Hence

\[ dU = \frac{V^2 dx}{2AG} \]

Therefore the elastic strain energy of shear for the entire beam is given by

\[ U = \int \frac{V^2 dx}{2AG} \]

(10)

**Example Problem 5**

Determine the strain energy of shear in the beam of Example Problem 4.

Solution:

The shear load diagram was drawn first.

Then

\[ U = \frac{1}{2} \int \frac{A^2 x dx}{2AG} \]

(11)

Eq. (9) may be used to compute the shear strain energy in a thin sheet. The element \( dx x dy \) is visualized as but one of many in the sheet and the total energy is obtained by summing. Thus

\[ U = \frac{1}{2} \int \int \frac{\gamma \tau^2 t dx dy}{E} \]

(11a)

Here a double integration is required, the summation being carried out in both directions over the sheet. In dealing with thin sheet the use of the shear flow \( q = \gamma \tau \) is often convenient so that eq. (11) rewrites

\[ U = \frac{1}{2} \int \int \frac{q^2 dx dy}{E} \]

(11a)

A very important special case occurs when a homogenous sheet of constant thickness is analyzed assuming \( q \) is constant everywhere. In this case one has

\[ U = \frac{1}{2} \int \int \frac{q^2 dx dy}{E} = \frac{q^2 E}{2AG} \]

(12)

where \( S = \int \int dx dy \) is the surface area of the sheet.

*The sketch is visualized as a side view of an element of length \( dx \) taken from a beam of total height \( dy \).*
Example Problem 7
Find the shear strain energy stored in a cantilever box beam of uniform rectangular cross section under the action of torque T. (Fig. A7.8).

Solution:

The shear flow was assumed given by Bredt’s formula (Ref. Chapter A-6)

\[ q = \frac{T}{2A} = \frac{T}{2bc}, \]

constant around the perimeter of any section. Then

\[ U = \sum q^t \frac{S}{b} = \frac{T^2}{2Gb^2c} \cdot 2L(b+c) \]

\[ = \frac{T^2L(b+c)}{4Gb^2c} \cdot E \]

THE TOTAL ELASTIC STRAIN
ENERGY OF A STRUCTURE

The strain energy by its definition is always a positive quantity. It also is a scalar quantity (one having magnitude but not direction) and hence the total energy of a composite structure, having a variety of elements under various loadings, is readily found as a simple sum.

\[ U_{TOT} = \int \frac{S^2dx}{2b^2} + \int \frac{M^2dx}{2EI} + \int \frac{T^2dx}{2Gb^2c} \]

\[ = \int \frac{V^2dx}{2A} + \int \frac{q^2dx}{2b} \]

The integral symbols and common use of "x" as an index of integration should not be taken too literally. It is probably best to read these terms as "the sum of so and so over the structure" rather than "the integral of", for quite often the terms are formed as simple sums without resort to the calculus. The calculus is only used as an aid in some applications.

It is seldom that all the terms of eq. (13) need be employed in a calculation. Many of the loadings, if actually present, may be of a localized or of a secondary nature and their energy contribution may be neglected.

A7.4 The Theorems of Virtual Work and Minimum Potential Energy.

An important relationship between load and deformation stems directly from the definitions of work and strain energy. Consider Fig. A7.9(a).

Thus

\[ dU = Pd\delta \]

\[ \frac{dU}{d\delta} = P \]

In words, the rate of change of strain energy with respect to deflection is equal to the associated load. Eq. (14) and the above quotation are statements of the Theorem of Virtual Work. The reader may find this theorem stated quite differently in the literature on rigid body mechanics but should be able to satisfy himself that the expressions are nevertheless compatible.

A useful restatement of the above theorem is obtained by rewriting eq. (14) as

\[ dU - Pd\delta = 0 \]

It is next assumed that if the change in displacement \( \delta \) is sufficiently small the load \( P \) remains sensibly constant and hence

\[ dU - d(P\delta) = 0 \]

The quantity \( U - P\delta \) is called the total potential of the system and eq. (15), resembling as it does the mathematical condition for the minimum value of a function, is said to be a statement of the Theorem of Minimum Potential.

From the foregoing it is clear that the Theorem of Minimum Potential is a restatement of the Theorem of Virtual Work.

In structural analysis the most important uses of these theorems are made in problems concerning buckling instability and other nonlinearities. No applications will be made at this point.

A7.5 The Theorem of Complementary Energy and Castigliano's Theorem.

Again in the case of an elastic body, examination of the area above the load-deformation curve shows that increments in this area (called
the complementary energy, \( U^* \), are related to the load and deformation by (see Fig. A7.8b).

\[
\frac{dU^*}{dP} = \delta \quad \text{------------- (16)}
\]

This is the Theorem of Complementary Energy.

Now for the linearly elastic body a very important theorem follows since (Fig. A7.9c)

\[
dU = dU^* \quad \text{so that}
\]

\[
\frac{dU}{dP} = \delta \quad \text{------------- (17)}
\]

In words, "The rate of change of strain energy with respect to load is equal to the associated deflection".

Eq. (17) and the above quotation are statements of Castigliano's Theorem.

For a body under the simultaneous action of several loads the theorem is written so as to apply individually to each load and its associated deflection, thus:

\[
\frac{dU_i}{dP_i} = \delta_i \quad \text{------------- (17a)}
\]

The partial derivative sign in eq. (17a) indicates that the increment in strain energy is due to a small change in the particular load \( P_i \), all other loads held constant.

Note that by "load" and "deflection" may be meant:

\[
\begin{array}{c|c}
\text{Load} & \text{Associated Deflection} \\
\hline
\text{Force (lbs.)} & \text{Translation (inches)} \\
\text{Moment (in. lbs.)} & \text{Rotation (radians)} \\
\text{Torque (in. lbs.)} & \text{Rotation (radians)} \\
\text{Pressure (lbs/in\(^2\))} & \text{Volume (in\(^3\))} \\
\text{Shear Flow (lbs/in)} & \text{Area (in\(^2\))}
\end{array}
\]

Any generalizations of the meanings of "force" and "deflection" are possible only so long as the units are such that their product yields the units of strain energy (in. lbs).

Once again for emphasis it is repeated that, while the complementary nature of eqs. (14) and (17) are clearly evident, the use of eq. (17) (Castigliano's Theorem) is restricted to linearly elastic structures. A brief example will serve for illustration of the possible pitfalls.

The strain energy stored in an initially straight uniform column under an axial load \( P \) when deflected into a half-sine wave is

\[
U = \frac{P^2aL^2}{8E} \\
Y = Y_0 \sin \frac{\pi x}{L}, \quad M = PY \\
\text{Consider Flexural Energy Only} \\
\frac{M^2}{2EI} = \frac{P^2y_0 aL}{8E} \\
\Rightarrow \quad \delta = \frac{1}{2} \left( \frac{dy}{dx} \right) = \frac{Y_0 aL}{4E} \\
\Rightarrow \quad U = \frac{P^2aL^2}{8E}
\]

where \( \delta \) is the end shortening due to bowing. Because the deflections grow rapidly as \( P \) approaches the critical (buckling) load the problem is non-linear. The details of the calculation of \( U \) are given with Fig. A7.10.

Now according to the Theorem of Virtual Work (eq. 14)

\[
\frac{dU}{d\delta} = P
\]

but

\[
\frac{dU}{d\delta} = \frac{P^2a}{8E}
\]

Therefore

\[
\frac{P^2a}{8E} = P
\]

or

\[
P = \frac{8E}{a}
\]

(Buckling formula for uniform column). The correct result.

Application of Castigliano's Theorem. eq. (17), leads to the erroneous result:

\[
\frac{dU}{dP} = 0
\]

\[
\frac{dU}{dP} = \frac{2P^2aL^2}{8E} = 0
\]

**See Art. A17.8, Chapter A-17 for detailed derivation of this equation.**

---

* The proof of the theorem for the case of multiple loads is generally formulated more rigorously, appeal to a simple diagram such as Fig. A7.8c being less effective. See, for example, "Theory of Elasticity" by S. Timoshenko.
**ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES**

\[
P = \frac{n}{2L} \text{ (incorrect)}
\]

Normal: Do not use Castigliano's theorem for non-linear problems.

Fortunately the above restriction upon the use of Castigliano's theorem is not a very severe one, the majority of every day structural problems being linear. Castigliano's theorem is quite useful in performing deflection calculations and a variety of applications will be made in the following sections.

### A7.6 Calculations of Structural Deflections by Use of Castigliano's Theorem

As the examples of Art. A7.3 have illustrated, the strain energy of a structure can be expressed as a function of the external loadings provided the internal load or stress distribution is calculable. Having the strain energy so expressed the deflections at points of load application can be determined with the aid of eq. (17), Castigliano's Theorem.

In the examples to follow the deflections of a variety of statically determinate structures are computed. Methods of handling redundant structures are considered in subsequent articles.

**Example Problem 8**

Find the vertical deflection at the point of load application of the crane of Fig. A7.11.

Cross sectional areas are given on each member. The stranded cables have effective moduli of 13.5 \( \times 10^6 \) psi. \( E = 29,000,000 \) for other members.

![Fig. A7.11](image)

**Solution:**

The strain energy considered here was that due to axial loading in each of the four members. The load distribution was obtained from statics and the energy calculation was made in tabular form as follows:

<table>
<thead>
<tr>
<th>MEMBER</th>
<th>S LBS</th>
<th>L FT</th>
<th>( E \times 10^6 ) LBS</th>
<th>( \frac{S}{AE} \times 10^6 )</th>
<th>( \sum_{i} ) ( \frac{S}{AE} \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>OA</td>
<td>-1.50 P</td>
<td>40.0</td>
<td>136</td>
<td>0.66 P</td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>2.50 P</td>
<td>50.0</td>
<td>11.3</td>
<td>26.48 P</td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>1.58 P</td>
<td>63.0</td>
<td>11.3</td>
<td>13.33 P</td>
<td></td>
</tr>
<tr>
<td>OC</td>
<td>-2.12 P</td>
<td>84.5</td>
<td>136</td>
<td>2.78 P</td>
<td></td>
</tr>
<tr>
<td>( \sum = 43.26 ) P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then

\[ 2U = 43.26 \times 10^{-6} \text{ P} \text{ LBS} \text{ FT} \]

\[ \frac{\partial U}{\partial P} = 43.26 \times 10^{-6} \text{ P} \text{ FT} \]

Note that in this problem the only use made of the calculus was in the differentiation. A simple sum was used to form \( U \), as per eq. (2).

**Example Problem 9**

Derive Bret's formula for the rate of twist of a hollow, closed, thin-walled tube:

\[ \Theta = \frac{T}{4A \bar{G}} \int \frac{ds}{t} \]


The strain energy is stored in shear according to

\[ U = \frac{1}{2} \int \frac{q^2 dx dy}{Gt} \]

This is the only energy stored, secondary effects neglected. Over a given cross section \( q \) is a constant and is given by Bret's equation

\[ q = \frac{T}{2A} \]

where \( A \) is the enclosed area of the tube (Ref. Chap. A.5).

**Solution:**

Since the twist per unit length was desired the strain energy per unit length only was written. Thus, assuming \( y \) in the axial direction, no integration was made with respect to \( y \). The integration in the remaining direction was to be carried out around the perimeter of the tube and so the index was changed from \( "y" \) to the more appropriate "s". Hence (compare with Example Prob. 7)

\[ U = \frac{1}{2} \oint \frac{q^2 ds}{Gt} = \frac{T}{4A \bar{G}} \oint \frac{ds}{t} \]

The symbol \( \oint \) means the integration is carried out around the tube perimeter. Therefore

\[ \Theta = \frac{dU}{dP} = \frac{T}{4A \bar{G}} \oint \frac{ds}{t} \]

**USE OF FICTITIOUS LOADS**

In the following example the desired deflection is at the free end of the bar where no load is applied. A fictitious load will be added for purposes of the calculation. The rate of change of strain energy with respect to this fictitious load will be found after which the load will be set equal to zero. This technique gives the desired result in as much as the deflection is equal to the rate of change of strain energy with respect to the load and such a rate exists even though the load itself be zero.

**Example Problem 10**

Compute the axial motion of the free end of the tapered bar of Fig. A7.12.
DEFLECTIONS OF STRUCTURES

\[ q = \text{constant} \]
\[ A = A_0 (1 - \frac{x}{2L}) \]
\[ q = 400 \text{ in.} \]
\[ A_0 = 4 \text{ in.}^2 \]
\[ L = 40 \text{ in.} \]
\[ E = 10 \times 10^6 \text{ psi} \]

Fig. A7.12

Solution:

After addition of the fictitious end load \( R \) the axial load from statics was found to be

\[ S(x) = R + q \left( L - x \right) \]

Hence, since loadings other than tensile are of a secondary nature

\[ U = \frac{1}{2} \int_0^L \frac{S^2 dx}{AE} \]
\[ = \frac{1}{2} \int_0^L \frac{\left[ R + q \left( L - x \right) \right]^2 \, dx}{A_0 E \left( 1 - \frac{x}{2L} \right)} \]
\[ = \frac{R^2}{2A_0 E} \int_0^L \frac{dx}{\left( 1 - \frac{x}{2L} \right)} + \frac{q^2}{2A_0 E} \int_0^L \frac{(L - x)^2 \, dx}{\left( 1 - \frac{x}{2L} \right)} \]

Before evaluating the integrals it was observed that the steps to follow, in which \( U \) was to be differentiated with respect to \( R \) and the subsequent setting of \( R = 0 \) would drop out both the first and last terms. Hence only the second term was evaluated.

\[ U = x + \frac{2L^2 q}{A_0 E} \left( 1 - \ln(2) \right) + x \]

Then

\[ \delta = \frac{du}{dx} = \frac{2L^2 q}{A_0 E} \left( 1 - \ln(2) \right) \]

DIFFERENTIATION UNDER THE INTEGRAL SIGN

An important labor savings may be had in the calculation of deflections by Castigliano's theorem.

In the strain energy integrals arising in this class of problems, the load \( P_i \), with respect to which the deflection is to be found, acts as an independent parameter in the integral. Provided certain requirements for continuity of the functions are met - and they invariably are in these problems - the differentiation with respect to \( P_i \) may be carried out before the integration is made. The resulting integrals generally are easier to evaluate.

Example Problem 11

Find the deflection at point \( S \) of the beam of Fig. A7.13.

\[ P \]
\[ EI \text{ constant} \]
\[ \frac{L}{3} \]
\[ \frac{L}{3} \]
\[ \frac{L}{3} \]

Fig. A7.13

Solution:

A fictitious load \( P_i \) was added at point \( B \) and the bending moment diagram was drawn in two parts.

\[ M_p \]
\[ \frac{2PL}{9} \]
\[ \frac{P_X}{3} \]
\[ \frac{P_Y}{3} \]
\[ \frac{P_1}{9} \]
\[ \frac{P_2}{3} \]
\[ \frac{P_3}{3} \]

Fig. A7.13a

Then neglecting the energy of shear as being small

\[ U = \frac{1}{2EI} \int_0^{L/3} \left[ \frac{L/3}{2} \left( 2P + P_i \right) \frac{x^2}{2} \right] \, dx \]
\[ + \frac{1}{2EI} \int_0^{L/3} \left[ \frac{P}{9} \left( 2L - 3y \right) + \frac{P_i}{9} \left( L - 3y \right) \right] \, dy \]
\[ + \frac{1}{2EI} \int_0^{L/3} \left[ \frac{P + 2P_i}{3} \left( L - 3\delta \right) \right] \, ds \]

Differentiation under the integral sign with respect to \( P_i \) gave

\[ \delta = \frac{du}{dx} = \frac{2L^2 q}{A_0 E} \left( 1 - \ln(2) \right) \]

* For beams of usual high length to depth ratios the shear strain energy is small compared to the energy of flexure. Neglecting the shear energy is equivalent to neglecting the shear deflection contribution. (see p. A7.14)
The fictitious load $P$, having served its purpose, was set equal to zero before completing the work.

$$\delta_B = \frac{1}{2!} \cdot 2P \int_0^{L/3} x^2 dx + \frac{1}{2} \frac{P}{EI} \int_0^{L/3} (2L - 3y)(L + 3y) dy + \frac{1}{2!} \frac{2P}{EI} \int_0^{L/3} (L - 3a) dy$$

Example Problem 12

Fig. A7.14a shows a cantilever round rod of diameter $D$ formed in a quarter circle and acted upon by a torque $T_o$. Find the vertical movement of the free end.

Solution:

Fig. A7.14a shows the vector resolution of the applied torque $T_o$ on beam elements. $T_o(\Theta) = T_o \cos \Theta$ and the moment $M(\Theta) = T_o \sin \Theta$. Application of a fictitious vertical load $P$ (down) at the point of desired deflection gave the loadings shown in Fig. A7.14a.

The total loadings were

$$M = M_1 - M_2 = (T_o - PR) \sin \Theta$$
$$T = T_1 + T_2 = T_o \cos \Theta + FR (1 - \cos \Theta)$$

Thus:

$$U = \frac{1}{2EI} \int_0^{\pi/2} (T_o - PR)^2 \sin^2 \Theta \, d\Theta$$

(Note: Use of "d" for the length of a differential beam element instead of "dx". Differentiating under the integral sign gives

$$\delta U = \frac{d}{dP} \left[ \frac{1}{2EI} \int_0^{\pi/2} (T_o - PR)^2 \sin^2 \Theta \, d\Theta \right]$$

Putting $P$, the fictitious load equal to zero and integrating gave

$$\delta_{VERT} = \frac{\delta U}{dP} = \frac{T_o R^2}{4} \left( \frac{\pi}{2} - \frac{\pi}{6} \right)$$

Since $J = 2I$ and $G = E/2.6$ the deflection was negative (UP).

A7.7 Calculation of Structural Deflections by the Method of Dummy-Unit Loads (Method of Virtual Loads).

The strict application of the calculus to Castigliano's theorem as in Art. A7.5, leads to a number of cumbersome techniques ill-suited to the solution of large complex structures. A more flexible approach, readily adapted to improved "book keeping" techniques is the Method of Dummy-Unit Loads developed independently by J. C. Maxwell (1884) and O. Z. Mohr (1874).

That the equations for the Method of Dummy-Unit Loads may be derived in a number of ways is attested to by the great variety of names applied to this method in the literature. Presented below are two derivations of the equations stemming from different viewpoints. One derivation obtains the equations by a reinterpretation of the symbols of Castigliano's theorem - essentially an appeal to the concepts of strain energy. The other derivation uses the principles of rigid body mechanics. Based as they are upon a common set of consistent assumptions, all the methods must, of course, yield the same result.

Derivation of Equations for the Method of Dummy-Unit Loads (Virtual Loads)

I - From Castigliano's Theorem

Beginning with the general expression for strain energy, eq. (13)

II - From the Principle of Virtual Work

When a system of forces whose resultant is zero (a system in equilibrium) is displaced a...
DEFORMATIONS OF STRUCTURES

\[ U = \frac{1}{2} \int \frac{S^2}{AE} + \frac{1}{2} \int \frac{M^2}{EI} + \frac{1}{2} \int \frac{T^2}{GJ} + \ldots \quad \text{etc.} \]

differentiate under the integral sign with respect to \( P_1 \) to obtain

\[ \delta_1 = \int \frac{S}{AE} \frac{\partial P_1}{\partial x} + \int \frac{M}{EI} \frac{\partial P_1}{\partial x} + \int \frac{T}{GJ} \frac{\partial P_1}{\partial x} + \ldots \quad \text{etc.} \]

Consider the symbols

\[ \frac{\partial S}{\partial P_1}, \frac{\partial M}{\partial P_1}, \frac{\partial T}{\partial P_1}, \ldots \quad \text{etc.} \]

Each of these is the "rate of change of so-and-so with respect to \( P_1 \)" or "how much so-and-so changes when \( P_1 \) changes a unit amount" OR EQUALLY, "the so-and-so loading for a unit load \( P_1 \)."

Thus, to compute these partial derivative terms one need only compute the internal loadings due to a unit load (the virtual load) applied at the point of desired deflection. For example, the term \( \frac{\partial M}{\partial P_1} \), could be computed in either of the two ways shown in Fig. A7.10a.

**RATE METHOD**

\[ \frac{M}{P_1} \]

\[ \frac{\partial M}{\partial P_1} = x \]

\[ x = \frac{\partial M}{\partial P_1} \]

**UNIT METHOD**

\[ \text{Fig. A7.10b} \]

\[ \text{Fig. A7.10a} \]

\[ M = P_1 x \quad m = \text{dummy loading} \]

\[ \frac{\delta M}{\delta P_1} = x \]

Likewise \( \frac{\partial S}{\partial P_1} \), where \( P_1 \) is a load (real or fictitious) applied at joint \( C \) of Fig. A7.10b, is given by the loadings for the unit load applied as shown.

In practice the use of the unit load is most convenient. Using the notation

\[ u = \frac{\partial S}{\partial P_1}, \quad m = \frac{\partial M}{\partial P_1}, \quad t = \frac{\partial T}{\partial P_1}, \quad \]

\[ v = \frac{\partial V}{\partial P_1}, \quad q = \frac{\partial q}{\partial P_1} \]

for the unit loadings, the deflection equation becomes

\[ \delta_1 = \int \frac{S}{AE} \frac{\partial u}{\partial x} + \int \frac{M}{EI} \frac{\partial m}{\partial x} + \int \frac{T}{GJ} \frac{\partial t}{\partial x} + \ldots \quad \text{etc.} \]

\[ + \int \frac{V}{AG} \frac{\partial v}{\partial x} + \int \frac{Q}{Gt} \frac{\partial q}{\partial x} + \ldots \quad \text{(18)} \]

The external virtual work is

\[ 1 \times \delta_0 = \text{External Virtual Work} = \int \text{ dummy loading} \]

The internal virtual work is the sum over the deflections of the products of the member virtual loads \( u \) by the member distortions \( \delta_1 \). That is,

\[ \text{Internal Virtual Work} = \int u \delta_1 \]

Then equating these works,

\[ 1 \times \delta_0 = \text{Internal Virtual Work} = \int u \delta_1 \]

If the deformations \( \delta_1 \) are the result of elastic strains due to real member loads \( S \) then

\[ \delta_1 = \frac{S L}{AE} \quad \text{for each member and one has} \]

\[ \delta_0 = \int u \frac{S L}{AE} \]

The argument given above may be extended quickly to include the internal virtual work of virtual bending moments (m), torsion loads (t), shear loads (v), and shear flows (q) doing work during deformations due to real moments (M), torques (T), shear loads (V), and shear flows (Q). The general expression becomes

\[ \delta_1 = \int \frac{S}{AE} + \int \frac{M}{EI} + \int \frac{T}{GJ} + \int \frac{V}{AG} + \int \frac{Q}{Gt} + \ldots \quad \text{(18)} \]

Note that the deformations are not restricted to those due to elastic strains only. They may be the result of plastic or inelastic strains, temperature strains or misalignment corrections.
In applying eq. (18) the labor of a deflection calculation divides conveniently into several steps:

1. Calculation of the real (actual) load distribution \(S_{H,T, etc.}\)

2. Calculation of the unit (virtual) load distribution \(u_{H,T, etc.}\) due to a unit (virtual) load applied at the point of desired deflection and reacted at the reference point(s).

3. Calculation of flexibilities, \(\frac{1}{AE}, \frac{1}{EI}, \text{etc.}\)

4. Summation and/or integration.

Here again note the general nature of the terms "load" and "deflection". (See p. A7.6)

The following examples apply the method of dummy-unit loads to the determination of absolute and relative deflections, both rotation and translation.

Example Problem 13

The pin-jointed truss of Fig. A7.16 is acted upon by the external loading system shown. The member loads are given on the figure. Member properties are given in Table A7.3. Find the vertical movement of joint G and the horizontal movement of joint H.

Solution:

Only the energy of axial loadings in the members was considered. Unit (virtual) loads were applied successively at joints \(G\) and \(H\) as shown in Figs. A7.16a and A7.16b. All \(S\) and \(u\) loadings were entered in Table A7.3 and the calculation for \(\delta = \sum \frac{S_u l}{AE}\) was carried out by forming the sum of \(\frac{S_u l}{AE}\) terms for the members of the truss, i.e. \(\delta = \sum \frac{S_u l}{AE}\).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{MEM.} & \text{L}_{\text{lb}} & \text{AE} \times 10^{-6} \text{ in}^3 & \text{K} \times 10^{-3} \text{ in/lb} & S & u & \frac{S_u l}{AE} \times 10^{-4} \text{ in} \\ \hline
\text{AB} & 30 & 4.735 & 6.270 & 10,500 & 1.5 & 0 & 99.75 \text{ in} \\ \text{BC} & 50 & 2.074 & 9.759 & 2,250 & 0 & 0 & 0 \\ \text{CD} & 30 & 2.074 & 9.759 & 2,250 & 0 & 0 & 0 \\ \text{EF} & 50 & 5.385 & 5.591 & -3,250 & -.125 & 1 & 21.21 \text{ in} \\ \text{FG} & 50 & 5.385 & 5.591 & -3,250 & -.125 & 1 & 21.21 \text{ in} \\ \text{GH} & 50 & 2.65 & 8.821 & 0 & 0 & 0 & 0 \\ \text{BE} & 50 & 10.15 & 6.250 & -4,750 & -.125 & 0 & 52.88 \text{ in} \\ \text{BG} & 50 & 2.68 & 14.268 & 5,000 & 1.25 & 0 & 89.80 \text{ in} \\ \text{DG} & 50 & 5.38 & 9.320 & -3,250 & 0 & 0 & 0 \\ \text{EF} & 50 & 2.074 & 13.012 & 2,000 & 0 & 0 & 0 \\ \text{CG} & 40 & 2.074 & 13.012 & -1,000 & 0 & 0 & 0 \\ \text{EH} & 40 & 2.074 & 13.012 & 2,000 & 0 & 0 & 0 \\ \hline
\end{array}
\]

\[
\sum = 286.4 \text{ in} = 48.72 \text{ in}
\]

Answers: \(\delta_{\text{VER}} = 0.286\text{ in}\)

\(\delta_{\text{HOR}} = -.0687\text{ in}\) (the negative sign means the joint moves to the LEFT since the unit load was drawn to the RIGHT in Fig. A7.16b).

Example Problem 14

For the truss of Fig. A7.16 find the following relative displacements of joints:

c) the movement of joint \(C\) in the direction of a diagonal line joining \(C\) and \(F\).

d) the movement of joint \(G\) relative to a line joining points \(F\) and \(H\).

Relative deflections are determined by applying unit (virtual) loads at the points where the deflections are desired and by supporting such unit load systems at the reference points of the motion. Thus, for solution to part (c) a unit load system was applied as shown in Fig. A7.16c and for the solution of
part (d) the system of unit loads of Fig. A7.16d was used. Table A7.4 completes the solution, the real loads and member flexibilities \( \frac{L}{AE} \) being the same as for example problem 13.

![Fig. A7.16c](image1)  
![Fig. A7.16d](image2)

Rotations, both absolute and relative are determined by applying unit (virtual) couples to the member or portion of structure whose rotation is desired. The unit couple is resisted by reactions placed on the line of reference for the rotation. Thus Figs. A7.16e and A7.16f show the unit (virtual) loadings for parts (e) and (f) respectively. Table A7.5 completes the calculation, the real loads and member flexibilities \( \frac{L}{AE} \) being the same as for example problem 13.

![Fig. A7.16e](image3)  
![Fig. A7.16f](image4)

**TABLE A7.4**

<table>
<thead>
<tr>
<th>MEM.</th>
<th>( \frac{L}{AE} \times 10^8 ) in/lb (See Table A7.3)</th>
<th>( S_{th} )</th>
<th>( u_s )</th>
<th>( u_a )</th>
<th>( \frac{S_{th}L}{AE} \times 10^8 ) inch</th>
<th>( \frac{S_{th}L}{AE} \times 10^8 ) inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>6.270</td>
<td>10,500</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>9.759</td>
<td>2,250</td>
<td>0</td>
<td>-375</td>
<td>0</td>
<td>-8.22</td>
</tr>
<tr>
<td>EF</td>
<td>5.591</td>
<td>-5,250</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>FG</td>
<td>5.591</td>
<td>-5,250</td>
<td>-8</td>
<td>0</td>
<td>17.61</td>
<td>0</td>
</tr>
<tr>
<td>GH</td>
<td>8.621</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BE</td>
<td>4.926</td>
<td>-8,750</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BG</td>
<td>14.386</td>
<td>5,000</td>
<td>1.0</td>
<td>625</td>
<td>71.94</td>
<td>44.9</td>
</tr>
<tr>
<td>DG</td>
<td>9.320</td>
<td>-3,750</td>
<td>0</td>
<td>625</td>
<td>0</td>
<td>-21.8</td>
</tr>
<tr>
<td>BF</td>
<td>13.012</td>
<td>2,000</td>
<td>-8</td>
<td>-50</td>
<td>-20.82</td>
<td>-12.0</td>
</tr>
<tr>
<td>CG</td>
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<td>-1,000</td>
<td>-8</td>
<td>0</td>
<td>-10.41</td>
<td>0</td>
</tr>
<tr>
<td>DH</td>
<td>13.012</td>
<td>2,000</td>
<td>0</td>
<td>-50</td>
<td>0</td>
<td>-13.0</td>
</tr>
</tbody>
</table>

\( Z = 68.97 \) \( \Sigma = 19.39 \)

Therefore the movement of joint c towards joint F was \( \delta = 0.06587 \) inches and the motion of joint G relative to a line between F and H was \( \delta = -0.0194 \) inches, the negative sign indicating an upward movement.

**Example Problem 15**

For the truss of Fig. A7.16 determine

e) the absolute rotation of member DG

f) the rotation of member BG relative to member CG.

**TABLE A7.5**

<table>
<thead>
<tr>
<th>MEM.</th>
<th>( \frac{L}{AE} \times 10^8 ) in/lb (See Table A7.3)</th>
<th>( S_{th} )</th>
<th>( u_s )</th>
<th>( u_a )</th>
<th>( \frac{S_{th}L}{AE} \times 10^8 ) inch</th>
<th>( \frac{S_{th}L}{AE} \times 10^8 ) inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>AE</td>
<td>6.270</td>
<td>10,500</td>
<td>0.025</td>
<td>0</td>
<td>1.85</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>9.759</td>
<td>2,250</td>
<td>0.025</td>
<td>-0.025</td>
<td>-0.55</td>
<td>-0.55</td>
</tr>
<tr>
<td>CD</td>
<td>9.759</td>
<td>2,250</td>
<td>0.025</td>
<td>0</td>
<td>0.55</td>
<td>0</td>
</tr>
<tr>
<td>EF</td>
<td>5.591</td>
<td>-5,250</td>
<td>-0.025</td>
<td>0</td>
<td>0.73</td>
<td>0</td>
</tr>
<tr>
<td>FG</td>
<td>5.591</td>
<td>-5,250</td>
<td>-0.025</td>
<td>0</td>
<td>0.73</td>
<td>0</td>
</tr>
<tr>
<td>GH</td>
<td>8.621</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BE</td>
<td>4.926</td>
<td>-8,750</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BG</td>
<td>14.386</td>
<td>5,000</td>
<td>0.015</td>
<td>0</td>
<td>1.08</td>
<td>0</td>
</tr>
<tr>
<td>DG</td>
<td>9.320</td>
<td>-3,750</td>
<td>-0.015</td>
<td>0</td>
<td>0.52</td>
<td>0</td>
</tr>
<tr>
<td>BF</td>
<td>13.012</td>
<td>2,000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CG</td>
<td>13.012</td>
<td>-1,000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DH</td>
<td>13.012</td>
<td>2,000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\( Z = 4.73 \) \( \Sigma = 0.53 \)

Therefore the absolute rotation of member DG was \( \Theta_{DG} = 0.00473 \) radians and the rotation of BG relative to CG was \( \Theta_{BG - CG} = 0.00053 \) radians.
Example Problem 16
Find the vertical deflection of point C for the cantilever beam of Fig. A7.17 carrying a concentrated load P at its end. Also find slope of elastic curve at C.

Solution:

\[ \delta_C = \int \frac{M_{max}}{EI} \]

With origin at B

\[ M = -Px \text{ (Fig. A7.17)} \]

For virtual loading (Fig. A7.17a)

\[ m = 0, \text{ for } x < b; \quad m = -1 \text{ (x > b), for } x > b \]

Hence \( M_{max} = -Px(-x + b)dx = (Px^2 - Pbx)dx \)

\[ \delta_C = \frac{P}{EI} \int_{b}^{L} (x^2 - bx)dx = \frac{P}{EI} \left[ \frac{x^3}{3} - \frac{bx^2}{2} \right]_{b}^{L} \]

\[ = \frac{P}{EI} \left[ \frac{L^3}{3} - \frac{3L^2b}{2} + \frac{b^3}{2} \right] \]

If \( b = 0 \), then \( \delta_B = \frac{PL^3}{3EI} \)

\[ \theta_C = \int \frac{M_{max}}{EI} \]

For virtual loading see Fig. A7.17b

\[ m = 0, \text{ } x < b, \quad m = -1, \text{ } x > b \]

Hence \( M_{max} = (Px)(-1)dx = Pxdx \)

\[ \therefore \theta_C = \int \frac{M_{max}}{EI} = \frac{P}{EI} \int_{b}^{L} xdx = \frac{P}{2EI} (L^3 - b^3) \]

If \( b = 0 \), \( \theta_B = \frac{PL^2}{2EI} \)

Example Problem 17
For the uniformly loaded cantilever beam of Fig. A7.18, find the deflection of point D relative to the line joining points C and E on the elastic curve of the beam. This is representative of a practical problem in aeronautics, in that AB might represent a rear wing beam and points C, D, E, the attachment points of an aileron or a flap. The wing beam deflection bends the aileron or flap structure by applying a load at D thru aileron supporting bracket. To know this force the deflection of the wing beam at D relative to line CE must be known.

Solution:

Origin at B:

\[ M = 30x \cdot \frac{x}{2} = 15x^2 \]

\[ \theta_C = \int \frac{M_{max}}{EI} \]

\[ = \frac{1}{EI} \int_{0}^{50} 15x^2 \left( -0.5x + 10 \right)dx + \frac{1}{EI} \int_{50}^{80} 15x^2 \left( -0.5x - 40 \right)dx \]

\[ = \frac{1}{EI} \left[ -7.5x^4 + \frac{150x^3}{3} \right]_{0}^{50} + \frac{1}{EI} \left[ \frac{7.5x^4}{4} - \frac{600x^3}{3} \right]_{50}^{80} \]

\[ = \frac{10^6}{EI} \left[ -11.72 + 6.25 + .300 - .400 + 76.9 - 102.4 - 11.72 + 25.0 \right] \]

\[ = -17.9 \times 10^6 \]

Therefore deflection of point D relative to line joining CE is down because result comes out negative and therefore opposite to direction of virtual load.

Example Problem 18
For the uniformly loaded cantilever beam of Fig. A7.19, find the deflection of point D relative to the line joining points C and E on the elastic curve of the beam. This is representative of a practical problem in aeronautics, in that AB might represent a rear wing beam and points C, D, E, the attachment points of an aileron or a flap. The wing beam deflection bends the aileron or flap structure by applying a load at D thru aileron supporting bracket. To know this force the deflection of the wing beam at D relative to line CE must be known.

Solution:

Origin at B:

\[ M = 30x \cdot \frac{x}{2} = 15x^2 \]

\[ \theta_C = \int \frac{M_{max}}{EI} \]

\[ = \frac{1}{EI} \int_{0}^{50} 15x^2 \left( -0.5x + 10 \right)dx + \frac{1}{EI} \int_{50}^{80} 15x^2 \left( -0.5x - 40 \right)dx \]

\[ = \frac{1}{EI} \left[ -7.5x^4 + \frac{150x^3}{3} \right]_{0}^{50} + \frac{1}{EI} \left[ \frac{7.5x^4}{4} - \frac{600x^3}{3} \right]_{50}^{80} \]

\[ = \frac{10^6}{EI} \left[ -11.72 + 6.25 + .300 - .400 + 76.9 - 102.4 - 11.72 + 25.0 \right] \]

\[ = -17.9 \times 10^6 \]

Therefore deflection of point D relative to line joining CE is down because result comes out negative and therefore opposite to direction of virtual load.
DEFLECTIONS OF STRUCTURES

Find the horizontal deflection of point C for the frame and loading of Fig. A7.19. Also, angular deflection of C with respect to line OD.

Solution:

Fig. b shows the static moment curve for the given loading and Fig. c the moment diagram for the virtual loading of a unit horizontal load applied at C and resisted at D.

\[ \delta_C = \int \left( \frac{Mdx}{EI} \right) ; \quad M = \frac{WL}{2} - \frac{wx^2}{2} \; ; \; m = h \]

hence

\[ \delta_C = \int_0^L \left( \frac{WLx}{2E} - \frac{wx^2}{2} \right) dx = \left[ \left( \frac{WLx^2}{4E} - \frac{wx^3}{6} \right) \right]_0^L = \frac{1}{12} \left( \frac{WL}{E} \right) \]

To find angular deflection at C apply a unit imaginary couple at C with reactions at C and D. Fig. A7.20 shows the virtual m diagram.

\[ \theta_C = \int_0^L \left( \frac{Mdx}{EI} \right) = \frac{1}{EI} \int_0^L \left( \frac{WLx}{2} - \frac{wx^2}{2} \right) dx = \frac{1}{EI} \left[ \frac{WLx^2}{4} - \frac{wx^3}{6L} \right]_0^L = \frac{1}{24} \left( \frac{WL^2}{E} \right) \]

Linear Deflection of Beams Due to Shear by Virtual Work.

Generally speaking, shear deflections in beams are small compared to those due to bending except for comparatively short beams and therefore are usually neglected in deflection calculations. A close approximation is sometimes made by using a modulus of elasticity slightly less than that for bending and using the bending deflection equations.

The expression for shear deflection of a beam is derived from the same reasoning as in previous derivations. The virtual work equation for the hypothetical unit load system for a shear deflection dy (Fig. A7.21) considering only dx elastic is 1 x 6-udy where y is shear on section due to unit hypothetical load at point 0, and dy is the shear detrusal of the element dx due to any given load system or any other cause.

\[ \text{Fig. A7.21} \]

\[ * \text{Sometimes "G", instead. See p. A7.4} \]

dy = \frac{\delta}{L} dx and \frac{\delta}{E} = \frac{V}{AE_S}, where \lambda = \text{cross sectional area of beam at section and } E_S = \text{modulus of rigidity}, and assuming that the shearing stress \( V \) is uniform over the cross-section.

Therefore \( L \times \delta = \frac{V}{AE_S} \). Then the total deflection for the shear slips of all elements of the beam equals

\[ \delta_{\text{total}} = \int_0^L \frac{V}{AE_S} \; \Box \; \text{(a)} \]

where \( V \) is the shear at any section due to given loads. \( V \) is shear at any section due to unit hypothetical load at the point where the deflection is sought and acting in the desired direction of the deflection. The reactions to the hypothetical unit load fix the line of reference for the deflection.

\( A \) is the cross-sectional area and \( E_S \) the modulus of rigidity. Equation (a) is slightly in error as the shearing stress is not uniform over the cross-section, e.g. being parabolic for a rectangular section. However, the average shearing stress gives close results.

For a uniform load of \( w \) per unit length, the center deflection on a simply supported beam is:

\[ \delta_{\text{center}} = 2 \int_0^L \frac{Vdx}{AE_S} = 2 \left( \frac{WL}{2} - \frac{wx^2}{2} \right) \left( \frac{L}{2} \right) = \frac{1}{4} \left( \frac{WL}{E} \right) \]

For bending deflection for a simply supported beam uniformly loaded the center deflection is

\[ \delta = \frac{5wL^4}{384EI} \]

Hence

\[ \frac{WL^2}{AE_S} = 24 \left( \frac{r}{L} \right)^2 \text{, using } E_S = .4E, \]

\[ \frac{5wL^4}{384EI} \text{.} \]

For I beams and channels \( r \) is approximately \( \frac{d}{2} \) and for rectangular sections \( r = \frac{d}{\sqrt{12}} \; (d = \text{depth}) \)

In aircraft structures a ratio of \( \frac{d}{L} \) is seldom greater than \( \frac{1}{12} \).

Thus the shearing deflection in percent of the bending deflection equals 4.1% for a \( \frac{d}{L} \) ratio of \( \frac{1}{12} \) for I-beam sections and 1.4 percent for rectangular sections.
Example Problem 19

Find the vertical deflection of free end A due to shear deformation for beam of Fig. A7.22 assuming shearing stress uniform over cross-section, and AE constant.

\[ \delta_A = \frac{V}{AE_s} \]

\[ V = 100\# \text{ for } x = 0 \text{ to } 10 \]
\[ V = 150\# \text{ for } x = 10 \text{ to } 20 \]
\[ v = 1 \text{ for } x = 0 \text{ to } 20 \]

hence

\[ \delta_A = \frac{1}{AE_s} \left[ 100 \cdot 10 - \frac{1}{AE_s} \int_{10}^{20} 150 \cdot 1 \, dx \right] \]

\[ = \frac{1}{AE_s} \left[ 1000 - \frac{1}{AE_s} \left( 150 \cdot 10 \right) \right] = 2500 \frac{\text{in}}{\text{AE}} \]

Method of Virtual Work Applied to Torsion of Cylindrical Bars.

The angle of twist of a circular shaft due to a torsional moment may be found by similar reasoning as used in previous articles for finding deflection due to bending or shear forces. The resulting expressions are:

\[ \theta = \int \frac{T \, dx}{E_s J} \quad \text{(A)} \]
\[ \theta = \int \frac{T \, dx}{E_s J} \quad \text{(B)} \]

In equation (A), for translation deflections, \( T \) = twisting moment at any section due to applied twisting forces.

\( t \) = torsional moment at any section due to a virtual unit 1 lb. force applied at the point where deflection is wanted and applied in the direction of the desired displacement. (in lbs/in)

\( E_s \) = shearing modulus of elasticity for the material. (also "G")

\( J \) = polar moment of inertia of the circular cross-section.

In equation (B), for rotational deflections, \( \theta \) = angle in twist at any section due to the applied twisting moments in planes perpendicular to the shaft axis. Angle in radians.

\[ t = \text{torsional moment at any section due to a unit virtual couple acting at section where angle of twist is desired and acting in the plane of the desired deflection. (inches lbs/inch lb)} \]

Example Problem

Fig. A7.22 shows a cantilever landing gear strut-axle unit ABC lying in XY plane. A load of 1000\# is applied to axle at point A normal to XY plane. Find the deflection of point A normal to XY plane. Assume strut and axle are tubular and of constant section.

Solution:

The loading shown causes both bending and twisting of the strut axle unit. First find bending and torsional moments on axle and strut due to 1000\# load.

\[ \delta = \int \frac{M \, dx}{E_1 I} + \int \frac{T \, dx}{E_s J} \]

Member AB
\[ M = 1000 \times, \text{ (for } x = 0 \text{ to } 3) \]
\[ T = 0 \]

Member BC
\[ M_{BC} = 3000 \sin 20^\circ + 1000 \times, \text{ (for } x = 0 \text{ to } 36) \]
\[ T_{BC} = 3000 \cos 20^\circ \text{ constant between } B \text{ and } C \]

Now apply a unit \# force at A normal to XY plane as shown in Fig. A7.24 and find bending and torsional moments due to this \# force.

Member AB
\[ m = 1 \times, \text{ (for } x = 0 \text{ to } 3) \]
\[ t = 0 \]

Member BC
\[ m = 3 \sin 20^\circ + 1 \times \text{ (for } x = 0 \text{ to } 36) \]
\[ t = 3 \cos 20^\circ \text{ constant between } B \text{ and } C \]
DEFORMATIONS OF STRUCTURES

\[
\delta = \left( \frac{EJ}{EI} \right) \int_0^L 1000x \cdot x \, dx + \frac{1}{EI} \int_0^L (1000x + 1026) (x + 1.026) \, dx + \frac{1}{EJ} \int_0^L (2820) (2.82) \, dx
\]

\[
= \frac{1}{EI} \left( \left[ 333 \times 3 \right] + 333 \times 3 \times 1026 \times 2 + 1050 \right) \int_0^L x \, dx + \frac{1}{EJ} \int_0^L 7962 \, dx = \frac{1}{EI} (16,82500) + \frac{1}{EJ} (286200)
\]

Note: A practical landing gear strut would involve a tapered or reinforced section involving a variable \( i \) and \( j \) and the integration would have to be done graphically or numerically.

Example Problem 21

For the thin-web aluminum beam of Fig. A7.25 determine the deflection at point \( C \) under the loading shown. Stringer section areas are given on the figure.

Solution:

It was assumed that the webs did not buckle and carried shear only. Fig. A7.26 is an exploded view of the beam showing the internal real loads carried as determined by statics.

The shear flows shown on the (nearly) horizontal edges of the web panels are average values. Fig. A7.27 is an exploded view of the beam showing the unit (virtual) loads.

Virtual loading. Fig. A7.27

Since both axial loads and shear flows were considered, the form of deflection equation used was

\[
\delta = \int \frac{Mdx}{AE} + \int \frac{Qdx}{EI} + \int \frac{Qq \, dx \, dy}{GT}
\]

Integrations in the flanges were made assuming linear load variations. Such an integration carried out over a uniform flange of length \( L \) whose real load varies from \( S_i \) to \( S_j \) and whose virtual load varies from \( u_i \) to \( u_j \) yields

\[
\int L \frac{Sdx}{AE} = \frac{L}{AE} \left( \frac{S_j u_i}{3} + \frac{S_i u_j}{3} \right) + \frac{S_j u_i}{3} + \frac{S_i u_j}{3}
\]

The integrations in the trapezoidal sheet panels were made using the shear flows on the (nearly) horizontal sides as average values, assumed constant over the panel. With this simplification

\[
\int q_{AV} q_{AV} \, dx \, dy = q_{AV} S_{AV} S_{GT}
\]

where \( S \) is the panel area.

The calculation was completed in Table A7.6.

*The equations of statics for tapered beam webs are derived in Art. A15.19, Ch. A-19.*
\[ \delta_g = \frac{(2710 + 13.4)}{2} \times 10^{-6} = 0.147 \times 10^{-3} \text{ in} = 0.265 \text{ in}. \]

### A7.3 Deflections Due to Thermal Stresses

As noted in the "virtual work derivation" of the dummy-unit load deflection equations, the real internal strains of the structure may be due to any cause including thermal effects. Hence, provided the temperature distribution and thermal properties of a structure are known, the dummy-unit load method provides a ready means for computing thermal deflections.

**Example Problem 22**

Find the axial movement at the free end of a uniform bar due to heat application to the fixed end, resulting in the steady state temperature distribution shown in Fig. A7.29. Assume material properties are not functions of temperature.

\[
T = \text{temperature above ambient temperature} \quad K = \text{an empirical constant depending upon thermal properties and rate of heat addition.}
\]

**Fig. A7.28**

Solution:

The thermal coefficient of expansion of the rod material was \( \alpha \). Hence a rod element of length \( dx \) experienced a thermal deformation \( \Delta = \alpha \cdot T \cdot dx \). Application of a unit load at the bar end gave \( u = 1 \). Therefore

\[
\delta_1 = \int u \cdot \alpha \cdot T \cdot dx = \alpha T_a \left[ L \left( 1 - \tanh \frac{KL}{L} \right) \right] = \alpha T_a \left[ 1 - \frac{1}{K} \left( \ln \cosh \frac{K}{L} \right) \right]
\]

**Example Problem 23**

The idealized two-flange cantilever beam of Fig. A7.23a undergoes rapid heating of the upper flange to a temperature \( T \), uniform spanwise, above that of the lower flange. Determine the resulting displacement of the free end.

**Fig. A7.29**

Solution:

The axial deformation of a differential element of the upper flange (subscript \( U \)) was assumed given by

\[
\Delta = \alpha T \cdot dx \quad \text{where} \quad \alpha \quad \text{was the material thermal coefficient of expansion.}
\]

The lower flange, having received no heating underwent no expansion.

*Inasmuch as a thermal expansion is uniform in all directions no shear strain can occur on a material element. Hence no shear strain occurs in the web. The apparent anomaly here - that web elements appear to undergo shear deformations \( \delta = \frac{ATX}{h} \) (Fig. A7.29b) - is explained as follows: The temperature varies linearly over the beam depth. The various horizontal beam "fibers" thus undergo axial deformations which vary linearly also in the manner of Fig. A7.29b giving the apparent shear deformation. No virtual work is done during the web deformation since no axial virtual stresses are carried in the web.*

With the addition of a unit (virtual) load to the free end, the virtual loadings obtained in the flanges were:

\[
u_U = \frac{L - X}{h} \quad \text{and} \quad \nu_L = -\frac{X}{h}
\]

Then the deflection equation was

\[
\delta = \int \nu_U dU + \int \nu_L dL = \left[ \frac{L - X}{h} \right] \cdot \alpha T \cdot dx + \int 0 = \frac{\alpha TL^2}{2h}
\]

**Example Problem 24**

The first step in computing the thermal stresses in a closed ring (3 times indeterminate) involves cutting the ring to make it statically determinate and finding the relative movement of the two cut faces.

Fig. A7.30a shows a uniform circular ring whose inside surface is heated to a temperature \( T \) above the outside surface. The temperature is constant around the circumference and is assumed
to vary linearly over the depth of the cross section. Find the relative movement of the cut surfaces shown in Fig. A7.30b.

\[ \delta_z = \int_0^{2\pi} (-\sin \phi) \frac{Rd\phi}{2} d\phi + \int_0^{2\pi} (-R \sin \phi) \frac{Rd\phi}{n} d\phi = 0 \]

Remarks:
In the three elementary examples given above no stresses were developed inasmuch as the idealizations yielded statically determinate structures which, with no loads applied, can have no stresses. Indeterminate structures are treated in Chapter A.3.

A7.9 Matrix Methods in Deflection Calculations.

Introduction. There is much to recommend the use of matrix methods for the handling of the quantity of data arising in the solutions of stress and deflection calculations of complex structures. The data is presented in a form suitable for use in the routine calculatory procedures of high speed digital computers; a flexibility of operation is present which permits the solution of additional related problems by a simple expansion of the program; the notation itself suggests new and improved methods both of theoretical approach and work division.

The methods and notations employed here and later are essentially those presented by Wehle and Lansing in adapting the Method of Dummy-Unit Loads to matrix notation. Other appropriate references are listed in the bibliography.

**Basis of Method**

Assume the structure to be analyzed has been idealized into a truss-like assembly of rods, bars, tubes and panels (sheets) upon which are acting the external loads applied as concentrated loads \( P_m \) or \( P_n \), each with a different numerical value.
subscript. Thus the system of Fig. A7.32a is idealized into that of Fig. A7.32b.

With the above idealization an improved scheme may be employed to systematize the computation of deflection calculations. The following steps summarize the procedure which is discussed in detail in succeeding sections.

I. A set of internal generalized forces, denoted by \( q_i \) or \( q_j \) (\( i, j \) are different numerical subscripts), is used to describe the internal stress distribution. The \( q \)'s may represent axial loads, moments, shears, etc. In conjunction with a set of member flexibility coefficients, \( a_{ij} \), the \( q \)'s are employed to express the strain energy \( U \). \( a_{ij} \) gives the displacement of point \( i \) per unit force at point \( j \).

II. Equilibrium conditions are used to relate the internal generalized forces \( q_i \), \( q_j \) to the external applied loads, \( P_i \) or \( P_j \). With this relationship the strain energy expression obtained in I. above is then transformed to give \( U \) as a function of the \( P \)'s.

III. Castigliano's Theorem is used to compute deflections.

**CHOICE OF GENERALIZED FORCES**

Consider for example the problem of writing the strain energy (of flexure) of the stepped cantilever beam of Fig. A7.32a, assuming external loads are to be applied as transverse point loads at \( A \) and \( B \). The set of internal generalized forces of Fig. A7.32b will completely determine the bending moment distribution in the beam elements and hence the strain energy. Set (b) then is a satisfactory choice of generalized forces.

It should be pointed out that set (b) is not a unique set. Other satisfactory choices (not an exhaustive display) are shown in Figs. A7.32c, d and e. The final selection may be made for convenience or personal taste.

Note that only as many generalized forces are used per element as are required to determine the significant loadings in that element.

**THE STRAIN ENERGY**

It is next desired to write the strain energy as a function of the \( q \)'s. Continuing the illustrative example, write

\[
U = \frac{1}{2} \int_0^L \frac{L_x x dx}{E I_1} \left[ q_i^2 \right]_0^L + \frac{1}{2} \int_0^L \frac{L_y y dy}{E I_2} \left[ q_j^2 \right]_0^L
\]

\[
+ 2 \times \frac{1}{2} q_i q_j \int_0^L \frac{L_x y dy}{E I_1} + \frac{1}{2} q_j \left[ q_j^2 \right]_0^L \frac{L_y y dy}{E I_2}
\]

Observe that each of the integral terms in the above expression is a property of the structural element (variation of \( E I \)) and of the nature of the associated generalized force (exponent on variable). Introducing the notation

\[
a_{11} = \int_0^L \frac{L_x x dx}{E I_1} \quad a_{22} = \int_0^L \frac{L_y y dy}{E I_2}
\]

\[
a_{12} = \int_0^L \frac{L_x y dy}{E I_1} \quad a_{21} = \int_0^L \frac{L_y y dy}{E I_2}
\]

the strain energy becomes

\[
U = \frac{1}{2} q_i^2 a_{11} + \frac{1}{2} q_j^2 a_{22} + 2 \times \frac{1}{2} q_i q_j a_{12} + \frac{1}{2} q_j \left[ q_j^2 \right]_0^L \frac{L_y y dy}{E I_2}
\]

(19)

Equation (19) is an expression for \( U \) which could have been written immediately from physical considerations. Each coefficient \( a_{ij} \) is the displacement at point \( i \) per unit change in force at point \( j \). This identity is easily seen by applying Castigliano's Theorem to eq. (19).

With this interpretation the first term in eq. (19), representing the strain energy in the outer beam portion, is written by analogy to eq. (2) of Art. A7.3 \( U = \frac{1}{2} \Delta V \). The remaining three terms, representing the energy stored in the inner beam segment by \( q_i \) and \( q_j \), are likewise readily written, with proper account taken for the cross influence of one force upon another (the \( a_{12} q_i q_j \) term).

Note that

\[
a_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} = \frac{\partial^2 U}{\partial q_j \partial q_i} = a_{ji} \quad (20)
\]

Hence

\[
a_{ij} = a_{ji} \quad (Maxwell's \ Reciprocal \ Theorem)
\]

**Fig. A7. 32. Some possible choices of generalized forces.**

w("Relative displacements in the individual member")
The general form of strain energy expression is (expanding eq. (19) by induction)

\[ 2U = q_1 q_a + q_1 q_a G_{1a} + \cdots + q_1 q_n G_{1n} \]

\[ + q_a q_1 G_{a1} + q_a q_a G_{aa} + \cdots + \]

\[ + q_a q_a G_{a1} + q_a q_a G_{aa} + \cdots + \] \[ + q_n q_1 G_{n1} + \cdots + q_n q_n G_{nn} \]

In matrix notation this equation is written (see appendix)

\[ 2U = \begin{bmatrix} q_1 & q_a & \cdots & q_n \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_a \\ \cdots \\ q_n \end{bmatrix} \]

or, more concisely,

\[ 2U = \begin{bmatrix} q_1 & q_a & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_a \\ \cdots \\ q_n \end{bmatrix} \]

In the matrix \( \begin{bmatrix} q_1 & q_a & \cdots & q_n \end{bmatrix} \) many (if not most) of the elements are zero. In the specific example, eq. (21) would be written

\[ 2U = \begin{bmatrix} q_1 & q_a & \cdots & q_n \end{bmatrix} \begin{bmatrix} G_{11} & 0 & \cdots & 0 \\ 0 & G_{aa} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_a \\ \cdots \\ q_n \end{bmatrix} \]

The problem of computing and tabulating various \( q_{ij} \)'s is considered in detail later.

RELATING THE INTERNAL GENERALIZED FORCES TO THE EXTERNAL APPLIED LOADS

For the statically determinate structures considered in this chapter the internal forces are related to the external loads by use of the equations of static equilibrium. A set of linear equations results. Thus, in the specific example considered, if \( P_1 \) and \( P_a \) are the external loads applied as in Fig. A7.32f, then by statics (refer to Fig. A7.32b)

\[ q_1 = P_1 \]

\[ q_a = P_1 + P_a \]

\[ q_a = P_1 L_1 \]

In matrix notation

\[ \begin{bmatrix} q_1 \\ q_a \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ L_1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_a \end{bmatrix} \]

Symbolically these relationships are written

\[ \begin{bmatrix} q_1 \\ q_a \end{bmatrix} = \begin{bmatrix} G_{1m} \\ G_{am} \end{bmatrix} \begin{bmatrix} P_m \end{bmatrix} \]

The matrix \( \begin{bmatrix} G_{1m} \end{bmatrix} \) is called the "unit load distribution" inasmuch as any one column of \( \begin{bmatrix} G_{1m} \end{bmatrix} \), say the \( m \)th column, gives the values of the generalized forces (the \( q \)'s) for a unit value of load \( P_m \), all other external loads zero.

THE STRAIN ENERGY IN TERMS OF APPLIED LOADS

If eq. (22) and its transpose are used to substitute for the \( q \)'s in eq. (21) one gets

\[ 2U = \begin{bmatrix} P_m \end{bmatrix} \begin{bmatrix} G_{1m} \end{bmatrix} \begin{bmatrix} G_{1j} \\ G_{mj} \end{bmatrix} \begin{bmatrix} P_n \end{bmatrix} \]

In the notation here \( i \) and \( j \) are used interchangeably as are \( m \) and \( n \). Also \( \begin{bmatrix} G_{1i} \end{bmatrix} \) is the transpose of \( \begin{bmatrix} G_{im} \end{bmatrix} \), i.e. interchange of subscripts denotes transposition (see appendix).

If the matrix triple product in eq. (23) is formed and defined as

\[ \begin{bmatrix} A_{mn} \end{bmatrix} = \begin{bmatrix} G_{mi} \end{bmatrix} \begin{bmatrix} G_{ij} \end{bmatrix} \begin{bmatrix} G_{jn} \end{bmatrix} \]

then

\[ \begin{bmatrix} A_{mn} \end{bmatrix} = \begin{bmatrix} G_{mi} \end{bmatrix} \begin{bmatrix} G_{ij} \end{bmatrix} \begin{bmatrix} G_{jn} \end{bmatrix} \]
\[ 2U = \begin{bmatrix} P_m \\ \mathbf{A}_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{P}_n \end{bmatrix} \quad (25) \]

Eq. (25) expresses the strain energy as a function of the external applied loads. In the specific example being used:

\[
\mathbf{A}_{mn} = \begin{bmatrix} 1 & 1 & L_1 \\ 0 & 1 & L_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ L_1 \end{bmatrix}
\]

DEFLECTIONS BY CASTIGLIONI'S THEOREM

Application of Castigliano's theorem to eq. (25),

\[ 2U = \begin{bmatrix} P_m \\ \mathbf{A}_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{P}_n \end{bmatrix} \quad (25) \]
yields

\[ \frac{\partial U}{\partial P_m} = \mathbf{s} = \begin{bmatrix} \mathbf{A}_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{P}_n \end{bmatrix} \quad (26) \]

The steps in passing to eq. (26) may be demonstrated readily by writing out eq. (25) for, say, a set of three loads \( m, n = 1, 2, 3 \), differentiating successively with respect to \( P_1, P_2, \) and \( P_3 \) and then re-collecting in matrix form.

The matrix \( \mathbf{A}_{mn} \) gives the deflection at the external points \( m \) for unit values of the loads \( P_n \) and is therefore, by definition, the matrix of influence coefficients.

COMPARISON WITH DUMMY-UNIT LOADS EQUATIONS

It is instructive to write eq. (26) out as

\[ \mathbf{s}_m = \begin{bmatrix} G_{m1} \\ G_{m2} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad (26a) \]

and compare the expression with a typical term from the dummy-unit load method equations, say \( S = 2 u \frac{SL}{AD} \). In the matrix equation (26a) the \( G_{m1} \) is the dummy-unit matrix corresponding to the symbol "m" in the simple sum. The \( G_{ij} \) are the member flexibilities corresponding to \( \frac{L}{AD} \). The matrix product \( \begin{bmatrix} G_{m1} & G_{m2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \) gives the member load distributions due to the applied loads, hence these are the "S" loads. Finally, the operation of matrix multi-

plication itself yields a summation to complete the calculation.

From this last discussion it is clear that eq. (26a) also may be derived by formulating the Dummy-Unit Load equations (Art. A7.7) in matrix notation.

To illustrate the application of the matrix methods presented thus far, a brief and elementary example is worked with those tools already developed.

Example Problem 25

Determine the influence coefficient matrix for transverse forces to be applied to the uniform cantilever of Fig. A7.33 at the three points indicated.

\[ \begin{array}{c}
\begin{array}{ccc}
\hline
& 2 & 1 \\
\hline
3 & & \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\hline
\hline
L_1 & L_2 & L_3 \\
\hline
\hline
\end{array}
\end{array} \]

\[ q_1 \rightarrow q_3 \]

Fig. A7.33

Solution:

The choice and numbering of generalized forces are shown on the figure. These forces were placed so that previously derived expressions for the \( a \)'s could be used. The following member flexibility coefficients were computed. Note that the only non-zero coefficients of mixed subscripts (i not equal to j) are those for loads common to an element.

\[ a_{11} = \int_0^{L/3} \frac{L}{EI} dx = \frac{L^2}{3EI} = a_{22} = a_{33} \]

This expression was adopted from that developed for a transverse shear force on a cantilever beam segment in the preceding illustrative example.

\[ a_{23} = a_{32} = \int_0^{L/3} \frac{L}{EI} dy = \frac{L}{3EI} \]

This expression is for a couple on the end of a cantilever segment (of length \( L/3 \)).

\[ a_{33} = a_{33} = \int_0^{L/3} \frac{L}{EI} dy = \frac{L}{3EI} \]

This expression is for the cross influence of a couple and a shear load on a cantilever segment. Collecting in matrix form,
DEFLECTIONS OF STRUCTURES

\[ \{q_{1j}\} = \frac{L^4}{3EI} \begin{bmatrix} \frac{1}{27} & 0 & 0 & 0 \\ 0 & \frac{1}{27} & \frac{1}{2L} & 0 \\ 0 & \frac{1}{2L} & \frac{1}{L^3} & 0 \\ 0 & 0 & \frac{1}{27} & \frac{1}{2L} \\ 0 & 0 & \frac{1}{2L} & \frac{1}{L^3} \end{bmatrix} \]

The unit load matrix \( \{G_{im}\} \) was computed by successively applying unit loads at points 1, 2 and 3 and computing the values of the \( q \)'s by statics.

\[ \{G_{im}\} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ L/3 & 0 & 0 \\ 1 & 1 & 1 \\ 2L/3 & L/3 & 0 \end{bmatrix} \]

Note that the first column of \( \{G_{im}\} \) gives the values of the \( q \)'s obtained for a unit load at point "1" with no other loads applied. The second column gives the \( q \)'s for a unit load at point "2" only, and so forth.

Finally,

\[ \{A_{mn}\} = \frac{L^4}{3EI} \begin{bmatrix} \frac{1}{27} & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{27} & \frac{1}{2L} & 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2L} & \frac{1}{L^3} & 0 & L/3 & 0 & 0 \\ 0 & 0 & \frac{1}{27} & \frac{1}{2L} & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{2L} & \frac{1}{L^3} & 2L/3 & 1 & 0 \end{bmatrix} \]

Multiplying (see appendix),

\[ \{A_{mn}\} = \frac{L^4}{3EI} \begin{bmatrix} 54 & 28 & 8 \\ 28 & 16 & 5 \\ 8 & 5 & 2 \end{bmatrix} \]

Should the deflections be desired at the three points one forms

\[ \{\delta_s\} = \frac{L^4}{152EI} \begin{bmatrix} 54 & 28 & 8 \\ 28 & 16 & 5 \\ 8 & 5 & 2 \end{bmatrix} \{P_s\} \]

\[ \{\delta_s\} = \frac{L^4}{152EI} \{P_s\} \]

as per eq. (26).

The matrix \( \{A_{mn}\} \) is seen to be symmetric about the main diagonal as it must be: from Maxwell's reciprocal theorem \( A_{mn} = A_{nm} \) (See eq. (20)).

A.7.10 Member Flexibility Coefficients:

Compilation of a Library.

Several member flexibility coefficients are derived below for various members and loadings. A more comprehensive listing is available in the paper by Wehle and Lansing referenced earlier.

**BARS**

The energy in a uniform bar under varying axial force (Fig. A.7.34) is

\[ U = \frac{1}{2AE} \int_0^L \left[ q_1 + \frac{q_1 - q_1}{L} x \right]^2 dx \]

Then referring to eq. (20),

\[ a_{11} = \frac{\delta^*U}{\delta q_1^2} = \frac{1}{L^4} \int_0^L (1 - \frac{x}{L})^2 dx \]

\[ = \frac{L}{2AE} \left( = a_{JJ} \right) \]

and,

\[ a_{1J} = \frac{\delta^*U}{\delta q_1 \delta q_J} = \frac{1}{L^5} \int_0^L (1 - \frac{x}{L}) dx \]

\[ = \frac{L}{2AE} \left( = a_{JJ} \right) \]

An equally likely choice of generalized forces for the above case is shown in Fig. A.7.34a. The strain energy is \( x \) measured from free end

\[ U = \frac{1}{2AE} \int_0^L \left[ q_1 + q_J x \right]^2 dx \]

\[ \frac{q_1}{L} \]

**Fig. A.7.34**

\[ a_{11} = \frac{\delta^*U}{\delta q_1^2} = \frac{1}{L^4} \int_0^L x^2 dx = \frac{1}{3AE} \]

\[ a_{JJ} = \frac{\delta^*U}{\delta q_J^2} = \frac{1}{L^5} \int_0^L x dx = \frac{L^2}{2AE} \]

\[ a_{1J} = \frac{\delta^*U}{\delta q_1 \delta q_J} = \frac{1}{L^5} \int_0^L x dx = \frac{1}{5AE} \]
In the case of tapered members the coefficients are determined by evaluating integrals of the form

\[ a_{ij} = \frac{1}{2EI} \int \frac{L_2(x)dx}{A(x)} \]

Such a quadrature can always be made in these problems. For the linearly tapered bar the results may be obtained as functions of the end area ratios. Thus, Wahl and Lansing give

\[ a_{11} = \frac{L}{3A_1 E} \cdot \phi_{11} \]
\[ a_{1j} = \frac{L}{3A_2 E} \cdot \phi_{1j} \]
\[ a_{jj} = \frac{L}{3A_2 E} \cdot \phi_{jj} \]

**BEAMS**

The energy in the uniform beam of Fig. A7.34d is given by

\[ U = \frac{1}{2EI} \int_0^L \left( q_1 + \left( \frac{q_1 - q_1}{L} \right) x \right)^2 dx \]

Then

\[ a_{11} = \frac{\delta U}{\delta q_1} = \frac{L}{3EI} \quad (= a_{jj}) \]
\[ a_{1j} = \frac{\delta U}{\delta q_j} = \frac{L}{5EI} \quad (= a_{jj}) \]

An alternate choice of generalized forces for the beam of Fig. A7.34d is shown in Fig. A7.34e.

\[ U = \frac{1}{2EI} \int_0^L \left( q_1 + q_j x \right)^2 dx \]

and from which

\[ a_{11} = \frac{L}{5EI} \]
\[ a_{1j} = a_{j1} = \frac{L^3}{2EI} \]
\[ a_{jj} = \frac{L^3}{2EI} \]

**SHEAR PANELS**

For the rectangular shear panel with a uniform shear flow \( q_1 \) on all edges (Fig. A7.34f)

\[ \dot{q}_{11} = \frac{S}{2Et} \]

\[ \dot{q}_{1j} = \frac{S}{2Et} \]

**The trapezoidal shear panel (Fig. A7.34g) is treated approximately by using the average shear flow on the non-parallel sides as though it were constant throughout the sheet. Thus**

\[ \dot{q}_{11} = \frac{S}{2Et} \quad S = \text{surface area} \]

Since by statics \( q_j = \frac{h_j}{h} \), one could use \( q_j \) as an alternate choice of generalized force and

\[ a_{jj} = \frac{\left( \frac{h_j}{h} \right)^2 S}{2Et} \]

**TORSION BAR**

A uniform shaft under torque \( q_1 \) has strain energy

\[ U = \frac{q_1^2 L}{2GI} \quad \text{Then} \quad a_{11} = \frac{L}{2GI} \]

**A7.11 Application of Matrix Methods to Various Structures.**

**Example Problem 26**

The tubular steel truss of Fig. A7.35 is to be analyzed for vertical deflections at points \( E \) and \( F \) under several load conditions in which vertical loads are to be applied to all joints excepting \( A \) and \( B \). The cross sectional areas of tube members are given on the figure.

Set up the matrix form of expression for the
deflections at points E and F.

Solution:

The member flexibility coefficient for a uniform bar under constant axial load is $L/AE$. Fig. A7.36a gives the numbering scheme applied to the members and the q's (these being one and the same, since q is constant in a given member). Fig. A7.36b shows the numbering scheme adopted for the external loading points.

Then the matrix triple product

$$A_{mn} = G_{m1} A_{11} G_{1n}$$

was formed giving, per eq. (26),

$$d_n = \begin{bmatrix} 440 \\ 359 \\ 257 \\ 389 \\ 257 \\ 252 \\ 257 \\ 789 \\ 789 \end{bmatrix}$$

The results here give the deflections of all four points. Since only the deflections of points 3 and 4 were desired the first two rows of $A_{mn}$ may be dropped out. The same result could have been achieved by leaving out the first two rows of $G_{m1}$ (the transpose of $G_{1n}$).

The matrix form of equation above is useful in organizing the computation of deflections for a number of different loading conditions. Thus, should there be several different sets of external loads $P_n$, corresponding to various loading conditions, each set is placed in column form giving the loads as the rectangular matrix

$$P_{nk}, k$$ different numerical subscripts for the load conditions. The matrix product
\[
\begin{bmatrix}
\delta_{mk}
\end{bmatrix} = \begin{bmatrix}
\delta_{mn}
\end{bmatrix} \begin{bmatrix}
P_{nk}
\end{bmatrix} - - - - - - - - - - - - - - - - (26b)
\]

now gives the deflections at each point (m) for the various load conditions (k).

Example Problem 27

Deflections at points 1, 2, 3 and 4 of the truss of Fig. A7.35 are desired for the following loading conditions:

<table>
<thead>
<tr>
<th>Condition No.</th>
<th>P_1</th>
<th>P_2</th>
<th>P_3</th>
<th>P_4 (see Fig. 7.35a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2500</td>
<td>2000</td>
<td>500</td>
<td>450 A7.35b</td>
</tr>
<tr>
<td>2</td>
<td>-1200</td>
<td>-900</td>
<td>-2100</td>
<td>-1750</td>
</tr>
<tr>
<td>3</td>
<td>1500</td>
<td>1470</td>
<td>-1200</td>
<td>-1100</td>
</tr>
</tbody>
</table>

Solution:

The matrix product formed per eq. (26b) was set up as

\[
\begin{bmatrix}
\delta_{mk}
\end{bmatrix} = \begin{bmatrix}
440 & 389 & 257 & 389
389 & 227 & 252 & 789
257 & 252 & 257 & 252
389 & 789 & 252 & 789
\end{bmatrix} \begin{bmatrix}
P_1
P_2
P_3
P_4
\end{bmatrix}
\]

Example Problem 28

For the landing gear unit of Example Problem 20, Fig. A7.23 find the matrix of influence coefficients relating deflections due to lift and drag loads acting at point A and torque about the axle A-B.

Solution:

The structure was divided into elements and the set of internal generalized forces applied as shown in Fig. A7.37a. (Torques and moments are shown vectorially by R.H. rules). Axial stresses were neglected in C-B.

\[
\begin{bmatrix}
G_{1m}
\end{bmatrix}
\]

The unit load distributions \([G_{1m}]\) were obtained by applying unit external applied loads, numbered and directed as in Fig. A7.37b.

\[
\begin{bmatrix}
\frac{G_{1m}}{	ext{EI}}
\end{bmatrix}
\]

At this point the engineer may consider the problem as solved, for the remaining computations is a routine operation:

\[
\begin{bmatrix}
A_{mn}
\end{bmatrix} = \begin{bmatrix}
G_{mn}
G_{1m}
G_{3m}
\end{bmatrix}
\]

Example Problem 29

The beam of example problem 21 is to be resolved by the matrix methods presented herein.
Influence coefficients for points \( F, G \) and \( H \) are to be found.

Solution:

Fig. A7.35 shows the choice and numbering of generalized forces.

No forces were shown applied to the lower flange elements as these were known to be equal to those of the upper flange due to symmetry. Entries were made for \( a_{ij} \) in matrix form as below. Entries for \( a_{xx} \) and \( a_{yy} \) were quadrupled as these occur in two identical members each on top and bottom. Entries for \( a_{xx}, a_{yy} \) and \( a_{zz} \) were doubled. (See Art. A7.10 for coefficient formulas.)

The matrix triple product

\[
\mathbf{a}_{mn} = \mathbf{G}_{mi} \cdot \mathbf{a}_{ij} \cdot \mathbf{G}_{jn}
\]

completes the calculation.

Example Problem 30

Deflections of statically indeterminate structures often may be computed successfully by the methods of this chapter provided that some auxiliary means is employed to obtain an approximation to the true internal force distribution. The exact internal force distribution is not necessarily required in making deflection calculations inasmuch as such a calculation amounts to an integration over the structure - an operation which tends to average out any errors. Thus one may use the engineering theory of bending (S.T.B.), experimental data, previous experience, etc. to obtain reasonable estimates of the internal force distribution for unit loadings.

In the following problem the matrix of influence coefficients is determined for a single cell, three-bay box beam (3 times indeterminate) by using the S.T.B.

Fig. A7.39a shows an idealized doubly symmetric single cell cantilever box beam having three bays. Determine the matrix of influence coefficients for the six point load indicated.

[Table and diagram for calculations]

Solution:

Fig. A7.39b is an exploded view of the beam showing the placement and numbering of the internal generalized forces. Note that only the upper side of the beam was numbered, the lower side being identical by symmetry.

Member flexibility coefficients were computed by the formulas of Art. A7.10 and entered in matrix form as below. Note that all entries for which there were corresponding loads on the lower surface of the beam were doubled. By this means the total strain energy of the beam...
Fig. A7.39b

was accounted for. Note also that entries for \( q_{44}, q_{55}, q_{66}, \) and \( q_{10,10} \) were re-doubled as each of these \( q \)'s act on two (identical) members.

<table>
<thead>
<tr>
<th>( q )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1320</td>
<td>1320</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>2</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>3</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>4</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>5</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>7</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>8</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>9</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
<tr>
<td>10</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
<td>20.26</td>
</tr>
</tbody>
</table>

Note: VOID SPACES INDICATE ZEROS.

Unit load distributions were obtained for successive applications of unit loads to points one through six (Fig. A7.39a). The internal forces predicted by the B.T.B., for a load through the shear center (center of beam, due to symmetry) were superposed on the uniform shear flow (\( q = \frac{P}{A} \)) due to the torque developed in transferring the load to one side.

The matrix triple product

\[
\begin{bmatrix}
A_{mn}
\end{bmatrix} = \begin{bmatrix}
G_{m1}
G_{1j}
G_{jn}
\end{bmatrix}
\]

completes the solution.

A7.12 Deflections and Angular Change of the Elastic curve of Simple Beams by the "Method of Elastic Weights" (Mohr's Method).

In the calculation of structural deflections there occur many steps involving simple integral properties of elementary functions. The Method of Elastic Weights (and the Area Moment Method to follow in Art. A7.14) owes its popularity in large measure to the fact that it enables the analyst to write down many of these integral properties almost by inspection, relying as it does upon the analyst's familiarity with the properties of simple geometric figures. For finding the deflection of a point on a simply supported beam relative to a line joining the supports, the Method of Elastic Weights states:

The deflection at point \( A \) on the elastic curve of a simple beam is equal to the bending moment at \( A \) due to the \( M_{EI} \) diagram acting as a distributed beam load.

Spelled out in steps:

1. The \( M_{EI} \) diagram is drawn just as it occurs due to the applied beam load.

2. This diagram is visualized as being the loading on a second beam (the conjugate beam) supported at the points of reference for the deflection desired.

3. The bending moment in this conjugate beam is found at the station where the deflection of the original beam was desired. This bending moment is equal to the desired deflection.

To prove the theorem, consider the dummy-unit load (virtual work) equation

\[
1^* \cdot \delta = m \cdot d \cdot \Theta = m \cdot \frac{M_{EI}}{EI}
\]

This expression equates the external virtual work done by a unit load, applied at a point deflecting an amount \( \delta \), to the internal virtual work on a beam element experiencing an angular change \( d \Theta = \frac{M_{EI}}{EI} \). The sum (integral) of such expressions throughout a beam gives the total
deflection at the point (cf. eq. 18). We now show that the deflection expression, using the above equation, is the same as the bending moment expression for a simple beam loaded by an "elastic weight" $\frac{M_0}{EL}$.

In Fig. A7-40, the loading of (a) produces the real moments of (b). Consider the deflections of points B and C due to the angular change $\frac{M_0}{EL}$ in a beam element at A* (Fig. A7-40c).

\[ P \quad A \quad B \quad C \]

\[ L/4 \quad L/4 \quad L/4 \quad L/4 \]

\[ M/EL \text { diagram} \]

\[ \frac{M_0}{EL} \text { diagram} \]

\[ \frac{M_0}{EL} dx \]

\[ M_0 \quad \text{unit load at point} \quad C \]

\[ 3 \frac{M_0}{4EI} \]

\[ 1 \frac{M_0}{4EI} \]

\[ \text{Fig. A7.40} \]

For a unit load at point b, Fig. d shows the m diagram. The value of m at the midpoint of dx (point a) = L/8. Hence

\[ \delta_b = \frac{M_0}{EI} \cdot \frac{L}{8} = \frac{M_0 L dx}{8 EI} \]

For deflection of point C, draw m diagram for a unit load at C (see Fig. e). Value of m on element dx = $\frac{L}{16}$

Hence

\[ \delta_c = \frac{M_0}{EI} \cdot \frac{L}{16} = \frac{M_0 L dx}{16 EI} \]

* For simplicity the points A, B and C were placed at the one-quarter span points. The reader may satisfy himself with the general character of the proof by substituting $x_A$, $x_B$ and $x_C$ for the point locations and then following through the argument once again.

In Fig. f consider $\frac{M_0}{EI}$ as a load on a simply supported beam and determine the bending moment at points b and c due to $\frac{M_0}{EI}$ acting at point a.

\[ M_b = \frac{M_0}{4EI} \cdot \frac{L}{2} = \frac{ML dx}{8EI} \]

\[ M_c = \frac{M_0}{4EI} \cdot \frac{L}{4} = \frac{ML dx}{16EI} \]

These values of the beam bending moments at points b and c are identical to the deflections at b and c by the virtual work equations. The moment diagram m for a unit load at b and c (Figs. d and e) is numerically precisely the same as the influence line for moment at points b and c.

Therefore deflections of a simple beam can be determined by considering the M curve as an imaginary beam loading. The bending moment at any point due to this M loading equals the deflection of the beam under the given loads.

Likewise it is easily proved that the angular change at any section of a simply supported beam is equal to the shear at that section due to the M diagram acting as a beam load.

A7.13 Example Problems

Example Problem 31. Find the vertical deflection and slope of points a and b for beam and loading shown in Fig. A7.41. The lower Fig. shows the moment diagram for load P acting at center of a simple beam.

\[ \frac{P}{2} \]

\[ \frac{PL^2}{16} \quad \frac{PL^3}{16} \quad \frac{PL^3}{16} \]

\[ \frac{PL^3}{64} \quad \frac{PL^3}{16} \quad \frac{PL^3}{16} \]

Deflection at point a equals bending moment due to M diagram as a load divided by EI. (See lower Fig. of Fig. A7.41)

\[ \delta_a = \left( \frac{PL^2}{16} \cdot \frac{L}{4} \cdot \frac{PL^2}{64} \cdot \frac{L}{12} \right) \frac{1}{EI} = \frac{11 PL^3}{768 EI} \]

\[ \delta_B = \left( \frac{PL^2}{16} \cdot \frac{L}{4} \cdot \frac{PL^2}{64} \cdot \frac{L}{8} \right) \frac{1}{EI} = \frac{1 PL^3}{48 EI} \]

The angular change of any point equals the shear due to M/EI diagram as a load.

\[ \delta_a = \left( \frac{PL^2}{16} - \frac{PL^3}{64} \right) \frac{1}{EI} = \frac{3 PL^2}{64 EI} \]
\[ \delta_0 = \left( \frac{PL^2}{12} - \frac{PL^2}{4} \right) \frac{1}{EI} = 0 \] (Slope is horizontal or no change from original direction of beam axis.)

Example Problem 32. Determine the deflection of a simple beam loaded uniformly as shown in Fig. A7.42. The bending moment expression for a uniform load \( M = \frac{wL^2 - \frac{wL}{2}}{2} \) or parabolic as shown in Fig. A7.42a. The deflection at mid-point equals the bending moment due to \( M \) diagram as a load.

\[ \frac{wL}{2} - \frac{wL}{2} \]

\[ \text{Fig. A7.42} \]

\[ \frac{wL^3}{24} = \frac{wL^3}{24} = \text{Area} \]

\[ \frac{wL^3}{24} = \frac{wL^3}{24} \]

\[ \text{Fig. A7.42a} \]

\[ \delta_{\text{center}} = \left( \frac{1}{24} \frac{wL^3}{2} - \frac{1}{24} \frac{wL^3}{2} \right) \frac{1}{EI} = \frac{1}{3} \frac{wL^4}{8EI} \]

\[ \delta_{\text{center}} = \left( \frac{1}{24} \frac{wL^3}{2} - \frac{1}{24} \frac{wL^3}{2} \right) \frac{1}{EI} = 0 \]

Slope at supports = the reaction = \( \frac{1}{24} \frac{wL^3}{EI} \).

Example Problem 33.

Fig. A7.43 shows the plan view of one-half of a cantilever wing. The aileron is supported on brackets at points D, E, and F with self-aligning bearings. The brackets are attached to the wing rear beam at points A, B, and C. When the wing bends under the air load the aileron must likewise bend since it is connected to wing at three points. In the design of the aileron beam and similarly for cases of wing flaps this deflection produces critical bending moments. Assuming that the running load distributed to the rear beam as the wing bends as a unit is as shown in the Fig., find the deflection of point B with respect to straight line joining points A and C, which will be the deflection of E with respect to line joining D and F if bracket deflection is neglected. The moment of inertia of the rear beam between A and C varies as indicated in the Table A7.6.

---

Fig. A7.43

Solution:- Due to the beam variable moment of inertia the beam length between A and C will be divided into 10 equal strips of 10 inches each. The bending moment \( M \) at the midpoint of each will be calculated. The elastic weight for each strip will equal \( Mgs \), where \( ds = 10^\circ \) and \( I \) the moment of inertia at midpoint of the strip. These elastic loads are then considered at loads on an imaginary beam of length AC and simply supported at A and C. The bending moment on this imaginary beam at point B will equal the deflection of B with respect to line joining AC.

The bending moment at \( C = 16 \times 30 \times 15 + 10 \times 15 \times 10 = 8250^\circ \#

The shear load at \( C = (15 + 25) \times 30 = 600^\circ \#

Bending moment expression between points C and A equals, \( M = 8250 + 600x + 12.5x^2 \), where \( x = 0 \) to 100.

Table A7.6 gives the detailed calculations for the strip elastic loads. The \( I \) values assumed are typical values for an aluminum alloy beam carrying the given load. The modulus of elasticity \( E = 10 \times 10^6 \) is constant and thus can be omitted until the final calculations. The figures below the table shows the elastic loads on the imaginary beam.

---

**Table A7.6**

<table>
<thead>
<tr>
<th>Strip No.</th>
<th>ds in.</th>
<th>( M_2 ) moment at mid-point</th>
<th>( I ) at mid-point</th>
<th>Elastic Load ( \frac{M_2}{I} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>11563</td>
<td>5.4</td>
<td>21000</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>20063</td>
<td>6.5</td>
<td>30000</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>31083</td>
<td>7.5</td>
<td>41400</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>44580</td>
<td>8.5</td>
<td>52300</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>59280</td>
<td>9.3</td>
<td>63400</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>79080</td>
<td>12.0</td>
<td>83000</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>1000050</td>
<td>16.0</td>
<td>82500</td>
</tr>
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<td>10</td>
<td>132650</td>
<td>20.0</td>
<td>82700</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>149550</td>
<td>24.0</td>
<td>83200</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>178150</td>
<td>28.0</td>
<td>82600</td>
</tr>
</tbody>
</table>
Bending moment at point B due to above elastic loading = 7,100,000 \text{ in}-\text{lb} \cdot \text{in} = .71 \text{ in}  

\text{EI} = 10,000,000

Example Problem 34

Fig. A7.43a shows a section of a cantilever wing of a plane. The wing beams are attached to the hull at points A and B. Due to wing loading the wing will deflect vertically relative to attachment points AB. Thus, installations such as piping, controls, etc., must be so located as not to interfere with the wing deflections between A and B. For illustrative purposes a simplified loading has been assumed as shown in the figure. EI has been assumed as constant whereas the practical case would involve variable I. For the given loading determine the deflection of point C with respect to the support points A and B. Also determine the vertical deflection of the tip points D and E.

Solution:- Fig. A7.43b shows the bending moment diagram for the given wing loading. To find the deflection of C normal to line joining A and B we treat the moment diagram as a load on an imaginary beam of length AB and simply supported at A and B (See Fig. A7.43c.) The deflection of C is equal to the bending moment on this fictitious beam.

Hence \( \Delta_{OC} = 25920 \times 40 - 25920 \times 20 \)

or \( \Delta_{OC} = 51,800 \text{ in} \)

To find the tip deflection, we place the elastic loads (area of moment diagram) on an imaginary beam simply supported at the tip D and E (See Fig. A7.43d). The bending moment on this imaginary beam at points A or B will equal numerically the deflection of these points with respect to the tip points D and E and since points A and B actually do not move this deflection will be the movement of the tip points with respect to the beam support points.

Bending moment at A = 183420 \times 700 - 40000 \times 435 = 127500 \times 124 = 102200000 \text{ in}-\text{lb} \cdot \text{in} \text{.} \text{Tip} = 102,200,000 \text{ in} \text{.}

A7.14 Deflections of Beams by Moment Area Method

For certain types of beam problems the method of moment areas has advantages and this method is frequently used in routine analysis. Angular Change Principle. Fig. A7.44 shows a cantilever beam. Let it be required to determine the angular change of the elastic line between the points A and B due to any given loading. From the equation of virtual work, we have

\[ \Delta_\alpha = \int_B^A \frac{M(x)}{EI} \, dx \]

where \( \alpha \) is the moment at any section, distant x from B due to unit hypothetical couple applied at B. But \( \alpha = \text{unity} \) at all points between B and A.

Therefore \( \Delta_\alpha = \int_B^A \frac{M(x)}{EI} \, dx \)

(*Moment area first developed by Prof. C. E. Greene and published in 1874.)
Referring to Fig. A7.44, this expression represents the area of the M diagram between points B and A. Thus the first principle: "The change in slope of the elastic line of a beam between any two points A and B is numerically equal to the area of the M diagram between these two points."

Deflection Principle

In Fig. A7.44 determine the deflection of point B normal to tangent of elastic curve at A. In Fig. A7.44 this deflection would be vertical since tangent to elastic line at A is horizontal.

From virtual work expression $\delta_B = \frac{A}{B} \frac{\text{Max.}}{\text{EI}} x$

where $m$ is the moment at any section A distance x from B due to a unit hypothetical vertical load acting at B. Hence $m = 1.x = x$ for any point between B and A.

$\delta_B = \frac{A}{B} \frac{\text{Max.}}{\text{EI}} x$

This expression represents the 1st moment of the M diagram about a vertical thru B. Thus the EI deflection principle of the moment area method can be stated as follows: "The deflection of a point A on the elastic line of a beam in bending normal to the tangent of the elastic line at a point B is equal numerically to the statical moment of the M area between points "A" and "B" about point A."

Illustrative Problems

Example Problem 35. Determine the slope and vertical deflection at the free end B of the cantilever beam shown in Fig. A7.46. EI is constant.

\[ \delta_B = \frac{-PL^2}{2EI} \]

Solution:

The moment diagram for given load is triangular as shown in Fig. A7.46. Since the beam is fixed at A, the elastic line at A is horizontal or slope is zero. Therefore true slope at B equals angular change between A and B which equals area of moment diagram between A and B divided by EI.

$\delta_B = \frac{-PL^2}{2EI}$

The vertical deflection at B is equal to the 1st moment of the moment diagram about point B divided by EI, since tangent to elastic curve at A is horizontal due to fixed support.

Hence

$\delta_B = \left( \frac{PL^2}{2} \right) \frac{1}{EI} = \frac{PL^3}{3EI}$

Example Problem 36

Fig. A7.46 illustrates the same simplified wing and loading as used in example problem 34. Find the deflection of point C normal to line joining the support points A and B. Also find the deflection of the tip points D and E relative to support points A and B.

Solution:

Due to symmetry of loading, the tangent to the deflected elastic line at the center line of airplane is horizontal. Therefore, we will find the deflection of points A or B away from the horizontal tangent of the deflected beam at point C which is equivalent to vertical deflection of C with respect to line AB.

Thus to find vertical deflection of A with respect to horizontal tangent at C take moments of the M diagram as a load between points A and C about point A.

Whence

\[ \delta_A = \frac{1}{EI} \left( \frac{650 + 564}{2} \right) 40 \times 20 = \]

\[ \frac{1}{EI} (518400) = \text{deflection of C normal to AB.} \]

To find the vertical deflection of the tip point D with respect to line AB, first find deflection of D with respect to horizontal tangent at C and subtract deflection of A with respect to tangent at C.

\[ \delta_D = \frac{1}{EI} \left( 400000 \times 267 + 127500 \times 576 + 25920 \times 720 \right) = \frac{1}{EI} (1027000000) \]

(See Fig. A7.46 for areas and arms of N/EI diagram). Subtracting the deflection of A with respect to C as found above we obtain

\[ \delta_D = \frac{1}{EI} \left( 1027000000 - 518400 \right) = \frac{1}{EI} (1021800000) \]
A.7.15 Beam Fixed End Moments by Method of Area Moments

From the two principles of area moments as given in Art. A.7.14, it is evident that the deflection and slope of the elastic curve depend on the amount of bending moment area and its location or its center of gravity.

Fig. A.7.47 shows a beam fixed at the ends and carrying a single load \( P \) as shown. The bending moment shown in (c) can be considered as made up of two parts, namely that for a load \( P \) acting on a simply supported beam which gives the triangular diagram with value \( Pa \) \((L-a)/L \) for the moment at the load point, and secondly a trapezoidal moment diagram of negative sign with values of \( MA \) and \( MB \) and of such magnitude as to make the slope of the beam elastic curve zero or horizontal at the support points A and B, since the beam is considered fixed at A and B.

The end moments \( MA \) and \( MB \) are statically indeterminate, however, with the use of the two moment area principles they are easily determined. In Fig. b the slope of elastic curve at A and B is zero or horizontal, thus the change in slope between A and B is zero. By the 1st principle of area moments, this means that the algebraic sum of the moment areas between A and B is equal zero. Hence in Fig. c

\[
\frac{-MA}{L} + \frac{(Pa)(L-a)}{L} \cdot \frac{L}{2} = 0 \quad \text{(A)}
\]

In Fig. b the deflection of B away from a tangent to elastic curve at A is zero, and also deflection of A away from tangent to elastic curve at B is zero.

Thus by moment area principle, the moment of moment diagrams of Fig. C about points A, or B is zero.

Taking moments about point A:

\[
\begin{align*}
MA &= \frac{P_a(L-a)}{2L} + \frac{P_a(L-a)}{2L} (a + L-a) + \frac{Ma}{2} \times \frac{L}{3} + \frac{Mb}{2} \times \frac{L}{3} \\
&= \frac{P_a(L-a)}{2} \times \frac{L}{2} + \frac{Ma}{2} \times \frac{L}{2} + \frac{Mb}{2} \times \frac{L}{2} \\
&= \frac{P_a(L-a)}{2} \times \frac{L}{2} \quad \text{(B)}
\end{align*}
\]

Solving equations A and B for \( MA \) and \( MB \)

\[
MA = -\frac{Pb^2}{L^2} \quad \text{and} \quad MB = -\frac{Pb^2}{L^2} \quad \text{where} \quad b = (L-a)
\]

To find the fixed end moments for a beam with variable moment of inertia use the M/I diagrams in place of the moment diagrams.

Example Problem 37

Fig. A.7.48 shows a fixed-ended beam carrying two concentrated loads. Find the fixed-end moments \( MA \) and \( MB \).

Solution: Fig. b shows the static moment diagram assuming the beam simply supported at A and B.

For simplicity in finding areas and taking moments of the moment areas the moment diagram has been divided into the 4 simple shapes as shown. The centroid of each portion is shown together with the area which is shown as a concentrated load at the centroids.

Fig. C shows the moment diagrams due to unknown moments \( MA \) and \( MB \). The area of these triangles is shown as a concentrated load at the centroids.

Since the change in slope of the elastic curve between A and B is zero, the area of these moment diagrams must equal zero, hence

\[
\begin{align*}
5265 + 14040 + 2160 + 6385 + 15MA + 15MB &= 0 \\
15MA + 15MB &= 28350 = 0 \quad \text{(1)}
\end{align*}
\]

The deflection of point A away from tangent to elastic curve at B is zero, therefore the first moment of the moment diagrams about point A equals zero. Hence,

\[
\begin{align*}
5265 \times 6 + 15 \times 14040 + 17 \times 2160 + 24 \times 6385 + 15MA + 300MB &= 0 \quad \text{or} \quad 15MA + 300MB = 444600 \quad 0 \quad \text{(2)}
\end{align*}
\]

Solving equations (1) and (2), we obtain

\[
\begin{align*}
MA &= -816 \text{ in. lb} \\
MB &= -1074 \text{ in. lbs}
\end{align*}
\]

With the end moments known, the deflection or slope of any point on the elastic curve between A and B can be found by use of the 2 principles of area moments.
A7.16 Truss Deflection by Method of Elastic Weights

If the deflection of several or all the joints of a trussed structure are required, the method of elastic weights may save considerable time over the method of virtual work used in previous articles of this chapter. The method in general consists of finding the magnitude and location of the elastic weight for each member of a truss due to a strain from a given truss loading or condition and applying these elastic weights as concentrated loads on an imaginary beam. The bending moment on this imaginary beam due to this elastic loading equals numerically the deflection of the given truss structure.

Consider the truss of diagram (1) of Fig. A7.49. Diagram (2) shows the deflection curve for the truss for a ΔL shortening of member bc, all other members considered rigid. This deflection diagram can be determined by the virtual work expression δ = uΔL. Thus for deflection of joint O, apply a unit vertical load acting down at joint O. The stress m in bar bc due to this unit load is 1/ΔL = P = 4. Therefore

δ₀ = ΔL₀ ⋅ 4P

The deflection at other lower chord joints could be found in a similar manner by placing a unit load at these joints. Diagram (2) shows the resulting deflection curve. This diagram is plainly the influence line for stress in bar bc multiplied by ΔL₀.

Diagram (3) shows an imaginary beam loaded with an elastic load ΔL₀ acting along a vertical line thru joint O, the moment center for obtaining the stress in bar bc. The beam reactions for this elastic loading are also given. Diagram (4) shows the beam bending moment diagram due to the elastic load at point O. It is noticed that this moment diagram is identical to the deflection diagram for the truss as shown in diagram (2).

The elastic weight of a member is therefore equal to the member deformation divided by the arm r to its moment center. If this elastic load is applied to an imaginary beam corresponding to the truss lower chord, the bending moment on this imaginary beam will equal to the true truss deflection.

Diagram 5, 6 and 7 of Fig. A7.49 gives a similar study and the results for a ΔL shortening of member ck. The moment stress center for this diagonal member lies at point O', which lies outside the truss. The elastic weight ΔL at point O' can be replaced by an equivalent system at points 0 and k on the imaginary beam as shown in Diagram (6). These elastic loads produce a bending moment diagram (Diagram 7) identical to the deflection diagram of diagram (5).

Table A7.7 gives a summary of the equations for the elastic weights of truss chord and web members together with their location and sign.

![Diagram of truss and deflection curves](image)

**Table A7.7 Equations for Elastic Weights**

<table>
<thead>
<tr>
<th>Elastic-Weight for Chord Members (See Member ab)</th>
<th>Lower Chord</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Chord</td>
<td>Lower Chord</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>h</td>
<td>h</td>
</tr>
<tr>
<td>w=1/2L</td>
<td>w=1/2L</td>
</tr>
<tr>
<td><img src="image" alt="Diagram of elastic weight equations" /></td>
<td><img src="image" alt="Diagram of elastic weight equations" /></td>
</tr>
</tbody>
</table>

The moment center 0 of a chord member is the intersection of the other two members cut by the section used in determining the load in that member by the method of moments.

The sign of the elastic weight w for a chord member is plus if it tends to produce downward deflection of its point of application. Thus for a simple truss compression in top chord or tension in bottom chord produces downward or positive elastic weight.
For a truss diagonal member the elastic weights P & Q have opposite signs and are assumed to be directed toward each other or away according as the member is in compression or tension. In fig. a, if x is greater than Q and P is located at the end of the diagonal nearest the moment center O. Downward elastic weights are plus.

\[ P = \frac{3L}{r} \]
\[ Q = \frac{1}{r} \]

In the case of web diagonal members, the elastic weights are given by:

\[ P = \frac{3L}{r} \]
\[ Q = \frac{1}{r} \]

**TABLE A7.7 (CONTINUED)**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (ft)</th>
<th>Area (in²)</th>
<th>Load (lb)</th>
<th>Elastic wt. (r)</th>
<th>Elastic wt. (Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>30</td>
<td>30.012</td>
<td>1000</td>
<td>0.008</td>
<td>0.00117</td>
</tr>
<tr>
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<td>30.012</td>
<td>1000</td>
<td>0.008</td>
<td>0.00117</td>
</tr>
<tr>
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<td>30</td>
<td>30.012</td>
<td>1000</td>
<td>0.008</td>
<td>0.00117</td>
</tr>
<tr>
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<td>30.012</td>
<td>1000</td>
<td>0.008</td>
<td>0.00117</td>
</tr>
</tbody>
</table>

**Fig. A7.50**

**Table A7.8 Web Member Elastic Weights.**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (ft)</th>
<th>Area (in²)</th>
<th>Load (lb)</th>
<th>Elastic wt. (r)</th>
<th>Elastic wt. (Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
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<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>AB</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>BB</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>CC</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>DD</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
</tbody>
</table>

**Fig. A7.51**

**Table A7.9 Web Member Elastic Weights.**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (ft)</th>
<th>Area (in²)</th>
<th>Load (lb)</th>
<th>Elastic wt. (r)</th>
<th>Elastic wt. (Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>AB</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>BB</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>CC</td>
<td>25</td>
<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
<tr>
<td>DD</td>
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<td>25.301</td>
<td>1000</td>
<td>0.018</td>
<td>0.0185</td>
</tr>
</tbody>
</table>

**A7.17 Solution of Example Problems.**

The method of elastic weights as applied to truss deflection can be best explained by the solution of several simple typical trusses.

**Example Problem 38**

Fig. A7.50 shows a simply supported truss symmetrically loaded. Since the axial deformations in all the members must be found, the first step is to find the loads in all the members due to the given loading. The results of this step are given in the figure, the stresses being written adjacent to each member. The next step or steps is to compute the member elastic weights, their location and their sense or direction. Tables A7.8 and A7.9 give these calculations.
The slope of the elastic curve at the truss joint points equals the vertical shear at these points for the beam of Fig. A7.51.

Example Problem 39

Find the vertical deflection of the joints of the Pratt truss as shown in Fig. A7.52. The member deformations, for each member due to the given loading are written adjacent to each member. Table A7.10 gives the calculation of member elastic weights. Fig. A7.53 shows the imaginary beam loaded with the elastic weights from Table A7.10. The deflections are equal numerically to the bending moments on this beam.

\[
\delta = 0.1855 \times 25 = 4.65''
\]

\[
\delta = \frac{0.465 + 0.093}{100} (\Delta P \text{ in } \text{bar} \text{ in } \text{bar}) = 0.518''
\]

\[
\delta = 0.1855 \times 50 - 0.00387 \times 25 = 0.833''
\]

\[
\delta = 0.853 + 0.093 \text{ (AL of } \text{CC}) = 0.864
\]

\[
\delta = 0.1855 \times 75 - 0.00387 \times 50 - 0.00623 \times 25 = 1.03''
\]

Fig. A7.52

Table A7.10

<table>
<thead>
<tr>
<th>Member</th>
<th>(\Delta L)</th>
<th>(r)</th>
<th>(w = \frac{AL}{r})</th>
<th>Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0.061</td>
<td>30</td>
<td>0.0203</td>
<td>B</td>
</tr>
<tr>
<td>BC</td>
<td>0.061</td>
<td>30</td>
<td>0.0203</td>
<td>C</td>
</tr>
<tr>
<td>CD</td>
<td>0.066</td>
<td>30</td>
<td>0.0222</td>
<td>D</td>
</tr>
<tr>
<td>CD</td>
<td>0.081</td>
<td>30</td>
<td>0.0277</td>
<td>E</td>
</tr>
</tbody>
</table>

Table A7.11

<table>
<thead>
<tr>
<th>Member</th>
<th>(\Delta L)</th>
<th>(f_1)</th>
<th>(P = \frac{AL}{f_1})</th>
<th>Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0.0450</td>
<td>0.00282</td>
<td>0.00282</td>
<td>A</td>
</tr>
<tr>
<td>BC</td>
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<td>B</td>
</tr>
<tr>
<td>CD</td>
<td>0.080</td>
<td>0.00417</td>
<td>0.00417</td>
<td>C</td>
</tr>
</tbody>
</table>

Fig. A7.53

Example Problem 40

Find the vertical joint deflections for the unsymmetrically loaded truss of Fig. A7.54. The \(AL\) deformations for all members are given on the figure. Table A7.11 gives the calculation of the elastic weights, their signs and points of application. Fig. A7.55 shows the imaginary beam loaded with the elastic weights from Table A7.11. Table A7.12 gives the calculation for the joint deflections.

**Fig. A7.54**

**Table A7.12**

<table>
<thead>
<tr>
<th>Panel</th>
<th>Panel Shear</th>
<th>(\Delta L)</th>
<th>Moment of (12.5 \times \text{Shear} + \text{deflection})</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>0.0158</td>
<td>0.00282</td>
<td>0</td>
<td>A</td>
</tr>
<tr>
<td>BC</td>
<td>0.00282</td>
<td>0.0158</td>
<td>0</td>
<td>B</td>
</tr>
<tr>
<td>CD</td>
<td>0.00417</td>
<td>0.00282</td>
<td>0</td>
<td>C</td>
</tr>
<tr>
<td>CD</td>
<td>0.00417</td>
<td>0.00282</td>
<td>0</td>
<td>D</td>
</tr>
<tr>
<td>CD</td>
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<td>0.00282</td>
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<td>0.00417</td>
<td>0.00282</td>
<td>0</td>
<td>F</td>
</tr>
</tbody>
</table>

*error*

Example Problem 41

Fig. A7.56 shows a simply supported truss with cantilever overhangs on each end. This simplified truss is representative of a cantilever wing beam the fuselage attachment points being
DEFLECTIONS OF STRUCTURES

at e and e'. The DL deformation in each truss member due to the given external loading is given on the figure. The complete truss elastic loading will be determined. With the elastic loading known the truss deflections from various reference lines are readily determined.

Fig. A7.56

Fig. A7.57

Table A7.13

<table>
<thead>
<tr>
<th>Member</th>
<th>(\Delta L)</th>
<th>(r)</th>
<th>(w = \frac{f}{r})</th>
<th>Apply at Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>-.098</td>
<td>20</td>
<td>.0040</td>
<td>a</td>
</tr>
<tr>
<td>AB</td>
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<td>20</td>
<td>.0035</td>
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<td>ef</td>
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<td>20</td>
<td>.0024</td>
<td>i</td>
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Table A7.14

<table>
<thead>
<tr>
<th>Member</th>
<th>(\Delta L)</th>
<th>(r_1)</th>
<th>(P = \frac{f}{r_1})</th>
<th>Apply at joint</th>
<th>(r_2)</th>
<th>(Q = \frac{f}{r_2})</th>
<th>Apply at joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>aA</td>
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<td>a</td>
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<td>.00346</td>
<td>a</td>
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<td>.00995</td>
<td>b</td>
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<td>.00703</td>
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<td>.00703</td>
<td>c</td>
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<td>.00638</td>
<td>d</td>
<td>8.95</td>
<td>.00638</td>
<td>d</td>
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<tr>
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<td>.00670</td>
<td>e</td>
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<td>e</td>
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<td>.00781</td>
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<td>.00781</td>
<td>f</td>
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<td>.00728</td>
<td>g</td>
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<tr>
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</tbody>
</table>

In finding deflections this overhang elastically loaded portion is considered as fixed at a and free at e. The bending moment at any point on this beam equals the magnitude of the vertical deflection at that point.

Thus to find the deflection of the truss and (Joint a) we find the bending moment at point a of the imaginary beam of Fig. A7.58. Hence the deflection at a = \(\frac{EM_a}{2}\) (calling counterclockwise positive), = \(-.01922 - .00512\) x 60 + .00245 x 70 + .00333 x 60 + .00239 x 50 + .00648 x 40 + .00234 x 30 + .00543 x 20 - .00149 x 10 = 2.13" upward. Due to symmetry of truss and loading of the truss we know the slope of the elastic curve at the center line of the truss is horizontal or zero. Thus to find the deflection of any point with reference to Joint a we can make use of the deflection principle of the moment area method. Thus in Fig. A7.59 the vertical deflection of any point for example Joint f, relative to Joint a equals the moment of all elastic loads between a and f about "a".

\[ R = .00312 \quad \text{(From Span ee') \quad \text{Fig. A7.59}} \]

\[ EM_a = 2.077" \quad \text{student should make calculations} \]

Previously the deflection of f with respect to e was found to be - .0586". Thus deflection of a with respect to point e = 2.077 + .0586 = 2.135" which checks value found above. Let it be required to find the deflection of Joint c relative to a line connecting joints b and d.

\[ R = .00312 \quad \text{Fixed} \]

\[ EM_c = 2.077" \quad \text{student should make calculations} \]

Find the vertical deflection of the canti-lever over-hang portion of the truss relative to support points e and e'.
For this problem we need only to consider the elastic loads between points b and d as loads on a simple beam supported at b and d. (See Fig. A7.61.) The deflection at C with respect to a line bd of the deflected truss equals the bending moment at point C for the loaded beam of Fig. A7.61.

\[ Q = 0.004743 \times 20 - 0.00239 \times 10 = 0.7 \text{ inches} \]

\[ R_d = 0.004743 \quad R_b = 0.004667 \]

Fig. A7.61

A7.18 Problems

(1) Find vertical and horizontal deflection of joint B for the structure in Fig. A7.62. Area of AB = 0.2 sq. in. and BC = 0.3 sq. in. E = 10,000,000 psi.

(2) For the truss in Fig. A7.63, calculate the vertical deflection of joint C. Use AE for each member equal to 2 x 10^-7.

(3) For the truss of Fig. A7.64 determine the horizontal deflection of joint E. Area of each truss member = 1 sq. in., E = 10,000,000 psi.

(4) Determine the vertical deflection of joint E of the truss in Fig. A7.64.

(5) Determine the deflection of joint D normal to a line joining joint CE of the truss in Fig. A7.64.

(6) Calculate the vertical displacement of joint C for the truss in Fig. A7.65 due to the load at joint B. Members a, b, c and h have areas of 20 sq. in. each. Members d, e, f, g and i have areas of 2 sq. in. each. E = 30,000,000 psi.

(7) For the truss in Fig. A7.66, calculate the deflection of joint C along the direction CE. E = 30,000,000 psi.

(8) For the truss in Fig. A7.67, find the vertical and horizontal displacement of joints C and D. Take area of all members carrying tension as 2 sq. in. each and those carrying compression as 5 sq. in. each. E = 30,000,000 psi.

(9) For the truss in Fig. A7.68, determine the horizontal displacement of points C and B. E = 28,000,000 psi.

(10) For the truss in Fig. A7.69, find the vertical deflection of joint B. Depth of truss = 180°. Width of each panel is 180°. The area of each truss member is indicated by the number on each bar in the figure. E = 30,000,000 psi. Also calculate the angular rotation of bar DE.

(11) For the truss in Fig. A7.70, calculate the vertical and horizontal displacement of joints A and B. Assume the cross-sectional area for members in tension as 1 sq. in. each and those in compression as 2 sq. in. E = 10,300,000 psi.

(12) For the truss in Fig. A7.70 calculate the angular rotation of member AB under the given truss loading.

(13) For the beam in Fig. A7.71 determine the deflection at points A and B using method of elastic weights. Also determine the slopes of the elastic curve at these points. Take E = 1,000,000 psi and I = 1266 in.²

(14) For the beam in Fig. A7.72 find the deflection at points A and E. Also the slope of the elastic curve at point C. Assume EI equals to 5,000,000 lb in. sq.
(15) Fig. A7.73 illustrates the airloads on a flap beams ABCDE. The flap beams is supported at B and D and a horn load of 500# is applied at C. The beam is made from a 1"-.049 aluminum alloy round tube. I = 0.1659 in^4; E = 10,300,000 psi. Compute the deflection at points C and E and and the slope of the elastic curve at point E.

![Fig. A7.73](image)

(16) For the beam of Fig. A7.74 determine the deflections at points C and D in terms of EI which is constant. Also determine slopes of the elastic curve at these same points.

(17) For the cantilever beam of Fig. A7.75 determine the deflections and slopes of the elastic curve at points A and B. Take EI as constant. Express results in terms of EI.

![Fig. A7.74](image)

(18) For the loaded beam in Fig. A7.76 determine the value of the fixed end moments MA and MB. EI is constant. Also find the deflection at points C and D in terms of EI.

(19) In Fig. A7.77 determine the magnitude of the fixed end moment MA and the simple support RB.

![Fig. A7.76](image)

(20) In Fig. A7.78 EI is constant throughout. Calculate the vertical deflection and the angular rotation of point A. 

(21) For the curved beam in Fig. A7.79 find the vertical deflection and the angular rotation of point A. Take EI as constant.

(22) For the loaded curved beam of Fig. A7.80, determine the vertical deflection and the angular rotation of the point A. Take EI as constant.

(23) In Fig. A7.81 find the vertical movement and the angular rotation of point A. Take EI = 12,000,000.

(24) Determine the vertical deflection of point A for the structure in Fig. A7.82. EI = 14,000,000.

(25) The cantilever beam of Fig. A7.83 is loaded normal to the plane of the paper by the two loads of 100# each as shown. Find the deflection of point A normal to the plane of the paper by the method of virtual work. The rectangular moment of inertia for the tube is 0.0277 in^4. E = 29,000,000.

(26) The cantilever landing gear strut in Fig. A7.84 is subjected to the load of 500# in the drag direction at point A and also a torsional moment of 2000 in. lb. at A as shown. Determine the displacement of point A in the drag direction. The tube size for portion CB is 2"-.083 and for portion BA, 2"-.065 round tube. Material is steel with E = 29,000,000 psi.

(27) Using the matrix equation 2U = [a_{ij}] [q_j] compute the strain energy in the truss of Fig. A7.63 (Problem 2). The member flexibility coefficient for a member under uniform axial load is L/3E (see Fig. A7.35a). Ans. U = 22.4 lb.in.

(28) Using matrix equation (23) compute the strain energy in the beam of Fig. A7.71. Note: the choice of generalized forces should be made so as to permit computation of the member flexibility coefficients by the equations of p. A7.19. Ans. U = 3533 lb. in.

(29) Re-solve the problem of example problem 25 for a stepped cantilever beam whose I doubles at point "2" and doubles again at "3". (Heaviest section at built-in end.)
(30) For the truss of Fig. A7 85 determine the influence coefficient matrix relating vertical deflections due to loads $P_1$, $P_2$, $P_3$, and $P_4$ applied as shown. Member areas are shown on the figure.

![Fig. A7.85](image)

Answer.

$$
\begin{bmatrix}
39.67 & 44.67 & 44.67 & 32.0 \\
44.67 & 105.0 & 99.34 & 38.0 \\
44.67 & 99.34 & 99.90 & 38.0 \\
32.0 & 38.0 & 38.0 & 33.0
\end{bmatrix}

A_{mn} = \frac{1}{P_1}

(31) For the truss of problem (30) determine which of the following two loading conditions produces the greatest deflection of point 4. (All loads in pounds).

<table>
<thead>
<tr>
<th>Condition No.</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1000</td>
<td>500</td>
<td>800</td>
<td>400</td>
</tr>
<tr>
<td>2</td>
<td>300</td>
<td>700</td>
<td>400</td>
<td>600</td>
</tr>
</tbody>
</table>

(32) Determine the matrix of influence coefficients relating drag load (positive aft), braking torque (positive nose up) and moment in the V-S plane (positive right wing down) as applied to the free end of the gear strut assembly of Problem 26.

Answer.

$$
A_{mn} = 10^{-6}

\begin{bmatrix}
2500 & -50.7 & 0 \\
-50.7 & 6.37 & 0 \\
0 & 0 & 6.00
\end{bmatrix}

(33) Find the deflection of the load applied to the cantilever panel of Fig. A7 86. (Assume the web does not buckle). Use matrix notation. Ans. $\delta = 7.34 \times 10^{-5}$ inches.

![Fig. A7.86](image)

$G = 3.85 \times 10^6$ psi

$AE = 7.5 \times 10^6$ lbs

$AE = 3 \times 10^6$ lbs

$500\#$

(34) Find the influence coefficients relating deflections at points 1 and 2 of the simply supported beam of Fig. A7 87. Use matrix methods.

![Fig. A7.87](image)

$G = 3.85 \times 10^6$ (Typ.)

$AE = 3 \times 10^6$ lbs

$AE = 8 \times 10^6$ lbs

Ans.

$$
A_{mn} = 10^{-6}

\begin{bmatrix}
12.34 & 7.102 \\
7.102 & 12.34
\end{bmatrix}

Note to student: It will be highly instructive to re-work problems 33 and 34 using the alternate choice of generalized forces in the stringers from those used in your first solution. See p. A7.22 for alternate generalized forces on a stringer.

References for Chapters A7, A8.


### Text Books with Matrix Applications


### Technical Papers


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Many problems involving calculation of deflections are encountered in the structural design of a large modern airplane such as the Douglas DC-8.
CHAPTER A8
STATICALLY INDETERMINATE STRUCTURES
ALFRED F. SCHMITT

A8.00 Introduction.

A statically indeterminate (redundant) problem is one in which the equations of static equilibrium are not sufficient to determine the internal stress distribution. Additional relationships between displacements must be written to permit a solution.

The "Theory of Elasticity" shows that all structures are statically indeterminate when analyzed in minute detail. The engineer however, is often able to make a number of assumptions and coarse approximations which render the problem determinate. In addition, auxiliary aids are available such as the Engineering Theory of Bending (EI) and the constant-shearflow rules of thumb (q = T/2A) (see Chaps. A-5, A-6 and A-13 through A-15). While these latter are certainly not laws of "statics", the engineer employs them often enough so that problems in which they are used to obtain stress distributions are often thought of as being "determinate".

It is frequently the case in aircraft structural analysis that, in view of the requirements for efficient design, one cannot obtain a determinate problem without sacrificing necessary accuracy. The Theory of Elasticity assures the existence of a sufficient number of auxiliary conditions to permit a solution in such cases.

This chapter employs extensions of the methods of Chapter A-7 to effect the solution of typical redundant problems. Special methods of handling particular structural configurations are shown in later chapters.

A8.0 The Principle of Superposition.

The general principle of superposition states that the resultant effect of a group of loadings or causes acting simultaneously is equal to the algebraic sum of the effects acting separately. The principle is restricted to the condition that the resultant effect of the several loadings or causes varies as a linear function. Thus, the principle does not apply when the member material is stressed above the proportional limit or when the member stresses are dependent upon member deflections or deformations, as, for example, the beam-column, a member carrying bending and axial loads at the same time.

A8.1 The Statically Indeterminate Problem.

Several characteristics (and interpretations thereof) of the statically indeterminate problem may be pointed out. These characteris-

ics are individually useful in forming the bases for methods of solution.

A8.1. There are more members in the structure than are required to support the applied loads. If n members may be removed (cut) while leaving a stable structure the original structure is said to be "n-times redundant".

-COROLLARY-

In an n-times redundant structure the magnitude of the forces in n members may be assigned arbitrarily while establishing stresses in equilibrium with the applied loads. Thus, in Fig. A8.1 (a singly redundant structure), the internal force distribution of (a) is in equilibrium with the external loads for any and all values of X, the force in member BD.

Fig. A8.1

Singly redundant stress distribution, (a) consisting of a stress in static equilibrium with the applied loads, (b), with one zero-resultant stress distribution, (c), superposed.

Only the system (b) is actually required to equilibrate the external loads (corresponding to X = 0). Note that the system (c) has zero external resultant.

Of all the possible stress (force) distributions satisfying static equilibrium the one correct solution is that one which results in kinematically possible strains (displacements), i.e., retains continuity of the structure.

Thus, for example, there are an infinite number of bending moment distributions satisfying static equilibrium in Fig. A8.1 (d) since \( M_0 \) can assume any value. Of these, only one will result in the zero deflection of the right hand beam tip necessary to maintain structural continuity with the support at that point.

A8.1
A8.2 STATICALLY INDETERMINATE STRUCTURES

Fig. A8.1d
Singly redundant beam with root bending moment $M_0$ undetermined by statics.

COROLLARY:

If $n$ member loads have been assigned arbitrarily while establishing equilibrium with the external loads, relative movements of the elements will result, violating continuity at $n$ points. $n$ zero-resultant stress (force) distributions may then be superposed to reduce the relative motions to zero. The resulting stress distribution is the correct one.

A8.2 The Theorem of Least Work.

A theorem extremely useful in the solution of redundant problems may be obtained from Castiglano's Theorem. Consider first the problem of redundant reactions such as in a beam over three supports (Fig. A8.2). One of the reactions cannot be obtained by statics.

![Diagram of a beam with redundant reactions](diagram)

A singly redundant beam with one reaction given an arbitrary value $(R_x)$.

If the unknown reaction (say that on the far right) is given a symbol, $R_x$, then the remaining reactions and the bending moments may be determined from statics. The strain energy $U$ may then be written as a function of $R_x$, i.e., $U = f(R_x)$. Next form

$$
\frac{\delta U}{\delta R_x} = 5R_x
$$

This is the deflection at $R_x$ due to $R_x$. But this must also be zero, since the support is rigid. Hence

$$
\frac{\delta U}{\delta R_x} = 0 \quad \text{(1)}
$$

Eq. (1) is true for all redundant reactions occurring at fixed supports. Because it corresponds to the mathematical condition for the minimum of a function, eq. (1) is said to state the Theorem of Least Work. In words; "the rate of change of strain energy with respect to a fixed redundant reaction is zero".

A8.2.1 Determination of Redundant Reactions by Least Work.

Example Problem A

By way of illustration, the problem posed by Fig. A8.2 was carried to completion. The bending moment was given by $(x, y, z)$ measured from the left ends of the three beam divisions

$$
M = (500 + R_x) x \quad 0 < x < L/2
$$

$$
M = \frac{(500 + R_x) L}{2} + (R_x - 500) y \quad 0 < y < L/2.
$$

Then

$$
U = \frac{1}{2} \int_{0}^{L/2} M^2 dx
$$

$$
+ \frac{1}{2EI} \int_{0}^{L} \left[ \frac{M^2}{2} (500 + R_x) x + (R_x - 500) y \right] dx
dy
$$

Differentiating under the integral sign, (see p. A7.8)

$$
\frac{\delta U}{\delta R_x} = \frac{1}{2EI} \int_{0}^{L/2} (500 + R_x) x^2 dx
$$

$$
+ \frac{1}{2EI} \int_{0}^{L} \left[ \frac{(500 + R_x) x}{2} + (R_x - 500) y \right] dx
$$

$$
+ \frac{1}{2EI} \int_{0}^{L} R_x (L - 2) y dy
$$

$$
\frac{\delta U}{\delta R_x} = 0 = (500 + R_x) \int_{0}^{L/2} x^2 dx
$$

$$
+ (500 + R_x) \int_{0}^{L/2} \frac{1}{2} (x^2 + y) dy + R_x \int_{0}^{L} (L - 2) y dy
$$

$$
(R_x - 500) \int_{0}^{L} y (\frac{L}{2} + y) dy + R_x \int_{0}^{L} (L - 2) y dy
$$
Completion of the computation gave
\[ R_x = \frac{-1500}{15} \text{ lbs.} = -93.3 \text{ lbs.} , \]
the negative sign indicating that \( R_x \) was down.

**Example Problem B**
Determine the redundant fixed end moments for the beam of Fig. A8.2(a).

![Diagram of a beam with reactions and moments](Fig. A8.2a)

A doubly redundant beam with two reactions given two arbitrary values.

**Solution:**

The redundant end moments were designated as \( M_L \) and \( M_R \) for the left and right beam ends respectively and were taken positive as shown. The moment equations for the two beam portions (\( x \) from left end, \( y \) from right) were

\[ M = M_L + \frac{M_R + 2PL - M_L}{3L} x \quad 0 < x < L \]
\[ M = M_R + \frac{M_L + PL - M_R}{3L} y \quad 0 < y < 2L \]

Then

\[ U = \frac{1}{2EI} \left[ \int_0^L \left( \frac{M_R + 2PL - M_L}{3L} X \right)^2 dx + \int_0^{2L} \left( \frac{M_L + PL - M_R}{3L} Y \right)^2 dy \right] \]

Differentiating under the integral sign (see remarks on p. A7-3)

\[ \frac{3U}{3P_L} = 0 = \left[ \int_0^L \left( \frac{M_L + M_R + 2PL - M_L}{3L} X \right) \left( 1 - \frac{x}{3L} \right) dx + \int_0^{2L} \left( \frac{M_L + M_R + PL - M_R}{3L} Y \right) \frac{y}{3L} dy \right] \]

\[ \frac{3U}{3P_R} = 0 = \left[ \int_0^L \left( \frac{M_L + M_R + 2PL - M_L}{3L} X \right) \left( 1 - \frac{x}{3L} \right) dx + \int_0^{2L} \left( \frac{M_L + M_R + PL - M_R}{3L} Y \right) \frac{y}{3L} dy \right] \]

**A8.2.2 Redundant Stresses by Least Work.**

The Theorem of Least Work may be applied to the problem of determining redundant member forces within a statically indeterminate structure. Thus, in an \( n \)-times redundant structure if the redundant member forces are assigned symbols \( X, Y, Z, \ldots \) etc., the values which these forces must assume for continuity of the structure are such that the displacements associated with these forces (the discontinuities) must be zero.

Hence, by an argument parallel to that used for redundant reactions, one writes,
\[
\begin{align*}
\frac{\partial U}{\partial x} &= 0 \\
\frac{\partial U}{\partial y} &= 0 \\
\text{etc.}
\end{align*}
\]

In words, "the rate of change of strain energy with respect to the redundant forces is zero".

Eqs. (2), like eq. (1), are statements of the Theorem of Least Work. They provide a equations for the n-times redundant structure. The simultaneous solution of these equations yields the desired solution of the problem.

Example Problem C

The cantilever beam and cable system of Fig. A8.3(a) is singly redundant. Find the member loadings by use of the Least Work Theorem.

![Image](image.png)

Fig. A8.3

A singly redundant structure with one member force given an arbitrary value (X).

Solution:

The tensile load in the cable was treated as the redundant load and was given the symbol X (Fig. A8.3(b)). The strain energies considered were those of flexure in portions AC, CD and BC and that of tension in the cable AB. Energies due to axial forces in the beam portions were considered negligible.

The bending moment in BC (origin at B) was

\[
M_{BC} = \left(1000 - \frac{30}{58.3} X\right) x
\]

In AC, (origin at A):

\[
M_{AC} = \frac{50}{58.3} X \cdot y
\]

In CD:

\[
M_{CD} = 50,000
\]

The strain energy was therefore

\[
U = \frac{X^2}{2} \left( \frac{L}{AB} \right)_{AB}
\]

\[
+ \frac{1}{2EI_{BC}} \left(1000 - \frac{30}{58.3} X\right) x^2 dx
\]

\[
+ \frac{1}{2EI_{AC}} \left(\frac{50}{58.3} X\right) y^2 dy
\]

\[
+ \frac{1}{2EI_{CD}} \left(50\right) \left[(50,000) y\right] ds
\]

Obviously there was no need to consider the energy in CD as its loading did not depend upon X and hence could not enter the problem. Differentiating under the integral sign.

\[
\frac{\partial U}{\partial x} = \frac{58.3 X}{EA_{AB}}
\]

\[
- \frac{30}{58.3} \left(1000 - \frac{30}{58.3} X\right) \int_0^{50} x^2 dx
\]

\[
+ \left(\frac{50}{58.3} \right) \frac{X}{EI_{AC}} \int_0^{30} y^2 dy = 0
\]

\[
\frac{58.3 X}{EA_{AB}} + \frac{11,032 X}{EI_{BC}} + \frac{6620 X}{EI_{AC}}
\]

\[
= 21.44 \times 10^6
\]

Putting

\[
A_{AB} = 0.028 \text{ in}^2
\]

\[
I_{BC} = 8.0 \text{ in}^4
\]

\[
I_{AC} = 10.0 \text{ in}^4
\]

gave

\[
X = 613 \text{ lbs.}
\]

Then

\[
M_{BC} = \left[1000 - \frac{30}{58.3} \left(613\right)\right] x = 655x
\]

\[
M_{AC} = \frac{50}{58.3} \times 613 \cdot y = 528 y
\]
Example Problem D

A semicircular pin-ended, uniform ring is supported and loaded as shown in Fig. A8.3(c). As a first approximation the horizontal floor tie is to be assumed rigid axially. Find the bending moment distribution in the ring.

Solution:

The axial load in the floor was taken to be the redundant (since the floor was assumed rigid, this could have been thought of as a redundant floor reaction from fixed supports). The loading is shown in Fig. A8.3(d).

The bending moment distribution was

\[ M = XR \sin \theta - FR \sin \theta \quad 0 \leq \theta < 90^\circ \]
\[ M = XR \sin \theta - FR/2 \quad 90^\circ \leq \theta < 180^\circ \]

The axial loadings were

\[ S = P \cos \theta + X \sin \theta \quad 0 \leq \theta < 90^\circ \]
\[ S = X \sin \theta \quad 90^\circ \leq \theta < 180^\circ \]

The strain energy (for only half the structure) was:

\[ U = \frac{1}{2EI} \int_0^{90^\circ} [XR \sin \theta - FR \sin \theta]^2 d\theta \]
\[ + \frac{1}{2EI} \int_{90^\circ}^{180^\circ} [XR \sin \theta - FR/2]^2 d\theta \]
\[ + \frac{1}{2AE} \int_0^{90^\circ} [P \cos \theta + X \sin \theta]^2 d\theta \]
\[ + \frac{1}{2AE} \int_{90^\circ}^{180^\circ} [X \sin \theta]^2 d\theta \]

* Zero strain energy in the rigid floor (AE → ∞).

Differentiating under the integral sign,

\[ \frac{AU}{AX} = \frac{R^2}{EI} \int_0^{90^\circ} \sin^2 \theta d\theta \]
\[ - \frac{R^2p}{2EI} \int_0^{90^\circ} (1 - \cos \theta) \sin \theta d\theta \]
\[ + \frac{R^2}{2IT} \int_0^{90^\circ} \sin^2 \theta d\theta \]
\[ - \frac{R^2p}{2EI} \int_0^{90^\circ} \sin \theta d\theta \]
\[ + \frac{FR}{AE} \int_0^{90^\circ} \cos \theta \sin \theta d\theta \]
\[ + \frac{XR}{AE} \int_0^{90^\circ} \sin \theta d\theta + \frac{XR}{AE} \int_{90^\circ}^{180^\circ} \sin \theta d\theta \]

Evaluating,

\[ \frac{AU}{AX} = 0 = \frac{\pi}{4} X \left( \frac{R^2}{EI} + \frac{R}{AE} \right) \]
\[ + \frac{3p}{4} \left( \frac{R}{AE} - \frac{R^2}{2IT} \right) \]

Therefore

\[ X = \frac{3p}{2R} \left( \frac{R}{EI} - \frac{R}{AE} \right) \]

Example Problem E

The portal frame of Fig. A8.3(e) is three times redundant. Set up the simultaneous equations in the redundant forces. The relative bending stiffnesses of the segments are given on the figure.

Solution:

The redundant forces selected were the bending moment, the transverse shear, and the axial force, all at point A. The four figures A8.3(f) through A8.3(i) show the bending moment diagrams of the structure due to applied loads and due to redundant forces.
acting individually (it being easier to compute the loads in this fashion). The complete loading was obtained by superposition.

![Diagram](Fig. A8.3f)

![Diagram](Fig. A8.3g)

![Diagram](Fig. A8.3h)

![Diagram](Fig. A8.3i)

The composite bending moments as functions of \( s, \varphi \) and \( s' \) were

\[
M_{AB} = M + Vs
\]

\[
M_{SC} = -50,000 \sin \varphi + M + 50V + 50V \sin \varphi - 50T (1 - \cos \varphi)
\]

\[
M_{CD} = 1000 s' + M - Vs' + 50V - 100 T
\]

Then since

\[
U = \frac{1}{E} \int \frac{M ds}{2I} \quad \text{and} \quad \frac{dU}{dV} = \frac{\partial U}{\partial V} = \frac{\partial U}{\partial T} = 0,
\]

one has,

\[
\frac{\partial U}{\partial M} = 0 = \int_{0}^{50} \left( \frac{M + Vs}{2.50} \right) ds
\]

\[
\frac{\partial U}{\partial Vs'} = \frac{50}{2.50}
\]

After evaluation of the integrals the equations obtained were

\[
\begin{align*}
0.1452M + 9.682V - 7.452T &= 1.112 \times 10^3 \\
9.682M + 763.1V - 494.07T &= 288.3 \times 10^3 \\
-7.452M + 484.0V + 614.97T &= 111.15 \times 10^3
\end{align*}
\]

**A8.3 Redundant Problems by The Methods of Dummy-Unit Loads.**

While the Theorem of Least Work may be made the basis of redundant problem analysis, its direct application by the calculus, as in Art. A8.2.1 and A8.2.2, is often impractical. For the majority of problems the work is facilitated if carried out by the techniques of the Method of Dummy-Unit Loads.

The following derivation is for a doubly redundant truss structure. The extension to a more general n-times redundant structure, in which other loadings in addition to axial (flexure, torsion and shear) are present, is indicated later.

Consider the doubly redundant truss of Fig. A8.4(a). It may be made statically determinate by "cutting" two members such as the diagonals indicated. Application of the external loads to this determinate ("cut") structure gives a load distribution, "g", computed by satisfying static equilibrium. At this time discontinuities appear at the cuts "x" and "y" due to the strains developed.

![Diagram](Fig. A8.4a)

![Diagram](Fig. A8.4b)

![Diagram](Fig. A8.4c)

To compute these and subsequent displacements the Method of Dummy-Unit Loads may be used (Art. A7-7). For this purpose virtual loads are placed alternately at the \( x \) and \( y \) cuts as in Figs. A8.4(b) and (c). From the dummy-unit load solutions:
**For continuity these net relative displacements must be zero. Equating the above expressions each to zero, and rearranging, gives the simultaneous equations**

\[
\begin{align*}
\delta_{x_0} &= \sum \frac{S_{x_x} L}{AE} \\
\delta_{y_0} &= \sum \frac{S_{y_y} L}{AE}
\end{align*}
\]

\[
\sum \frac{u_x^* L}{AE} + \sum \frac{u_y^* L}{AE} = -\sum \frac{S_{x_x} L}{AE}
\]

\[
\sum \frac{u_x^* L}{AE} + \sum \frac{u_y^* L}{AE} = -\sum \frac{S_{y_y} L}{AE}
\]

Eqs. (4) are two simultaneous equations in the two unknowns X and Y. Upon solution for X and Y the true stress distribution may be computed as

\[
S_{TRUE} = S + X u_x + Y u_y
\]

For a structure which is only singly redundant, eqs. (4) and (5) are applied by setting

\[Y = 0\] giving

\[
\sum \frac{u_x^* L}{AE} = -\sum \frac{S_{x_x} L}{AE}
\]

or, simply,

\[
\sum \frac{S_{y_y} L}{AE}
\]

\[X = -\frac{\sum \frac{u_x^* L}{AE}}{\sum \frac{S_{y_y} L}{AE}}\]  
\[
\sum \frac{u_y^* L}{AE}
\]

and,

\[
S_{TRUE} = S + X u_x
\]

**A8.4 Example Problems - Trusses With Single Redundancy.**

**Example Problem #1**

Fig. A8.5 shows a single bay pin connected truss. The truss is statically determinate with respect to external reactions, but statically indeterminate with respect to internal member loads, since at any joint there are 3 unknowns with only two equations of statics available for a concurrent force system. The truss is therefore redundant to the first degree. The general procedure for solution is to make the truss statically determinate by cutting one of the members; on Fig. A8.6, member bc has been selected as the redundant member, and it is cut as shown. The member stresses S for the truss of
Fig. A8.6 are then determined, the results being recorded on the members also entered in Table A8.1. In Fig. A8.7, a unit 1# tensile dummy load has been applied at the cut section of the redundant member bc, and the loads in all the members due to this unit load are calculated. The results are recorded on the figure and also in Table A8.1 under the head of \( u \) stresses. The solution for the redundant load \( X \) in the redundant member bc is given at the bottom of Table A8.1. The true load in any member equals the \( S \) stress plus \( X \) times its \( u \) stress.

<table>
<thead>
<tr>
<th>Member</th>
<th>( L )</th>
<th>( A )</th>
<th>( S )</th>
<th>( u )</th>
<th>( \text{Sul}_{L} ) ( A )</th>
<th>( u_{F} ) ( A )</th>
<th>True Stress ( = S + u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>30</td>
<td>1</td>
<td>0</td>
<td>-7.07</td>
<td>0</td>
<td>15</td>
<td>395</td>
</tr>
<tr>
<td>bc</td>
<td>30</td>
<td>1</td>
<td>-1000</td>
<td>-7.07</td>
<td>2120</td>
<td>15</td>
<td>605</td>
</tr>
<tr>
<td>cd</td>
<td>30</td>
<td>1</td>
<td>0</td>
<td>-7.07</td>
<td>0</td>
<td>15</td>
<td>395</td>
</tr>
<tr>
<td>cd</td>
<td>30</td>
<td>1</td>
<td>0</td>
<td>-7.07</td>
<td>0</td>
<td>15</td>
<td>395</td>
</tr>
<tr>
<td>cb</td>
<td>42.4</td>
<td>1.5</td>
<td>1414</td>
<td>1.0</td>
<td>40000</td>
<td>28.3</td>
<td>435</td>
</tr>
</tbody>
</table>

\[ X = \text{True load is redundant member bc} \]
\[ X = \sum \frac{\text{Sul}_{L}}{A} = \frac{61410}{108.3} = 569.1 \]

Example Problem #2

Fig. A8.8 shows a singly redundant 3-member frame. Find the member loadings. Member areas are shown on the figure.

Solution: Member CC was selected as the redundant and was cut in figuring the S-loads, as in Fig. A8.9. Fig. A8.10 shows the u-load calculation. The table completes the calculation.

<table>
<thead>
<tr>
<th>Mem</th>
<th>( L )</th>
<th>( A )</th>
<th>( S )</th>
<th>( u )</th>
<th>( \text{Sul}_{L} ) ( A )</th>
<th>( u_{F} ) ( A )</th>
<th>True Load ( = S + u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>141.4</td>
<td>0.2</td>
<td>0</td>
<td>1.224</td>
<td>0</td>
<td>1059</td>
<td>335.5</td>
</tr>
<tr>
<td>BC</td>
<td>100.0</td>
<td>0.2</td>
<td>1000</td>
<td>-1.366</td>
<td>-6.83x10^5</td>
<td>933</td>
<td>625.6</td>
</tr>
<tr>
<td>CO</td>
<td>200.0</td>
<td>0.4</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
<td>500</td>
<td>274.1</td>
</tr>
</tbody>
</table>

\[ \sum = -6.83x10^5 \]

\[ X = \sum \frac{\text{Sul}_{L}}{A} = 2492 \]

\[ X = \sum \frac{u_{F}}{A} = 274.1 \text{ lb.} \]
<table>
<thead>
<tr>
<th>Mem.</th>
<th>L</th>
<th>A</th>
<th>S*</th>
<th>u**</th>
<th>SuL/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>30</td>
<td>1</td>
<td>395</td>
<td>.395</td>
<td>4,660</td>
</tr>
<tr>
<td>bd</td>
<td>30</td>
<td>1</td>
<td>-605</td>
<td>-.605</td>
<td>10,980</td>
</tr>
<tr>
<td>dc</td>
<td>30</td>
<td>1</td>
<td>395</td>
<td>.395</td>
<td>4,660</td>
</tr>
<tr>
<td>ca</td>
<td>30</td>
<td>1</td>
<td>395</td>
<td>.395</td>
<td>4,660</td>
</tr>
<tr>
<td>cb</td>
<td>42.4</td>
<td>2</td>
<td>-559</td>
<td>-.559</td>
<td>6,620</td>
</tr>
<tr>
<td>ad</td>
<td>42.4</td>
<td>1.5</td>
<td>855</td>
<td>.855</td>
<td>20,660</td>
</tr>
<tr>
<td>Σ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>52,300</td>
</tr>
</tbody>
</table>

\[ \sigma = \frac{52,300}{E} \]

- Identical with the "true stress" of Table A8.1.

** simply 1/1000th of the "S-loads" since the dummy-unit load is applied exactly as is the 1000# real load.

** Example Problem #2-A.

Find the horizontal deflection of point O of Example Problem #2 under application of the vertical 1000# load shown in Fig. A8.3.

Solution: To compute the deflection use

\[ \delta = \frac{E}{A} \frac{\text{SuL}}{a} \]

Again the symbols "S" and "u" are to be re-interpreted for a deflection calculation as explained above in Example Problem #1-A. The "S-loads" are now the "true loads" computed in Example Problem #2, above. The "u-loads" are loads due to placing a dummy-unit load acting horizontally on the structure at point O. Since this load acts on a redundant structure it would appear that another redundant stress calculation is required. However, this is not necessary.

** Theorem: For the u-loads in a deflection calculation any set of stresses (loads) in static equilibrium with the dummy-unit load may be used, even from the simplest of "cut" structures.

This theorem says that to get the "u-loads" for this deflection calculation we may "cut" any one of the three members and get a satisfactory set of u-loads by simple statics. Before proving the theorem we complete the calculation in tabular form as shown. The "u-loads" were obtained by cutting member OC and applying a unit load horizontally at O.

<table>
<thead>
<tr>
<th>Mem.</th>
<th>L</th>
<th>A</th>
<th>S*</th>
<th>u</th>
<th>SuL/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>AO</td>
<td>141.4</td>
<td>0.2</td>
<td>335.5</td>
<td>√2</td>
<td>3.355x10⁵</td>
</tr>
<tr>
<td>BO</td>
<td>100</td>
<td>0.2</td>
<td>625.6</td>
<td>-1</td>
<td>-3.128x10⁵</td>
</tr>
<tr>
<td>CO</td>
<td>200</td>
<td>0.4</td>
<td>274.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Σ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.227x10⁵</td>
</tr>
</tbody>
</table>

\[ \delta = \frac{22,700}{E} \]

* true loads from Example Problem #2.

By way of demonstration another set of u-loads, called u', were found for this same problem, this time by cutting member OA. The corresponding calculations follow:

<table>
<thead>
<tr>
<th>S</th>
<th>u'</th>
<th>SuL/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>335.5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>625.6</td>
<td>.578</td>
<td>1.808x10⁵</td>
</tr>
<tr>
<td>274.1</td>
<td>-1.155</td>
<td>-1.568x10⁵</td>
</tr>
<tr>
<td>Σ</td>
<td></td>
<td>0.225x10⁵</td>
</tr>
</tbody>
</table>

The results are identical (allowing for round-off errors).

** Proof of Theorem

To prove the theorem above we return to the virtual work principle and the argument from which the dummy-unit loads deflection equation, Eq. (18) of Chapter A-7, was derived (refer to p. A7.10). It will be remembered that the deflection was shown to be equal to the work done by the internal virtual loads (u-loads) moving through the distortions (δ) due to the real loads, i.e., \[ \delta = \sum u \delta \]. The internal virtual loads are those loads due to a unit load acting at the point of desired deflection.

Now for the statically indeterminate structure these internal virtual loads (u-loads) are, in general, indeterminate since the dummy-unit load is applied at an external point of the structure. However, we recall that,

1 - any stress distribution in static equilibrium with the "applied load" (for the moment now we are thinking of the dummy-unit load as the "applied load") differs from the correct (true) distribution only by a stress distribution having zero external resultant (p. A8.1).

ii - a zero-resultant stress distribution moving through a set of displacements does zero work.
Mathematically expressed these points are;

\[ 1 - u_{\text{true}} = u_{\text{static}} + u_{R=0} \]

where \( u_{\text{static}} \) is a u-load distribution obtained from statics in a simple "cut" structure under the action of the externally applied dummy-unit load and \( u_{R=0} \) is the zero-resultant u-load system which must be superposed to give the true u-load distribution

\[ 11 - \sum \Delta x u_{R=0} = 0 \]

It follows, therefore, that any set of u-loads in static equilibrium with the externally applied dummy-unit load will do the same amount of virtual work when the structure undergoes its distortion as would a "true" set of u-loads computed by an indeterminate stress calculation. That is,

\[ \delta = \sum \Delta x u_{\text{true}} = \sum \Delta x (u_{\text{static}} + u_{R=0}) = \sum \Delta x u_{\text{static}} + \sum \Delta x u_{R=0} = \sum \Delta x u_{\text{static}} \]

Q.E.D.

A8.5 Trusses With Double Redundancy

Trusses with double redundancy are handled directly by Eqs. (4). By way of illustration, the structure of Fig. A8.4, from which Eqs. (4) were derived, will be solved for a loading \( P = 2000\) and \( P = 1000\).

Choices of redundants were made identical with those of Fig. A8.4a. Figs. A8.11, A8.12 and A8.13 show the S, \( u_X \) and \( u_Y \) load calculations respectively.

We note here the rule by which the degree of redundancy of a planar pinjointed truss can be determined. For a truss of \( m \) members with \( p \) joints, the truss is \( n \) times redundant where \( n = m - (2p - 3) \). For a spatial truss (3 dimensional truss) \( n = m - (3p - 6) \).

In the present problem \( n = 11 - (12 - 3) = 2 \).

The calculation is carried out in tabular form in Table A8.2. The member dimensions are given in the table.

### Table A8.2

<table>
<thead>
<tr>
<th>Member</th>
<th>L</th>
<th>A</th>
<th>S</th>
<th>( u_X )</th>
<th>( u_Y )</th>
<th>( S_{u_X} )</th>
<th>( S_{u_Y} )</th>
<th>( S_{u_{XY}} )</th>
<th>True Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>30</td>
<td>5</td>
<td>750</td>
<td>-6</td>
<td>0</td>
<td>27,000</td>
<td>0</td>
<td>31.6</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>30</td>
<td>25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>43.2</td>
<td>0</td>
<td>560</td>
</tr>
<tr>
<td>CD</td>
<td>40</td>
<td>5</td>
<td>-1000</td>
<td>0</td>
<td>0</td>
<td>84,000</td>
<td>0</td>
<td>51.2</td>
<td>0</td>
</tr>
<tr>
<td>ED</td>
<td>50</td>
<td>25</td>
<td>1250</td>
<td>0</td>
<td>0</td>
<td>250,000</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>CE</td>
<td>50</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BF</td>
<td>40</td>
<td>7</td>
<td>-3000</td>
<td>-8</td>
<td>0</td>
<td>137,000</td>
<td>43.6</td>
<td>36.6</td>
<td>-1015</td>
</tr>
<tr>
<td>ED</td>
<td>30</td>
<td>5</td>
<td>-750</td>
<td>0</td>
<td>0</td>
<td>27,000</td>
<td>0</td>
<td>21.5</td>
<td>0</td>
</tr>
<tr>
<td>BF</td>
<td>50</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>42.5</td>
<td>0</td>
<td>-1550</td>
</tr>
<tr>
<td>AE</td>
<td>50</td>
<td>5</td>
<td>3750</td>
<td>0</td>
<td>0</td>
<td>137,000</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>EF</td>
<td>20</td>
<td>7</td>
<td>-3000</td>
<td>-8</td>
<td>0</td>
<td>77,200</td>
<td>0</td>
<td>15.4</td>
<td>0</td>
</tr>
<tr>
<td>AF</td>
<td>40</td>
<td>25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>100.5</td>
<td>0</td>
<td>1240</td>
</tr>
</tbody>
</table>

\[ \sum_{i=1}^{n} S_{u_{XY}} = 562,200(478,000+941.1+452.6) \]

Substituting from the table into Eqs. (4) gives (common factor of \( \Sigma \) divided out)

\[
\begin{align*}
X & \sum \frac{u_{XY}}{A} + Y \sum \frac{u_{XYZ}}{A} = - \Sigma \frac{S_{u_{XY}}}{A} \\
X & \sum \frac{u_{XY}}{A} + Y \sum \frac{u_{XYZ}}{A} = - \Sigma \frac{S_{u_{XY}}}{A} \\
341.1 X + 36.6 Y & = - 562,000 \\
36.6 X + 452.6 Y & = - 478,000
\end{align*}
\]

Solving,

\[ X = -1550 \] \[ Y = -932 \]

Finally (see Table A8.2)

True Load = \( S + Xu_X + Yu_Y \).
A8.5 Trusses With Double Redundancy, cont'd.

Example Problem 3

Fig. A8.14 shows a structure composed of four co-planar members supporting a 2000# load. With only two equations of statics available for the concurrent force system the structure, relative to loads in the members, is redundant to the second degree.

Solution:

Fig. A8.15 shows the assumed statically determinate structure; the two members CE and DE were taken as the redundants and were cut at points X and Y as shown. The member stresses for this structure and loading are recorded on the members. Figs. A8.16 and A8.17 give the ux and uy member stresses due to unit (1#) tensile loads applied at the cut faces x and y. Table A8.3 gives the complete calculations for eqs. (4) and (5). The load in the redundant member CE was designated X and that in DE as Y.

\[ X \sum \frac{ux^* L}{AE} + Y \sum \frac{uy^* L}{AE} = \Sigma \frac{Suy^* L}{AE} \]

and after solving for X, Y, \( \delta \),

True Stresses = \( S + Xu_x + Yu_y + Zu_z \) - - - - - (7)

A8.6 Trusses With Multiple Redundancy.

By induction, eqs. (4) may be extended for application to trusses which have three or more times redundant. Thus for a triple redundancy,

\[ X \sum \frac{ux^* L}{AE} + Y \sum \frac{uy^* L}{AE} + Z \sum \frac{uz^* L}{AE} = \Sigma \frac{Suz^* L}{AE} \]

Solving the two equations for X and Y, one obtains X = 521# and Y = 416#. The true load in any member = \( S + Xu_x + Yu_y \) which gave the values in the last column of the table.

A8.7 Redundant Structures With Members Subjected to Loadings in Addition to Axial Forces.

Eqs. (6) are extended readily to cover problems in which flexural, torsional, and shear loadings occur. Thus, for a three times redundant structure

\[
\begin{align*}
Xa_{xx} + Ya_{xy} + Za_{xz} &= -\delta_{x0} \\
Xa_{yx} + Ya_{yy} + Za_{yz} &= -\delta_{y0} \\
Xa_{xz} + Ya_{zy} + Za_{zz} &= -\delta_{z0}
\end{align*}
\]

where

\[
a_{xx} = \Sigma \frac{ux^* L}{AE} + \int \frac{Mx dx}{EI} + \int \frac{tx^* dx}{GJ} + \int \frac{Lx^* dx dy}{Gl} \\
a_{yy} = a_{yx} = \Sigma \frac{uy^* L}{AE} + \int \frac{My dy dx}{EI} + \int \frac{tly dy dx}{GJ} + \int \frac{Ly^* dx dy}{Gl} \\
a_{zz} = \Sigma \frac{uz^* L}{AE} + \int \frac{Mz dx}{EI} + \int \frac{txz dx dy}{GJ} + \int \frac{lzx dy dx}{Gl} + \int \frac{Lz^* dx dy}{Gl} + \text{etc.}
\]

and
\[ \delta_x = \frac{1}{AE} \left( \int \frac{M_x}{E} \, dx + \int \frac{T}{G} \, dx \right) + \int \frac{q_x}{G} \, dx \, dy \]
\[ \delta_y = \frac{1}{EA} \left( \int \frac{M_y}{E} \, dy + \int \frac{T_y}{G} \, dy \right) + \int \frac{\bar{q}_y}{G} \, dx \, dy \]

and where

- \( S, M, T, q \) are the real loads in the determinate structure;
- \( u_x, m_x, t_x, q_x \) are the unit (virtual) loadings due to a unit load at cut \( x \);
- \( u_y, m_y, t_y, q_y \) are for a unit load at cut \( y \);

The redundant force(s) need not be an axial force but may be a moment, torque etc. After solution for the redundants, True Axial Forces

\[ S_x = X u_x + Y u_y + Z u_z \]

True Bending Moments

\[ M_x = X m_x + Y m_y + Z m_z \]

Example Problem 4

The symmetric sheet stringer panel of Fig. A8.18 is to be analyzed for distribution of load \( P \) between stringers. As a first approximation, assume constant shear flow in the sheet panels. All stringers have the same area.

Solution:

The shear flow in the sheet panels was chosen as redundant. Because of symmetry the problem was only singly redundant. Fig. A8.19 shows the \( u_x \) and \( \bar{q}_x \) loadings due to the redundant shear flow \( X = 1 \). The real loading in the determinate structure consisted of a constant load \( P \) in the central stringer alone. The equation solved was (ref. eqs. (5)).

\[ x \left( \int \frac{u_x \, dx}{AE} + \int \frac{\bar{q}_x \, dx \, dy}{G} \right) = \]
\[ - \left( \int \frac{S_x \, dx}{AE} + \int \frac{q \bar{q}_x \, dx \, dy}{G} \right) \]

where

**REAL LOADS**

- \( S = P = \) constant, in central stringer
- \( S = 0 \) in side stringers
- \( q = \) zero

**VIRTUAL LOADS**

- \( u_x = L - x \) in side stringers
- \( \bar{q}_x = 2(x - L) \) in central stringers
- \( \bar{q}_x = 0 \)

When evaluated, (note that the double integrals simply reduce to a constant times the panel area)

\[ x \left( \frac{L}{AE} \int \frac{2L}{G} \right) = - \left( \frac{-PL^2}{AE} \right) \]

\[ X = \frac{P}{2L} \left( \frac{1}{l + \frac{bAE}{GtL^4}} \right) \]

Therefore the true stresses were

- \( P_{ROOT} = P - 2LX = \frac{P}{1 + \frac{L^2}{AE}} \) in central stringer
- \( P_{ROOT} = LX = \frac{P}{1 + \frac{AE}{GtL^4}} \) in side stringers

Example Problem 5

The problem of Fig. A8.19 is doubly redundant as shown. Determine the bending moment distribution. Both members have equal sectional properties.

- \( S = 2P \) \( \frac{1}{2} \) constant
- \( S = 2P \) \( \bar{x} \)

- \( "Pinned" \)

- \( "Clamped" \)
Solution:

The bending moments at point C in member 3BD and at point B in member AB were selected as redundants yielding (when cut) the pin-jointed indeterminate structure of Fig. A8.20. The virtual loadings were as shown in Figs. A8.21 and A8.22.

\[ X \left( \frac{L^2}{AE} + \frac{L^3}{6EI} \right) + Y \left( \frac{L}{2AE} + \frac{(1 + \sqrt{2})L^3}{3EI} \right) = -PL \left( \frac{2L^2}{AE} \right) \]

For a specific case it was assumed that

\[ \frac{AE}{L} = 100 \frac{EI}{L^3} \]

giving

\[ \begin{cases} 0.3716 X + 0.1526 Y = 0.09011 PL \\ 0.1526 X + 0.3121 Y = 0.3516 PL \end{cases} \]

\[ \begin{cases} X = 0.0645 PL \\ Y = 0.456 PL \end{cases} \]

Then as usual

\[ \text{True Stresses} = S = X u_x + Y u_y \]
\[ \text{True Moments} = M = X m_x + Y m_y \]

A8.8 Initial Stresses.

In a redundant structure initial stresses are developed if, upon assembly, certain members must be forced into place because of lack of fit. In some situations intentional misfits are employed to obtain more favorable stress distributions under load ("prestressing").

If, in Fig. A8.4(a), the redundant member with the "X cut" was initially oversized (too long) an amount \( \delta x_1 \) (an oversize, corresponding to a distortion in the positive X direction, is a positive \( \delta_x \)), the modified condition for continuity at the X cut would be (compare with the equations just preceding eqs. (4)).

\[ \delta x_0 + \delta x_1 + \delta xx + \delta xy = 0 \]

Similarly if the Y redundant member were too long

\[ \delta y_0 + \delta y_1 + \delta yx + \delta yy = 0 \]

Then using the previous notations, the appropriate equations for the redundant forces are

\[ \begin{cases} X \left( \frac{u_x}{AE} + Y \left( \frac{u_y}{AE} \right) \right) = -\left( \frac{Su_x}{AE} - \delta x_1 \right) \\ X \left( \frac{u_y}{AE} \right) = -\left( \frac{Su_y}{AE} - \delta y_1 \right) \end{cases} \]

The "S loads" of eq. (10) are present because of applied external loads. These may or may not be zero depending upon the problem.
Example Problem 6

If in example problem 3 member CE was 0.01 inches too short before assembly, determine the stress distribution after assembly and load application.

Solution:

Data obtained from the previous problem was substituted into eqs. (10) along with

\[ \delta_{x1} = -0.01 \text{"} \text{(negative because "too short")} \]

\[ \delta_{y1} = 0 \]

to give

\[
\begin{align*}
2446 \, X + 2350 \, Y &= 2.263 \times 10^5 \pm 0.01E \\
2350 \, X + 3039 \, Y &= 2.458 \times 10^5 \\
\end{align*}
\]

with \( E = 29 \times 10^6 \) the redundant forces were

\[ X = 985 \text{ lbs.} \]
\[ Y = 57 \text{ lbs.} \]

Then, as usual,

True stresses \( = S + Xu_x + Yu_y \).

Example Problem 7

Assume that in the structure of example problem 5 an angular misalignment occurred between members AB and CBD at joint B such that the end of member AB had to be rotated 2.7° clockwise to fit upon assembly. Determine the moments developed without external loads applied.

Solution:

The initial imperfection was \( \delta_{y1} = -2.7/\theta_2 = -0.0471 \text{ radians} \)

The sign was determined by noting that the original misalignment was in the negative direction of the redundant couple \( Y \).

The equations used from the previous problem were (noting the equations there had been multiplied by \( L^2 \times EI/L^2 = EI/L \)).

\[
\begin{align*}
0.3716 \, X + 0.1526 \, Y &= 0 \\
0.1526 \, X + 0.8121 \, Y &= 0.0471 \frac{EI}{L} \\
\end{align*}
\]

Solving,

\[ X = -0.0258 \frac{EI}{L} \]
\[ Y = 0.0630 \frac{EI}{L} \]

True initial stresses and moments were determined as usual.

If, as is sometimes the case, the number of misalignments exceeds the number of redundancies, or if the misalignment does not coincide with the redundant cut chosen but occurs elsewhere, one may use the virtual work principle to compute the effect of these misalignments on the redundant cuts proper. Thus, referring to the "virtual work" derivation of the Dummy-Unit load equations, (Chap. A7) one has

\[ \delta_{x1} = 2 \, u_x \Delta_1 = \cdots = -0.391" \]

where \( \delta_{x1} \) is the initial misalignment in the \( X \)-determinate structure at the \( X \)-redundancy-cut due to initial imperfections (equivalent to initial strains \( \Delta_1 \) throughout the structure).

\( u_x \) as before, is the unit loading due to a \( X \)-virtual load at cut \( X \). Eq. (11) and similar expressions for the \( Y \), \( z \), etc. cuts may be inserted in eqs. (10).

Example Problem 8

Referring back to example problems numbers 3 and 6, assume that member BB is .025" too long. Determine the initial stresses if the other members are of proper length and no external load is applied.

Solution:

To employ the same equations as those of example problem 3, the initial imperfections occurring at the same \( X \) and \( Y \) cuts used there were computed, in this case due to the initial elongation of BE. Thus

\[ \delta_{x1} = 2 \, u_x \Delta_1 = (-1.564)\times(0.025") = -0.391" \]
\[ \delta_{y1} = 2 \, u_y \Delta_1 = (-1.729)\times(0.025") = -0.432" \]

Then, use of those previously computed coefficients in eqs. (10) gave,

\[ \begin{align*}
2446 \, X + 2350 \, Y &= 0.391 \, E \\
2350 \, X + 3039 \, Y &= 0.432 \, E \\
\end{align*} \]

With \( E = 29 \times 10^6 \) psi

\[ X = 263 \text{ lbs.} \]
\[ Y = 209 \text{ lbs.} \]

Finally,

True Initial Stresses \( = S + Xu_x + Yu_y \)

A8.9 Thermal Stresses.

Stresses induced in redundant structures by thermal strains may be computed by application of methods presented above. The problem may be approached from the point of view of computing the
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relative motions at the cuts of the determinate structure caused by the thermal strains and then restoring continuity by applying redundant member forces to the cuts.

Specifically, consider a doubly redundant truss such as that of Fig. A8.4(a). After making cuts \( x \) and \( y \) to render the structure determinate, the application of the temperature distribution is visualized. Relative displacements occur at the cuts, denoted by \( \delta_x \) and \( \delta_y \).

These displacements may be computed by the Dummy-Unit Load method as shown in Art. A7.8 of Chap. A7. After this calculation is accomplished, the problem proceeds as for initial strains, Art. A8.8. Thus, the continuity condition at the cut gives (compare with the equations immediately preceding eqs. (4) and assume for simplicity that the external loads are absent, making \( \delta_x = \delta_y = 0 \))

\[
\begin{align*}
\delta_x + \delta_y + \delta_{xx} &= 0 \\
\delta_y + \delta_{yy} &= 0
\end{align*}
\]

In a truss, the thermal strains produce relative displacements at the cuts given by the "virtual work" derivation of the Dummy-Unit Load equations (ref. Chap. A7, Arts. A7.7 and A7.8) as

\[
\begin{align*}
\delta_x &= \int u_x \cdot a \cdot T \, dx \\
\delta_y &= \int u_y \cdot a \cdot T \, dx
\end{align*}
\]

where \( a \) is the material thermal coefficient of expansion, \( T \) is the temperature above the ambient temperature and \( u_x \) and \( u_y \) are the unit load distributions due to virtual loads at the \( x \) and \( y \) cuts, respectively. The sums in eqs. (12) are written as integrals rather than finite sums to allow for possible variation in \( a \) and \( T \), along the members as well as from member to member. Then the final equations for thermal stresses in a doubly redundant truss become

\[
\begin{align*}
X \left\{ \frac{u_x u_L}{E} + Y \right\} + Y \left\{ \frac{u_x u_L}{E} \right\} &= - \left\{ u_x a \cdot T \right\} \\
X \left\{ \frac{u_x u_L}{E} + Y \right\} + Y \left\{ \frac{u_y u_L}{E} \right\} &= - \left\{ u_y a \cdot T \right\}
\end{align*}
\]

Equations (13) may, of course, be extended for application to structures other than trusses. The expressions appropriate to other loadings have been developed previously (eqs. (8) et seq. in this chapter and other equations in Art. A7.8).

Example Problem 9

The end upright of the truss of Fig. A8.23 is heated to the temperature distribution shown. Determine the stresses and reactions developed.

Solution:

The structure was made determinate by cuts \( x \) and \( y \) as in Fig. A8.24. The unit loadings are shown in Fig. A8.25.

\[
\begin{align*}
\delta_{xy} &= 0 \\
\delta_{yy} &= 0
\end{align*}
\]

The thermal coefficient \( a \) was assumed constant. The calculation was set up in tabular form.

<table>
<thead>
<tr>
<th>TABLE A8.4</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Member</th>
<th>( L \times 10^3 )</th>
<th>( E \times 10^6 )</th>
<th>( A \times 10^2 )</th>
<th>( F \times 10^5 )</th>
<th>( T \times 10^3 )</th>
<th>( T \times 10^3 )</th>
<th>( T \times 10^3 )</th>
<th>( T \times 10^3 )</th>
<th>( T \times 10^3 )</th>
<th>( T \times 10^3 )</th>
</tr>
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<tbody>
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<td>1.0</td>
<td>1.0</td>
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<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>AC</td>
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<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
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<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Substituting into eqs. (13)

\[
\begin{align*}
5.008 X + .90 Y &= 9 a \cdot T \times 10^5 \\
- .90 X + 1.50 Y &= 0
\end{align*}
\]

Solving,

\[
\begin{align*}
X &= 2.01 a \cdot T \times 10^5 \\
Y &= 1.21 a \cdot T \times 10^5
\end{align*}
\]

True stresses are given in Table A8.4.

Example Problem 10

The upper surface of the built-in beam of Fig. A8.26 is heated to a uniform temperature \( T \).
Through the depth of the beam the temperature varies linearly to normal (T = 0) at the lower surface. Determine the end moments developed, neglecting axial constraint and influence of axial forces.

\[ T \]

**EI constant**  
Fig. A8.26a

**Virtual Loading**  
Fig. A8.26b

Then the equations corresponding to eqs. (13) were written (see also eqs. (8)).

\[
X \left( \frac{u_x^2 ds}{AE} + \frac{m_x^2 ds}{EI} \right) + Y \left( \frac{u_y ds}{AE} + \frac{m_y ds}{EI} \right) + \frac{\theta}{\partial} \left( \frac{u_x ds}{AE} + \frac{m_x ds}{EI} \right) = - \delta_{XT}
\]

\[
X \left( \frac{u_x ds}{AE} + \frac{m_y ds}{EI} \right) - Y \left( \frac{u_y ds}{AE} + \frac{m_y ds}{EI} \right) + \frac{\theta}{\partial} \left( \frac{u_y ds}{AE} + \frac{m_y ds}{EI} \right) = - \delta_{YT}
\]

Evaluating, the equations were

\[
\begin{align*}
\frac{1}{EI} X + \frac{R}{EI} Y + 0.5 \theta &= \frac{d^2 T}{R} \\
\frac{-R}{EI} X + \left( \frac{1}{AE} + \frac{R}{EI} \right) Y + 0.5 \theta &= \frac{d_{RT}}{R}
\end{align*}
\]

Note that from the last of these equations \( \theta = 0 \), as it must be because of the symmetry of the ring. Solving the first two equations

\[
X = \frac{d_{TEI}}{h} Y = 0
\]

A non-zero value of \( Y \) would produce a varying bending moment which cannot be because of symmetry. Hence this result too, is rational.

**Example Problem 11**  
Complete the problem begun in Example Problem 24 Art. A7.8, viz, that of computing the thermal stresses in a closed ring whose inner surface is uniformly heated to a temperature \( T \) above the outside.

Solution:

The ring was made determinate by cutting at the top as in Fig. A7.30(b). The unit loadings and thermal deflections were determined in the referenced example. The results of deflection calculations made previously were

\[
\begin{align*}
\delta_{XT} &= \frac{2T R a T}{h} \\
\delta_{YT} &= -\frac{2T R^2 a T}{h} \\
\delta_{ST} &= 0
\end{align*}
\]

A8.10 Redundant Problem Stress Calculations by Matrix Methods.

In the following section the indeterminate structural problem is formulated in matrix notation. The reader is assumed to be familiar with the matrix applications of Art. A7.9 and the elements of matrix notation and arithmetic (see Appendix).

The stress distribution of the structure is specified by a set of internal generalized forces, \( q_i, q_j \) * (Ref. Art. A7.8). Unlike the case of the determinate structure, these \( q_i, q_j \) cannot be

*In the case of indeterminate structures, wherein some of the support reactions may also be redundant, these reactions also are denoted by \( q \)'s. (see Example Problem 13a).
related immediately to the external loads by the equations of statics. Thus a certain subset of the \( q_i', q_j' \) are redundant and are denoted by \( q_r, q_s' \), \((r, s) \text{ different numerical subscripts}) \). When, finally, the redundant forces, \( q_r', q_s' \), are computed (by satisfying continuity) the true values of all the \( q_i', q_j' \) may be found by statics.

**SYMBOLS**

- \( q_i', q_j' \) - internal generalized forces acting on structural elements and reactions at support points.
- \( q_r, q_s' \) - redundant generalized forces and redundant reactions.
- \( P_m, P_n \) - applied external loads.
- \( S_{im} (=S_{jn}) \) - the value of \( q_i' \), \( q_j' \) in the determinate structure under application of a unit external load \( P_m = 1 \) \((P_n = 1)\).
- \( S_{ir} (=S_{js}) \) - the value of \( q_i \), \( q_j \) in the determinate structure due to application of a unit redundant force \( q_r = 1 \) \((q_s = 1)\).
- \( G_{im} (=G_{jn}) \) - the true value of \( q_i' \), \( q_j' \) in the redundant structure due to application of load \( P_m = 1 \) \((P_n = 1)\).
- \( G_{rn} (=G_{ln}) \) - the true value of \( q_r \), \( q_s \) for a unit value of applied load \( P_n = 1 \) \((P_m = 1)\).
- \( a_{ij} \) - member flexibility coefficient: deflection at point \( i \) for a unit force, \( q_j = 1 \) \( \text{(see Arts. A7.9, 10).} \)
- \( a_{mn} \) - influence coefficient for the determinate structure: displacement at external loading point \( m \) for a unit applied load, \( P_n = 1 \).
- \( a_{rn} (=a_{sm}) \) - influence coefficient for the determinate structure: displacement at redundant cut \( r \) (s) for a unit applied load, \( P_n = 1 \) \((P_m = 1)\).
- \( a_{rs} \) - influence coefficient for the determinate ("cut") structure: displacement at redundant cut \( r \) for a unit redundant force \( q_s = 1 \).
- \( A_{mn} \) - influence coefficient for complete redundant structure: deflection at external loading point \( m \) for a unit applied load, \( P_n = 1 \).

\[ (A_{mn} = A_{nm}) \]

**SYMBOLS** - continued

- \( T_i', T_j \) - the temperature (above normal) at points \( i, j \).
- \( \Delta_{IT}, \Delta_{JT} \) - the member thermal distortions associated with \( q_i', q_j' \).

In the notation of Arts. A8.3 et seq., the final true values of the stresses were expressed as (eq. (5) Art. A8.3)

\[ S_{true} = S + X_{ux} + Y_{uy} \]

In the notation to be employed here, this equation is restated as

\[ \{q_i\} = \begin{bmatrix} S_{im} \end{bmatrix} \{P_m\} + \begin{bmatrix} S_{ir} \end{bmatrix} \{q_r\} \]

Here:

\[ \begin{bmatrix} S_{im} \end{bmatrix} \begin{bmatrix} S_{jn} \end{bmatrix} \]

is the matrix of unit-load stress distributions in the determinate ("cut") structure found by the application of unit (virtual) loads at the external loading points. The product \( \begin{bmatrix} S_{im} \end{bmatrix} \{P_m\} \) then gives the real loads in the determinate structure, corresponding to the "S" loads of eq. (5).

\[ \begin{bmatrix} S_{ir} \end{bmatrix} \begin{bmatrix} S_{js} \end{bmatrix} \]

is the matrix of unit-load stress distributions found by application of unit (virtual) forces at the redundant cuts in the determinate ("cut") structure. Hence this is the matrix of \( u_x, u_y \), etc. loads.

The \( \{q_r\} \) of course, correspond to \( X, Y \), etc.

Note that the \( \begin{bmatrix} S_{im} \end{bmatrix} \) and \( \begin{bmatrix} S_{ir} \end{bmatrix} \) matrices are load distributions computed and arranged in much the same fashion as was \( \begin{bmatrix} G_{im} \end{bmatrix} \) of Art. A7.9. The small letter "s" is used to indicate load distribution in the "cut" structure.

By way of illustration, the final result for Example Problem 3, Art. A8.3 is expressed below. FIRST, in the form of eq. (5):

**True Stresses**

\[ S = X (u_x) + Y (u_y) \]

\[ \begin{align*}
AE &= 0 + 521 (.806) + 416 (1.154) \\
BE &= 2000 + 521 (-1.564) + 416 (-1.722) \\
CE &= 0 + 521 (1.00) + 416 (0) \\
DE &= 0 + 521 (0) + 416 (1.00)
\end{align*} \]

*Note that within each of the sets of subscript symbols \((i, j)\), \((r, s)\), \((m, n)\) the symbols may be used interchangeably.*
SECOND, in the form of eq. (14):

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} \cdot \begin{bmatrix}
2000 \\
-1.564 & -1.729 \\
1.00 & 0 \\
0 & 1.00
\end{bmatrix} \begin{bmatrix}
.806 \\
1.154 \\
-1.564 \\
-1.729 \\
1.00 \\
0 \\
0 \\
1.00
\end{bmatrix}
\]

Note that in this case \([\delta_{1m}]\) consisted of only one column, inasmuch as there was only a single external load.

In Art. 7.9 the strain energy was written

\[2U = \mathbf{q}^T \mathbf{[a]} \mathbf{q} \quad \ldots \ldots \ldots (15)\]

where \([a]\) is the matrix of member flexibility coefficients (Art. 7.10). If now eq. (14) and its transpose are used to substitute into (15) the expression becomes (note the use of \((i,j),(r,s),\) and \((m,n)\) interchangeably)

\[2U = \mathbf{q}^T \mathbf{[p]} \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} + \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (16)\]

Multiplying out

\[2U = \mathbf{p}^T \mathbf{[e]} \mathbf{q} + \mathbf{q}^T \mathbf{[j]} \mathbf{p} + \mathbf{q}^T \mathbf{[e]} \mathbf{q} + \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (19)\]

The reader may satisfy himself that the "cross product" term in the middle of the above result is correct by observing that, because of the symmetry of \([a]\),

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} \quad \ldots \ldots \ldots (21)\]

The various matrix triple products occurring above are assigned the following symbols, each having the interpretation given (compare with eq. (24) of Art. 7.9)

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} \quad \ldots \ldots \ldots (16)\]

- the matrix

of external-point influence coefficients in the "cut" structure: deflection at point \(n\) per unit load at point \(n\).  

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} \quad \ldots \ldots \ldots (17)\]

- the matrix

of influence coefficients relating relative displacements at the "cuts" to external loads:  

\[\mathbf{q}^T \mathbf{[j]} \mathbf{p} = \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (18)\]

- the matrix

of influence coefficients relating relative displacements at the "cuts" to redundant loads at the "cuts":  

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = \mathbf{q}^T \mathbf{[e]} \mathbf{q} \quad \ldots \ldots \ldots (19)\]

- the matrix

With the above notation one may write

\[2U = \mathbf{p}^T \mathbf{[e]} \mathbf{q} + \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (19)\]

Now according to the Theorem of Least Work  

\[\frac{\delta U}{\delta \tau} = 0 \quad \ldots \ldots \ldots (20)\]

This last result may be verified by writing eq. (19) out in expanded form, differentiating and then recombing in matrix form. Rearranged, eq. (20) gives

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = - \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (22)\]

Eq. (21) is a set of simultaneous equations for the redundant internal forces \(q_r\), \(q_s\).  

It may be compared with eq. (6) of Art. 7.6, to which it corresponds.  

Eq. (21) may be solved directly from the form there displayed or its solution may be obtained by computing \[\mathbf{q}^{-1}\]

the inverse of the matrix of coefficients, giving

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = - \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (22)\]

The matrix product \[\mathbf{q}^{-1} \mathbf{[j]} \mathbf{p}\] gives the values of the redundant forces for unit values of the external loads.  

This may be given the symbol \[\mathbf{q}^{-1} \mathbf{[j]} \mathbf{p}\] so that

\[\mathbf{q}^T \mathbf{[e]} \mathbf{q} = - \mathbf{q}^T \mathbf{[j]} \mathbf{p} \quad \ldots \ldots \ldots (22)\]
If now eq. (22) is substituted into eq. (14) one gets (with exchange of \( r \) for \( s \) and \( m \) for \( n \))

\[
\{q_1\} = \begin{bmatrix} \delta_{1m} \end{bmatrix} \{P_m\} - \begin{bmatrix} \delta_{1r} & \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \{P_r\} = \begin{bmatrix} \delta_{1m} - \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \{P_m\} = \begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \{P_m\}
\]

The matrix set off in parentheses above, gives the internal force distribution per unit value of the external loads. It is given the symbol

\[
\delta_{1m} = \begin{bmatrix} \delta_{1m} - \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \quad \text{--- (23)}
\]

so that

\[
\{q_1\} = \begin{bmatrix} \delta_{1m} \end{bmatrix} \{P_m\} \quad \text{--- (24)}
\]

Eqs. (23), (24) constitute the major result, inasmuch as they present the means for computing the internal force distribution in a redundant structure.

Example Problem 13

The doubly redundant beam of Fig. A8.27 (a) is to be analyzed for the bending moment distribution. The beam is loaded by couples over the supports as shown.

\[
\text{Fig. A8.27}
\]

Solution:

The choice of internal generalized forces is shown in Fig. A8.27 (b). The appropriate member flexibility coefficients were arranged in matrix form as (ref. Art. A7.10 for coefficient expressions).

\[
\begin{bmatrix} L^2/3 & L/2 & 0 & 0 \\
L/2 & 1 & 0 & 0 \\
0 & 0 & L^2/3 & L/2 \\
0 & 0 & L/2 & 1 \\
\end{bmatrix}
\]

The moments \( q_1 \) and \( q_4 \) were taken as the redundants. With these set equal to zero, the internal force distributions due to application of unit values of \( P_r \) and \( P_n \) were determined, giving

\[
\begin{bmatrix} 1/L & 0 \\
0 & 0 \\
0 & 1/L \\
0 & 0 \end{bmatrix}
\]

With the applied loads set equal to zero, unit values of the redundants were applied yielding

\[
\begin{bmatrix} -1/L & 0 \\
1 & 0 \\
1/L & -1/L \\
0 & 1 \end{bmatrix}
\]

Note that redundant load \( q_2 \) was applied as a self-equilibrating internal couple, acting on both beam halves.

The following matrix products were formed:

\[
\begin{bmatrix} L^2/3 & L/2 & 0 & 0 \\
L/2 & 1 & 0 & 0 \\
0 & 0 & L^2/3 & L/2 \\
0 & 0 & L/2 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 1/L & 0 \\
0 & 0 \\
0 & 1/L \\
0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} -1/L & 0 \\
1 & 0 \\
1/L & -1/L \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 4 & 1 \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} -1/L & 0 \\
0 & 1/L \\
0 & -1/L \\
0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 1/L & 0 \\
0 & 0 \\
0 & 1/L \\
0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\
0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\
0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \delta_{1m} + \delta_{1r} \delta_{sr}^2 & \delta_{sm} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\
0 & 0 \end{bmatrix}
\]
The inverse of $G_{rs}$ was found (ref. appendix)

$$G_{rs}^{-1} = \frac{6EI}{PL} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

Next, the unit redundant load distribution was found (eq. 22).

$$G_{sn} = -G_{rs}^{-1}G_{rn} = -\frac{1}{P} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.286 & -0.428 \\ 0.143 & -0.286 \end{bmatrix}$$

Finally, the true unit stress distribution was computed.

$$G_{tm} = G_{sm} + G_{tr} G_{rm}$$

$$= \frac{1}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.286 & -0.428 \\ 0.143 & -0.286 \end{bmatrix}$$

$$= \begin{bmatrix} 1.286 & 0.428 \\ -0.286L & -0.428L \\ 0.143L & -0.286L \end{bmatrix}$$

(Though the matrix of member flexibility coefficients was expanded to a $6 \times 6$, the coefficients for $q_s$ and $q_r$ were zero. Thus,

$$\begin{bmatrix} L^*/3 & L/2 & 0 & 0 & 0 & 0 \\ L/2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & L^*/3 & L/2 & 0 & 0 \\ 0 & 0 & 0 & L/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_s & 0 \\ 0 & 0 & 0 & 0 & 0 & q_r \end{bmatrix}$$

With the redundants $q_s$ and $q_r$ set equal to zero, successive applications of unit external couples $P_s$ and $P_r$ gave the stress distribution

$$G_{tm} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and, with $P_s$ and $P_r$ zero, successive applications of unit redundant forces $q_s$ and $q_r$ gave

$$G_{tm} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, multiplying out per eqs. (17) and (18)

$$G_{tm} = \begin{bmatrix} -1 & 0 \\ L & 0 \\ -1 & -1 \\ 2L & L \end{bmatrix}$$

The inverse was found:

$$G_{rs}^{-1} = \frac{6EI}{PL} \begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}$$

Finally,

$$G_{tm} = G_{tm} - G_{tr} G_{rs}^{-1} G_{sm}$$
Example Problem 14

The continuous truss of Fig. A8.28 is twice redundant. It is desired to analyze it for stress distributions under a variety of loading conditions consisting of concentrated vertical loads applied at the four external points indicated.

Solution:

The internal generalized forces \(q_1, q_2\) employed were the axial loads in the various members. These were numbered from one to thirty-one as shown on the figure. The member flexibility coefficients in this case were of the form \(q_1 = L/6E\) (Ref. Fig. A7.35a). The coefficients are written as a column matrix below. (They were employed as the diagonal elements of a square matrix in the matrix multiplications, but are written here as a column to conserve space.)

Member loads \(q_3\) and \(q_4\) were selected as redundants. With \(q_3\) and \(q_4\) set equal to zero ("cut"), unit loads were applied successively at external loading points one through four, the four stress distributions thus found being arranged in four columns giving the matrix \(\mathbf{E}_{im}\) (below). By way of illustration, the loading figure used to obtain the second column of \(\mathbf{E}_{im}\) is shown in Fig. A8.29.

Next, unit forces were applied successively at the redundant cuts "three" and "five" as shown in Figs. A8.30a and A8.30b. These loads were arranged in two columns to give the matrix \(\mathbf{E}_{tr}\)
STATICALLY INDETERMINATE STRUCTURES

\[
\left[ g_m \right] = \begin{bmatrix}
1 & 1.0 \\
2 & .50 & -.50 \\
3 & \\
4 & \\
5 & \\
6 & \\
7 & \\
8 & -.50 & .50 \\
9 & \\
10 & \\
11 & \\
12 & \\
13 & \\
14 & \\
15 & \\
16 & \\
17 & \\
18 & \\
19 & \\
20 & \\
21 & \\
22 & \\
23 & \\
24 & \\
25 & \\
26 & \\
27 & \\
28 & \\
29 & \\
30 & \\
31 & \\
\end{bmatrix}
\]

\[
\text{The calculation was completed as per eq. (23) to give } \left[ g_{1m} \right], \text{ the values of the member forces for unit applied external loads.}
\]

\[
\left[ g_{1m} \right] = \begin{bmatrix}
1.0 & -1.22 \\
.50 & .111 & .0065 & .0035 \\
.0035 & .0065 & .111 & .089 \\
-1.22 & .0035 & .0065 & .111 & .089 \\
\end{bmatrix}
\]

NOTE: VOIDS INDICATE ZEROS

Multiplying out gave

\[
\left[ a_{rs} \right] = \left[ g_{1m} \right] \left[ g_{1a} \right] \left[ g_{ja} \right] = \frac{1}{136} \begin{bmatrix}
136 & -8.0 \\
-8.0 & 136 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
\left[ a_{rn} \right] = \left[ g_{1m} \right] \left[ g_{1a} \right] \left[ g_{jn} \right]
\]

\[
= \frac{1}{136} \begin{bmatrix}
-8.0 & -15.0 & 0 & 0 \\
0 & 0 & -15.0 & -8.0 \\
\end{bmatrix}
\]

The inverse of \( \left[ a_{rs} \right] \) was found (ref. appendix)

\[
\left[ a_{rs}^{-1} \right] = \frac{1}{136} \begin{bmatrix}
136 & 8.0 \\
-8.0 & 136 \\
\end{bmatrix}
\]

Next, the values of the redundant forces, for unit values of the applied loads, were found per eq. (22).

\[
\left[ g_{sn} \right] = - \left[ a_{rs}^{-1} \right] \left[ a_{rn} \right]
\]

Example Problem 14a

Fig. A8.31 shows the two bays of a steel tubular tail fuselage truss which is loaded by tail air loads to be resolved into three concentrated loads applied as shown. The fuselage bulkhead at the attack-points station (A-E-F-K) is heavy enough so that it may be assumed to be rigid in its own plane. Hence, the truss may be analyzed as if cantilevered from A-E-F-K as shown. All members are steel tubes, their lengths and areas being tabulated below.

Solution:

The generalized forces were taken to be the member axial loads, these being numbered as in the table below. Member flexibility coefficients, \( E/A \), (\( E \) set equal to unity for convenience) were also tabulated.
than apply successive unit loads and forces at points \( m = 1, 2, 3 \) and \( r = 22, 23, 24 \) and then carry each loading through the structure, another procedure, often better adapted to large complex structures, was employed.

In the method used, the equations of static equilibrium were written for each of the seven joints. Summation of forces in three directions on each joint gave 21 equations in 24 \( q \)'s (the unknowns) and the three applied loads \( P_1, P_2, \) and \( P_3 \). Some typical equations obtained were:

From \( \Sigma F \) on Joint C.

\[
.21368 q_{11} - .21368 q_{16} + .18334 q_{18} - .18334 q_{19} - P_1 
\]

\[
q_{11} + q_{16} = 1.0386 P_1 
\]

From \( \Sigma F \) on Joint B.

\[
.081759 q_1 - .18334 q_{19} - .07614 q_{18} = .5227 q_{16} 
\]

\[
q_1 - q_{19} - .15689 q_{18} = 0 
\]

\[
-.1410 q_1 + q_{16} + .4146 q_{17} = -.5227 q_{16} 
\]

And so forth, for the other joints.

Note that in each case the equations were arranged with the applied loads \( (P_n) \) and the redundant \( q \)'s \( (q_{11}, q_{16}, q_{18}) \) grouped on the right hand side of the equal sign. This arrangement was observed for all 21 equations, after which the equations were placed in matrix form as

\[
[C] [q] = [D] \begin{bmatrix} D_{11} & \cdots & D_{1s} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{ns} \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} 
\]

\[
1, j = 1, 2, \ldots, 24 
\]

\[
n = 1, 2, 3 
\]

\[
s = 22, 23, 24 
\]

(Note that there were 24 equations here, the additional three equations being the identities)

\[
q_{11} = q_{11} 
\]

\[
q_{16} = q_{16} 
\]

\[
q_{18} = q_{18} 
\]

On the right hand side of the above matrix equation the matrices are shown "partitioned". The first three columns of \([D]\) are the coefficients of \( P_n \) and the last three are the

The structure was three times redundant.

In a space framework of \( p \) joints, \( 3p-6 \) independent equations of statics may be written (p. A8.10). Here, however, stress details in the plane \( AEKF \) are to be sacrificed; six equations are lost thereby since only net forces in two directions in this plane can be summed.

\( 3 \times 11 - 6 - 6 = 21 \) equations, 24 member unknown. Members 22, 23 and 24 were cut.

The next step was to compute the unit stress distributions \([\sigma_{1m}]\) and \([\sigma_{1r}]\). Rather
coefficients of the $q_s$.

The matrix equation was solved for the $q_j$ by finding the inverse of $[C_{ij}]$ (see appendix). Thus,

$$\{q_j\} = [S_{jn} : S_{js}]^{-1} \{P_n\}$$

where

$$[S_{jn} : S_{js}] = [C_{ij}]^{-1} \begin{bmatrix} D_{1n} & D_{1s} \end{bmatrix}.$$  

Thus, the unit stress distributions in the determinate structure were found by a procedure having as its main advantage the reduction of the work to a routine mathematical operation. In the conduct of this work appropriate standardized techniques may be employed.*

The result in this case was

<table>
<thead>
<tr>
<th>$a_{ij}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5416</td>
<td>2.767</td>
<td>5.374</td>
<td>9.409</td>
<td>6.949</td>
<td>6.949</td>
</tr>
</tbody>
</table>

$$[S_{im} : S_{ip}] =$$

The member flexibility coefficients $L/\alpha$ were arranged as the diagonal elements of the matrix $[\alpha_{ij}]$. Then, multiplying out according to eqs. (17) and (18),

$$[G_{im}] = [G_{im}] + [G_{ip}]$$

<table>
<thead>
<tr>
<th>$G_{im}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5555</td>
<td>.2980</td>
<td>.2975</td>
</tr>
<tr>
<td>2</td>
<td>.2980</td>
<td>3.0412</td>
<td>2.3015</td>
</tr>
<tr>
<td>3</td>
<td>.2975</td>
<td>2.3015</td>
<td>3.022</td>
</tr>
</tbody>
</table>

The inverse was found (see appendix)

$$[G_{rs}]^{-1} = 10^{-3} [ .18 543 2239 ]$$

$$[G_{en}] = [G_{rs}]^{-1} [G_{en}] = 10^{-3} [ -136.4 -238 -201.0 ]$$

$$[G_{sn}] = [G_{rs}]^{-1} [G_{sn}] = 10^{-3} [ -31.4 17.6 121.3 ]$$

Finally, the complete unit stress distribution was obtained as

$$[S_{im}] = [S_{im}] + [S_{ip}]$$

Example Problem 15

The doubly symmetric four flange idealized box beam of Fig. A6.32 is to be analyzed for stresses due to load application at the six points indicated. Flange areas taper linearly from root to tip while sheet thicknesses are constant in each panel. The beam is mounted rigidly at the root, providing full restraint.

* These are the "effective areas", being the flange area plus adjacent effective cover sheet area plus one-sixth of the web area. (The factor of one-sixth provides the same moment of inertia as the distributed web area).
against warping of the root cross section due to any torsion loadings.

The member flexibility coefficients were collected in matrix form as shown below. Note that entries for $\bar{a}_{11}$, $\bar{a}_{22}$, $\bar{a}_{33}$, $\bar{a}_{12}$, and $\bar{a}_{31}$ were collected from two stringers each (as well as being doubled as discussed above). Coefficients for these tapered stringers were computed from the formulas of Art. A7.10.

$\bar{a}_{ij}$ matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>.400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.006</td>
<td>4.006</td>
<td>4.006</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.150</td>
<td>.150</td>
<td>.150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5.341</td>
<td>2.671</td>
<td>2.671</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.0889</td>
<td>.0889</td>
<td>.0889</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>.0444</td>
<td>.0444</td>
<td>.0444</td>
<td>2.006</td>
<td>2.006</td>
<td>2.006</td>
</tr>
<tr>
<td>8</td>
<td>.0444</td>
<td>.0444</td>
<td>.0444</td>
<td>2.006</td>
<td>2.006</td>
<td>2.006</td>
</tr>
</tbody>
</table>

NOTE: VOIDS INDICATE ZEROS. $\bar{a}_{ij}$

Cover sheet shear flows $q_s$, $q_v$, and $q_u$ were selected as redundants. With these set equal to zero, and with unit loads placed successively at loading points "one" through "six", the $[g_{im}]$ matrix was obtained. When the cover sheets are "cut" the two webs act independently as plane-web beams. The details of the stress calculation for such beams are similar to those of Example Problem 21, Art. A7.7 and are not shown here.

$[g_{im}]$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.150</td>
<td>.150</td>
<td>.150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.006</td>
<td>4.006</td>
<td>4.006</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5.341</td>
<td>2.671</td>
<td>2.671</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.0889</td>
<td>.0889</td>
<td>.0889</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.0444</td>
<td>.0444</td>
<td>.0444</td>
<td>2.006</td>
<td>2.006</td>
<td>2.006</td>
</tr>
</tbody>
</table>

NOTE: VOIDS INDICATE ZEROS.

Calculations for $[g_{im}]$ were made by successively assigning unit values to the redundants $q_s$, $q_v$, and $q_u$. The calculations are illustrated by the exploded view of the end bay in Fig. A8.34 showing the calculation in that part of the structure for $q_s = 1$. Note that $q_v = 1$ was applied as a self-equilibrating pair of shear flows acting one on each side of the "cut". The ribs were considered rigid in their own planes.

Put $q_s = 1$

From equilibrium of end rib: $(\Sigma M, \Sigma F)$
The following matrix products were formed:

Per eq. (18):

\[
\mathbf{G}_{rs} = \frac{E}{B} \begin{bmatrix}
    .3774 & .1789 & .0520 \\
    .1789 & .2676 & .07322 \\
    .0520 & .07322 & .1254
\end{bmatrix}
\]

Per eq. (17):

\[
\mathbf{G}_{rs} = \frac{E}{B} \begin{bmatrix}
    .03936 & .03936 & -.00016 & .00016 & -.00016 \\
    -.00016 & .03936 & -.00016 & .00016 & -.00016 \\
    -.00016 & -.00016 & .03936 & -.00016 & .00016 \\
    -.00016 & -.00016 & -.00016 & .03936 & -.00016 \\
    -.00016 & -.00016 & -.00016 & -.00016 & .03936
\end{bmatrix}
\]

The inverse of \( \mathbf{G}_{rs} \) was found (ref. appendix).

\[
\mathbf{G}_{rs}^{-1} = \frac{E}{B} \begin{bmatrix}
    3.862 & -2.562 & -.1137 \\
    -2.562 & 6.137 & -2.521 \\
    -.1137 & -2.521 & 9.499
\end{bmatrix}
\]

Then

\[
\mathbf{G}_{sn} = \mathbf{G}_{rs}^{-1} \mathbf{G}_{rn}
\]

\[
= \begin{bmatrix}
    .0699 & -.0699 & .00568 & -.00568 & .00016 & -.00016 \\
    .03593 & -.03593 & .03409 & -.03409 & .00043 & -.00043 \\
    .01568 & -.01568 & .01572 & -.01572 & .01578 & -.01578
\end{bmatrix}
\]

Finally, the true stresses were (per eq. 23)

\[
\mathbf{G}_{im} = \mathbf{E}_{im} - \mathbf{G}_{ir} \mathbf{G}_{mr}
\]

\[
= \begin{bmatrix}
1 & .330 & .0599 & .0060 & .0059 & .00332 & .00332 \\
2 & .0599 & .0599 & .0060 & .0060 & .00332 & .00332 \\
3 & .0599 & .0599 & .0060 & .0060 & .00332 & .00332 \\
4 & 2.23 & 2.23 & .152 & .152 & .00460 & .00460 \\
5 & 1.75 & 1.75 & .152 & .152 & .00460 & .00460 \\
6 & .0599 & .0121 & .152 & .152 & .00460 & .00460 \\
7 & .0599 & .0599 & .0121 & .152 & .00460 & .00460 \\
8 & .0141 & .0141 & .0141 & .152 & .00460 & .00460 \\
9 & .0141 & .0141 & .0141 & .152 & .00460 & .00460 \\
10 & .01568 & .01568 & .01568 & .01568 & .01568 & .01568 \\
11 & .01568 & .01568 & .01568 & .01568 & .01568 & .01568 \\
12 & .01568 & .01568 & .01568 & .01568 & .01568 & .01568 \\
14 & 2.47 & 2.47 & 2.47 & 2.47 & .610 & .610 \\
15 & 2.47 & 2.47 & 2.47 & 2.47 & .610 & .610
\end{bmatrix}
\]

The reader will observe that the result displays the "bending stresses due to torsion"; that is, the buildup of axial flange stresses near the root of a beam under torsion when the root is restrained against warping. The solution for application of a torque is readily
found by superposing stresses for \( P_i = 1 \) and \( P_j = -1 \). Thus, one finds that under this condition there is a root flange load of
\[
q_{1*} = 3.47 - 2.76 = 0.71 \text{ lbs.}
\]
for a torque of 15 inch lbs.

A8.11 Redundant Problem Deflection Calculations by Matrix Methods.

Deflections (and in particular, the matrix of influence coefficients) are readily computed from the results of Art. A8.10.

Assume that the redundant forces \( \{ q_s \} \) have been determined from eq. (22). The total deflection of external loading points is then easily computed as the sum of, ONE, the deflection due to external loads acting on the "cut" structure, \( \{ q_m \} \{ P_n \} \) (ref. eq. 16) and, TWO, the deflections due to the redundant forces acting on this same cut structure, \( \{ q_m \} \{ q_s \} \) (ref. eq. 17). Thus,
\[
\{ \delta_m \} = \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ P_n \} + \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ q_s \}.
\]

Substituting from eq. (22)
\[
\{ \delta_m \} = \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ P_n \} - \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ \Gamma_m \} \left[ \begin{array}{c} q_s \\ \hline \end{array} \right] \{ P_n \}
\]
\[
= \left( \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] - \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ \Gamma_m \} \left[ \begin{array}{c} q_s \\ \hline \end{array} \right] \right) \{ P_n \}
\]

The matrix expression set off in parentheses above, giving as it does the deflections for unit values of the applied loads, is the matrix of influence coefficients. Let
\[
A_{mn} = \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] - \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \left[ \begin{array}{c} q_m \\ \hline \end{array} \right] \{ \Gamma_m \} \left[ \begin{array}{c} q_s \\ \hline \end{array} \right]
\]
so that
\[
\{ \delta_m \} = \left[ \begin{array}{c} A_{mn} \\ \hline \end{array} \right] \{ P_n \} - - - - - - - - - - - - - - - - - - (25)
\]

Example Problem 16

Determine the matrix of influence coefficients for the redundant truss of example problem 14.

Solution:

From the previous work the following matrix products were formed

\[
\begin{bmatrix} 144 & -15.0 & 0 & 0 \\ -15.0 & 38.0 & 0 & 0 \\ 0 & 0 & 38.0 & -15.0 \\ 0 & 0 & -15.0 & 144 \end{bmatrix}
\]

\[
\begin{bmatrix} 0.472 & 0.368 & 0.525 & 0.368 \\ 0.368 & 1.665 & 0.975 & 0.525 \\ 0.525 & 0.975 & 1.665 & 0.868 \\ 0.368 & 0.525 & 0.868 & 0.472 \end{bmatrix}
\]

Finally, the sum of the above two matrices gave

\[
\begin{bmatrix} 144 & -15.9 & -0.52 & -0.28 \\ -15.9 & 35.3 & -0.98 & -0.62 \\ -0.52 & -0.98 & 36.3 & -15.9 \\ -0.28 & -0.62 & -15.9 & 144 \end{bmatrix}
\]

Example Problem 17

Determine the matrix of influence coefficients for the box beam of example problem 15.

Solution:

An alternate procedure to that shown by eq. (25) was followed. The influence coefficient matrix was formed as in Chapter A-7, Art. A7.9, by the product

\[
A_{mn} = G_{m1} G_{13} G_{3n}
\]

This product was formed readily, inasmuch as
\[
G_{m1}
\]
was available from example problem 15.

The result was

\[
\begin{bmatrix} 43.63 & 35.00 & 15.22 & 124.3 & 39.7 & -39.7 \\ 35.00 & 43.63 & 124.3 & 124.3 & 39.7 & -39.7 \\ 15.22 & 124.3 & 997.8 & 994.2 & 36.7 & -36.7 \\ 124.3 & 124.3 & 997.8 & 994.2 & 36.7 & -36.7 \\ 39.7 & 39.7 & 36.7 & 36.7 & 36.8 & -36.8 \\ -39.7 & -39.7 & -36.7 & -36.7 & -36.8 & 36.8 \end{bmatrix}
\]

A8.12 Precision and Accuracy in Redundant Stress Calculations.

Matters of precision are dependent upon the number of significant figures obtained and retained in dealing with the geometry of the structure and in the care with which arithmetic operations are performed. In the discussion to follow it is assumed that all due caution is exercised with regard to the precision of the work.

Matters of accuracy have to do with the number of significant figures finally obtained in the answer as influenced by the manner of formu-
lotion of the problem. The accuracy of the result may be affected by a number of factors, two of the most important of which are discussed here.

Two factors influencing accuracy often are considered together under the heading "choice of redundants". They are:

- the number of significant figures which may be retained in the solution for the inverse of the redundant force coefficient matrix, \([Q_{rs}]\); (eq. 21).

- the magnitude of the redundant forces relative to the size of the determinate forces. These two factors are concerned respectively with the left hand side and the right hand side of eq. 21, viz.

\[ [Q_{rs}] \{ q_{s} \} = - [Q_{rn}] \{ p_{n} \}; \]

they are discussed in detail below.

- ACCURACY OF INVERSION OF \([Q_{rs}]\);

THE CONDITION OF THE MATRIX.

The characteristic of the matrix \([Q_{rs}]\) which determines the accuracy with which its inverse can be computed is its condition. The condition of the matrix is an indication of the magnitude of elements off the main diagonal (upper left to lower right) relative to those on. The smaller the relative sizes of elements off the main diagonal, the better is the condition of the matrix. A well-conditioned matrix is more accurately inverted than a poorly conditioned one. Two extreme cases are now given for illustration:

a) the diagonal matrix. Its off-diagonal elements are zero, so that it is ideally conditioned. Thus, the inverse of

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 7 & 0 \\
0 & 0 & -3
\end{bmatrix}
\]

is easily and accurately obtained as

\[
\begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1/7 & 0 \\
0 & 0 & -1/3
\end{bmatrix}
\]

b) a matrix all of whose elements are equal in each row. All the elements of the main diagonal are equal to those on. The determinant of such a matrix is zero and hence its inverse cannot be found (ref. appendix). Its condition is terrible. Example:

\[
\begin{bmatrix}
2 & 2 & 2 \\
7 & 7 & 7 \\
-3 & -3 & -3
\end{bmatrix}
\]

In the matrix \([Q_{rs}]\) the off-diagonal elements are measures of the structural cross-coupling of one redundant force with another. The strength of this cross-coupling is dependent upon the choice of redundants made in "cutting" the structure to make it statically determinate.

FOR EXAMPLE, the doubly redundant beam of Fig. A8.35(a) may be made statically determinate by "cutting" any two constraints.

\[
\begin{array}{c}
\begin{array}{c}
L \\
L \\
L \\
\end{array}
\end{array}
\]

(a)

(b)

(c)

(d)

Fig. A8.35

Fig. A8.35(b) shows the choice of generalized forces. Only two \((q_1\) and \(q_2\) are required to describe the strain energy, but the central support reactions were also given symbols as it was desired to consider them in the discussion. Then

\[
\begin{bmatrix}
2/3 & 1/6 & 0 & 0 \\
1/6 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

FIRST, suppose the beam was made determinate by selecting the support reactions \(q_3\) and \(q_4\) as redundants. The "cut structure" in this case may be visualized as the beam of Fig. A8.35(c) whose central supports have been removed. Application of unit redundant forces \(q_3 = 1\) and \(q_4 = 1\) gave

\[
\begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 1/7 & 0 \\
0 & 0 & -1/3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 4 \\
1 & 2 & 3 \\
2 & 1 & 0 \\
4 & 0 & 1
\end{bmatrix}
\]
Multiplying out,

\[
\begin{bmatrix}
\psi_s \\
\eta_s
\end{bmatrix} = \begin{bmatrix}
\psi_1 & \psi_2 \\
\eta_1 & \eta_2
\end{bmatrix} \begin{bmatrix}
\psi_f \\
\eta_f
\end{bmatrix} = L \begin{bmatrix}
3 & 7 \\
8 & 11
\end{bmatrix}
\]

\[
r, s = 3, 4
\]

The condition of this matrix is poor. Physically, a unit load at point "3" causes almost as much deflection at point "4" as at "3" itself. The cross-coupling is large.

SECOND, suppose the moments \( q_1 \) and \( q_2 \) had been chosen as redundants. The "cut structure" in this case being visualized as in Fig. A8.35(d). Application of unit redundant forces \( q_1 = 1 \) and \( q_2 = 1 \) gave

\[
\begin{array}{c|cc}
\hline
& 1 & 2 \\
\hline
1 & 1 & 0 \\
2 & 0 & 1 \\
3 & 2 & 1 \L \\
4 & 1 & 2 \L \\
\hline
\end{array}
\]

Multiplying out,

\[
\begin{bmatrix}
\psi_s \\
\eta_s
\end{bmatrix} = \begin{bmatrix}
\psi_1 & \psi_2 \\
\eta_1 & \eta_2
\end{bmatrix} \begin{bmatrix}
\psi_f \\
\eta_f
\end{bmatrix} = \frac{L}{2} \begin{bmatrix}
4 & 1 \\
3 & 1
\end{bmatrix}
\]

The condition of this redundant matrix is obviously better than that obtained with the first choice of redundants. There is less cross-coupling between the redundant forces.

Thus, the analyst, by choice of redundants, determines the condition of the matrix. The choice may be critical in the case of a highly redundant structure, for it may prove impossible to invert a large, ill-conditioned matrix with the limited number of significant figures available from the initial data. The following statements and rules-of-thumb may be useful in the treatment of highly redundant problems:

1. It is always possible to find a set of redundants for which the cross-coupling is at a minimum zero, in fact. Theoretically, then, by proper choice of redundants, the matrix \( [\psi_s] \) may be reduced to a diagonal matrix (ideally conditioned).

2. The choice of redundants which gives zero cross-coupling ("orthogonal functions") is not readily found in general. In some special structures, such as rings and frames, orthogonal redundants may be chosen simply. In most structures, however, the additional labor involved in seeking an orthogonal set of redundants is not warranted.

3. In choosing a set of redundants which will yield a well-conditioned redundant matrix, it is best to make such "cuts" as will leave a statically determinate structure retaining as many of the characteristics of the original structure as possible. Thus, one may consider that the structure of Fig. A8.35(d) retains more of the features of the original continuous-beam structure than does that of Fig. A8.35(c).

(3) The degree to which one redundant influences another (extent of cross-coupling) can be visualized by observing how much their individual unit-load diagrams overlap.

To elaborate on this last point, refer once again to the above illustrative example. The cross-coupling of \( q_3 \) with \( q_4 \) may be expected to be large if their unit-load diagrams are drawn as in Fig. A8.35(a). This deduction follows easily if it is recalled that the dummy unit-load equation for such a cross-coupling term is of the form \( \int \frac{m_2 q_3}{\eta_1} dx \), obviously large for \( m_2 \) large.

Strictly, the comparison is with the terms

\[
\int \frac{m_1 q_3}{\eta_1} dx, \int \frac{m_2 q_4}{\eta_1} dx
\]

which form the off-diagonal elements of the matrix \( [\psi_s] \).

Study of the unit-load diagrams for the redundant choice \( q_1, q_4 \) (Fig. A8.35(b)) reveals that the cross-coupling should be small here since an integral of the form \( \int \frac{m_2 q_3}{\eta_1} dx \) can have a contribution from the center span only. Hence, the \( \int \frac{m_2 q_3}{\eta_1} dx \) is obviously considerably smaller than \( \int \frac{m_1 q_3}{\eta_1} dx \) or \( \int \frac{m_2 q_4}{\eta_1} dx \). Thus, a visual inspection of Fig. A8.36 reveals that \( q_1, q_4 \) is a better choice of redundants than is \( q_3, q_4 \).

* see eq. (8) Art. A8.7.
Combinations of redundants may be employed to yield new unit-redundant stress distributions which do not "overlap" as extensively as those of the individual redundants originally chosen.

For example, suppose that in the previous illustrative problem, the choice $q_3$, $q_4$ for redundants had been made originally, leading to the unit-load diagrams of Fig. A8.37(a). Inspection of the diagrams leads one to anticipate a strong degree of cross-coupling and hence a new set of redundants is sought. Rather than return to the structure to choose new "outs", combinations of the $m_3$ and $m_4$ diagrams are looked for which will have less "overlap" and hence less cross-coupling.

It is observed by inspection that two new stress distributions which have the desired property may be formed from the $m_3$, $m_4$ diagrams by proper combination. Thus, if one half the $m_3$ diagram is subtracted from the $m_4$ diagram to form one new stress distribution and one half the $m_4$ diagram is subtracted from the $m_3$ diagram to form the other new stress distribution, the results are as shown in Fig. A8.37(b). There is obviously less "overlap" of the diagrams for these new combinations.

\[
\begin{align*}
4 \times 10 - \frac{1}{2} L & \quad 4 \times 10 - \frac{1}{2} m_4 \\
4 \times 10 - \frac{1}{2} L & \quad 4 \times 10 - \frac{1}{2} m_3
\end{align*}
\]

**Fig. A8.37**

New unit-redundant-force stress distributions (b) obtained by combining previous distributions (a).

In this way two new unknowns are introduced by linear combination. In matrix notation, the old stress distributions $[s_{1r}]$ are transformed to a new set $[s_{1p}]$ by forming

\[
[s_{1p}] = [s_{1r}] [s_{rp}]
\]

\[
\begin{bmatrix}
-\frac{1}{2} & L & 0 \\
0 & -\frac{1}{2} & L \\
1 & -\frac{1}{2} & 1
\end{bmatrix}
\]

Now form the matrix of redundant coefficients for the new unknowns (subscripts $\rho, \sigma$):

\[
[s_{1\sigma}] = [s_{1\rho}]
\]

\[
\begin{bmatrix}
\frac{L^3}{24EI} & 0 & \frac{L}{4} & 1
\end{bmatrix}
\]

The condition of this matrix is greatly improved over that obtained for $q_3, q_4$ alone (previously computed), viz.,

\[
[s_{rs}] = [q_{\rho \sigma}]
\]

(That the $[q_{\rho \sigma}]$ matrix obtained here happens to be similar to that obtained for $q_1, q_2$ in a previous example, is coincidental.)

Once a transformation has been performed, leading to a new unit redundant matrix $[s_{1p}]$, the problem may be completed in the $\rho, \sigma$ system. The appropriate equations are obtained from eqs. (14), (21), (23) and (25) simply by replacing all "$r, s" by "$\rho, \sigma". Thus

\[
\{q_1\} = [s_{1m}] \{P_m\} + [s_{1\sigma}] \{q_{\sigma}\} - - - - - - (29)
\]

where the redundants $q_\rho (= q_\sigma)$ are the solutions of
and where
\[
\mathbf{q}_{\sigma} = \mathbf{F}_{\sigma} \mathbf{q}_{1} \mathbf{F}_{\sigma} \mathbf{q}_{n} \quad \quad \quad (30a)
\]
\[
\mathbf{q}_{\sigma} = \mathbf{F}_{\sigma} \mathbf{q}_{1} \mathbf{F}_{\sigma} \mathbf{q}_{n} \quad \quad \quad (30b)
\]
The final unit load distribution is
\[
\mathbf{u}_{\text{im}} = \mathbf{u}_{\text{1m}} - \mathbf{u}_{\text{1p}} \mathbf{q}_{\sigma} ^{-1} \mathbf{u}_{\text{on}} \quad \quad \quad (31)
\]
and the matrix of influence coefficients is given by
\[
\mathbf{A}_{\text{im}} = \mathbf{A}_{\text{1m}} - \mathbf{A}_{\text{1p}} \mathbf{q}_{\sigma} ^{-1} \mathbf{A}_{\text{on}} \quad \quad \quad (32)
\]

---

**THE MAGNITUDE OF THE REDUNDANT FORCES; SIZE OF ELEMENTS IN \( \mathbf{q}_{\text{rn}} \)**

Inspection of eqs. (14) or (29) reveals that the redundant forces act in the nature of corrections to the determinate stress distributions originally assumed (the \( \mathbf{q}_{\text{im}} \)). It is apparent that if these corrections are large, then any inaccuracies in the redundants (arising from the difficulties inherent in accurately inverting \( \mathbf{q}_{\text{rs}} \)) or \( \mathbf{q}_{\sigma} \)) will have an important effect on the accuracy of the final stress distribution. Thus, as a matter of general practice, it is desirable to keep the magnitudes of the redundant forces as small as possible.

It follows immediately that one should use for the determinate stress distribution one requiring a minimum of correction, i.e., Select as the determinate stress distribution one which approximates the true stress distribution as closely as possible.

The rule given in connection with the "choice of redundants" (Rule 3, above) is an aid in making a good selection for the determinate stress distribution. If, as suggested, a determinate structure is obtained by making "cuts" which leave a system having properties similar to the original, the stress distribution obtained therein by statics should be a fair approximation to the final true stress distribution.

However, it is even more important to realize that the determinate stress distribution only need be in static equilibrium with the external applied loads and that it may be determined with the aid of any appropriate auxiliary rules, test information, or even intuitive guess-work which leads to a distribution close to the final true distribution. There is no need to set the redundant (cut) member forces equal to zero in establishing the determinate distribution. Instead, reasonable approximate values may be employed for them. (The corrections to these values become the unknown redundants!)

Mathematically, the magnitudes of the redundant forces are directly dependent upon the magnitudes of the elements in the matrices \( \mathbf{q}_{\text{rn}} \) or \( \mathbf{q}_{\text{pn}} \) on the right-hand side of eqs. (21) or (30). (The right-hand side of such a set of simultaneous equations is called the non-homogeneous part.) Thus, the relative merits of several possible determinate stress distributions may be judged by forming the matrix product \( \mathbf{q}_{\text{rn}} \) with each and comparing results.

**FOR EXAMPLE**, if the doubly redundant structure of Example Problem 3, art. A8.6 were formulated in matrix form the \( \mathbf{q}_{1\text{r}} \) and \( \mathbf{q}_{1\text{j}} \) matrices would be (see Fig. A8.38 for numbering scheme)

![Fig. A8.38](image)

Generalized force numbering scheme in illustrative problem. \( q_3 \) and \( q_4 \) selected as redundants.

\[
\mathbf{q}_{1\text{r}} = \begin{bmatrix}
432 & 0 & 0 & 0 \\
0 & 720 & 0 & 0 \\
0 & 0 & 402 & 0 \\
0 & 0 & 0 & 312
\end{bmatrix}
\]

\[
\mathbf{q}_{1\text{j}} = \begin{bmatrix}
0.806 & 1.154 \\
-1.564 & -1.729 \\
1.0 & 0 \\
0 & 1.0
\end{bmatrix}
\]

Several possible determinate stress distributions \( \mathbf{q}_{\text{im}} \) will now be tried. FIRST, the stress distribution obtained by statics alone in the "cut" structure \( \mathbf{q} = q_4 = 0 \)

\[
\mathbf{q}_{\text{im}} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

CUT
(This is certainly a poor approximation to the final stress distribution). Multiplying out

\[
\begin{bmatrix}
  q_{\text{cut}} \\
  q_{\text{cut}}
\end{bmatrix}
= \begin{bmatrix}
  G_{1} \\
  G_{2}
\end{bmatrix}
\begin{bmatrix}
  e_{1} \\
  e_{2}
\end{bmatrix} = \begin{bmatrix}
  -1126 \\
  -1246
\end{bmatrix}
\frac{1}{E}
\]

SECOND, a stress distribution in which loads of 0.40 lbs were guessed at for \( q_1 \) and \( q_2 \). (In a two times redundant structure any two forces may be assigned values arbitrarily while satisfying static equilibrium.) \( q_a, q_s \) were found by statics. Thus,

\[
\begin{bmatrix}
  e_{1m} \\
  e_{2m}
\end{bmatrix} = \begin{bmatrix}
  0.40 \\
  0.256 \\
  0.40 \\
  0.0672
\end{bmatrix}
\]

Multiplying out one finds

\[
\begin{bmatrix}
  q_{\text{cut}} \\
  q_{\text{cut}}
\end{bmatrix}
\begin{bmatrix}
  e_{1} \\
  e_{2}
\end{bmatrix} = \begin{bmatrix}
  9.49 \\
  -101
\end{bmatrix}
\]

Note that a reasonable guess at a stress distribution resulted in non-homogenous terms only one-tenth as great as those obtained by use of the "cut" distribution. The magnitudes of redundants are correspondingly reduced.

THIRD, as a matter of interest, the true stress distribution, obtained in Example Problem 3, was used. The result:

\[
\begin{bmatrix}
  e_{1m} \\
  e_{2m}
\end{bmatrix} = \begin{bmatrix}
  0.40 \\
  0.256 \\
  0.40 \\
  0.260
\end{bmatrix} \begin{bmatrix}
  q_{\text{cut}} \\
  q_{\text{cut}}
\end{bmatrix} = \begin{bmatrix}
  0.16 \\
  -0.16
\end{bmatrix}
\]

The non-homogenous terms are practically zero, as they should be*. The redundant forces would be zero also.

Example Problem 18

It is desired to increase the accuracy of the calculation in the case of the box beam, Example Problem 15. It will be assumed for this purpose that the initial data of that problem were sufficiently precise to warrant an increase of accuracy.

Solution:

The first step taken was an examination of the unit-redundant stress distributions obtained with the choice previously made of \( q_a, q_f \) and \( q_{\text{cut}} \) as redundants. The three unit-redundant stress distributions were represented graphically as in Fig. A8.39.

![Fig. A8.39](image)

Unit-redundant-force stress distributions - axial flange forces.

Inspection of the figures showed that the following combinations of distributions should give new distributions likely to have considerably less "overlap".

(1) \( 1 \times \{ e_{1a} \} - 20.87 \{ e_{1f} \} + 0 \times \{ e_{1s} \} \)

(2) \( 0 \times \{ e_{1a} \} + 1 \times \{ e_{1f} \} - 26.44 \{ e_{1s} \} \)

(3) \( 0 \times \{ e_{1a} \} + 0 \times \{ e_{1f} \} + 1 \times \{ e_{1s} \} \)

In matrix form the transformation was

\[
\begin{bmatrix}
  e_{1f} \\
  e_{1s}
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 \\
  -71.03 & 1 & 0 \\
  0 & -8332 & 1
\end{bmatrix}
\]

Using \( \{ e_{1f} \} \) as previously computed, the multiplication gave

* The demonstration here suggests a useful check upon the final result of a redundant stress calculation. After obtaining the final true stresses \( \{ G_{1m} \} \) (\( \{ G_{1n} \} \)), one forms

\[
\begin{bmatrix}
  q_{\text{cut}} \\
  q_{\text{cut}}
\end{bmatrix}
\begin{bmatrix}
  e_{1} \\
  e_{2}
\end{bmatrix} = \begin{bmatrix}
  G_{1} \\
  G_{2}
\end{bmatrix}
\begin{bmatrix}
  e_{1} \\
  e_{2}
\end{bmatrix}
\]

and compares the result element-by-element with the matrix previously computed,

\[
\begin{bmatrix}
  q_{\text{cut}} \\
  q_{\text{cut}}
\end{bmatrix} = \begin{bmatrix}
  e_{1} \\
  e_{2}
\end{bmatrix} \begin{bmatrix}
  G_{1} \\
  G_{2}
\end{bmatrix}
\]

The "true-matrix" elements ought to be zero, or nearly so, if \( \{ G_{1m} \} \) is error-free.
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[
\begin{array}{c|c|c|c}
\hline
j & 1 & 2 & 3 \\
\hline
1 & -1.00 & & \\
2 & 1.00 & & \\
3 & -1.00 & & \\
4 & -25.04 & & \\
5 & 25.04 & & \\
6 & -1.00 & -1.00 & \\
7 & 0.7103 & 1.00 & \\
8 & 1.00 & 1.00 & \\
9 & -29.38 & & \\
10 & 25.38 & & \\
11 & -1.00 & -1.00 & \\
12 & -0.8330 & 1.00 & \\
13 & -1.00 & -1.00 & \\
14 & -0.010 & -0.028 & -31.71 & \\
15 & -0.010 & -0.028 & 31.71 & \\
\hline
\end{array}
\]

\[\{\sigma_p\} = \begin{bmatrix} 0.5777 & 0.1789 & 0.0520 \\ 0.1789 & 0.2673 & 0.07322 \\ 0.0520 & 0.07322 & 0.1254 \end{bmatrix}\]

It remained to select a determinate stress distribution which would reduce the magnitude of the redundants. For this purpose, the engineering theory of bending was employed to compute stress distributions satisfying equilibrium for each application of a unit external load. The result was (refer to Example Problem 30, Art. A7.11).

\[
\begin{array}{c|c|c|c|c|c}
\hline
j & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & -10 & 10 & & & \\
2 & -10 & -10 & & & \\
3 & 2.0 & 2.0 & & & \\
4 & 2.0 & 2.0 & & & \\
5 & 2.0 & 2.0 & & & \\
6 & 0.50 & 0.15 & -0.15 & -0.25 & \\
7 & 0.15 & -0.25 & 0.15 & -0.25 & \\
8 & 0.15 & -0.25 & 0.15 & -0.25 & \\
9 & 0.15 & -0.25 & 0.15 & -0.25 & \\
10 & 0.15 & -0.25 & 0.15 & -0.25 & \\
11 & 0.15 & -0.25 & 0.15 & -0.25 & \\
12 & 0.15 & -0.25 & 0.15 & -0.25 & \\
13 & 0.15 & -0.25 & 0.15 & -0.25 & \\
14 & 0.15 & -0.25 & 0.15 & -0.25 & \\
15 & 0.15 & -0.25 & 0.15 & -0.25 & \\
\hline
\end{array}
\]

\[\{\sigma_p\} = \begin{bmatrix} 1.30 & 1.00 & & & \\
2 & 1.00 & -1.00 & & & \\
3 & 1.00 & -1.00 & & & \\
4 & 1.00 & -1.00 & & & \\
5 & 1.00 & -1.00 & & & \\
6 & 0.50 & 0.15 & 0.15 & -0.25 & \\
7 & 0.15 & -0.25 & 0.15 & -0.25 & \\
8 & 0.15 & -0.25 & 0.15 & -0.25 & \\
9 & 0.15 & -0.25 & 0.15 & -0.25 & \\
10 & 0.15 & -0.25 & 0.15 & -0.25 & \\
11 & 0.15 & -0.25 & 0.15 & -0.25 & \\
12 & 0.15 & -0.25 & 0.15 & -0.25 & \\
13 & 0.15 & -0.25 & 0.15 & -0.25 & \\
14 & 0.15 & -0.25 & 0.15 & -0.25 & \\
15 & 0.15 & -0.25 & 0.15 & -0.25 & \\
\end{bmatrix}
\]

\[\{\sigma_p\} = \begin{bmatrix} 0.0003 & 0.0005 & 0.0008 & 0.0008 & 0.0002 \\
0.0005 & 0.0008 & 0.0008 & 0.0008 & 0.0002 \\
0.0008 & 0.0008 & 0.0008 & 0.0008 & 0.0002 \\
0.0008 & 0.0008 & 0.0008 & 0.0008 & 0.0002 \\
0.0002 & 0.0002 & 0.0002 & 0.0002 & 0.0002 \end{bmatrix}
\]

The new redundant coefficient matrix \( [\sigma_{\infty}] \) was obtained by multiplying out per eq. (30b).

\[
\begin{bmatrix} 0.2584 & -0.0132 & 0.0000175 & & & \\
-0.0132 & 0.2586 & -0.03124 & & & \\
0.0000175 & -0.03124 & 0.1254 & & & \\
& & & & & \\
\end{bmatrix}
\]

This matrix is very well conditioned, being a considerable improvement over that of \([I_{rs}]\), found originally in Example Problem 15, viz.,

Next the matrix \([\sigma_{pn}]\) was obtained by multiplying out per eq. (30b).

\[
\begin{bmatrix} 3495 & -3495 & -1841 & 1841 & 0 & 0 \\
1167 & -1167 & 1554 & -1554 & -1460 & 1460 \\
1082 & -1082 & 1445 & -1445 & -1802 & 1802 \\
\end{bmatrix}
\]

This result compared very favorably with \([\sigma_{rn}]\) previously obtained, the elements being from one- to one-tenth as large.

The solution may be carried to completion in the "p, r, s" system using the matrices \([\sigma_{p}], [\sigma_{\infty}], [\sigma_{pn}]\) and \([\sigma_{rn}]\) in eqs. (31) and (32).


The thermal stress problem is conveniently formulated in matrix notation by an extension of the techniques presented above.

First, consider eq. (21), written in the form

\[
\begin{bmatrix} 3495 & -3495 & -1841 & 1841 & 0 & 0 \\
1167 & -1167 & 1554 & -1554 & -1460 & 1460 \\
1082 & -1082 & 1445 & -1445 & -1802 & 1802 \\
\end{bmatrix}
\]
\[
\left[ q_{rs} \right] \{ q_s \} + \left[ q_{rn} \right] \{ p_n \} = 0.
\]

In light of the physical interpretations given to \( q_{rs} \) and \( q_{rn} \), the equation is seen to be a matrix-form statement of the condition for continuity at the redundant cuts, viz., the displacement at each cut caused by the redundant forces plus the displacement at each cut caused by the external loads, must be equal to zero. To modify this equation for thermal stresses, the appropriate expressions for thermal displacements at the cuts must be added. Following the argument used in Art. A8.9 one writes

\[
\{ \delta_{tt} \} + \left[ q_{rs} \right] \{ q_s \} + \left[ q_{rn} \right] \{ p_n \} = 0
\]

where \( \delta_{tt} \) is the displacement at the \( r \)th cut due to thermal straining in the determinate structure. The explicit form for this term will be derived below. Rewritten,

\[
\left[ q_{rs} \right] \{ q_s \} = - \left[ q_{rn} \right] \{ p_n \} - \{ \delta_{tt} \} \quad - - - - - - (33)
\]

Eq. (33) is a modified form of eq. (21), giving as its solution the redundant forces in an indeterminate structure under the application of both external loads and a temperature distribution.

To derive an explicit expression for \( \delta_{tt} \), the virtual work concept may be employed to advantage. Thus, following the argument of the "virtual work" derivation for deflections (Arts. A7.7, A7.8), the thermal deflection at the \( r \)th redundant cut must be equal to the total internal virtual work done by the \( r \)th redundant force virtual stresses (due to a unit \( r \)th-redundant force) moving through distortions caused by thermal strains.

Since the internal stress distribution is expressed in terms of the internal generalized forces \( q_1, q_2 \), it is convenient to employ these \( q \)'s in writing the virtual work of straining. If one lets \( \delta_{tt} \) be the displacement of internal generalized force \( q_1 \) due to thermal straining, then the virtual work done by a single generalized force is \( q_1 \delta_{tt} \). The quantity \( \delta_{tt} \) will be called the member thermal distortion. The total virtual work throughout the structure is obtained by summing, giving the deflection at the \( r \)th cut as the matrix product

\[
\delta_{tt} = \left[ q_{rs} \right] \{ q_s \} \quad - - - - - - (34)
\]

where \( q_{rs} \) is the value of the \( q_1 \) due to a unit (virtual) load at cut \( r \). Note that the term \( \delta_{tt} \) may consist of the sum of several contributions should \( q_1 \) act on more than one member.

Because \( \delta_{tt} \) is desired at each of the redundant cuts, eq. (34) is expanded by writing

\[
\{ \delta_{tt} \} = \left[ q_{rs} \right] \{ \delta_{tt} \} - - - - - - - - - - - (35)
\]

\( \left[ q_{rs} \right] \) is, of course, the transpose of \( \left[ q_{rs} \right] \), the unit redundant force stress distribution in the determinate structure. Substitution of eq. (35) into eq. (33) gives

\[
\left[ q_{rs} \right] \{ q_s \} = - \left[ q_{rn} \right] \{ p_n \} - \left[ q_{rs} \right] \{ \delta_{tt} \} \quad - - - - - - (36)
\]

Solution of eq. (36) gives the values of the redundants \( q_s \), after which the problem may be completed in the usual fashion, viz,

\[
\{ q_1 \} = \left[ q_{rs} \right] \{ q_s \} \quad - - - - - - (37)
\]

It is obvious that the use of combinations of redundants (the "p, q" system of Art. A8.12) is possible. One makes a direct substitution of \( \left[ q_{rs} \right] \) for \( \left[ q_{rs} \right] \) and \( \left[ q_{rn} \right] \) for \( \left[ q_{rn} \right] \), etc. into eq. (36).

MEMBER THERMAL DISTORTIONS

It remains to establish the forms for \( \delta_{tt} \).

Thermal strains on an infinitesimal element of homogeneous material can cause uniform normal extensions only, so that no shear strains develop. Hence only normal (as opposed to shear) virtual stresses need be considered in computing the internal virtual work. Note that normal stresses associated with flexure must be included. It follows that only virtual work in axially loaded bars and in beams in flexure need be considered. Hence \( \delta_{tt} \) is zero for all \( q_1 \) which are shear flows on panels or torques on shafts.

BARS

The general expression for the virtual work done by virtual axial loads \( u \) in a bar under varying temperature \( T \) is

\[
W = \int u \cdot \alpha \ T \ dx
\]

where \( \alpha \) is the material thermal coefficient of expansion.

* It will be convenient later to designate by \( \delta_{tt} \) the solution to eq. (36) when the mechanical loads \( p_n \) are zero, the stresses in such a case being purely "thermal".
Several specific cases are now treated.

A bar under linearly varying load with linearly varying temperature is shown in Fig. A8.40.

In this case

\[ u = q_j + \frac{q_1 - q_j}{L} x \]

\[ T = T_j + \frac{T_1 - T_j}{L} x \]

Then, assuming a constant

\[ \omega = \int_0^L \left( q_j + \frac{q_1 - q_j}{L} \right) \left( T_j + \frac{T_1 - T_j}{L} x \right) dx \]

\[ = aL \left( \frac{2T_1 + T_j}{6} q_j + aL \frac{T_1 + 2T_j}{6} q_j \right) \]

This expression may be put in the form

\[ \omega = q_j \Delta_{IT} + q_j \Delta_{JT} \]

where

\[ \Delta_{IT} = aL \frac{2T_1 + T_j}{6} \]

\[ \Delta_{JT} = aL \frac{T_1 + 2T_j}{6} \]

Note that variation in the cross-sectional area of the bar does not affect the distortions \( \Delta_{IT}, \Delta_{JT} \).

The alternate choice of generalized forces for the bar under varying axial load is shown in Fig. A8.41. By a derivation similar to that above one finds

\[ \omega = q_j \Delta_{IT} + q_j \Delta_{JT} \]

where

\[ \Delta_{IT} = aL \frac{T_1 + T_j}{2} \]

\[ \Delta_{JT} = aL \frac{T_1 + 2T_j}{6} \]

The simpler cases of uniform load \( q_j = u = \) constant and uniform \( T \) (\( T_1 = T_j \)) follow immediately by specialization of the above forms. For example, for a bar under constant load \( q_j = q_j \), and constant temperature, \( T_1 = T_j = T \), one has \( \Delta_{IT} = aLT \).

For a beam the general form of expression for the virtual work done during thermal straining by a top-to-bottom-surface temperature difference \( \delta T \), varying linearly over the beam depth, is (see Art. A7.8, Ex. Prob. 24).

\[ \omega = a \int m \cdot \frac{\delta T}{h} dx \]

where

\[ m = \text{virtual moment (positive for compression on top fiber)} \]

\[ \delta T = T_{\text{BOTTOM}} - T_{\text{TOP}} \]

\[ h = \text{beam depth}. \]

Applied to the case of Fig. A8.42 one gets

\[ \omega = a \int_0^L \left( q_j + \frac{q_1 - q_j}{L} x \right) \left( T_j + \frac{\delta T_1 - \delta T_j}{L} x \right) dx \]

\[ = aL \left( \frac{2\delta T_1 + \delta T_j}{6} q_j + aL \frac{\delta T_1 + 2\delta T_j}{6} q_j \right) \]

\[ \delta T_j \]

\[ \delta TT \]

\( h = \text{constant} \)

Fig. A8.42

or

\[ \omega = q_j \Delta_{IT} + q_j \Delta_{JT} \]

where

\[ \Delta_{IT} = \frac{aL}{h} \left( \frac{2\delta T_1 + \delta T_j}{6} \right) \]

\[ \Delta_{JT} = \frac{aL}{h} \left( \frac{\delta T_1 + 2\delta T_j}{6} \right) \]

Special forms of the thermal distortion expressions for beams of varying depth may be derived readily as required.

Example Problem 19

The upper surface of the beam of Fig. A8.43 is subjected to a temperature \( \delta T \) above that of the lower surface, varying linearly as shown (i.e.,

\[ \delta T_0 \]

Fig. A8.43

\[ \delta T \]

\( h = \text{constant} \)

Fig. A8.43
the temperatures are equal at the left end and
differ by OT₀ at the right end). Determine the
center reactions assuming a is constant.

Solution:

In the illustrative example of Art. A8.12
this structure was analyzed by employing as
generalized internal forces the bending moments
Q₁ and Q₄ over the central supports (see Fig.
A8.35). The two central reactions were denoted
by Q₄ and Q₄. The matrix of redundant coef-
fi cients, considering Q₁ and Q₄ as redundants (the
better choice, it will be recalled) was (ref.
Art. A8.12)

\[
\begin{bmatrix}
1 & 1 \\
1 & 4
\end{bmatrix}
\]

The unit redundant load distribution for Q₁ = 1
and Q₄ = 1 was

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 0 \\
2 & 0 & 1 \\
3 & -2 & 1 \\
4 & 1 & -2
\end{bmatrix}
\]

Member thermal distortions were computed. (Note
that OT was negative according to the conven-
tion adopted earlier).

\[
\Delta T = \frac{dL}{h} \left( \frac{2 \times \frac{1}{3} OT₀ + \frac{1}{3} OT₄}{5} \right)
\]

\[
= -\frac{dL}{h} \cdot OT₀
\]

\[
\Delta r = \frac{dL}{h} \left( \frac{2 \times \frac{1}{3} OT₀ + \frac{1}{3} OT₄}{5} \right) + \frac{dL}{h} \left( \frac{2 \times \frac{1}{3} OT₄}{5} \right)
\]

\[
= \frac{-2 \frac{dL}{h} \cdot OT₀}{5}
\]

Then from eq. (36)

\[
\begin{bmatrix}
1 & 1 \\
1 & 4
\end{bmatrix}
\begin{bmatrix}
Q₁ \\
Q₄
\end{bmatrix}
= \begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{-dL}{3h} \cdot OT₀ \\
\frac{-2dL}{3h} \cdot OT₄
\end{bmatrix}
\]

Solving

\[
\begin{bmatrix}
Q₁ \\
Q₄
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{-2dL}{3h} \cdot OT₀ \\
\frac{-2dL}{3h} \cdot OT₄
\end{bmatrix}
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

The final stress distribution was

\[
\begin{bmatrix}
Q₁ \\
Q₄
\end{bmatrix}
= \frac{EI \cdot a \cdot OT₀}{h} \begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

\[
= 0 \begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

Thus the reactions were

Q₁ = 0.399 \( \frac{EI \cdot a \cdot OT₀}{h} \)

Q₄ = -1.60 \( \frac{EI \cdot a \cdot OT₀}{h} \) (negative indicates DOWN)

Example Problem 20

The symmetric sheet-stringer panel of Fig.
A8.44(a) is to be analyzed for thermal stresses
developed by heating the two outside stringers
to a uniform temperature T above the center
stringer. Assume G = 0.338E.

Solution:

The panel was divided for convenience into
three bays. The numbering and placing of gener-
alized forces is shown in Fig. A8.44(b). The
Transverse members (ribs, not shown on figure)
were considered rigid in their own planes - a
satisfactory assumption for symmetric panels.
Because of symmetry only one half the panel was
handled. All member flexibility coefficients and
thermal distortions were doubled where appro-
priate.

The matrix of flexibility coefficients was
set up as (VOIDS DENOTE ZEROS)
The matrix of member thermal distortions was

\[
\{ \delta_{1T} \} = a \ T
\]

Here, for example \( \delta_{1T} = 2 \times 2 \times 10 = 40 \) (doubled once because \( q_1 \) acts on two stringer ends and doubled again to account for the other half of the panel).

![Fig. A8.44](image)

The structure was three times redundant. Stringer loads \( q_s \), \( q_e \) and \( q_r \) were selected as redundants. Fig. A8.45 shows the unit redundant load sketches for \( q_s = 1 \), \( q_e = 1 \) and \( q_r = 1 \).

Then

\[
\begin{bmatrix}
-0.5 & 0 & 0 \\
0 & -0.5 & 0 \\
0 & 0 & -0.5 \\
1.0 & 0 & 0 \\
0 & 1.0 & 0 \\
0 & 0 & 1.0 \\
-0.025 & 0 & 0 \\
0.025 & -0.025 & 0 \\
0 & 0.025 & -0.025
\end{bmatrix}
\]

Multiplying out per eq. 18,

\[
[q_{rs}] = \frac{1}{5}
\begin{bmatrix}
172.2 & 13.8 & 0 \\
13.8 & 172.2 & 13.8 \\
0 & 13.8 & 86.1
\end{bmatrix}
\]

and

\[
[q_{re}] = \begin{bmatrix} 40 \\ 40 \\ 40 \\ 40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

Then the redundant equations, per eq. (36), were

\[
\begin{bmatrix}
172.2 & 13.8 & 0 \\
13.8 & 172.2 & 13.8 \\
0 & 13.8 & 86.1
\end{bmatrix} [q_{e}] = 10 a \ T
\]

Solving,

\[
[q_{e}] = a \ T [0.1082, 0.0944, 0.1002]
\]

Finally, the complete thermal stress distribution was

\[
[q_{T}] = E [q_{e}] = a \ T E \times 10^{-8}
\]

\[
\begin{bmatrix}
-54.1 \\
-49.7 \\
-50.1 \\
108.2 \\
-54.1 \\
-49.7 \\
-50.1 \\
108.2 \\
-54.1 \\
-49.7 \\
-50.1 \\
108.2 \\
-54.1 \\
-49.7 \\
-50.1 \\
108.2
\end{bmatrix}
\]
The result compares favorably with "exact" solutions made under the same assumptions (ref. NACA TN 2240) as far as the stringer loads are concerned. The shear flow result is not very satisfactory, primarily due to the use of too few "bays" in this analysis.

Example Problem 21

The uniform four-flange box beam of Fig. A8.46(a) is to be analyzed for the thermal stresses developed upon heating one flange to a temperature T, uniform spanwise, above the other three flanges.

Solution:

The beam was divided into four equal bays giving a four times redundant problem. Four self-equilibrating (zero-resultant) independent stress distributions were taken as the unknowns, these being shown in Fig. A8.46(b). Such zero-resultant stress distributions are the only ones possible in a structure having no applied loads*.

The matrix of member flexibility coefficients was formed by collecting coefficients from the several members. Unit redundant stress distributions were prepared, taking q₁, q₂, q₃, and q₄ as the redundants and setting these equal to unity successively.

\[
\begin{array}{c|c|c|c|c}
1 & 1 & 1 & 1 \\
L/2 & L/2 & L/2 & L/2 \\
L/2 & L/2 & L/2 & L/2 \\
L/2 & L/2 & L/2 & L/2 \\
\end{array}
\]

\[
\begin{align*}
\left[ G_t \right] &= L \left( \begin{array}{c}
.657^* & .1647^* \\
.1647^* & .657^* \\
.1647^* & .657^* \\
.1647^* & .657^* \\
\end{array} \right)
\end{align*}
\]

where \( k^* = Gt / AE (b + c) \)

Multiplying out per eq. (18),

\[
\begin{align*}
\left[ G_t \right] &= \frac{Gt}{L (b + c)} \\
\end{align*}
\]

For a specific case let \( k^* L^2 = 1 \). The inverse was computed to be

\[
\begin{align*}
1.070 & = 1.5291 - 2337 - .06657 \\
- .5291 & = 1.366 - .3613 - .1018 \\
- .2337 & = 1.497 - .1934 \\
- .06657 & = 1.934 - 1.792
\end{align*}
\]

Member thermal distortions \( \Delta_T \) were computed for loads \( q_1, q_2, q_3 \) and \( q_4 \) and were collected from the one heated flange only.

\[
\begin{align*}
\left\{ \Delta_T \right\} &= \frac{q L T}{G} \\
\end{align*}
\]

(Note that if two adjacent flanges are heated equally, one must set the corresponding \( \Delta_T \) equal to zero; this because the virtual loads being of opposite sign in adjacent flanges, the virtual work must cancel).

Multiplying out:

\[
\begin{align*}
\left[ G_t \right] \left\{ \Delta_T \right\} &= \frac{q L T}{G} \\
\end{align*}
\]

Then the solution to eq. (26) was written as
\[ \{ \sigma_{\text{ST}} \} = - \left[ \bar{q}_{\text{ST}} \right] \left[ \sigma_{\text{T}} \right] \{ \Delta_{\text{T}} \} \]

which, when multiplied out gave

\[ \{ \sigma_{\text{ST}} \} = - \frac{G_{\text{ST}} \sigma_{L}}{18} \begin{bmatrix} 4.079 \\ 1.938 \\ 3.008 \\ 3.437 \\ 3.588 \end{bmatrix} \left( b + c \right) \]

Finally the complete set of thermal stresses were (still for the case \( L^2K = 1 \))

\[ \{ q_{\text{T}} \} = \left[ \bar{q}_{\text{T}} \right] \{ q_{\text{ST}} \} = - \frac{G_{\text{ST}} \sigma_{L}}{18} \left( b + c \right) \begin{bmatrix} 4.079 \\ 2.039 \\ 1.938 \\ 3.008 \\ 3.437 \\ 3.588 \end{bmatrix} \]

The result compares favorably with an exact solution (NACA TN 2240) insofar as flange stresses are concerned, an error of less than 1 percent being present at the root. Shear stress values in the sheet are less satisfactory due to the (relatively) crude assumption of constant shear in each beam quarter.


A. STATICALLY DETERMINATE STRUCTURES

The problem of the thermal deflections of a statically determinate structure was considered earlier in Art. A7.6 in non-matrix form. It should be apparent from the derivation, that the matrix method presented for the calculation of redundant-cut thermal deflections may be applied equally well to the problem of computing the thermal deflections of external points of a determinate structure.

Thus

\[ \{ \Delta_{\text{mT}} \} = \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} \]

where \( \Delta_{\text{mT}} \) is the thermal deflection of external point \( m \), and \( \left[ \bar{q}_{\text{mT}} \right] \) is the transpose of \( [q_{\text{T}}] \), the unit-applied load stress distribution (compare with eq. 36).

B. REDUNDANT STRUCTURES

In the case of the redundant structure, additional strains are present due to the thermal stresses set up; the effect of these strains upon the deflection of external points must be included in the calculation.

The appropriate equation is most easily derived by visualizing the action in two stages. FIRST, the redundant structure is "cut" making it determinate, after which the temperature distribution is applied producing thermal deflections

\[ \{ \Delta_{\text{mT}} \} = \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} \]

at the external points (compare with eqs. (36) and (37)). Simultaneously the redundant cuts will experience relative displacements.

SECOND, the redundant cuts are restored to zero displacement by the application of redundant forces (this problem was solved in Art. A8.13). The \( q_{\text{mT}} \) are given by eq. (36); they produce additional deflections at points \( m \)

\[ \{ \Delta_{\text{mT}} \} = \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} \]

The total deflection of point \( m \) is then

\[ \{ \Delta_{\text{mT}} \} = \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} + \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} \]

But

\[ \left[ \bar{q}_{\text{mT}} \right] \left[ \bar{q}_{\text{mT}} \right] = \left[ \bar{q}_{\text{mT}} \right] \left[ q_{\text{T}} \right] \left[ q_{\text{T}} \right] \]

Therefore

\[ \{ \Delta_{\text{mT}} \} = \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} + \left[ \bar{q}_{\text{mT}} \right] \{ q_{\text{T}} \} - - - \] (36)

The matrix quantity in parentheses is the total strain (thermal plus "mechanical").

For definite reasons the equation for thermal deflections has been left in the form of eq. (36) rather than the more polished form which might be obtained by substitution from eq. (36). First, the \( q_{\text{T}} \), the thermal stresses, will probably have been solved for previously and will be readily available in explicit form. Second, and far more important from a labor saving standpoint, the unit load distribution \( [q_{\text{T}}] \) (whose transpose is used in eq. 36) may be any convenient stress distribution satisfying statics in the simplest of "cut" structures. One need not even use the same \( [q_{\text{T}}] \) distribution (and same choice of "cuts") as employed in the redundant thermal stress calculation; a more convenient choice of cuts may be employed. In principle, any stress distribution statically equivalent to the unit applied load \( [q_{\text{T}}] \) may be used for \( [q_{\text{T}}] \) in eq. (36). (See p. A8.9).
Example Problem 22

Compute the rotation occurring at the right hand end of the beam of Fig. A8.43.

Solution:

To compute the rotation a unit couple was applied (positive counterclockwise) at the right hand end. An additional generalized force, called \( q_s \), also was added at that point. Then,

\[
\Delta_{ST} = \frac{q_s}{h} \left( \frac{2L \delta T_o + 25T_o}{6} \right) = \frac{4aL \delta T_o}{6h}
\]

and, using some results from Example Prob. 19,

\[
\left\{ \Delta_{IT} \right\} = \frac{aL \delta T_o}{6h} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}
\]

Note that \( q_s \) and \( q_s \), the intermediate support reactions, were omitted from consideration. They do not enter into any expression for the internal virtual work of the structure; or, equally, they are not used to describe the strain energy of the structure. Hence they are not included in writing the total strain. (Their \( \Delta_{IT} \) are zero.)

The member flexibility coefficient matrix was

\[
\left[ q_{11} \right] = \frac{L}{6EI} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}
\]

From Example Problem 19, the true thermal stress distribution was

\[
\left\{ q_1 \right\} = \frac{EI \alpha \delta T_o}{h} \begin{bmatrix} 0.267 \\ 0.533 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \end{bmatrix}
\]

\( q_s \) was zero by inspection.

To determine \( \left[ z_{1m} \right] \), a unit couple \( q_s = 1 \) was applied. Taking as the "cut" structure one where \( q_{1s} = 0 \) one has simply

\[
\left\{ z_{1m} \right\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \left[ z_{m1} \right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Then substituting into eq. (38),

\[
\left[ q_{1m} \right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left( \frac{aL \delta T_o}{6h} \right) \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 410 \\ 012 \\ 012 \end{bmatrix} \frac{EI \alpha \delta T_o}{h} \begin{bmatrix} 0.267 \\ 0.533 \\ 0 \end{bmatrix}
\]

It is apparent that the values of the redundant moments \( q_i \), \( q_s \) could have been chosen arbitrarily in \( \left[ z_{1m} \right] \) above, without affecting the result. Here, clearly, \( \left[ z_{1m} \right] \) for any "cut" structure visualized will lead to the same result, \( q_s \) being equal to unity in all cases.

Example Problem 23

Compute the axial movement of the free end of the central stringer of the panel of Fig. A8.44.

Solution:

An additional generalized force, \( q_{1s} \), was added axially to the free end of the central stringer (ref. Fig. A8.44b). Then

\[
\left[ z_{1m} \right] = \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}
\]

Using the \( q_{1m} \) as obtained in Example Problem 20 (with \( q_{1s} = 0 \)), the following product was formed:

\[
\left[ q_{1m} \right] = \frac{aL \delta T_o}{h} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 410 \\ 012 \\ 012 \end{bmatrix} \begin{bmatrix} 0.267 \\ 0.533 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} aL \delta T_o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.289 \\ -0.289 \end{bmatrix}
\]

\[
\delta_T = -0.289 \frac{aL \delta T_o}{h}
\]

177.6 177.6 177.6
44.4 44.4 44.4
44.4 44.4 44.4

31,300 31,300 31,300
22.2 22.2 22.2
18.6 22.2 22.2
18.6 22.2 22.2

\[
\begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}
\]
Then, substituting into eq. (33),

\[ \delta_m = a T \begin{bmatrix} g_{1m} \\ g_{2m} \\ \vdots \end{bmatrix} = \begin{bmatrix} 28.19 \\ 25.35 \\ 13.34 \\ 11.51 \\ 13.46 \\ 6.556 \\ -84.24 \\ 6.833 \\ 0.9228 \\ 2.402 \end{bmatrix} \]

It remained to find the determinate stress distribution for \( \delta_m = 1 \).

As the determinate distribution \( \{g_{1m}\} \), the stresses due to a unit load \( \delta_m = 1 \) were computed in the "cut" structure obtained when \( q_1 = q_2 = q_3 = 0 \). (Note that this is a different choice of redundant cuts from that employed in computing the thermal stresses in Example Problem 20.) Thus

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \]

and finally,

\[ \delta_m = a T \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 28.19 \\ 25.35 \\ 13.34 \\ 11.51 \\ 13.46 \\ 6.556 \\ -84.24 \\ 6.833 \\ 0.9228 \\ 2.402 \end{bmatrix} \]

\[ = 34.3 a T \text{ (ANSWER)} \]

As a matter of interest, a different unit stress distribution was employed with a different choice of "cuts". If the forces \( q_1, q_2, \) and \( q_3 \) are set equal to zero ("cut"), application of a unit load \( q_4 = 1 \) gives (writing the transpose of \( \{g_{1m}\} \))

\[ \begin{bmatrix} g_{1m} \\ g_{2m} \end{bmatrix} = \begin{bmatrix} .50 & .50 & .50 & 0 & 0 & 0 & .025 & 0 & 0 & 1 \end{bmatrix} \]

Then multiplying out for the thermal deflection:

\[ \delta_m = 34.3 a T \text{ (same answer).} \]

It is apparent from the above result that the simplest determinant ("cut") structure should be used to compute \( \{g_{1m}\} \); it is completely adequate.

**CLOSURE**

The general thermal stress problem is complicated by the fact that the material properties \( E, G, \) and \( \alpha \) vary with temperature. The problem created thereby is primarily one of bookkeeping - computing the member flexibility coefficients \( (a_{1j}) \) and the member thermal distortions \( (\Delta_{jT}) \) for a structure whose properties vary from point to point with the temperature. The variations of \( E, G, \) and \( \alpha \) with \( T \) will, of course, have to be known from test data.

Two additional complications, not considered here, are the lowering of the yield point with heating (and the attendant increased likelihood of developing inelastic strains) and the phenomenon of "creep" (the time-dependent development of inelastic strains under steady loading).

Should it prove necessary to analyze for thermal stresses under more than one temperature distribution, the member thermal distortion matrix \( \{\Delta_{jT}\} \) may be generalized easily into a rectangular form such as

\[ \begin{bmatrix} \Delta_{1R} \end{bmatrix} = [C_{1j}] [T_{jR}] \]

where \( \Delta_{1R} \) is the member thermal distortion associated with force \( q_j \) from thermal loading condition \( R \). The matrix \( [C_{1j}] \) of member thermal coefficients consists of the constant coefficients in the \( \Delta_{1T} \) expressions previously presented, while \( T_{jR} \) would be the temperature associated with \( q_j \) for condition \( R \). (Compare with eq. (26b), Art. A7.11).
A8.15 Problems.

Note: Problems (1) through (9) below may be worked by either the Least Work or Dummy Unit Load Methods. The student will be well advised to try some problems both ways for comparison.

(1) Determine the load in all the members of the loaded truss shown in Fig. (a). Values in ( ) on members represent the cross-sectional area in sq. in. for that member. All members of same material.

(2) For the structure in Fig. (b), determine the load in each member for a 700# load at joint B. Areas of members are given by the values in ( ) on each member. All members made of same material.

(3) For the loaded truss in Fig. c, determine the axial load in all members. Values in parenthesis adjacent to members represent relative areas. E is constant for all members.

(4) For structure in Fig. d, calculate the axial loads in all the members. Values in parenthesis adjacent to each member represent relative areas. E is constant or same for all members.

(5) For the "King post" truss in Fig. e, calculate the load in member BD. Members AB, BC and BD have area of 2 sq. in. each. The continuous member ABC has an area of 9.25 sq. in. and moment of inertia of 215 in^4. E is same for all members.

(6) In Fig. f, AB is a steel wire both 0.50 sq. in. area. The steel angle frame CBD has a 4 sq. in. cross section. Determine the load in member AB. E = 30,000,000 psi.

(7) In Fig. g find the loads in the two tie rods BD and CE. $A_{ac} = 72$ in.; $A_{bc} = 0.05$ sq. in. $A_{de} = 0.15$ sq. in. E is same for all members.

(8) For the structure in Fig. h, determine the reactions at points A, B. Members CE and CD are steel tie rods with areas of 1 sq. in. each. Member AB is a wood beam with 12" x 12" cross section. $E_{steel} = 30,000,000$ psi. $E_{wood} = 1,300,000$ psi.

(9) For the structure in Fig. i, determine the axial loads, bending moments and shears in the various members. The structure is continuous at joint D. Members AB, BC are wires. The member areas are $A_{AB} = 1.2$, $BC = 0.6$; $CD = 6.0$; BDE = 10.0. The moment of inertia for members CD = 60.0 in.^4; for BDE = 140 in.^4.

(10) Re-solve Example Problem 10, p. A8.15 using as redundants the two restraints at one end (couple and transverse force). Solved in this way the problem is doubly redundant as no advantage is made of the symmetry of the structure. (11) Add two additional members, diagonals FB and EC (each with areas 1.0 in.^2) to the truss of Fig. A7.35, chapter A-7. Find the matrix of
influence coefficients.

\[
[A_{mn}] = \begin{bmatrix}
21.8 & 27.2 & 27.5 & 18.3 \\
27.2 & 68.5 & 65.4 & 26.4 \\
27.5 & 65.4 & 68.4 & 26.2 \\
18.3 & 26.4 & 26.2 & 20.5 \\
\end{bmatrix}
\]

(12) Re-solve the doubly redundant beam of Example Problem 8, page A8.3 by matrix methods. The redundant reactions should be given "q" symbols. (See Example Problem 13a, page A8.20).

(13) Re-solve Example Problem 5, page A8.12 by matrix methods. For simplicity, make your choice of generalized forces including those designated as X and Y in the example so that Figs. A8.21 and A8.22 can be used to give the \( \delta_{11} \) loadings.

(14) By matrix methods re-solve Example Problem 4, p. A8.12 using 3 equal bay divisions along the panel (3 times redundant). Use the same structural dimensions as in Example Problem 20, p. A8.26. Compare the results with those obtained from the formulas developed in Example Problem 4.

(15) Show that the matrix equation eq. (21) is modified to cover the initial stress problems of Art. A8.8 by writing

\[
\begin{bmatrix}
\delta_{rs} \\
\delta_{rn} \\
\end{bmatrix} \begin{bmatrix}
q_s \\
q_n \\
\end{bmatrix} = \begin{bmatrix}
\delta_{rn} \\
\delta_{rl} \\
\end{bmatrix} \begin{bmatrix}
P_n \\
q_r \\
\end{bmatrix} - \begin{bmatrix}
\delta_{rs} \\
\delta_{rl} \\
\end{bmatrix} \begin{bmatrix}
q_t \\
\end{bmatrix} - \begin{bmatrix}
\delta_{rs} \\
\delta_{rl} \\
\end{bmatrix} \begin{bmatrix}
q_r \\
\end{bmatrix}
\]

where \( q_t \) is the initial imperfection associated with force \( q_r \). Refer to the argument leading to eq. (11) of Art. A8.8.


(19) For the doubly symmetric four flange box beam of Example Problem 15, p. A8.24, determine the redundant stresses \( q_s \), \( q_r \), and \( q_{1s} \) if one flange is heated to a temperature \( T \), uniform spanwise, above the remainder of the structure.

\[
\begin{bmatrix}
q_s \\
q_r \\
q_{1s} \\
\end{bmatrix} = \begin{bmatrix}
-\frac{E \cdot a \cdot T}{10^5} \\
1599 \\
741 \\
\end{bmatrix}
\]

REFERENCES.

See references at the end of Chapter A-7.

Douglas DC-8 airplane. Photograph showing simulated aerodynamic load being applied to main entrance door of fuselage test section.
DOUGLAS DC-8 AIRPLANE. An outboard engine pylon mounted on a section of wing for static and flutter tests. The steel box represents the weight and moment of inertia of the engine.
CHAPTER A9
BENDING MOMENTS IN FRAMES AND RINGS
BY
ELASTIC CENTER METHOD

A9.1 Introduction

In observing the inside of an airplane fuselage or seaplane hull one sees a large number of structural rings or closed frames. Some appear quite light and are essentially used to maintain the shape of the body metal shell and to provide stabilizing supports for the longitudinal shell stringers. At points where large load concentrations are transferred between body and tail, wing, power plant, landing gear, etc., relatively heavy frames will be observed. In hull construction, the bottom structural framing transfers the water pressure in landing to the bottom portion of the hull frames in turn transfers the load to the hull shell.

In general the frames are of such shape and the load distribution of such character that these frames or rings undergo bending forces in transferring the applied loads to the other resisting portions of the airplane body. These bending forces produce frame stresses in general which are of major importance in the strength proportioning of the frame, and thus a reasonable close approximation of such bending forces is necessary.

Such frames are statically indeterminate relative to internal resisting stress and thus consideration must be given to section and physical properties to obtain a solution of the distribution of the internal resisting forces.

General Methods of Analysis:

There are many methods of applying the principles of continuity to obtain the solution for the redundant forces in closed rings or frames and bents. The author prefers the one which is generally referred to as the "Elastic Center" method and has used it for many years in routine airplane design. The method was originated by Muller-Breslau. The main difference in this method as compared to most other methods of solution is that the redundant forces are assumed acting at a special point called the elastic center of the frame which gives resulting equations for the redundants which are independent of each other.

Assumptions

In the derivations which follow the distortions due to axial and shear forces are neglected. In general these distortions are small compared to frame bending distortions and thus the error is small.

In computing distortions plane sections are assumed to remain plane after bending. This is not strictly true because the curvature of the frame changes this linear distribution of bending stresses on a frame cross-section. Corrections for curvature influence are given in Chapter A13.

Furthermore it is assumed that stress is proportional to strain. Since the airplane stress analyst must calculate the ultimate strength of a frame, this assumption obviously does not hold with heavy frames where the rupturing stresses for the frame are above the proportional limit of the frame material.

This chapter will deal only with the theoretical analysis for bending moments in frames and rings by the elastic center method. Practical questions of body frame design are covered in a later chapter.

The following photographs of a portion of the structural framing of the hull of a seaplane illustrate both light and heavy frames.

---

A9.2 Derivation of Equations. Unsymmetrical Frame.

Fig. A9.1 shows an unsymmetrical curved beam fixed at ends (A) and (B) and carrying some external loading $P_1$, $P_2$, etc. This structure is statically determinate to the third degree because the reactions at (A) and (B) have three unknown elements, namely, magnitude, direction and line of action, making a total of six unknowns with only three equations of static equilibrium available.

Consider a small element ds of the curved beam as shown in Fig. A9.2. Let $M_s$ equal the bending moment on this small element due to the given external load system. The total bending moment on the element ds thus equals,

$$ M = M_s + M_A - X_A Y + Y_A X = 0 $$  \hspace{1cm} (1)

(Moments which cause tension on the inside fibers of the frame are regarded as positive moments.)

The following deflection equations for point (A) must equal zero:-

$$ \Theta = 0, \text{ (angular rotation of (A) = zero)} \hspace{1cm} (2) $$

$$ \Delta x = 0, \text{ (movement of (A) in x direction) = 0} \hspace{1cm} (3) $$

$$ \Delta y = 0, \text{ (movement of (A) in y direction) = 0} \hspace{1cm} (4) $$

From Chapter A7, which dealt with deflection theory, we have the following equations for the movement of point (A):

$$ \Theta = \sum \frac{M ds}{EI} \hspace{1cm} (2) $$

$$ \Delta x = \sum \frac{M ds}{EI} \hspace{1cm} (3) $$

In equation (2) the term $m$ is the bending moment on a segment ds due to a unit moment applied at point (A) (See Fig. A9.3). The bending moment is thus equal one or unity for all ds elements of frame.

Then substituting in equation (2) and using value of M from equation (1) we obtain

$$ \Theta = \sum \frac{M ds}{EI} + M_A \frac{ds}{EI} - X_A \frac{Y ds}{EI} + Y_A \frac{X ds}{EI} \hspace{1cm} = 0 $$

whence,

$$ \Theta = \sum \frac{M ds}{EI} + M_A \frac{ds}{EI} - X_A \frac{Y ds}{EI} + Y_A \frac{X ds}{EI} \hspace{1cm} = 0 \hspace{1cm} (5) $$

In equation (3) the term $m$ represents the bending moment on a segment ds due to a unit load applied at point (A) and acting in the x direction, as illustrated in Fig. A9.4.

The applied unit load has a positive sign as it has been assumed acting toward the right. The distance y to the ds element is a plus distance as it is...
measured upward from axis x-x through (A). However the bending moment on the ds element shown is negative (tension in top fibers), thus the value of \( m = - (1) y = -y \). The minus sign is necessary to give the correct bending moment sign.

Substituting in Equation (3) and using \( M \) from Equation (1):

\[
\Delta x = -\frac{Mxds}{EI} - M_A \frac{xds}{EI} + X_A \frac{x^2ds}{EI} - Y_A \frac{yds}{EI} = 0 \quad -(6)
\]

In equation (4) the term \( m \) represents the bending moment on a element ds due to a unit load at point (A) acting in Y direction as illustrated in Fig. A9.5. Hence, \( m = l(x) = x \)

Substituting in equation (4) and using \( M \) from equation (1), we obtain:

\[
\Delta y = \frac{Mxdys}{EI} + M_A \frac{xys}{EI} - X_A \frac{y^2ds}{EI} + \frac{Y_A x^2ds}{EI} = 0 \quad -(7)
\]

Equations 5, 6, 7 can now be used to solve for the redundant forces \( M_A, X_A \) and \( Y_A \). With these values known the true bending moment at any point on structure follows from equation (1).

**Referring Redundants to Elastic Center**

For the purpose of simplifying equations 5, 6, 7, let it be assumed that end A is attached to an inelastic arm terminating at a point (o) as illustrated in Fig. A9.6. The point (o) coincides with the centroid of the ds/EI values for the structure. Reference axes x and y will now be taken with point (o) as the origin. The redundant reactions will now be placed at point (o) the end of the inelastic bracket, as shown in Fig. A9.6. Since point A suffers no movement in the actual structure, then we can say that point (o) must undergo no movement since (o) is connected to point (A) by a rigid arm.

Thus equations 5, 6, and 7 can be rewritten using the redundants \( X_0, Y_0 \), and \( M_0 \) in place of \( X_A, Y_A \) and \( M_A \) respectively.

The axes x and y through the point (o) are centroidal axes for the values ds/EI of the structure. This fact means that the summations:

\[
\sum \frac{yds}{EI} = 0 \quad \text{and} \quad \sum \frac{xds}{EI} = 0
\]

The expressions \( \sum \frac{xds}{EI}, \sum \frac{yds}{EI} \) and \( \sum \frac{xys}{EI} \) also appear in equations 6 and 7. These terms will be referred to as elastic moments of inertia and product of inertia of the frame about y and x axes through the elastic center of the frame, and for simplicity will be given the following symbols.

\[
\sum \frac{x^2ds}{EI} = I_y, \quad \sum \frac{y^2ds}{EI} = I_x, \quad \sum \frac{xys}{EI} = I_{xy}
\]

Equations 5, 6 and 7 will now be rewritten using the redundant forces at point (o).

\[
\phi = \sum \frac{M_0ds}{EI} + M_0 \sum \frac{ds}{EI} = 0
\]

hence,

\[
M_0 = -\frac{\sum \frac{ds}{EI}}{\sum \frac{ds}{EI}} \quad -(8)
\]

\[
\Delta x = -\frac{M_0ys}{EI} + X_0I_x - Y_0I_{xy} = 0 \quad -(9)
\]

The term \( M_0 \sum \frac{yds}{EI} \) is zero since \( \sum \frac{yds}{EI} \) is zero, thus \( M_0 \) drops out when substituting in Equation (8).

\[
\Delta y = \frac{M_0dsx}{EI} - X_0I_y + Y_0I_{xy} = 0 \quad -(10)
\]

The term \( M_0ds/EI \) represents the angle change between the end faces of the ds element when acted upon by a constant static moment \( M_0 \). This angle change which actually is equal in value to the area of the \( M_0/\text{EI} \) diagram on the element ds will be given the symbol \( \phi_0 \), that is, \( \phi_0 = M_0/\text{EI} \). With this symbol substitution, equations 8, 9, 10 can now be rewritten as follows:

\[
\phi_0 = -\frac{\sum \phi_0}{\sum \frac{ds}{EI}} \quad -(11)
\]

\[
\sum \phi_0I_x + X_0I_y - Y_0I_{xy} = 0 \quad -(12)
\]

\[
\sum \phi_0I_x - X_0I_{xy} + Y_0I_y = 0 \quad -(13)
\]

Solving equations (12) and (13) for the redundant forces \( X_0 \) and \( Y_0 \) we obtain,
A9.4 \textbf{BENDING MOMENTS IN FRAMES AND RINGS}

\[ X_0 = \frac{2 \delta_0 Y - 2 \delta_0 x \left( \frac{1}{x^2} \right)}{I_x \left( 1 - \frac{1}{x^2} \right)} \quad \text{(14)} \]

\[ Y_0 = \frac{2 \delta_0 x - 2 \delta_0 y \left( \frac{1}{y^2} \right)}{I_y \left( 1 - \frac{1}{y^2} \right)} \quad \text{(15)} \]

A9.3 Equations for Structure with Symmetry About One Axis through Elastic Center.

If the structure is such that either the x or y axis through the elastic center is a axis of symmetry then the product of inertia \( ZXy/SE = I_{xy} = 0 \). Thus making the term \( I_{xy} = 0 \) in equations 11, 14 and 15 we obtain,

\[ M_0 = \frac{2 \delta_0}{2SE/1} \quad \text{(16)} \]

\[ X_0 = \frac{2 \delta_0 Y}{I_x} \quad \text{(17)} \]

\[ Y_0 = \frac{-2 \delta_0 x}{I_y} \quad \text{(18)} \]

A9.4 Example Problem Solutions. Structures with at least One Axis of Symmetry.

Example Problem 1

Fig. A9.7 shows a rectangular frame with fixed supports at points A and D, and carrying a single load as shown. The problem is to determine the bending moment diagram under this loading.

The first step in the solution is to find the location of the elastic center of the frame and the elastic moments of inertia \( I_x \) and \( I_y \).

Due to symmetry of the structure about the Y axis the centroidal Y axis is located midway between the sides of the frame, then the elastic center (o) lies on this axis.

Table A9.1 shows some of the necessary calculations to determine the location of the elastic center and the elastic moments of inertia. The reference axes used are \( x'x' \) and \( y'y' \).

### Table A9.1

<table>
<thead>
<tr>
<th>Port-ion</th>
<th>( w = 30 )</th>
<th>( w = 30 )</th>
<th>( w = 30 )</th>
<th>( w = 30 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{310} )</td>
<td>15</td>
<td>-12</td>
<td>-120</td>
<td>+150</td>
</tr>
<tr>
<td>( \frac{24}{2} )</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>360</td>
</tr>
<tr>
<td>( \frac{3}{310} )</td>
<td>15</td>
<td>12</td>
<td>120</td>
<td>150</td>
</tr>
<tr>
<td>Sum</td>
<td>32</td>
<td>0</td>
<td>660</td>
<td>16800</td>
</tr>
</tbody>
</table>

The terms \( I_x \) and \( I_y \) are the elastic moment of inertia of each portion of the frame about its centroidal x and y axes. Since I is constant over each portion the centroidal moment of inertia of each portion is identical to that of a rectangle about its centroidal axis.

To explain for member AB:–

Referring to Fig. a, –

\[ I_x = \frac{1}{12} bh^3 = \frac{1}{12} \times \frac{1}{3} \times 30^3 = 750 \]

Referring to Fig. b, –

\[ I_y = \frac{1}{12} bh^3 = \frac{1}{12} \times 30 \times \frac{1}{3} \times 30 \times \frac{1}{3} = \frac{30}{2} \] (negligible)

The distance from the two reference axes to the elastic center can now be calculated:

\[ y = \frac{2w}{3} = \frac{660}{3} = 20.666 \text{ in.} \]

Having the moment of inertia about axis \( x'x' \) we can now find its value about the centroidal axis \( xx \) of the frame, by use of the parallel axis theorem.

\[ I_x = I_x - 2w(\bar{y}^2) = 16800 - 32 \times 20.666^2 = 3188 \]

\[ I_y = I_y - 2w(\bar{x}^2) = 3456 - 32(3456) = 3456 \]

The problem now consists in solving equations (15), (17) and (18) for the redundants at the elastic center, namely.
$M_o = -\frac{\sum a}{zds/1} = \text{Area of static M/I diagram}$

$X_o = \frac{\sum a x}{lx} = \text{Moment of static M/I diagram about x axis}$

$Y_o = -\frac{\sum a y}{ly} = \text{Moment of static M/I diagram about y axis}$

Thus to solve these three equations we must assume a static frame condition consistent with the given frame and loading. In general there are a number of static conditions that can be chosen. For example, in the problem we might select one of the statically determinate conditions illustrated in Fig. A9.8 cases 1 to 5.

![Fig. A9.8](image)

To illustrate the use of different static conditions, three solutions will be presented with each using a different static condition.

**Solution No. 1**

In this solution we will use Case 3 as the static frame condition. The bending moment on the frame for this static frame condition is given in Fig. A9.9. The equations

$M_o = -\sum a = -\frac{(270)}{32} \text{ (from Table A9.1)} = -8.437 \text{ in.lb.}$

$X_o = \frac{\sum a y}{lx} = \frac{270(9.375)}{3165} = 0.7939 \text{ lb.}$

$Y_o = -\frac{\sum a x}{ly} = -\frac{(270)(-2)}{3452} = 0.1562 \text{ lb.}$

![Fig. A9.11](image)

Thus,

$X_o = \frac{270}{3165} = 0.7939 \text{ lb.}$

$Y_o = \frac{5452}{3452} = 0.1562 \text{ lb.}$

Fig. A9.12 shows these values of the redundants acting at the elastic center.

![Fig. A9.12](image)

![Fig. A9.13](image)

The bending moments due to these redundant forces will now be calculated.

$M_A = -8.437 - 0.1562 \times 12 + 9.375 \times 0.2625 = 5.06 \text{ in.lb.}$

$M_B = 8.437 - .7939 \times 9.375 - .1562 \times 12 = -17.75 \text{ in.lb.}$

$M_C = -8.437 + .7939 \times 9.375 - .1562 \times 12 = -14.00 \text{ in.lb.}$

$M_D = -8.437 + .7939 \times 20.625 + .1562 \times 12 = 9.81 \text{ in.lb.}$

These resulting values are plotted on Fig. A9.12 to give the bending moment diagram due to the redundant forces at the elastic center.
Adding this bending moment diagram to the static bending diagram of Fig. A9.9 we obtain the final bending moment diagram of Fig. A9.13.

The final bending moments could also be obtained by substituting directly in equation (1) using subscript (o) instead of (A). Thus,

\[ M = M_o + M_e - X_o y + Y_o x \]  \[ -(19) \]

For example, determine bending moment at point A.

For point B, \( x = -12 \) and \( y = 9.375 \), \( M_B = 0 \) substituting in (19)

\[ M_B = 0 + (-8.437) \times 9.375 = -1.562 \times (-12) = -17.75 \text{ as previously found} \]

AT POINT D. \( x = 12, y = -20.625 \), \( M_e = 0 \).

\[ M_D = 0 + (-8.437) \times (-20.625) + 1.562 \times 12 = 9.81 \text{ in.lb.} \]

Solution No. 2

In this solution we will use Case 4 (See Fig. A9.8) as the assumed static condition, which is two cantilever beams with half the external load or 5 lb. acting on each cantilever. Fig. A9.14 shows the static bending moment diagram and Fig. A9.15 the \( M_e/I \) diagram.

Fig. A9.15 also shows the results of calculating the \( M_e/I \) value for each portion of the \( M_e/I \) diagram and its centroid location. Substituting in the equations for the redundants we obtain,

\[ M_o = -\frac{E \delta_o}{L s} = \frac{(-45-405-300-900)}{32} = 51.56 \text{ in.lb.} \]

\[ X_o = \frac{E \delta_y}{L x} = \frac{(-45-405)9.375 + (-300-300)(-5.625)}{3158} = 0.7539 \text{ lb.} \]

\[ Y_o = \frac{-E \delta x}{L y} = \frac{-45(-10)-405(-8)-300(-12)-500(12)}{3456} = \frac{-9180}{3456} = 2.656 \text{ lb.} \]

The final moments at any point can now be found by equation (19).

Consider point B:-

\[ M_B = -30 \text{ from Fig. A9.14} \]

\[ x = -12, y = 9.375 \]

Subst. in (19)

\[ M_B = -30+51.56.-7939x.9.375+2.656(-12) = -17.75 \text{ in.lb. which checks first solution.} \]

Consider point D:-

\[ M_D = -90, x = 12, y = -20.625 \]

Subst. in (19)

\[ M_D = -90+51.56.-7939(-20.625)+2.656 \times 12 = 9.80 \text{ in.lb.} \]

Solution No. 3

In this solution we will use Case 5 (Fig. A9.8) as the assumed static condition, namely a frame with 3 hinges at points A, D and E, as illustrated in Fig. A9.16.

Before the bending moment diagram can be calculated the reactions at A and D are necessary.

To find \( V_D \) take moments about point A.

\[ 1=HA \quad A \quad D \]

\[ E \]

\[ V_A = 7.5 \quad V_D = 2.5 \]

hence, \( V_D = 2.5 \)

To find \( V_A \) take \( EZ_y = 0 = -10 + 2.5 + V_A \)

\[ V_A = 7.5 \]

To find \( H_p \) take moments about hinge at E of all forces on frame to right side of E and equate to zero.

\[ EZ_y = -2.5 \times 12 + 30H_p = 0, \text{ hence } H_p = 1. \]

Then using \( EZ_x = 0 \) for entire frame, we obtain \( EZ_x = -1 + H_A = 0 \), hence \( H_A = 1. \)
The frame static bending moment diagram can now be calculated and drawn as shown in Fig. A9.17.

The moment diagram is labeled in 6 parts 1 to 6 as indicated by the values in the small circles on each portion. Most of the calculations from this point onward can be done conveniently in table form as illustrated in Table A9.2.

Table A9.2

<table>
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<th>5</th>
<th>6</th>
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</thead>
<tbody>
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<td>MOM. DIAG.</td>
<td>MOM. DIA.</td>
<td>AREA</td>
<td>1/2</td>
<td>DIST. FROM Y</td>
<td>DIST. FROM X</td>
<td>1/2</td>
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<tr>
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<td>PORTION</td>
<td>DIAM.</td>
<td>Beam</td>
<td>Sect.</td>
<td>Y</td>
<td>X</td>
<td>C.G.</td>
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<td>-90</td>
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<td>9.375</td>
<td>-720</td>
<td>-844</td>
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<td>3</td>
<td>-150</td>
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<td>-0.625</td>
<td>-1800</td>
<td>93</td>
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<td>2590</td>
<td>6</td>
<td>-420</td>
<td>-340</td>
<td>-658</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In order to take moments of the \( \theta_x \) values in column (4) of the table, the centroid of each portion of the diagram must be determined. For example, the centroid of the two triangular bending moment portions marked 1 and 6 is \( 0.567 \times 30 \) from the lower end or 20 inches as shown in Fig. A9.17. Thus the distances \( x \) and \( y \) from this \( \theta_x \) location to the \( y \) and \( x \) axes through the frame elastic center are then calculated as -12 and -0.656 inches respectively.

\[
M_\alpha = \frac{-20\theta_x}{2113/1} = \frac{-360}{2113} = 12.19 \text{ in. lb.}
\]

\[
X_\alpha = \frac{20\theta_x}{1} = \frac{-648}{3158} = -0.203 \text{ lb.}
\]

\[
Y_\alpha = \frac{-20\theta_x}{1} = \frac{-540}{3466} = 0.1562 \text{ lb.}
\]

The final moments at any point can now be found by use of equation (19), namely

\[
M = M_\alpha + M_\beta - X_\alpha Y + Y_\alpha X
\]

Consider point B:

\[
x = -12, \quad y = 9.375, \quad M_\beta = -30
\]

Substituting

\[
M_\beta = -30+12.19-(-0.203)(9.375+0.1562(-12)) = -17.76 \text{ in. lb. (checks previous solutions)}
\]

Consider point D:

\[
x = 12, \quad y = -20.625, \quad M_\beta = 0
\]

Sub. in (19)

\[
M_\beta = 0+12.19-(-0.203)(-20.625+0.1562x12) = 9.82 \text{ in. lb. (checks previous solutions)}
\]

The final complete bending moment diagram would of course be the same as drawn in Fig. A9.18 for the results of Solution No. 1.

Example Problem 2.

Fig. A9.18 shows a rectangular closed frame supported at points A and B and carrying the external loads as shown. The reaction at B due to rollers is vertical. The frame at point A is continuous through the joint but the reaction is applied through a pin at the center of the joint. The problem is to determine the bending moment diagram.
The next step is to choose a static frame condition and determine the static (M₀) bending moment diagram. In this solution the frame is assumed cut on member AB just above point A as illustrated in Figs. A9.19, 18 and 20. For simplicity the static moment curve has been drawn in three parts, with each part considering only one of the three external given loads on the structure. Figs. A9.18, 19, 20 show these resulting bending moment curves. The portions of these bending moment diagrams numbered 1 to 10 are shown in the parenthesis on each portion.

The next step in the solution consists of finding the area of the M₀/I diagrams and the first moment of these diagrams about the x and y axes through the elastic center. These simple calculations can best be done in tabular form as illustrated in Table A9.3.

<table>
<thead>
<tr>
<th>Portion of M₀ diagram:</th>
<th>Area of M₀ / I = A</th>
<th>( \frac{\sum M₀}{I} )</th>
<th>( x ) dist. to y axis</th>
<th>( y ) dist. to x axis</th>
<th>( Oₓ ) ( x )</th>
<th>( Oᵧ ) ( y )</th>
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<td>1</td>
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<td>-747225</td>
<td>-4645</td>
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</tr>
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</table>

Solving for the redundants at the elastic center,

\[ M₀ = \frac{\sum M₀}{I} = \frac{-(-77700)}{33.5} = 2312 \text{ in}.lb. \]

\[ X₀ = \frac{\sum X}{I_x} = \frac{-1645}{3797} = -0.43 \text{ lb}. \]

\[ Y₀ = \frac{\sum Y}{I_y} = \frac{-747225}{5289} = 140.28 \text{ lb}. \]

Fig. A9.21 shows these redundant forces acting at the elastic center. Fig. A9.22 shows the bending moment diagram due to these redundant forces. The calculations with reference to Fig. A9.21 are -
Total elastic weight of ring =

\[ \sum \frac{ds}{t} = \pi \left( \frac{36}{1} \right) = 113.1 \]

The elastic moment of inertia about x and y axes through center point of ring are the same for each axis and equal

\[ I_x = I_y = n^2a = n \times 18^2 = 13500 \]

The next step in the solution is to assume a static ring condition and determine the static (M_e) diagram. In general, it is good practice to try and assume a static condition such that the M_e diagram is symmetrical about one or if possible about both x and y axes through the elastic center, thus making one or both of the redundants \( X_0 \) and \( Y_0 \) zero and thus reducing considerably the amount of numerical calculation for the solution of the problem.

In order to obtain symmetry of the M_e diagram and also the M_e/I diagram since I is constant, the static condition as shown in Fig. A9.24 is assumed, namely, a pin at (a) and rollers at (c). The static bending moment at points (a), (b), (c) and (d) are the same magnitude and equal,

\[ M_e = 50(18 - 18 \cos 45^\circ) = 265 \text{ in.lb} \]

The sign is positive because the bending moment produces tension on the inside of the ring.

The next step is to determine the \( \theta_g \) and \( \theta_gx \) values.

\( \theta_g \) is the area of the M_e/I diagram, however since I is unity it is the area of the M_e diagram. The static M_e diagram of Fig. A9.24 is divided into similar portions labeled (1) and (2). Hence

\[ \theta_g(1) = \text{area of portion (1)} = Pr^2(\sin c - \sin a) \text{, where } P = 50 \text{ lb. and } c = 45^\circ \]

Substituting and multiplying by 4 since there are four portions labeled (1),

\[ \theta_g(1) = 4 \left[ 50 \times 18^2(0.707 - 0.707) \right] = 5052 \]

The area of portion labeled (2) equals,

\[ Pr^2(1 - \cos c) \]

Since there are two areas (2) we obtain,

\[ \theta_g(2) = 2 \left[ 50 \times 18^2 \times \frac{7}{2} (1 - 0.707) \right] = 15000 \]

Hence, total \( \theta_g = 15000 + 5052 = 20052 \)

Since the centroid of the M_e diagram due to symmetry about both x and y axes coincides
with the center or elastic center of the frame, the terms $E_{e}a_{x}$ and $E_{e}a_{y}$ will be zero.

Substituting to determine the value of the redundants at the elastic center we obtain,

$$M_{0} = \frac{-E_{e}a_{x}}{12} = - \frac{(20052)}{113.1} = -177 \text{ in.} \text{lb}.$$  

$$X_{0} = \frac{E_{e}a_{y}}{113.1} = \frac{20052(0)}{113.1} = 0$$  

$$Y_{0} = \frac{-E_{e}a_{y}}{113.1} = \frac{(-20052)(0)}{113.1} = 0$$

Fig. 10 shows the values acting at the elastic center and the bending moment diagram produced by these loads. Adding the bending moment diagram of Fig. 10 which is a constant value over entire frame of $-177$ to the static moment diagram of Fig. 10 gives the final bending moment diagram as shown in Fig. 10.

Example Problem 4. Hull Frame

Fig. 10 shows a closed frame subjected to the loads as shown. The problem is to determine the bending moment diagram.

Solution:

The first step is to determine the elastic center of the frame and the elastic moments of inertia. Table 10 shows the calculations. A reference axis $X'$-X has been selected at the midpoint of the side AB. Since a static frame condition has been selected to make the $M_{0}$ diagram symmetrical about y axes through the elastic center (see Fig. 10), it is not necessary to determine $I_{y}$ since the redundant $Y_{0}$ will be zero due to this symmetry.

### Table 10

| Member | Length $d_s$ | $I_{y}$ | $w$ | $dy$ | $I_{x} + I_{y}$ | $I_{x} + I_{y}

| DB | 94.25 | 1.5 | 62.8 | 1.5 | 3.080 | 54000 + 151200 = 156600 |
| AB | 60.0 | 1.0 | 60.0 | 0 | 18000 + 0 = 18000 |
| A'B' | 60.0 | 1.0 | 60.0 | 0 | 18000 + 0 = 18000 |
| AC | 38.4 | 2.0 | 19.2 | -42 | -806922 + 33580 = 34772 |
| CA' | 38.4 | 2.0 | 19.2 | -42 | -806922 + 33580 = 34772 |
| Totals | 221.2 | 1468 | 54000 + 151200 = 156600 |

In the last column of Table 10 the term $I_{x}$ is the moment of inertia of a particular member about its own centroidal x axis. Thus for member DB,

$$I_{x} = \frac{3b^{3}}{12} = 0.3 \times 30^{3}/1.5 = 5400$$

For members AC and CA,

$$I_{x} = \frac{1}{12} b L h^{2}$$

$$I_{x} = \frac{1}{12} \times 24^{3} = 33.4x2.4 = 922$$

Let $\bar{y}$ = distance from X'X ref. axis to centroidal elastic axis X-X.

$$\bar{y} = \frac{E_{w}w}{E_{w}} = \frac{1468}{221.2} = 6.64 \text{ in.}$$

By parallel axis theorem,

$$I_{x} = I_{x}' - 6.64b(2w)$$

$$= 262140 - 6.64(221.2) = 252400$$

The next step in the solution is to compute the static moment elastic weights $a_{e}$ and their centroidal locations. In Fig. 10, the static frame condition assumed is a pin at point A and rollers at point A', which gives the
general shape of static moment curve as shown in the figure.

Consider member BDB'.

The bending moment curve will be considered in two parts, namely (1) and (2). The term \( \phi_3 \) represents the area of the \( M_e/I \) diagram.

Thus for portion (1) and (1')

\[
\phi_3(a) + \phi_3(a') = 2 \left[ \frac{Pr}{I} (a - \sin a) \right] = 2 \left[ 6000 \times 30^* \times \frac{0.524 - 0.5}{1.5} \right] = 172800
\]

The vertical distance from the line BB' to centroid of \( M_e \) curve for portion (1) and (1')

\[
\begin{align*}
V & = \frac{r(1 - \cos a - \frac{\sin a}{a'})}{a - \sin a} \\
& = 30(1 - 0.867 - \frac{0.5}{2}) \\
& = 10 \text{ in.}
\end{align*}
\]

For portion (2) of the \( M_e \) diagram the area of the \( M_e/I \) diagram which equals \( \phi_3 \) is

\[
\phi_3(a) = \frac{Pr}{I} (1 - \cos a) = \frac{6000 \times 30^* \times 2.11}{1.5} (a - 0.867)
\]

\[
= 1,007,000
\]

Consider member AC.

From free body diagram of bottom portion of frame (Fig. A9.31), the equation for bending moment

\[
6000 \quad 6000
\]

\[
\begin{align*}
\text{A} & \quad \text{A'} \\
\text{9.4} & \quad 200 \text{in/m}
\end{align*}
\]

\[\text{Fig. A9.31}\]

on member AC equals:

\[
M_X = 6490 \times -100 \text{ x}^2
\]

Area of \( M/e \) curve between A and C when \( I = 2 \) equals,

\[
\begin{align*}
\frac{1}{2} \int_{0}^{38.4} (6490 \times -100 \text{ x}^2) dx \\
= \frac{1}{2} \left[ \frac{6490 \times -100 \times x^3}{2} \right] \bigg|_{0}^{38.4} \\
= 784000
\end{align*}
\]

Distance to centroid of \( M/e \) curve along line AC from A.

Vertical distance from line AA' to centroid = 21.77 x 24 / 38.4 = 13.7°. The static moment weight for A'C' is same as for AC, thus

\[
\phi_{AC} + \phi_{A'C'} = 2 \times 784000 = 1568000
\]

Fig. A9.28 shows the frame with the moment weights \( \phi_3 \) located at the centroids, together with the redundant forces \( M_o \) and \( X_o \) at the elastic center. It makes no difference where the frame is cut to form our residual cantilever, if one of the cut faces is attached to elastic center and the other is considered fixed. With the elastic properties and moment weights known the redundant can be solved for:

\[
M_o = \frac{E_o I}{x} = \frac{(1007000 + 172800 + 1568000)}{221.2} = 12430\#
\]

\[
X_o = \frac{E_o x}{I} = \frac{1007000 \times 48.16 + 172800 \times 33.36 + 1568000 \times -6.64}{252400} = -98.5\#
\]

\( Y_o \) is zero because of the symmetrical frame and loading. The final or true bending moment at any point equals

\[
M = M_e + M_o - X_o Y
\]

Thus for point B

\[
M = 0 + (12430) - (-98.5 \times 23.36) = -10130\#
\]

For point C

\[
M_e \text{ at point C} = 6490 \times 38.4 - 200(38.4)^2/2 = 32700
\]

Hence \( M_o = 32700 - 12430 - (-98.5 \times -60.64) = 14300\#

Fig. A9.30 shows the general shape of the true frame bending moment diagram.

Example Problem 5

Fig. A9.31 shows the general details of one
half of a symmetrical hull frame that was used in an actual seaplane. The main external load on such frames is the water pressure on the hull bottom plating. The hull bottom stringers transfer the bottom pressure as concentrations on the frame bottom as shown. The resistance to this bottom upward load on the frame is provided by the hull metal covering which exerts tangential loads on the frame contour. The question as to the distribution of these resisting forces is discussed in later chapters. In this problem the resisting shear flow in the hull sheet has been assumed constant between the chine point and the upper heavy longeron. For analysis purposes the frame has been divided into 20 strips. The centroids of these strips located on the neutral axis of the frame sections are numbered 1 to 20 in Fig. A9.21. The tangential skin resisting forces are shown as concentrations on frame strips #6 to #16. On the figure these tangential loads have been replaced by their horizontal and vertical components. The sum of the vertical components should equal the vertical component of the bottom water pressure.

Table A9.5 shows the complete calculations for determining the bending moment on the frame. Columns 1 to 7 give the calculations for the elastic properties of the frame, namely the elastic weight of the frame; the elastic center location, and the elastic centroidal moment of inertia about the horizontal centroidal elastic axis. A reference horizontal axis X'X' has been selected as shown. All distances recorded in the table have been obtained by scaling from a large drawing of the frame.

The static condition assumed for computing the $M_x$ moments is a double symmetrical cantilever beam as illustrated in Fig. A9.32. The frame is cut at the top to form the free end of the cantilever beam, and the fixed end has been taken at the centerline bottom frame section. The static bending moment diagram will be symmetrical about the y axis through the elastic center of the frame and thus the redundant $Y_x$ at the elastic center will be zero since the term $\Delta Y_x$ will be zero. The calculations for determining the static moments $M_x$ in Column 8 of Table A9.5 are not shown. The student should refer to Art. A5.9 of Chapter A5 to refresh his thinking relative to bending moment calculations on curved beams.

Columns 9 and 10 give the calculations of the $\Delta x$ values (area of $M_x$/I diagram) and the first moment ($\Delta x$ values). The summations of columns 3, 9, 10 permits the solution for the redundants $M_y$ and $X_y$ as shown below the table. The final bending moment $M$ at any point on the frame is by simple statics equals,

$$M = M_y + M_z - X_y Y.$$
### Table A9.5

<table>
<thead>
<tr>
<th>Station</th>
<th>Strip</th>
<th>Length</th>
<th>Moment of Inertia</th>
<th>Elastic Weight</th>
<th>Arm to Ref. Axis</th>
<th>w''</th>
<th>w''^2</th>
<th>Arm to Axis XX</th>
<th>Static Moment M_S</th>
<th>Moment Weight Q_S</th>
<th>OS y</th>
<th>M_0</th>
<th>-X_0y</th>
<th>*Total Moment M</th>
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<td>18.82 -58.4 -32</td>
<td>98900 -73.7 32237</td>
<td>383940 -383940</td>
<td>-X_0y</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*Total Moment M at any station = M_S + M_0 - X_0y</td>
<td></td>
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</tbody>
</table>

\[ \gamma = 7228/811.48 = 8.9'' \]

\[ I_{XX} = 423237 - 811.48 \times 8.9^2 = 358940 \]

\[ M_0 = \frac{-Q_S}{S_w} (17315.7 \times 1000) = -9020'' \times 3113.5 \]

\[ X_0 = \frac{Q_S \gamma}{I_{XX}} \times \left( \frac{348106 \times 1000}{358940} \right) = -968'' \]

### A9.3 Unsymmetrical Structures. Example Problem Solutions.

#### Example Problem 1

![Fig. A9.34](image_url)

The elastic weight of frame = Eds/1 = (15/1) + (12/2) + 10/1 = 31
BENDING MOMENTS IN FRAMES AND RINGS

The distance $x$ from the line $AB$ to the elastic center is,

$$x = \frac{(15)0 + 6 \times 6 + 10 \times 12}{31} = 5.032 \text{ in.}$$

The distance $y$ from line $BC$ to elastic center equals,

$$y = \frac{15 \times 7.5 + 6 \times 0 + 10 \times 5}{31} = 5.242 \text{ in.}$$

The elastic moments of inertia $I_x$ and $I_y$ and the product of inertia $I_{xy}$ are required.

$$I_x = \frac{(15)^3}{12} + (15)(2.258)^2 + \frac{(12)^3}{12}(5.242)^2 + \frac{(10)^3}{12}$$

$$+ (10)(0.242)^2 = 605.51$$

$$I_y = (15)(5.032)^2 - \frac{(12)^3}{2 \times 12} + \frac{(12)^3}{2}(0.968)^2 +$$

$$+ (10)(6.968)^2 = 942.96$$

$$I_{xy} = (15)(-5.032)(-2.258) + \frac{(12)^3}{2}(0.968)(5.242)$$

$$+ (10)(6.968)(0.242) = 217.74$$

The next step is to assume some static frame condition and draw the static bending moment diagram. Fig. A9.33 shows that the frame has been assumed cut near point $C$ which gives two cantilever beams. The bending moment diagram in three parts for this static condition is also shown on Fig. A9.35.

Fig. A9.35

The $\bar{M}_a$ values which equal the area of the $\bar{M}_a$ diagram divided by the $I$ values of the particular portion will be calculated.

$$\bar{M}_{a1} = \frac{-2160}{5.242} = -411.2$$

$$\bar{M}_{a2} = \frac{9.758}{6.750} = 1.451$$

$$\bar{M}_{a3} = \frac{-4660}{5.332} = -872.9$$

The bending moment at any point from eq. (19) equals,

$$M = M_a + M_o - X_o Y + Y_o X,$$ for example,

Consider point $A$.

$$x = -5.032, \ y = -9.758, \ M_a = -2520$$

$$M_a = -2520 + 223 - 68.46 (-9.758) +$$

$$\frac{(-132.36)(-5.032)}{-223 \text{ in. lb.}} = -289 \text{ in. lb.}$$

Point $B$.

$$x = -5.032, \ y = 5.242, \ M_a = -1440$$

$$M_a = -1440 + 223 - 58.46 x 5.242 + (-132.36)$$

$$\frac{-5.032}{-289 \text{ in. lb.}} = -209 \text{ in. lb.}$$

Point $C$.

$$x = 6.968, \ y = 5.242, \ M_a = 0$$

$$M_a = 0 + 223 - 58.46 x 5.242 + (-132.36)$$

$$\frac{6.968}{-357 \text{ in. lb.}} = -209 \text{ in. lb.}$$

Thus, the complete bending moment diagram could be determined by computing several more values such as point $D$ and the external load points.
Example Problem 2. Fig. A9.37 shows an unsymmetrical closed frame. The bending moment diagram will be determined under the given frame loading.

\[ P_1 = 20 \text{#}, \quad P_2 = 100 \text{#} \]

\[ B \quad \quad \quad P_2 = 100 \text{#} \quad \quad \quad C \]

\[ P_1 = 20 \text{#} \quad \quad \quad \quad \quad 30^\circ \]

\[ A \quad \quad \quad \quad \quad 30^\circ \quad D \]

\[ E \quad \quad \quad \quad \quad 20^\circ \quad F \]

\[ x = \frac{30}{1.5} + \frac{31.67}{3} + \frac{20}{1} + \frac{30}{2} = 65.58 \]

The location of the elastic axes will be determined with reference to assumed axes \( x'x' \) and \( y'y' \) as shown on Fig. A9.37.

\[ \bar{x} = \frac{(30)(0) + \frac{31.67}{3}(15) + \frac{20}{1}(30) + \frac{30}{2}(15)}{65.58} = 15^\circ \]

\[ \bar{y} = \frac{(30)(15) + \frac{31.67}{3}(25) + \frac{20}{1}(10) + \frac{30}{2}(0)}{65.58} = 11.657^\circ \]

These distances \( \bar{x} \) and \( \bar{y} \) locate the \( x \) and \( y \) elastic axes as shown in Fig. A9.37.

The elastic moments of inertia and the elastic product of inertia will now be calculated.

Calculation of \( I_x \):

Member AB, \( I_x = \frac{1}{3} \times \frac{1}{1.5} (15.343^2 + 11.657^2) = 1723.51 \)

Member CD, \( I_x = \frac{1}{3} \times \frac{1}{1} (8.343^2 + 11.657^2) = 721.58 \)

Member BC, \( I_x = \frac{(31.67)(13.343^2)}{3} + \frac{1}{12} (31.67) (10^3) = 1967.48 \)

Member AD, \( I_x = \frac{30}{2} (11.657^3) + \frac{1}{12} \times 30 \times (0.5)^3 = 2038.50 \)

Total \( I_x = 6451.07 \)

Calculation of \( I_y \):

Member AB, \( I_y = \frac{30}{1.5} \times 15^3 + \frac{1}{12} \times 30(0.667)^3 = 4500.7 \)

Member BC, \( I_y = \frac{1}{12} \times \frac{31.67}{3} \times 30^3 = 791.7 \)

Member CD, \( I_y = \frac{20}{1} \times 15^3 + \frac{1}{12} \times 20(1)^3 = 4501.7 \)

Member AD, \( I_y = \frac{1}{12} \times \frac{1}{2} \times 30^3 = 1125.0 \)

Total \( I_y = 10919 \)

Calculation of \( I_{xy} \):

Member AB, \( I_{xy} = \frac{30}{1.5}(-15)(3.343) = -1002.9 \)

Member BC, \( I_{xy} = \frac{31.67}{3}(13.343)(0) - \frac{1}{12} \times \frac{31.67}{3} (30)(10) = -263.62 \)

Member CD, \( I_{xy} = \frac{20}{1}(15)(-1.657) = -497.1 \)

Member AD, \( I_{xy} = 0 \)

Total \( I_{xy} = -1763.9 \)

The next step in the solution is to assume a static frame condition and draw the \( M_x \) diagram. Fig. A9.38 shows the assumed static condition, namely pinned at point \( A \) and supported as rollers at point \( D \). The bending moment diagram is drawn in parts as shown.

The next step is to compute the value of \( \theta_3 \) for each portion of the moment diagram. \( \theta_3 \) is the area of the \( M_x/I_x \) diagram. For reference the portions of the \( M_x \) diagram have been labeled 1 to 4.
The bending moment at any point equals the original static $M_a$ plus the moment due to the redundant forces as shown in Fig. A9.40.

Consider point A: $M_a = 0$

$$M_B = 0 - 124.8 - 4.674 \times 15 + 31.07 \times 11.657 = 167 \text{ in. lb.}$$

Point B. $M_a = 600$

$$M_B = 600 - 124.8 - 4.674 \times 15 - 31.07 \times 18.343 = -165 \text{ in.lb.}$$

Point C. $M_a = 0$

$$M_B = 0 - 124.8 - 31.07 \times 8.343 + 4.574 \times 15 = -314 \text{ in.lb.}$$

Point D. $M_a = 0$

$$M_B = 0 - 124.8 - 4.674 \times 15 + 31.07 \times 11.657 = 307 \text{ in.lb.}$$

Fig. A9.41 shows the true bending moment diagram.

A9.6 Analysis of Frame with Pinned Supports.

Fig. A9.42 shows a rectangular frame and loading. This frame is identical to example problem 1 of Art. A9.3, except it is pinned at points A and D instead of fixed.

The first step will be to determine the elastic weight of the frame, the elastic center location and the elastic moments of inertia about axes through the elastic center.

The term $ds/EI$ of a beam element of length $ds$ represents the angle change between its two end faces when the element is acted upon by a unit moment. In this chapter this term has been called the elastic weight of the element. Physically, the elastic weight is the ability of the element to cause rotation when acted upon by a unit moment. When a unit moment is applied to a rigid support, the support suffers no rotation since the support is rigid, therefore a rigid support has zero elastic weight and therefore does not figure in the frame elastic properties.
If a support is pinned or hinged it has no resistance to rotation and thus a unit moment acting on a hinge would have infinite angle change or rotation and therefore a hinge or pin possesses infinite elastic weight.

![Diagram Image]

Due to symmetry of structure about the centerline y axis the elastic center will lie on this axis. Since the two hinges at A and B have infinite elastic weight, the centroid or elastic center of the frame will obviously lie midway between A and B. Fig. A9.43 shows the elastic center D connected to the point A by a rigid bracket.

\[ E d s / E I \] for frame is infinite because of the hinges at A and B.

The elastic moment of inertia about a y axis through elastic center is infinite since the hinge supports have infinite elastic weight.

\[ I_x \] is calculated as follows:

For Portion AB = \( \frac{1}{3} \times \frac{1}{3} \times 30^2 = 3000 \)

For Portion CD = \( \frac{1}{3} \times \frac{1}{3} \times 30^2 = 3000 \)

For Portion BC = \( 24 \times \frac{1}{2} \times 30^2 = 10800 \)

\[ I_x = 16800 \]

Fig. A9.44 shows the static frame condition assumed to obtain the \( M_d \) values.

The value of \( \theta_s \) for member BC equals the area of the \( M_d \) curve divided by \( I \) for BC, hence \( \theta_s = 45 \times 24 \times \frac{1}{2} = 270 \). The centroid of this \( \theta_s \) value is 10 inches from point B. The redundant forces at the elastic center can now be solved for:

\[ M_0 = \frac{-2 \theta_s}{E I} \frac{270}{\text{Infinity}} = 0 \]

\[ Y_0 = \frac{-2 \theta_s}{I_y} \frac{270}{\text{Infinity}} = 0 \]

\[ X_0 = \frac{2 \theta_s}{I_x} \frac{270 \times 30}{18000} = 0.482 \text{ lb.} \]

Fig. A9.45 shows the bending moment diagram due to the redundant \( X_0 \). Adding this-diagram to the original static diagram gives the final bending moment curve in Fig. A9.46.

A9.7 Analysis of Frame with One Pinned and One Fixed Support.

Fig. A9.47 shows the same frame and loading as in the previous example but point D is fixed instead of hinged.

The support D has zero elastic weight and the pin at A has infinite elastic weight, therefore the elastic center of the frame lies at point A. The total elastic weight of frame is infinite because of pin at A.

The elastic moments of inertia will be calculated about x and y axes through A.

\[ I_x = 16800 \text{ (Same as previous example)} \]

\[ I_y = \left( \frac{30}{3} \times 24^2 \right) + \left( \frac{1}{3} \times \frac{1}{3} \times 24^2 \right) = 8064 \]

\[ I_{xy} = \left( \frac{30}{3} \times 15 \times 24 \right) + \left( \frac{24}{2} \times 30 \times 12 \right) = 7920 \]

The static frame condition will be assumed the same as in the previous example problem, hence \( \theta_s = 270 \) and acts 10° from B (Fig. A9.48).

Since the frame is unsymmetrical, the x and y axes through the elastic center at A are not principal axes, hence
CHAPTER A10
STATICALLY INDETERMINATE STRUCTURES
SPECIAL METHOD - THE COLUMN ANALOGY METHOD

A10.1 General. The Column Analogy* method is a method that is widely used by engineers in determining the bending moments in a bent or ring type structure. The method considers only distortions due to bending of the structure.

The numerical work in using the column analogy method is practically identical to that carried out in applying the elastic center method of Chapter A9.

A10.2 General Explanation of Column Analogy Method.

Fig. A10.1 shows a short column loaded in compression by a load P located at distances (a) and (b) from the principal axes x and y of the column cross-section.

![Fig. A10.1](image)

To find the bearing stress between the supporting base and the lower end of the column, it is convenient to transfer the load P to the column centroid plus moments about the principal axes. Then if we let \( \sigma \) equal the bearing stress intensity at some point x distance x and y from the yy and xx axes, we can write

\[
\sigma = \frac{P}{A} + \frac{(Pa)y}{I_x} + \frac{(Pb)x}{I_y}
\]

Where A is the area of the column cross-section and Pa and Pb the moment of the load P about the xx and yy axes respectively. If we let \( Pa = M_x \) and \( Pb = M_y \), the above equation can be written,

\[
\sigma = \frac{P}{A} + \frac{M_y y}{I_x} + \frac{M_x x}{I_y}
\]  

(1)


Now assume we have a frame whose centerline length and shape is identical to that of the column section in Fig. A10.1. The width of each portion of this frame will be proportional to 1/EI of the member. Fig. A10.2 shows this assumed frame. Furthermore, assume that end B of the frame is fixed and that a rigid bracket is attached to the end A and terminating at point (O) the elastic center of the frame. The frame is subjected to an external loading, \( w_1, w_2, \) etc.

![Fig. A10.2](image)

This cantilever structure will suffer bending distortion under the external load system \( w_1, w_2, \) etc., and point (O) will be displaced. Point (O) can be brought back to its original undeflected position by applying a couple and two forces at (O), namely, \( M_0, X_0 \) and \( Y_0 \), as shown in Fig. A10.2. Since point (O) is attached to frame end A by the rigid bracket these three forces at the elastic center (O) will cause point A to remain stationary or in other words to be fixed. Therefore, for the frame in Fig. A10.2 fixed at A and B, the moment and two forces acting at the elastic center cause the statically indeterminate moments \( M_0 \) when resisting a given external loading causing static moments \( M_s \). The final true bending moment \( M \) at a point on the frame then equals

\[
M = M_s + M_t.
\]

From Fig. A10.2 we can write for a point on the frame such as B that the indeterminate bending moment \( M_t \) equals,

\[
M_t = M_0 + X_0 y + Y_0 x \quad \quad \quad \quad \quad (2)
\]

In Chapter A9, Art. A9.3, the equations for \( M_0, X_0 \) and \( Y_0 \) were derived. They are,
A10.2 THE COLUMN ANALOGY METHOD

\[ M_o = \frac{E \Delta s}{\varepsilon s/1}, \quad Y_o = \frac{E \Delta s x}{I_y}, \quad X_o = \frac{E \Delta s y}{I_x} \]

The term \( E \Delta s \) represents the area of the static \( M_o/1 \) curve. \( E \) has been assumed constant and therefore omitted. Let the term \( E \Delta s \) be called the elastic load and give it a new symbol \( P \). The term \( \varepsilon s/1 \) equals the elastic weight of the frame and equals the sum of the length of each member times its width which equals \( 1/f \). Let this total frame elastic weight be given a new symbol \( A \).

In the expressions for \( Y_o \) and \( X_o \), the terms \( E \Delta s x \) and \( E \Delta s y \) represent the moment of the static \( M/1 \) curve acting as a load about the \( y \) and \( x \) axes respectively passing through the frame elastic center. Therefore let \( E \Delta s \) be given a new symbol \( M_y \) and \( E \Delta s y \) a new symbol \( M_x \). With these new symbols, equation (2) can now be rewritten as follows:

\[ M_x = \frac{P}{x} + \frac{M_y}{I_x} + \frac{M_y}{I_y} \quad \text{---} \quad (3) \]

Comparing equations (1) and (3) we see they are similar. In other words, the indeterminate bending moments \( M_i \) in a frame are analogous to the column bearing pressures \( \sigma \), hence the name "Column Analogy" for the method using equation (1). With this general explanation, the method can now be clearly explained by giving several example problem solutions.

A10.3 Frames with One Axis of Symmetry.

From the previous discussion we can write,

(1) The cross-section of the analogous column consists of an area, the shape of which is the same as that of the given frame and the thickness of any part equals \( 1/EI \) of that part.

(2) The loading applied to the top of the analogous column is equal to the \( M_o/1 \) diagram, where \( M_o \) is the static moment in any basic determinate structure derived from the given frame. If \( M_o \) causes bending compression on the inside face of the frame it is a positive bending moment and the analogous load \( P \) on the column acts downward.

(3) The indeterminate bending moment \( M_i \) at a given frame point equals the base pressure at this same point on the analogous column. Thus the indeterminate moment at any point on the frame equals (from eq. 3),

\[ M_i = \frac{P}{x} + \frac{M_y}{I_x} + \frac{M_y}{I_y} \quad \text{---} \quad (4) \]

(4) The final or true bending moment \( M \) at any point then equals,

\[ M = M_o - M_i \quad \text{---} \quad (5) \]

Example Problem 1.

Fig. A10.3 shows a rectangular frame with fixed supports at points A and B. The bending moments at points A, B, C and D will be determined by the column analogy method. This frame and loading is identical to example problem 1 of Art. A6.4 of Chapter A9 where the solution was made by the elastic center method.

\[ \text{Fig. A10.3} \]

\[ \text{Fig. A10.4} \]

SOLUTION NO. 1

We first consider the frame centerline shape as shown in Fig. A10.3 as the cross-section of a short column (see Fig. A10.5). The width of each portion of the column section is equal to \( 1/EI \) of the member cross-section. Since \( E \) is constant, it will be made unity and the widths will then equal \( 1/I \).

\[ \text{Fig. A10.5} \]

\[ \text{Fig. A10.6} \]

The first step in the calculations is to compute the area \( A \) of the column cross-section in Fig. A10.5 and the moments of inertia of the column cross-section about \( x \) and \( y \) centroidal axes.

\[ \text{Area } A = \sum \frac{d}{2} = \frac{30}{3} + \frac{30}{3} + \frac{24}{2} = 32 \]

The calculation of the location of the centroidal axes and the moments of inertia \( I_x \) and \( I_y \)......
and \( I_y \) would be identical to the complete calculations given in Art. A9.4, and Table A9.1 where this same problem is solved by the elastic center method. These calculations will not be repeated here. The results were, \( I_x = 3188 \) and \( I_y = 3468 \).

Since the frame is statically indeterminate, the next step is to assume a static frame condition consistent with the given frame and loads. Fig. A10.4 shows the condition assumed for this solution, namely, pinned at point A and a pin with rollers at point D. The static \( M_y \) diagram is therefore as shown in Fig. A10.5. We now load the column cross-section with the \( M_y/I \) diagram as a load as shown in Fig. A10.6. The static moment sign is positive because the static condition causes tension on the inside face of the frame. In the column analogy method a positive \( M_y/I \) loading is a downward or compressive load on top of column, and therefore a negative \( M_y/I \) value would be an upward or tension load on the column.

Equation (3) requires the values of the \( M_y \) and \( M_y \), the moment of the \( M_y/I \) diagram as a load about the x and y axes. Equation (3) also requires the total column load \( P \) which equals the area of the \( M_y/I \) diagram.

For this problem the value of \( P \) from Fig. A10.5 equals,

\[
P = 22.5 \times 24/2 = 270
\]

The centroid of this triangular loading is 2 inches to left of y axis. Fig. A10.6 shows the column section with this resultant load \( P \).

We now use equation (3) to find the indeterminate moments \( M_1 \) which are equal to the base pressures on the column. Equation (3) involves bending moments \( M_x \) and \( M_y \) and distances \( x \) and \( y \), all of which must have signs. The signs will be determined as follows:

When moment \( M_x \) produces compression on base on that portion above x axis, then \( M_x \) is positive.

When moment \( M_y \) produces compression on base on that portion to right of y axis, then \( M_y \) is positive.

A distance \( y \) measured upward from x axis is positive; measured downward is negative.

A distance \( x \) measured to right from y axis is positive, to left negative.

From Fig. A10.6:

\[
P = 270
\]

\[
M_x = 270 \times 9.375 = 2530 \text{ (positive because base pressure is compressive on column portion above x axis)}
\]

\[
M_y = -(270 \times 2) = -540 \text{ (negative because base pressure is tension on column portion to right of y axis)}
\]

Substitution in equation (3).

Frame Point A. \( x = -12", \ y = -20.625" \)

\[
M_1 = \frac{P}{\frac{I_x}{I_x}} + \frac{M_y}{\frac{I_y}{I_y}}
\]

\[
M_1 = \frac{270}{\frac{32}{3188}} + \frac{2530(-20.625)}{-3468} = \frac{8.44 - 16.38 + 1.88}{\text{in.} \text{lb.}} = -6.06 \text{ in.} \text{lb.}
\]

The true bending moment from equation (5),

\[
M = M_y - M_1
\]

\[
M_y = 0, \text{ see Fig. A10.5}
\]

whence, \( M_y = 0 - (-6.06) = 6.06 \text{ in.} \text{lb.} \)

Frame Point B. \( x = -12", \ y = 9.325" \)

\[
M_1 = \frac{270}{\frac{32}{3188}} + \frac{2530 \times 9.375}{-3468} = \frac{8.44 + 7.44 + 1.88}{17.77} = 17.77 \text{ in.} \text{lb.}
\]

\[
M_y = M_y - M_1 = 0 - (17.77) = -17.77 \text{ in.} \text{lb.}
\]

Frame Point C. \( x = 12", \ y = 9.325 \)

\[
M_1 = \frac{270}{\frac{32}{3188}} + \frac{2530 \times 9.375}{-3468} = \frac{8.44 + 7.44 - 1.88}{14.0} = 14.0 \text{ in.} \text{lb.}
\]

\[
M_y = M_y - M_1 = 0 - (14.0) = -14.0 \text{ in.} \text{lb.}
\]

Frame Point D. \( x = 12", \ y = -20.625 \)

\[
M_1 = \frac{270}{\frac{32}{3188}} + \frac{2530(-20.625)}{-3468} = \frac{8.44 - 16.38 - 1.88}{9.82} = 9.82 \text{ in.} \text{lb.}
\]

Fig. A10.7 shows the final bending moment diagram, which of course checks the solution by the elastic center method in Art. A9.4. The student should note that the numerical work in the column analogy method is practically the same as in the elastic center method.
Solution No. 2

In this solution a different static frame condition will be assumed as shown in Fig. A10.8, namely the frame is cut under the load and one-half the 10 lb. load will be assumed as going to each cantilever part. Fig. A10.9 shows the static moment diagram and Fig. A10.10 the static M/EI diagram with centroid locations of each portion of diagram which are numbered (1), (2), (3) and (4). The area of each of these portions will represent a load P₁, P₂, etc. on the column in Fig. A10.11. Since the static moment is negative on each portion the load on the column section will be upward.

\[
P_1 = -10(30) = -300, \quad P_2 = -45 \times 9 = -405
\]

\[
P_3 = -15(6)/2 = -45, \quad P_4 = -30 \times 30 = -900
\]

\[
P = ZP = -300 - 45 - 405 - 900 = -1650
\]

From Fig. A10.11

\[
M_x = -(45 + 405) \times 9.375 + (300 + 900) 	imes 5.625 = 2331
\]

\[
M_y = 300 \times 12 + 45 \times 10 - 405 \times 6 - 900 - 12 = -9180
\]

\[
M_t = \frac{P}{A} + \frac{M_x}{I_x} + \frac{M_y}{I_y}
\]

POINT A. \( x = -12, \quad y = -20.625 \)

\[
M_t = -1650 + \frac{2531(20.625)}{3138} + \frac{-9180(-12)}{3456} = -36.06
\]

\[
M_A = M_0 - M_t = -30 - (-36.06) = 6.06 \text{ in.lb.}
\]

which checks solution 1.

Frame Point B. \( x = -12, \quad y = 9.375 \)

\[
M_t = -1650 \times 9.375 + \frac{-9180(-12)}{3456} = -12.23
\]

\[
M_B = M_0 - M_t = -30 - (-12.23) = -17.77 \text{ in.lb.}
\]

Frame Point C. \( x = 12, \quad y = 9.375 \)

\[
M_t = -1650 \times 9.375 + \frac{-9180(12)}{3456} = -76.0
\]

\[
M_C = M_0 - M_t = -90 - (-76.0) = -14.0 \text{ in.lb.}
\]

Frame Point D. \( x = 12, \quad y = -20.625 \)

\[
M_t = -1650 \times (-20.625) + \frac{-9180(12)}{3456} \neq -99.82
\]

\[
M_D = M_0 - M_t = -90 - (-99.82) = 9.82 \text{ in.lb.}
\]

Thus solution 2 checks solution 1. The student should solve this problem using other static conditions.

A10.4 Unsymmetrical Frames or Rings.

In applying the column analogy method to unsymmetrical frames and rings, the moment of the M/EI diagram must be taken about principal axes and the moments of inertia with respect to principal axes.

However, as explained for the elastic center method in Chapter A9, the moments and section properties with regard to centroidal axes can be used if \( M_x, M_y, I_x \) and \( I_y \) are modified to take care of the axis-symmetry of the structure. In Art. A9.2 it was shown that the redundant forces at the elastic center to unsymmetrical frame sections was, (see equations 11, 14, 15 of Art. A9.2),

\[
M_0 = \frac{Zy}{Zx} = \frac{P}{A} \text{ (same as for symmetrical frame)}
\]

\[
X_0 = \frac{Z_y y - Z_x x}{I_x} \left( 1 - \frac{I_y}{I_x + I_y} \right) = \text{----- (a)}
\]

\[
Y_0 = \frac{Z_x x - Z_y y}{I_y} \left( 1 - \frac{I_x}{I_x + I_y} \right) = \text{----- (b)}
\]

\[
M_A = M_0 - M_t = -30 - (-36.06) = 6.06 \text{ in.lb.}
\]

Frame Point B. \( x = -12, \quad y = 9.375 \)

\[
M_t = -1650 \times 9.375 + \frac{-9180(-12)}{3456} = -12.23
\]

\[
M_B = M_0 - M_t = -30 - (-12.23) = -17.77 \text{ in.lb.}
\]

Frame Point C. \( x = 12, \quad y = 9.375 \)

\[
M_t = -1650 \times 9.375 + \frac{-9180(12)}{3456} = -76.0
\]

\[
M_C = M_0 - M_t = -90 - (-76.0) = -14.0 \text{ in.lb.}
\]

Frame Point D. \( x = 12, \quad y = -20.625 \)

\[
M_t = -1650 \times (-20.625) + \frac{-9180(12)}{3456} \neq -99.82
\]

\[
M_D = M_0 - M_t = -90 - (-99.82) = 9.82 \text{ in.lb.}
\]

Thus solution 2 checks solution 1. The student should solve this problem using other static conditions.

A10.4 Unsymmetrical Frames or Rings.

In applying the column analogy method to unsymmetrical frames and rings, the moment of the M/EI diagram must be taken about principal axes and the moments of inertia with respect to principal axes.

However, as explained for the elastic center method in Chapter A9, the moments and section properties with regard to centroidal axes can be used if \( M_x, M_y, I_x \) and \( I_y \) are modified to take care of the axis-symmetry of the structure. In Art. A9.2 it was shown that the redundant forces at the elastic center to unsymmetrical frame sections was, (see equations 11, 14, 15 of Art. A9.2),

\[
M_0 = \frac{Zy}{Zx} = \frac{P}{A} \text{ (same as for symmetrical frame)}
\]

\[
X_0 = \frac{Z_y y - Z_x x}{I_x} \left( 1 - \frac{I_y}{I_x + I_y} \right) = \text{----- (a)}
\]

\[
Y_0 = \frac{Z_x x - Z_y y}{I_y} \left( 1 - \frac{I_x}{I_x + I_y} \right) = \text{----- (b)}
\]
As previously done in Art. A10.2 let,

\[ M_y = \Delta \bar{y} x \quad \text{and} \quad M_x = \Delta \bar{x} y \]

Furthermore, let,

\[ M_y = M_y \left( \frac{I_x}{I_y} \right) \quad \text{and} \quad M_x = M_x \left( \frac{I_y}{I_x} \right) \]

and \( k = \left( 1 - \frac{I_y}{I_x} \right) \)

Then substituting in equations (a) and (b) we obtain,

\[ X_0 = \frac{M_y - M_y}{k \frac{I_y}{I_x}}, \quad Y_0 = \frac{M_y - M_y}{k \frac{I_x}{I_y}} \]

From equation (2) we have,

\[ M_1 = M_0 + X_0 y + Y_0 x \]

Substituting values of \( X_0 \) and \( Y_0 \) into this equation, we obtain as the equation for \( M_1 \) the indeterminate moment in the column analogy method, the following -

\[ M_1 = \frac{P}{A} + \left( \frac{M_y - M_y}{k \frac{I_y}{I_x}} \right) y + \left( \frac{M_y - M_y}{k \frac{I_x}{I_y}} \right) x \]

The true moment is the same as for the symmetrical section, namely,

\[ M = M_0 - M_1 \]

Thus the solution of an unsymmetrical frame by the column analogy method follows the same procedure as for a symmetrical section except that equation (6) is used instead of equation (5).

A10.5 Example Problem - Unsymmetrical Section.

Fig. A10.12 shows a loaded unsymmetrical frame fixed at points A and D. Required, the true bending moments at points A, B, C and D. This problem is identical to example problem 1 of Art. A9.5 where a solution was given by the elastic center method.

\[ P_1 = -1440 \times 3/2 = -2160 \]
\[ P_2 = -1440 \times 15/1 = -21600 \]
\[ P_3 = -1080 \times 4.5/1 = -4860 \]
\[ ZP = -28620 \]
These loads act on the centerline of the frame members and through the centroid of the geometrical moment diagram shapes. These centroid locations are indicated by the heavy dots in Fig. A10.14 and their locations are given with respect to the centroidal x and y axes. The loads P1, P2, and P3 are now placed on the column in A10.13, acting upward because they are negative.

We now find the moments $M_x$ and $M_y$ which equal the moments of the loads P about the centroidal axes.

\[
M_x = -2160 \times 5.242 + 21600 \times 2.258 + 4860 \times 6.758 = 70290
\]

\[
M_y = 2160 \times 3.032 + 21600 \times 5.032 + 4860 \times 5.032 = 139700
\]

To solve equation (6) we must have the terms $M'_x$, $M'_y$, and k.

\[
M'_x = M_x \left( \frac{I_y}{I_x} \right) = 70290 \times 217.74/606.51 = 25534
\]

\[
M'_y = M_y \left( \frac{I_x}{I_y} \right) = 139700 \times 217.74/942.96 = 32258
\]

\[
k = \left(1 - \frac{I_y}{I_x} \right) = \left(1 - \frac{217.74}{606.51 \times 942.96} \right) = .9171
\]

Substituting values in equation (6) we obtain,

\[
M_1 = \frac{-28620 + (70290 - 32258) y}{0.9171 \times 606.51} + \frac{(139700 - 25534) x}{0.9171 \times 942.96}
\]

whence,

\[
M_1 = -923 + 68.37 y + 132.36 x
\]

For Frame Point A, \( x = -5.032, \ y = -9.758 \)

\[
M_1 = -923 + 68.37 (-9.758) + 132.36 (-5.032)
\]

\[
= -923 - 657.15 - 666.0 = -2256
\]

\[
M_A = M_0 - M_1 = -1440 - 1080 = -2520 \text{ in.lb.}
\]

For Frame Point B, \( x = -5.032, \ y = 5.242 \)

\[
M_1 = -923 + 68.37 \times 5.242 + 132.36 \times (-5.032)
\]

\[
= -923 + 358.39 - 666.0 = -1230.9
\]

\[
M_B = M_0 - M_1 = -1440 - (-1230.6) = -209.4 \text{ in.lb.}
\]

Frame Point C, \( x = 6.968, \ y = 8.242 \)

\[
M_1 = -323 + 68.37 \times 5.242 + 132.36 \times 6.968 = 357.7
\]

\[
M_C = M_0 - M_1 = 0 - (357.7) = -357.7 \text{ in.lb.}
\]

Frame Point D, \( x = 6.968, \ y = -4.758 \)

\[
M_1 = -323 + 68.37 (-4.758) + 132.36 \times 6.968 = -326
\]

\[
M_D = M_0 - M_1 = 0 - (-326) = 326 \text{ in.lb.}
\]

The above results check the solution of this same problem by the elastic center method in Art. A9.5 of Chapter A9. The student should solve this problem by choosing other static frame conditions.

A10.6 Problems

1. Determine the bending moment diagram for the loaded structures of Figs. A10.15 to A10.20.

For Frame A: \( w = 10 \text{ lb/in.} \)

For Frame B: \( w = 300 \text{ lb} \)

For Frame C: \( w = 400 \text{ lb} \)

2. Solve problems (2) and (3) at the end of Chapter A9, Art. A9.9.
CHAPTER AII
CONTINUOUS STRUCTURES-MOMENT DISTRIBUTION METHOD

All.1 Introduction. The moment distribution method was originated by Professor Hardy Cross.* The method is simple, rapid and particularly adapted to the solution of continuous structures of a high degree of redundancy, where it avoids the usual tedious algebraic manipulations of numerous equations. Furthermore, it possesses the merit of giving one a better conception of the true physical action of the structure in carrying its loads, a fact which is usually quite obscure in some methods of solution.

The method of procedure in the Cross method is in general the reverse of that used in the usual methods where the continuous structure is first made statically determinate by removing the redundant features and the value of the redundant then solved for which will provide the original continuity. In the Cross method each member of the structure is assumed in a definite restrained state. Continuity of the structure is thus maintained but the statics of the structure are unbalanced. The structure is then gradually released from its arbitrary assumed restrained state according to definite laws of continuity and statics until every part of the structure rests in its true state of equilibrium.

The general principles of the Cross method can best be explained by reference to a specific structure.

Fig. All.1 shows a continuous 2 span beam. Let it be required to determine the bending moment diagram. We first arbitrarily assume that each span is completely restrained against rotation at its ends. In the example selected ends A and C are already fixed so no restraint must be added to these points. Joint B is not fixed so this joint is imagined as locked so it cannot rotate. The bending moments which exist at the ends of each member under the assumed condition are then determined. Fig. All.2 shows the moment curves for this condition. (For calculation and formulas for fixed end moments see following article.) Fig. All.3 shows the general shape of the elastic curve under this assumed condition. It is noticed that continuity of the structure at B is maintained, however from the moment curves of Fig. All.2 it is found that the internal bending moments in the beams over support B are not statically balanced, or specifically there is an unbalance of 270. The next step is to statically balance this joint, so it is unlocked from its imaginary locked state and obviously joint B will rotate (See Fig. All.4) until equilibrium is established, that is, until resisting moments equal to 270 have been set up in the two beams at B. The question is now much of this moment is developed by each beam. The physical condition which establishes the ratio of this distribution to the two beams at B is the fact that the B end of both beams must rotate through the same angle and thus the unbalanced moment of 270 will be distributed between the two beams in proportion to their ability of resisting the rotation of their B end thru a common angle. This physical characteristic of a beam is referred to as its stiffness. Thus let it be considered that the stiffness factors of the beam BA and BC are such that 162 is distributed to BC and -108 to BA as shown in Fig. All.4. (The question of stiffness factors is discussed in a following article.)

Referring to Fig. All.4 again it is evident that when the elastic curve rotates over joint B that it tends to rotate the far ends of the beams at A and C, but since these joints are fixed, this rotation at A and C is prevented or moments at A and C are produced. These moments produced at A and C due to rotation at B are referred to as carry-over moments. As shown by the obvious curvature of the elastic curves (Fig. All.4), the carry-over moment is of opposite sign to the distributed moment at the rotating end. The ratio of the carry-over moment to the distributed moment, referred to as the carry-over factor, depends on the physical properties of the beam and the degree of restraint of its far end. (Carry-over factors are discussed in a following article. For a beam of constant section and fixed at the far end, the carry-over factor is -1/2). In figure All.4 a factor of -1/2 has been assumed which gives carry-over moments of 54 and -81 to A and C respectively. To obtain the final end moments we add the original fixed end moments, the distributed balancing moments and the carry-over moments as shown in Fig. All.4. With the indeterminate moments thus determined, the question of shear, reactions and span moments follow as a matter of statics.

All.2 Definitions and Derivations of Terms

1. Fixed-end moments:
   By "fixed end moment" is meant the moment which would exist at the ends of a member if those ends were fixed against rotation.

2. Stiffness Factor:
   The stiffness factor of a member is a value proportional to the magnitude of a couple that must be applied at one end of a member to cause unit rotation of that end, both ends of the member being assumed to have no movement of
THE MOMENT DISTRIBUTION METHOD

3. **Carry-over Factor:**
If a beam is simply supported at one end
and restrained to some degree at the other, and
a moment is applied at the simply supported end,
a moment is developed at the restrained end.
The carry-over factor is the ratio of the moment
at the restrained end to that at the
simply supported end. For a prismatic beam
without axial load the carry-over factor is
0.5 when far end is fixed. The letter \( C \)
will be used to designate carry-over factor.

4. **Distribution Factor:**
If a moment is applied at a joint where two
or more members are rigidly connected the dis-
tribution factor for each member is the propor-
tional part of the applied moment that is resisted
by that member. The distribution factor for any
member which will be given the symbol \( D \) equals
\( K/2X \), where \( K \) equals stiffness factor of a par-
ticular member and \( 2X \) equals the sum of \( K \) values
for all the joint members. The sum of the \( D \)
values for any joint must equal unity.

5. **Sign Convention:**
Due to the fact that in many problems where
members come into a joint from all directions as
is commonly found in airplane structure, the cus-
tomary sign convention for moments may produce
confusion in applying the moment distribution
method. The following sign convention is used in
this book: a clockwise moment acting on the end
of a member is positive, a counterclockwise one
is negative. It follows that a moment tending to
rotate a joint clockwise is negative. It should
be understood that when indeterminate continuity
moments are determined by the moment dis-
tribution method using the above adopted sign convention,
that they should be transferred into the conven-
tional signs before proceeding with the design of
the member proper.

The following sketches illustrate the adapted
sign convention.

**Illustrations of Sign Conventions for**
**End Moments.**

**Example 1**
Fixed end beam with
lateral loads.

**Conventional**
+

+ tension in bottom fibers is positive bending
moment.

**Adopted sign**

---

**convention**

moments which tend to
rotate end of member
clockwise are positive.

**Example 2**
Translation of supports
of fixed ended beam.

**Conventional**

---

**Adopted sign**

---
Example 3

External applied moment at joint in structure

adopted sign

Moments which tend to rotate joint counterclockwise are positive.

### A11.3 Calculation of Fixed End Moments

Since the fixed end moments are statically indeterminate, additional facts must be obtained from the laws of continuity in order to solve for them. In this book the theorem of area moments will be used to illustrate the calculation of the fixed end moments as well as the other terms which are used in the moment distribution method. (Ref. Chapter A7)

The following well known principles or theorem of area moments will be used:-

1. The deflection of any point "A" on the elastic curve of a beam away from a tangent to the elastic curve at another point "B" is equal to the moment of the area of the M diagram between the points A and B about point "A".
2. The change in slope between two points "A" and "B" on the elastic curve of a beam is equal to the area of the M diagram between the two points "A" and "B".

The "area moment" theorems will be illustrated by the applications to the solution of a simple problem. Fig. A11.5 shows a simply supported beam of constant moment of inertia and modulus of elasticity carrying a single concentrated load. Figs. A11.6 and A11.7 show the static moment curve and the shape of the beam elastic curve. Now assume that the ends are fixed as shown in Fig. A11.8 and let the value of the fixed end moments be required. Fig. A11.9 shows the shape of the final moment curves made up of the static moment curve and the unknown trapezoidal moment curve formed by the unknown end moments. Fig. A11.10 shows the shape of the elastic curve, the slope at the two supports being made zero by fixity at these points.

### Figs. A11.11 and A11.12 show the static and continuity moment areas, the total area of each portion and its c.g. location.

Substituting in Equation (1)
$$M = \frac{Pab}{2} + \frac{M_{a}}{2} + \frac{M_{b}}{2} = 0$$

and from equation (2)
$$M_{a} \text{ about left end} = \frac{Pab}{2} \cdot \frac{L + a}{3} + \frac{M_{a}}{2} \cdot \frac{L + b}{3}$$

The values of $M_1$ and $M_a$ for any value of $a$ or $b$ can now be found by solving equations (3) and (4)

Table A11.1 gives a summary of beam fixed end moments for most of the loadings encountered in routine design and analysis.

<table>
<thead>
<tr>
<th>Table A11.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$wl^2$</td>
</tr>
<tr>
<td>$12$</td>
</tr>
<tr>
<td>$wa^2 (6l^2 - 3aL + 3a^2)$</td>
</tr>
<tr>
<td>$Wla$</td>
</tr>
<tr>
<td>$L^2$</td>
</tr>
<tr>
<td>$u$</td>
</tr>
<tr>
<td>$12C$</td>
</tr>
</tbody>
</table>
Table All.1 - Continued

\[ \begin{align*}
\frac{wL^2}{20} & \quad \frac{wL^2}{30} \\
\frac{wa^2}{10L^2 - 10aL + 3a} & \quad \frac{wa^2}{20L^2 - 15aL + 6a} \\
\frac{M_B}{L} (3a^2 - 1) & \quad \frac{M_B}{L} (2b^2 - 1) \\
\frac{1}{L} \int_0^L wx(L-x)^n dx & \quad \frac{1}{L} \int_0^L wx(L-x)^n dx
\end{align*} \]

(Ref. N.A.C.A. T.N. #534)

All.4 Stiffness Factor; Carry over Factor: -

Derivation of:

(For definitions of these terms see page All.2)

Consider the beam of Figure All.13. By Mohr’s theorem (see Art. A7.12), the slope of a tangent to the elastic curve at point A relative to a line AB is equal to the shear at A on a simply supported beam AB due to the M curve between A and B acting as a load.

Thus

\[ \theta_A = - \left( \frac{ML}{2EI} \right) x / 3 \]  \[ \text{(Positive Shear = Neg. Slope)} \]

\[ \theta_B = \frac{ML}{2EI} \times 2/3 = \frac{ML}{2EI} \]

Let \( \theta_B = \text{unity} \)

Then \( L = \frac{ML}{2EI} \) or \( M_B = \frac{3EI}{L} = \text{stiffness factor of} \)

Beam BA of Fig. All.13. A moment applied at B produces no moment at A since end A is freely supported. Thus the carry-over factor for a beam freely supported at its far end is obviously zero. Consider the beam of Fig. All.14. Due to complete fixity at end A, the slope of the elastic curve at A is zero.

\[ \theta_A = \frac{MaL}{2EI} . 1/3 + \frac{MaL}{2EI} . 2/3 = 0 \]

or

\[ M_A = - \frac{Ma}{2} \]

Thus the carry over factor for a beam fixed at its far end is 1/2. Using the conventional moment signs, the carry over moment is of the opposite sign as shown by the above equation. However, for our adopted sign convention inspection of the shape of the elastic curve as shown in Fig. All.14 tells us that the sign of the carry-over moment is of the same sign as the rotating moment at the near end. That is, the moment acting on each end of the member is in the same direction, and therefore of the same sign.

\[ \begin{align*}
\theta_A = 0 & \quad \theta_B = \frac{MaL}{2EI} . \frac{2}{3} + \frac{MaL}{2EI} . \frac{1}{3}
\end{align*} \]

Then \( \theta_B = \frac{MaL}{2EI} - \frac{MaL}{2EI} = \frac{MaL}{2EI} \)

Let \( \theta_B = \text{unity}, \) then \( M_B = \frac{4EI}{L} = \text{stiffness factor of} \)

beam BA of Fig. All.14.

A comparison of the stiffness factor of this beam to that of Fig. All.13 shows that the stiff-
ness factor of a beam freely supported at its far end is 3/4 as great as one fixed at its far end. Furthermore, in one case the carry-over factor is zero and in the other case it is 1/2. It is therefore obvious that the values of these two terms depend in part upon the restraint of degree of fixation of the far end of the beam.

### 11.5 General Expressions for Stiffness and Carry-over Factor in Terms of Fixation Factor (F)

In the beam of Fig. 11.4(F) fixation factor at A was unity since beam had been taken as completely fixed at A. It was found that:

\[ M_B = \frac{4EI\theta_B}{L} \text{ and } M_A = -\frac{2EI\theta_B}{L} \]

Take \( \theta_B = \) unity and let \( EI = K \) for simplicity.

Then \( M_B = 4K \)
\[ M_A = 2K \]

Likewise the results for the beam of Fig. 11.13 give

\[ M_B = 3K \]
\[ M_A = 0 \]

Figs. 11.15 and 11.16 show these results. Fig. 11.17 shows the general case, the fixation factor at A being F. The difference between Figs. 11.15 and 11.17 is that the slope at end A has changed but \( \theta_B \) the slope at end B remains the same.

\[ F = 1 \text{ to } 0 \quad M_B = 4K \]
\[ M_A = 2K \]

Figs. 11.15 and 11.16

\[ F = 0 \text{ to } 1 \quad M_B = 3K \]
\[ M_A = 0 \]

Figs. 11.16

The change in moment at end A when changing beam 11.15 to that of 11.17 = 2K - 2KF = 2K(1-F). Since \( \theta_B \) is kept the same value, one-half of the moment change at end A appears at end B but of opposite sign, or

\[ M_B = 4K - 2KF = 2K(1-F) \]
\[ M_A = 2K(3+F) \]

Figs. 11.17

Thus the general expression for the stiffness factor of a beam of constant section equals \( \frac{EI}{L} \) (3+F). The carry-over factor from 3 to A =

\[ M_B = -\frac{2KF}{K(3+F)} = -\frac{2F}{3+F} \]

which is the general expression for carry-over factor for a degree of fixation F.

### 11.5a Example Problems

To obtain a definite conception of the true mechanics of the "cross" method, the reader is advised to follow thru the detailed solution of the following simple problems. In these problems, the moment of inertia in any span has been taken as constant and all joints have been assumed to undergo no translation. Problems involving variables I and joint translation will be considered later.

#### Example

<table>
<thead>
<tr>
<th>Problem #1</th>
<th>54/In.</th>
<th>54/In.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>51.5</td>
<td>B</td>
</tr>
<tr>
<td>E</td>
<td>51.5</td>
<td>51.5</td>
</tr>
</tbody>
</table>

- **Stiffness Factor** = 6 + K
- **Distribution Factor** = 1
- **Carry Over Factor** = 5/3
- **Fixed End Moments** = 0
- **1st Balancing** = 115
- **Carry Over 1/2**
- **End Balancing** = 0
- **Final Moments** = 0
- **Conventional Moment Signs**

*Overhang Moment: 24 x 51.5 = 123.75 + 28 x 51.5 = 1717.5*
end of a member clockwise is positive.

We now begin the solution proper by first unlocking joint B from its assumed fixed state. We find a moment of -863 on one side and 768 on the other side of joint or a static unbalance of -115. Joint B therefore will rotate until a resisting moment of 115 is set up in the members BA and BC.

The resistance of these members to a rotation of Joint B is proportional to their stiffness. The distribution factor based on the stiffness factors is 0 for BA and 1 for BC. Thus 1 x 115 = 115 is distributed to BC at B and 0 x 115 = 0 to BA. Joint B is now imagined as again locked against rotation and we proceed to Joint C, which is now released from its assumed locked state. Since the joint is already statically balanced, no rotation takes place and the distributing balancing moment to each span is zero. Next proceed to joint D, and release it. The unbalanced moment is, 115 so the joint is balanced by distributing -115 between DB and DC as explained above for joint B.

As pointed out in Art. 11.5, when we rotate one end of a beam it tends to rotate the far fixed end of the beam by exerting a moment equal to some proportion of the moment causing rotation at the near end. For beams of constant section and fixed at their far ends, the carry over factor is 1/2 as explained before. Thus the distributing balancing moments in line 4 produce the carry-over moments as shown in line 5 of the table. This completes one cycle of the moment distribution method, which is repeated until there is nothing to balance or carry-over, or in other words until all artificial restraints have been removed and the structure rests in its true state of equilibrium.

To continue with the second cycle, go back to joint B and release it again from its assumed locked state. There is an unbalance since the carry over moment from point C was zero, thus there is nothing to distribute or carry-over. Proceed to point C, releasing the joint, we find it balanced under the carry-over moments of 57.5 and -57.5. Thus the distributing balancing moments are zero. Joint D is likewise in balance since the carry-over moment from C is zero. All joints can now be released without any rotation since all joints are in equilibrium. To obtain the final moments we add the original fixed end moments plus all distributed balancing and carry over moments.

Example Problem #2

### Example Problem #2

<table>
<thead>
<tr>
<th>1k/m.</th>
<th>1k/m.</th>
<th>EZ is Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

#### Table

<table>
<thead>
<tr>
<th>Slope Factor</th>
<th>EZ g</th>
<th>5.314</th>
<th>6.354</th>
<th>4.319</th>
<th>0.094</th>
<th>3.219</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrimination Factor</td>
<td>1.0</td>
<td>0.5</td>
<td>0.50</td>
<td>0.50</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Carry-Over C</td>
<td>0.5</td>
<td>0.5</td>
<td>0.50</td>
<td>0.50</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Fixed End Moments</td>
<td>-325 766</td>
<td>642 762</td>
<td>762 410</td>
<td>410 410</td>
<td>410 410</td>
<td>410 410</td>
</tr>
<tr>
<td>1st Balancing</td>
<td>0.115</td>
<td>1.059</td>
<td>1.059</td>
<td>1.059</td>
<td>1.059</td>
<td>1.059</td>
</tr>
<tr>
<td>Carry-Over</td>
<td>-1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
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<tr>
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<td>1.060</td>
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<tr>
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</tr>
<tr>
<td>Carry-Over</td>
<td>-1.50</td>
<td>1.50</td>
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<tr>
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<tr>
<td>5th Balancing</td>
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</tbody>
</table>

### Example Problem #2

Problem 2 is similar to problem 1 but two spans have been added. We first assume all joints locked against rotation. The stiffness factor of each span is proportional to EI/L or 1/L since EI is constant. The carry-over factor is 1/2 as in previous example. Fixed end moments are calculated as shown. Unlock joint B, the unbalanced moment is -115. Balance the joint by distributing 1 x 115 to BC and zero to BA. Proceed to joint C,
and unlock, all other joints remaining fixed against rotation. The unbalanced moment is 
\((-768 + 432) = -336\). Balance by distributing 
\(428 \times 336 = 144\) to CB and \(372 \times 336 = 126\) to
CD. Proceed to joint B, and release. The un
balanced moment is zero which means that joint B 
is in equilibrium, thus no distribution is 
necessary. Proceed to joint E and F and balance 
in a similar manner. The distributed moments 
will be the same as the values for joints B and 
C due to symmetry of structure and loading, 
evertheless, the signs will be opposite under our 
adopted sign conventions. The next step is the 
carry-over moments which are equal to 1/2 the 
distributed balancing moments. This operation 
is shown clearly in the table. Values of all 
 moments are given only to first decimal place. 
The first cycle has now been completed. Cycle 
 two is started by again releasing joint B. We 
find that joint B has been unbalanced, the carry-
over moment of 72. Balance the joint by distri-
buting \(72 \times 1 = 72\) to BC and zero to BA. 
Proceed to joint C. The unbalanced moment is 
57.5. Balance by distributing \(-57.5 \times 428 = 
-24.6\) to CB and the remainder \(-32.9\) to CD. 
Proceed to joint D. There is no unbalance at 
this joint since the carry-over moments are in 
balance, thus no distribution is necessary.
Proceed to joints E and F in a similar manner.

The carry over moments equal to 1/2 of the 
2nd set of balancing distributed moments are now 
carried over as shown in the table. The second 
cycle has now been completed. This operation 
has been repeated five times in the solution 
shown, or until the values of the balancing and 
carry over moments are quite small or negligible. 

The final moments equal the algebraic summation 
of the original fixed end moments plus all dis-
tributed and carry over moments. One require-
ment of the final end moments at any joint is 
that the algebraic sum must equal zero. The 
other requirement consistent with the common 
slope to all members at any joint is given by 
equation (5) of Art. All.8. The results at 
joint C will be checked using this equation.

\[
\begin{align*}
\Delta M_{BC} & = \frac{1}{2} \Delta M_{C} = \frac{1}{2} K_{CB} \\
\Delta M_{CB} & = -\frac{1}{2} \Delta M_{C} = -\frac{1}{2} K_{CB} \\
\Delta M_{AB} & = -\frac{1}{2} \Delta M_{C} = -\frac{1}{2} K_{CB} \\
\end{align*}
\]

Subt. values
\[
\begin{align*}
(610.4 - 432) & = 0.344 - (-432) \quad 178.6 - 44 \\
-(-768) & = -0.838 - 768 \quad 157.8 - 57.5 = \]
\[
1.343
\]

Ratio of stiffness factors \(K_{CB} = 0.0139 \quad K_{CB} = 0.0104 \quad 1.34\). Thus the distribution is according to the K 
ratios of the adjacent members.

Simplifying Modifications - Example Problem #3

The solution as given in Problem #2 represents 
the "original" method in its fundamental and 
most elementary detailed form. Many modific-
ations of the general method have been presented, 
in the most part for the purpose of eliminating 
part of the arithmetic or the number of cycles
necessary. These modifications usually involve 
rather long expressions for expressing the stiffness 
and carry over factors of a member in terms 
of the fixation given by adjacent members. It is 
 felt that it is best to keep the method in its 
simpest form which means that very little is to 
be remembered and then the method can be used in 
frequently without refreshing one's mind as to 
many required formulas or equations.

There are however several quite simple modifi-
cations which are easily understood and re-
membered and which reduce the amount of arithmetic 
required considerably.

For example in Problem #2, joints B and F 
are in reality freely supported, thus it is need-
less arithmetic to continue locking and unlocking 
a joint which is definitely free to rotate. 
Likewise due to symmetry of structure and load-
ing it is only necessary to solve one half of the 
structure. Due to symmetry joint D does not ro-
tate and thus can be considered fixed, which 
eliminates the repeated locking and unlocking of 
this joint.

A second solution of Problem #2 is given in 
Example Problem #3. As before we assume each 
joint locked and calculate the fixed end moments. 
Now release or unlock joint B and balance as ex-
plained in previous example #2. Before proceed-
ing to joint C, carry over to C from B the car-
yover moment equal to 135 \(x\) 1/2 = 97.5. Joint 
D is now left free to rotate or in its natural 
condition. Proceed to joint C and unlock. The 
unbalanced moment \((-768 + 432 + 57.5) = -278.5 
or 278.5 is necessary for equilibrium. This moment 
is distributed between two beams, CD which is 
fixed at its far end D and CB which is freely 
supported at the far end B. The stiffness fac-
tor is equal to \((3 + F) / E / L\) (See Art. All.8).

Hence for CD stiffness factor \(= (3 + 1)
\)
\(E / L = 4\) \(E / L\). For CB stiffness factor \(= (3 + 0)
\)
\(E / L = 3\) \(E / L\) or in other words the stiffness of 
a beam freely supported at its far end is 3/4 
as great as when fixed at its far end. Thus the 
stiffness factor of CB at C is .75 x .0078 = 
.0075. The carry-over factor C to B is zero 
since B is left free to rotate. (See Art. All.8)

Example Problem #3. Simplified Solution of 
Problem #2

[Diagram showing forces and measurements]

[Table showing forces and measurements]

Stiffness Factor (A) 0.0064 0.0046 0.0019 0.0019 / 1/72 = 0.0019
Distribution Factor 0.010 0.010 0.010 0.010
Carry-over factor (C) 0.5 0.5 0.5 0.5
Fixed end moments -687,712 -788,712 -481
Sum of carry-over 100 100 100 100
Carry-over to C 0.5 0.5 0.5 0.5
Carry-over to B 0.5 0.5 0.5 0.5
Sum of carry-over 1 1 1 1

Conventional right 1 1 1 1
The stiffness factor of the fixed support is infinite, that is a rigid support has infinite resistance to rotation.

Joint C is now balanced by distributing 278.5 x \( \frac{36}{100} = 101.1 \) to CB and 278.5 x \( \frac{54}{178.4} = 89.2 \) to CD. Now carry over to joint B, \( 0 \times 100.1 = 0 \) and \( 0.5 \times 178.4 = 89.2 \) to D. Proceed to joint D and release it from its assumed locked state. The unbalanced moment is \( -432 + 89.2 = 342.8 \), we balance by distributing 342.8 between DC which has a stiffness of 0.159 and the support E which has an infinite stiffness or zero goes to DC and 342.8 goes to rigid support. The carry over moment to C from D is zero since \( 0.5 \times 0 = 0 \). The final moments thus equals the summations as shown which of course are equal to the results shown in Problem #2.

**Example Problem #4**

Problem 4 is similar to Problem #2 and #3, except that the support at D is assumed as having 50 percent fixity. Thus 50 percent of any moment at this point produces rotation of the member DC at D.

In continuous wing beams, which fasten together by fittings on a support it is commonly required that the beam be considered as being fully continuous and also that the degree of continuity be taken as 50 percent. Solution 1 of Problem 4 is a detailed solution. The only change that has been made is in the stiffness factor of the support E, which has been taken as equal to the beam DC, thus any unbalanced moment at this point is equally divided between the beam and the support.

Solution 2 is a modified solution which eliminates considerable arithmetic. Thus it is unnecessary to lock and unlock joint B and D since we know definitely that one is freely supported and the other 50 percent fixed. Therefore, once we have released these points from their assumed fixed state, we leave them in their natural state. The stiffness and carry-over factors for beams CB and CD must then be determined for these beams with their modified end conditions.

By reference to the fundamental equations for stiffness and carry-over factors in Art. 11.5, it is readily seen that the stiffness factor for CB is \( \frac{3}{4} \) as much as when fixed at its far end B and the carry over moment is zero. For beam CD the stiffness is \( \frac{7}{8} \) as much as when fixed at end D, and the carry over factor is \( \frac{1}{4} \) or \( \frac{3}{4} \times \frac{7}{8} = \frac{3}{8} \). With these modifications the solution is carried thru with a relatively small number of steps. Thus in solution 2, joint B is unlocked. The unbalanced moment of \(-115\) is balanced statically by distributing 115 to BC. The carry-over moment of \( \frac{3}{8} \times 115 = 57.5 \) is carried over to C as shown. Joint B is now left unlocked or free to rotate. Joint D is unlocked next. The unbalanced moment is \(-432\). It is balanced by distributing 216 to DC and 216 to E since support at E is considered to give 50% fixity. The carry-over moment of equals \(-768 + 432 = 57.5 + 108\) = -170.5, or 170.5 is necessary for equilibrium. Joint C is balanced by distributing \(-326 \times 170.5 = 66.3\) to CB and the remainder of 103.7 to CD. The carry-over moment to B is zero and to D it equals 103.7 \times \( \frac{1}{4} \times 170.5 = 29.5 \). The final moments in solution 2 are slightly different than solution 1. If another cycle had been added in solution 1 the discrepancy would be considerably smaller.

**Example Problem #2**

- CB \( \frac{3}{4} \times 100.1 \), BC \( \frac{3}{4} \times 178.4 \), CD \( \frac{3}{4} \times 89.2 \), E \( \frac{3}{4} \times 342.8 \) 50% Fixed.
- Fixed Moments: 89.2, 66.3
- Carry-Over Factor: 1/4
- Moment Distribution: BC = 103.7, CD = 29.5
- Conventional Signs: +

**Solution #1**

- CB \( \frac{3}{4} \times 100.1 \), BC \( \frac{3}{4} \times 178.4 \), CD \( \frac{3}{4} \times 89.2 \), E \( \frac{3}{4} \times 342.8 \) 50% Fixed.
- Fixed Moments: 89.2, 66.3
- Carry-Over Factor: 1/4
- Moment Distribution: BC = 103.7, CD = 29.5
- Conventional Signs: +

**Continuous Beams with Yielding or Deflected Supports**

In wing, elevator and rudder beams the support points usually deflect due to the deformation of the supporting struts or wires in the case of a wing, or to the deflection of the stabilizer or fin in the case of an elevator and rudder beams. If these beams are continuous this deflection of their support points causes additional bending moments in the beams. The moment distribution method can of course be used to find the additional moments due to this deflection. Thus Example Problem #5, shows a solution illustrating a problem which involves the deflecting of the supports of a continuous beam. Due to symmetry of structure and loading, the slope at B is zero or the beam may be considered fixed at
Joint D. Since the moment of inertia is constant and the spans are constant, the relative stiffness factor of the beam is 1. In the solution shown since beam is freely supported at B this joint is left free to rotate after releasing and thus the stiffness factor of beam CB is 3/4 x 1 = 3/4, when compared to one having full fixity at B.

Since the first step in the solution proper is to assume the joints fixed against rotation, it is evident that deflecting one support relative to an adjacent support will produce moments at the ends which are assumed fixed against rotation.

![Diagram](image)

**Fig. A11.18**

Principal deflection

\[ d = \frac{M L}{4EI} \quad \text{or} \quad d = \frac{ML^2}{8EI} \]

hence

\[ M = \frac{6EPd}{L^2} \]

the magnitude for the fixed end moment due to a transverse support settlement of \( d \).

Example Problem #5. Continuous beam with deflected supports.

### General Data:

<table>
<thead>
<tr>
<th>Data</th>
<th>Value</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 #/in.</td>
<td>4 #/in.</td>
<td>3 #/in.</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Fixed due to symmetry</td>
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<td></td>
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</table>

Solution:

<table>
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<tr>
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<th>Value</th>
<th>Notes</th>
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</thead>
<tbody>
<tr>
<td>Stiffness Factor X</td>
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</tr>
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<tr>
<td>Carry-over Factor</td>
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<td>1/2</td>
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<td>Fixed End Moments Due to Lateral Load</td>
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<td>Due to support deflection</td>
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<td>390</td>
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<td></td>
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<tr>
<td>Balance joint C</td>
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<td></td>
</tr>
<tr>
<td>Balance joint D</td>
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<td>-322</td>
</tr>
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<tr>
<td>Final Moments</td>
<td>50</td>
<td>580</td>
</tr>
</tbody>
</table>

### Fixed End Moments Due to Support Movement

From Art. A11.8 \( M = \frac{6EPd}{L^2} \)

For Span BC:

\[ M_{bc} = \frac{40^2}{60} (5 \times 3 + 1) = 426 \text{ in. lb.} \]

\[ M_{cb} = \frac{40^2}{60} (5 \times 3 + 1.5) = 440 \text{ in. lb.} \]

For Span CD:

\[ M_{dc} = \frac{40^2}{60} (5 \times 3.5 + 1.5) = 507 \text{ in. lb.} \]

Since the fixed end moments are due to both lateral loads and support deflection, the values as listed in the solution table will be explained in detail.

**Fixed End Moment For Lateral Beam Loading**

The distributed airload is trapezoidal in shape. The fixed-end moments for a trapezoidal loading from Table A11.4 are:

\[ M_{l2} = \frac{L^2}{60} (5u + 2v) \quad \text{(See Fig. (a))} \]

\[ M_{l1} = \frac{L^2}{60} (5u + 3v) \quad \text{(See Fig. (a))} \]

For Span BC:

\[ M_{bc} = \frac{40^2}{60} (5 \times 3 + 1) = 426 \text{ in. lb.} \]

\[ M_{cb} = \frac{40^2}{60} (5 \times 3 + 1.5) = 440 \text{ in. lb.} \]

For Span CD:

\[ M_{dc} = \frac{40^2}{60} (5 \times 3.5 + 1.5) = 507 \text{ in. lb.} \]

For signs of the moments due to these deflections see Art A11.2. Having determined the fixed end moments the general distributing and carrying over process follows as indicated in the solution table. Thus at joint B, the unbalanced moment = (50-426+390) = 14. Balance by distributing -14 x 1 = -14 to BC and zero to BA. Carry over .5 x -14 = -7 to C. Considering joint C, the unbalanced moment = (440+390-234-494-7) = 563. Balance by distributing -563 x .571 = 322 to CD and -563 x 429 = 241 to CB. Carry over .5 x -322 = -161 to D. At joint D the unbalanced moment = (507+234+161) = 800. This is balanced by distributing zero to DC and -800 to the fixed support.

are not practicable because of the large number of equations that must be solved to obtain values for the many unknowns.

Solution for Condition I. Fixed at Ends A and B

Referring to Fig. All.20, all joints are assumed locked against rotation or fixed. The vertical axle load of 6000 lb. produces a counterclockwise moment of 3 x 6000 = 18000 in. lb. about joint 0. The sign is positive (See Art. All.2). Release or unlock joint 0, the unbalanced moment is 18000 or -18000 is required for static equilibrium of joint 0. Joint 0 is balanced by distributing -18000(.464/.464 + .369) = -10630 to member OB and the remainder or -18000(.369/.464 + .369) = -7970 to OA. These distributed balancing moments at 0 produce carry over moments at A and B.

Thus carry over to B, .5 x - 10630 = -5315
and carry over to A, .5 x -7970 = -3985
Proceeding to joint A which is a fixed joint, the unbalanced moment of -3985 is balanced entirely by the rigid support, or no rotation takes place when joint is released from its imaginary fixed state. Similar action takes place at joint B. The final end moments are as shown in the Figure.

Solution for Condition II. Ends A and B Pinned

For this condition the ends A and B are freely supported. Instead of locking and unlocking these joints which are definitely known to be free to be freely supported, they will be left in their true state. Thus the carry over moments from end 0 will be zero. Since A and B are both pinned, the relation between the relative stiffness factors of members OA and OB remain the same as in condition I, thus the same K or stiffness factors that were used in condition I can be used in distributing moments at joint 0. Joint 0 is balanced in same manner as condition I but with zero carry over moments to A and B.

Solution for Condition III. A is 50% Fixed and B is Pinned.

Since each member has a different degree of fixity at its upper end, the stiffness and carry over factors will be considered in detail. In condition I since both members were fixed at their upper ends the relative stiffness factor of each member was proportional to 1/L for the member and this ratio was used. The general expression for stiffness factor is \[ K = \frac{EI}{L} \cdot (3 + F) / L \] carry over factor = \[ 2F - \frac{3 + F}{3} \]

For member OA, \[ K = \frac{EI}{L} \cdot (3 + 0.5) = \frac{3.5 EI}{L} \cdot 3.5 \cdot 1105 E = 0.0129 E \]

C.O. Factor from 0 to A = \[ 2 \times 0.5 = 0.286 \]

For member OB

\[ K = \frac{EI}{L} \cdot (3 + 0) = \frac{3 \cdot 135 E \cdot X}{40} = 0.0139 E \]

C.O. Factor from 0 to B = \[ 2 \times 0 = 0 \]

Considering joint 0 in Fig. All.20 the external moment of 18000 in. lb. is balanced by distributing -18000 between the two members in proportion to their stiffness factors. Hence -18000(.0129/ .0129 + .0139) = -9350 in lbs. is resisted by OA and the remainder of -18000(.0139/.0129 + .0139) = -9350 to OB. The carry over moment from 0 to A = \[ 0.286 \times -9350 = -2475 \] and zero from 0 to B. (See Fig. All.20)

Example Problem #7

Fig. All.21 shows a structure composed of 3 members. Member AO is subjected to a transverse load of 120#. Joint A is fixed, B is freely supported C is 25 percent fixed and joint O is considered to maintain continuity between all members at 0. The end moments on the three members due to the transverse loading on member AO will be determined.

Solution #1. Fig. All.21 gives a solution using the "Cross" method in its fundamental unmodified state. The solution is started by assuming all three members are fixed-ended. The relative stiffness factor K of each member is therefore proportional to 1/L of each member. These K values are listed in Fig. All.21. The distribution
factor D for each member at each joint which equal K/2K is recorded in on each member around each joint. Thus any balancing moment is distributed between the joint members as per these distribution factors. The carry over factors for all members is 1/2. The fixed end moments due to external loading are computed for the three members. For member AO, the fixed end moments equal \( FL/8 = 120 \times 20/8 = 300 \text{ in} \cdot \text{lb} \). The other two members having no transverse loading, the fixed end moments are zero.

In this solution the order of joint consideration has been AOBC and repeat. Starting with joint A the joint is released but since the member AO is actually held by a fixed support, no rotation takes place and the balancing moment of 300 is provided entirely by the support and zero by the member AO. The carry over moment C to O is zero. Releasing joint O, the unbalanced moment of 300 is balanced by distributing -300 between the three members according to their D values, thus -300 x .416 = -125 to OA; -300 x .168 = -50 to OB and -125 to OC. To prevent confusion it is recommended that a line be drawn under all distributed balancing moments, thus any values above these lines need not be given further consideration and any values below the lines need be considered in later balancing of the joints.

Immediately after distributing the moments at joint O the proper carry over moments should be taken over to the far end of each member, thus -62.5 to A, -62.5 to B and -25 to C. Joint B is next considered. The unbalanced moment is -62.5 and it is balanced by distributing 62.5 to 80 since the pin support has zero stiffness, or no resistance to rotation. A line is drawn under the 62.5 and the carry over moment of 31.25 is placed at O. Joint C is considered next. The unbalanced moment of -25 is balanced by distributing .75 x 25 = 18.75 to OC and the remainder of 6.25 to the support, since the fixity of the support at C has been assumed as 25 percent. A line is drawn under the 18.75 and the carry over moment of 9.37 is taken over to O. One cycle has now been completed. Returning to joint A, we find -62.5 below the line. This is balanced by distributing zero to OA and 62.5 to the fixed support. A line is drawn under the zero distributed moment to AO and the carry over moment of zero is placed at O. Considering joint O for the second time the unbalanced moment is 9.37 + 31.25 = 40.62 or the sum of all values below the column horizontal lines. The joint is balanced by distributing -17 to OA and OB and -6.25 to OC. Lines are drawn under these balancing moments as shown in Fig. A11.21 and the carry over moments are taken over to the far ends before proceeding to joint B.

This general process is repeated until joint A has been balanced 5 times and the other joints 4 times each, as indicated in the figure the distribution values have become quite small and it is evident that a high degree of accuracy has been obtained. The final end moments at each joint equal the algebraic sum of the values in each column. A double line is placed above the final moments as a distinguishing symbol. In the figure the letters b and c refer to balancing and carry over moments, the subscripts referring to the member of the balancing or carry over operation. Any order of joint consideration can be used in reaching the same result.

Solution #2 of Problem 7

Fig. A11.22 gives a second solution. With the end conditions known at A, B and C, the modified stiffness factors of the members can be found together with the modified carry over factors, thus making it necessary to balance joint O only once and carry over this final far end
moments of each member. The figure gives the
calculation of the modified or actual stiffness
and carry over factors. With these known the
solution is started as before by computing the
fixed end moments due to transverse loading on
member A0. Joints B and C are released and
since no fixed end moments exist, no balancing
is required and the joints are left in their
true state of restraint instead of locking and
unlocking as in solution #1. Releasing joint O
from its imaginary fixed state the unbalanced
moment is 300 which is balanced by distributing
-300 between the 3 connecting members according
to the new distribution factors at joint O.
Thus - 300 x .482 = - 145 to OA; - 300 x .361 =
- 108.2 to OB and - 300 x .157 = - 47 to OC.
The carry over moment to A = - 145 x .5 = - 72.5
to B = - 108.2 x 0 = 0 and - 47 x .154 = - 7.2 to
C.

Example Problem #3

Figure A11.23 shows the forward portion of a
fuselage side truss. Due to eccentricity of
engine mount and landing gear members, external
moments are produced in joints A, B and D as
shown. Furthermore, lateral loads due to equip-
ment installation are shown acting on members BE
and CD. Assuming the fuselage welded joints
produce rigid continuity of members thru the
joint, the problem is to find the end moments in
all the members due to the eccentric joint
moments and two lateral loads. The effect of joint
translation and secondary moments due to deflec-
tions and axial loads is to be neglected in this
element.

Solution:

Table A11.2 gives the calculation of the
stiffness factor for each truss element. The
fuselage truss aft of joints I and H have been
assumed to give 50% fixity to these joints. In
Table A11.2 a modified stiffness factor is cal-
culated for members GI, PI, and FH using a 50
percent fixity at their far ends. The last
column of Table A11.2 gives the summation of the
member stiffness factors for members intersect-
ing at each joint.

**Table A11.2**

<table>
<thead>
<tr>
<th>Member</th>
<th>Size Tube</th>
<th>Length</th>
<th>I</th>
<th>1 x 1000</th>
<th>Stiffness Factor</th>
<th>x for each Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>BE</td>
<td>1-1/8</td>
<td>-249 1.272</td>
<td>1.400</td>
<td>0.982</td>
<td>0.662</td>
<td>A</td>
</tr>
<tr>
<td>GI</td>
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<td>-249 1.272</td>
<td>0.240</td>
<td>0.982</td>
<td>0.662</td>
<td>B</td>
</tr>
<tr>
<td>CD</td>
<td>1-1/8</td>
<td>-249 1.272</td>
<td>0.240</td>
<td>0.982</td>
<td>0.662</td>
<td>C</td>
</tr>
</tbody>
</table>

(*stiffness factor - 7/8 because of 50% fixity*
Fig. A11.24 gives the solution of the problem. The procedure in this solution was as follows:

The stiffness factor $K$ for each member as computed in Table A11.2 is recorded in the circles adjacent to each truss member. The carry over factors for all members is 1/2 except for modified members GI, FI and FH for which the carry over factor to the 50% fixed ends is .286. The distribution factor for each member at each joint is recorded at the end of each member, and equals $K/EK$.

The next step in the solution is to compute the fixed end moments due to the transverse loads on members.

For member BE

$$MB_E = Fb/2L = 120 \times 29.26 \times 15^2/41.25^2 = -266^*$$

$$MB_B = 120 \times 29.26 \times 12/41.25^2 = 725^*$$

$$MC_D = 100 \times 20 \times 10/30 = -445$$

$$MD_O = 100 \times 10 \times 20/30^2 = 222$$

These moments are placed at the ends of the members on Fig. A11.23 together with the eccentric joint moments. The process of unlocking the joints, distributing and carrying over moments can now be started. In the solution as given the order of joint consideration is ABCDEFG and repeat, and each joint has been balanced three times.

Consider Joint A:-

Unbalanced moment = 2400. Balance by distributing - 2400 as follows:-

To AC = - 2400 x 527 = -1268. Carry over to C = - 634.

To AB = - 2400 x .473 = -1132. Carry over to B = - 566.

Proceed to Joint B:-

Unbalanced moment = (-566 + 3200 - 289) = 2256. Joint is balanced by distributing - 2256 to connecting members as follows:-

To BA = - 2256 x .569 = - 1330. Carry over to A = - 665.

To BC = - 2256 x .310 = - 724. Carry over to C = - 362.

To BE = - 2256 x .121 = - 282. Carry over to E = - 141.

The convenient device of drawing a line under all balancing moments is used to prevent confusion in later balances of the joint.

Proceed to Joint C:-

Unbalanced moment = (-434 - 362 - 445) = -1441. The joint is balanced by distributing 1441 as follows:-

To CA = 1441 x .44 = 635. Carry over to A = 318.

CB = 1441 x .214 = 309. Carry over to B = 155.

CE = 1441 x .064 = 92. Carry over to E = 46.

CD = 1441 x .282 = 406. Carry over to D = 203.

This process is continued for the remainder of the trusses. After all joints have been balanced once, on returning to joint A we find below the lines an unbalanced moment of (313 - 665) = -347. The joint is balanced a second time by distributing 183 to AC and 154 to AB with carry over moments of half these values to ends C and B respectively.

The student should now be able to check the rest of the solution as given on Fig. A11.24. The solution could be made with any order of joint consideration. If any particular joint appears to be nearly balanced, it is best to skip it for the time being and consider those joints which are considerably unbalanced.

The final moments at the end of each member are given below the double lines.

Example Problem #9

Fig. A11.25 represents a cross section of a welded tubular steel fuselage. The top and bottom members which are web members in the top and bottom fuselage trusses are subjected to the equipment installation transverse loads as shown. Let it be required to determine the end bending moments in the rectangular frame due to these transverse loads assuming full continuity thru joints.

Solution:

Fig. A11.26 shows the solution. The distribution factors based on the member stiffness factors are shown in at ends of each member. The first step is to compute the fixed end moments due to transverse loads, on members AB and CD using equations from Table A11.1. The magnitudes are 1890# for AB and 2225# for CD.

Joint B is now released from its assumed fixed state. The unbalanced moment of 1890 is balanced by distributing - 1890 x .247 = - 467 to BA and the remainder of -1423 to BD. The carry over moment to A = - 465 x .5 = - 232. Due to symmetry of structure and loading only one half of frame need be considered and hence these carry over moments to A are not recorded. However, in balancing joint A it will throw over to B the same magnitude of carry over moments as thrown over to A from B but of opposite sign since the original fixed end moment at B is minus. Thus 233 comes to B from first balance of A as shown in the figure. The distributing moment to B of -1423 produces a carry over moment of -1423 x .5 = -712 at D.

![Fig. A11.25](image-url)
Joint D is considered next: The unbalanced moment of \((2025 - 712) = 1313\) is balanced by distributing \(-1313\) to the connecting members.

To member BB = \(-1315 \times 0.678 = -922\) and the remainder of \(-421\) to DC. The carry-over moment to B is \(-446\). The carry-over from C to D is one-half the balancing moment \(B = -481\) but of opposite sign or 210, due to symmetry as explained before for member AB.

Returning to joint B, the unbalanced as recorded below the single lines is \((233 - 446) = -213\). To balance 160 is distributed to BD and 53 to SA. Carry-over 50 to D and bring over from A to B \(0.6x -53 = -27\). Continue this process until joints A and D have been balanced 4 times or 4 cycles have been completed. The final moments are shown below the double lines. Fig. All.26a shows the resulting moment diagram on frame.

![Final Moment Diagram](image)

Fig. All.26a

A11.10 Continuous Structures with Members of Variable Moment of Inertia

In Arts. All.3, 4 and 5 consideration was given to the derivation of expressions for fixed end moments, stiffness and carry-over factors for beams of uniform cross-section. Many cases occur in routine design where members have a variable cross section. This article will illustrate the calculation of the fixed end moments, stiffness and carry-over factor for a beam with variable moment of inertia. The effect of axial load on these fac-

tors will not be considered here but will be treated in a later article.

(A) Calculation of Fixed End Moments for a Given Beam Load

Fig. All.27 shows a fixed end beam with a variable moment of inertia and carrying a single concentrated load of 100°. The beam moment diagram for this load is considered in three parts. Fig. All.28 shows the static moment curve assuming the beam simply supported at A and B and carrying the load of 100°. Fig. All.29 shows the other two parts, namely the triangular moment diagram due to the unknown moments \(M_A\) and \(M_B\) which produce fixity at the two ends. In this figure \(M_A\) and \(M_B\) have arbitrarily been taken as 100, instead of unity.

Fig. All.30 shows the M/I diagram for the three moment diagrams of Fig. All.29. The beam is divided into ten equal strips, and the M/I curves are obtained by dividing the moment values at the end of each strip or portion by the corresponding I value from Fig. All.27.

From the conditions of fixity at the beam ends, we know that the slope of the beam elastic curve is zero at each end. Likewise the deflection of one end of the beam away from a tangent at the other end is zero. Stating these facts in terms of the moment area principles, we obtain

\[
\begin{align*}
\int_0^L \frac{M_{max}}{I} &= 0.0 - (5) \text{ (Area of M/I diagrams equal zero)} \\
\int_0^L M_{max} &= 0.0 - (6) \text{ (Moment of the M/I diagram as a load about either end equals zero)}
\end{align*}
\]

(Note: Since \(E\) is usually considered constant it has been omitted from denominator of the above equations.)

<table>
<thead>
<tr>
<th>Table All.3</th>
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<tbody>
<tr>
<td>Beam Portion</td>
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<td>10</td>
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<tr>
<td>Total</td>
</tr>
<tr>
<td>(\Sigma)</td>
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</tbody>
</table>

* Actual area is 4 times the results shown. For calculation of fixed end moments since factors only relative values are necessary.
identical to Fig. 11.3.30c. For the conditions of support assumed, the deflection of A away from a tangent to the elastic curve at B is zero. Thus by the deflection principle of area moments, the moment of the M/I diagrams of Figs. 11.3.30 (a and b) about end A equals zero. Thus,

\[ 845.7 \times 14.1 = \frac{743.8}{100} \times 24.4 = 0 \]

or

\[ M_B = -56\# \]

Since \( M_A \) was assumed 100, then carry over factor from A to B = -56 = - .56.

To find carry over factor from 3 to A, take moments about B and equate to zero.

Hence

\[ 743.8 \times 15.6 + \frac{845.7}{100} \times 25.9 = 0 \]

whence

\[ M_A = 53.2\# \]

Therefore carry over factor B to A = - 53.2 = - .532

(100)

(Note: For the moment sign convention used in this book carry over factor would be plus.)

(C) Calculation of Beam Stiffness Factors

When a beam is freely supported at one end A and fixed at the far end B, the stiffness factor at the A end is measured by the moment necessary at A to produce unit rotation of the elastic curve at A.

In Art. 11.4 it was proved for beams of uniform section that \( E I = \frac{M_A L}{4 I} \) or \( E I = \frac{4 E I}{L} \). In a continuous structure at any joint all members have the same \( E I \), thus \( 4 E I \) is constant and the stiffness \( K \) of any prismatic beam is proportional to \( I/L \). For beams of variable section the stiffness factor \( K \) may be written:

\[ K = \frac{c I_0}{L} \]

where \( I_0 \) is the moment of inertia at a particular reference section and \( c \) is a constant to be found for each non-uniform member. Thus for non-uniform members

\[ E I = \frac{M_A L}{4 c I_0} \]

(10)

By the moment area slope principle, the slope at A when A is fixed equals the area of the M/I diagram between A and B.

Taking \( M_A = 100 \), \( M_B \) was found to equal -56\#. Thus \( E I_A = \frac{(845.7 \times .56 \times 743.8)}{1412} \) = 1412.

The value + is due to the strip width since true area is wanted and only average ordinate was used in Table 11.3.

Equating this result to Equation (10)
1419 = \frac{100 \times 40}{4c^2 L}, \text{ whence } c = \frac{707}{100} \text{ in.}

I_0 \text{ was taken as the value at the A end or } L. \text{ (See Fig. A11.27).}

Therefore \( K_{AB} = \frac{707}{100} I_A/L \).

Similarly for end B:

\( 2W_B = 4(745.8 - 845.7) = \frac{100 \times 40}{4c^2 L} \)

\text{ whence }
\[ c = 0.85 \text{ and } K_{BA} = 0.85 I_A/L \]

**A11.11 Frames with Unknown Joint Deflections Due to Sway**

In the example problems so far treated the joints of the particular structure were assumed to rotate without translation or with a definite amount of translatory movement. Translation of the joints may however, be produced by shortening and lengthening of the members due to axial loads and by lateral sway due to lack of diagonal shear members. The problem relative to the effect of joint translation due to axial stresses is treated in a later article. In this article only the effect of sway of rectangular frames on the frame bending moments will be considered.

Fig. A11.31 illustrates cases of frames where only rotation of joints takes place (neglecting axial deformation) whereas Fig.A11.32 illustrates conditions in which sway takes place and the joints suffer translation as well as rotation.

![Schematic Diagram A11.32](image)

**Fig. A11.32**

Symmetry of Structure, Loading Unsymmetrical

There are several methods of determining the bending moments due to sway. Only one method will be presented here and it can best be explained by the solution of example problems.

**Example Problem 1.**

Fig. A11.33 shows a single bay rectangular bent carrying a distributed side load on one leg as shown. The K or I/L values for each member are given on the figure.

Under the given loading it is obvious that the frame will sway to the right or in other words, joints B and C will undergo considerable horizontal movement. The moment distribution method assumes that only joint rotation takes place. To make this assumption true for this structure we will add an imaginary support at joint C which will prevent sway of the frame as illustrated in Fig. A11.34. The end moments in the frame will then be found by the moment distribution process. Fig. A11.34 shows the results of this process. To explain, the solution begins with computing the fixed end moment on member AB = WL/12 = (300 \times 25)/12 = 62.5 thousand of foot lbs. This value with the proper sign is written at the head of a column of figures on member AB as shown in Fig. A11.34.

Now considering Joint B, the unbalanced joint moment of 15.63 is distributed as follows: -To Member BA = -15.63 (40/190) = -3.27. Carry over to Joint A = -3.27/2 = -1.65.

To Member BC = -25.63 (150/190) = -13.36. Carry over to Joint C = -13.36/2 = -6.18.

![Schematic Diagram A11.34](image)

**Fig. A11.34**
THE MOMENT DISTRIBUTION METHOD

Now consider Joint C:

The unbalanced moment is -6.18. To balance we distribute to CB = 6.18 x 150/190 = 4.86. Carry over to B = 2.44; to CD = 6.18 x 40/190 = 1.30. Carry over to D = 0.65

The balancing and carry over procedure is now repeated for joints B and C, until the unbalanced moments become of negligible magnitude. Fig. A11.34 shows that 4 cycles have been carried through. By keeping the ratio 150/190 set on the slide rule the unbalanced moments at joints B and C are distributed and chased back and forth as rapidly as one can write them down. Since joints A and D are assumed fixed, they absorb moments but do not give out any.

Fig. A11.35 shows a free body diagram of each portion of the frame. The end moments are taken from the results in Fig. A11.34. Consider member AB as a free body. To find the horizontal reaction Hg we take moments about point A.

\[ 3517 \times 25 \times 12.5 + 11750 - 17570 - 25H_g = 0 \]

hence \( H_g = \frac{87530}{25} = 3517 \text{ lb} \).

Now consider member CD as a free body. To find \( H_c \) take moments about D.

\[ 1530 + 770 - 25H_c = 0, \text{ hence } H_c = 69 \text{ lb} \]

We now place these horizontal forces on the top member BC as a free body as shown in the upper portion of Fig. A11.35. The unknown imaginary reaction \( R_c \) at point C that was added in Fig. A11.34 to prevent sideways is also shown. To find \( R_c \) take \( 2R_c = 0 \)

\[ ZP_h = 3517 + 89 - R_c = 0, \text{ hence } R_c = 3606 \text{ lb} \]

Since the external reaction of 3606 does not exist, we must eliminate it and find the bending moments due to the sideways of the frame. In other words, the frame will sway sideways until bending of the frame develops a horizontal shear reaction at the upper end of the vertical legs of the frame equal to 3606 lb.

Bending Moments Due to Sideways.

Assume the structure sways sideways as illustrated in Fig. A11.36, but with no rotation of the upper joints. It was proven in Art. A11.8 that the end moments produced by the lateral movement of one end of a beam whose ends are fixed against rotation are equal to \( \delta EI d/L^2 \) where \( d \) is the lateral movement. In Fig. A11.36 the load P causes both vertical members to suffer the same horizontal displacement at their upper ends, hence their end moments due to this displacement are proportional to \( EI/L^2 \).

Since \( EI \) and \( L \) are the same for each vertical member in our structure, the fixed end moments will be the same magnitude for each member. Therefore, for convenience we will assume fixed end moments of 10,000 ft.lbs. are produced by the sideways. We now use the moment distribution process in permitting the upper joints to rotate as illustrated in Fig. A11.37. The procedure is similar to that in Fig. A11.34. For example the solution is started by considering joint B. The unbalanced moment is -10. This is balanced by distributing 10 x 150/190 = 7.89 to BC and the remainder of 2.11 to BA. The carry over moments are 7.89/2 = 3.95 to C and 2.11/2 = 1.06 to A. Due to symmetry of loading and structure, the distributing and carry over moments at joint C will be the same as at joint B, hence it is needless work to show calculations at these joints. The carry over moments from C to B will be identical to those from D to C. Fig. A11.37 shows the 5 cycles have been carried out to obtain the final end moments as shown below the double lines.
Fig. A11.38 shows a free-body diagram of the frame vertical members with the end moments as found in Fig. A11.37. The shear reactions \( N_B \) and \( N_C \) are each equal to \((8470 + 9230)/65 = 708 \) lb. These shear reactions are then placed on the free body of member BC in Fig. A11.38. For equilibrium the summation of the horizontal forces must equal zero. Thus \( P = 708 + 708 = 1416 \) lb, and acting to the right as shown in Fig. A11.38. Since the reaction \( R_C = 3606 \) in Fig. A11.35 must be liquidated and since the liquidating force \( P \) produced by the moments in Fig. A11.37 is 1416 lb. it is obvious the values in Fig. A11.37 must be multiplied by a factor equal to 3606/1416 = 2.545. Therefore, the final bending moment values equal those of Fig. A11.34 plus 2.545 times those in Fig. A11.37. Fig. A11.39 shows the results and Fig. A11.40 the final moment diagram.

**Fig. A11.38**

**Fig. A11.39**

**Fig. A11.40**

**Example Problem 2:** Bent with Unequal Length Legs and Pinned Ends.

Fig. A11.41 shows a loaded unsymmetrical frame. The final bending moments at B and C will be determined.

**Fig. A11.41**

**SOLUTION:**

Relative Stiffness Factors: -

\[ K_{BA} = \frac{21}{L} = \frac{3 \times 24}{24} = 3, \quad K_{BC} = \frac{41}{24} = \frac{4 \times 16}{16} = 4 \]

\[ K_{CD} = \frac{31}{L} = \frac{3 \times 10}{15} = 2 \]

The distribution factors for each member at joints B and C, which equal \( K/2K \) are recorded in the small box on Fig. A11.41a.

Fixed End Moments:

- For Member AB = PL/8 = 100 x 24/8 = 300 in.lb.
- For Member BC = PL/8 = 100 x 16/8 = 200

As explained in example problem 1, the moment distribution is carried out in two steps, one for joint rotation only and the other for effect of sideways or horizontal translation of joints B and C. Fig. A11.41a shows the moment distribution for no sideways by placing an imaginary reaction \( R_C \) at joint C. The process is started at joint A and the order of joint balancing is ABCBC. As soon as a joint is balanced the carry over moments are immediately carried over before proceeding to the next joint. When a joint is balanced a horizontal line is drawn.

**F.E.M.** - 200 200 F.E.M.

| \( B_a \) | 143 | -71.5 |
| \( B_b \) | 42.3 | -55.7 |
| \( B_c \) | 42.4 | 12.2 |
| \( B_d \) | -4.1 | -8.1 |
| \( B_e \) | 363.2 | 0.7 |
| \( B_f \) | -47.3 |

**Fig. A11.41a**
With the end moments known the reaction \( R_C \) can be calculated by a consideration of the free body of each member as shown in Fig. All.41b.

\[
\begin{align*}
55 \quad B & \quad 3.15 \quad C \quad -R_C = 61.85 \\
363.2 \quad H_B = 65 \quad H_C = 3.15 \quad 47.3 \\
100 \\
35 \quad A \\
\end{align*}
\]

Fig. All.41b

To find \( H_B \) take moments about \( A \),

\[100 \times 12 + 362.3 - 24H_B = 0, \quad H_B = 65 \text{ lb.}\]

To find \( H_C \) take moments about \( D \),

\[-47.3 + 15H_C = 0, \quad H_C = 3.15\]

Placing these shear reactions on member BC and writing equilibrium of horizontal forces we find that \( R_C = 65 - 3.15 = 61.85 \text{ lb.} \) We now must liquidate this reaction \( R_C \) and permit frame to sideways.

We will assume that the frame is fixed ended and that an unknown horizontal force \( P \) at \( C \) will deflect the frame sideways. The fixed ends for equal horizontal deflection of \( B \) and \( C \) will be proportional to \( EI/L^2 \) of the vertical member.

For member AB, \( \frac{I}{L^2} = \frac{24}{24^2} = 0.0417 \)

For member DC, \( \frac{I}{L^2} = \frac{10}{15^2} = 0.0445 \)

For convenience we will assume 417 in lb. as the fixed end moments on AB and 446 on DC. The assumed fixed ends will now be eliminated by the moment distribution process as shown in Fig. All.41c. The order of joint balance was ABCDCBC.

Fig. All.41d shows the shear reaction on the vertical members at \( B \) and \( C \). These forces reversed on the top member show an unbalanced force of \( 6.02 = 10.85 = 16.87 \text{ lb.} \) Therefore a force \( P = 16.87 \) was necessary to produce the bending moments that resulted on the frame due to sideways.

![Diagram](image)

Example Problem 3. Bending Moments in Truss involving One Panel Without Diagonal Shear Member.

Frequently, in aircraft structures a truss is used in which a diagonal member must be left out of one or more truss bays. The external shear load on such bays must be carried by the truss chord members in bending. Fig. All.42 shows a 3 bay truss with no diagonal member in the center bay. The bending moments on the truss members will be determined for the truss
loading as shown in the figure.

![Diagram of truss structure]

Fig. A11.42

**SOLUTION:**

The relative moment of inertia values for each member are given in the circles on the truss in Fig. A11.42.

The distribution factor to each member at each joint is then computed and equals $K/ZK$. The values are recorded in the same column in each member in Fig. A11.45. The stiffness factor $K$ is proportional to $1/L$ for the member. For example for joint c:

$$ZK = \frac{1}{L} = \frac{4}{60} + \frac{2}{60} + \frac{1}{60} = \frac{7}{30}$$

Then the distribution factors are:

- To Member $cb = \frac{4}{7} = .57$
- To Member $cf = \frac{1}{7} = .14$
- To Member $cd = \frac{2}{7} = .29$

In this problem there are no loads applied to the members between joints. The external shear load on the truss to the right of the center truss panel which equals $10 + 10 = 20$ must be carried through the center panel by the truss chord members in bending. This bending causes the center truss to sway or deflect until a resisting shear force equal to 20 is developed.

We will assume the truss center panel is fixed at joints b, c, f, and g. The right end of this assumed fixed end truss will be given an upward deflection. This deflection will cause fixed end moments in members $bc$ and $cf$ which are proportional to $1/L^2$ for each member. Since $L$ is the same for each member, the fixed end moments will be proportional to $I$ of the member.

$$I_{bc} = 4, \quad I_{cf} = 3.$$
Therefore the fixed end moments on member gf will be 75 percent of those on member bc.

Assume 100 as the fixed end moments on bc. Then a 75 x 10 = 75 is the accompanying fixed end moment on gf.

We now remove the imaginary fixed supports on the center truss and let them rotate to equilibrium by the moment distribution process as shown in detail in Fig. All.43.

The first step is to record the assumed fixed end moments of 100 at each end of member bc and 75 at each end of member gf with due regard for sign. The moment distribution process will be started at joint C. The unbalanced moment is 100 or -100 is necessary for static balance. Using the distribution factors on joint C, we find -57 goes to cb, -29 to cd and -14 to cf. Short lines are drawn below each of these members. Fifty percent of these balancing moments are carried over to the far end of each of these members. This process is repeated at each joint until the remaining balancing moments are negligible. In Fig. All.43 the order of joint balance was cfdebghacdefghacdefbg. If the student will follow this order he should be able to check the figures in Fig. All.43.

Fig. All.44 shows free bodies of the truss members bc and gf with end moments from Fig. All.45. The shear reaction at ends c and f by statics equals

\[
H_c + H_f = \frac{36.2 + 45.1}{60} = \frac{26.7 + 34}{60} = 1.377 + 1.045 = 2.422 \text{ lb.}
\]

The external truss shear at line cf = 20 lb. Therefore it will take 20/2.422 = 8.25 times the final bending moments as found in Fig. All.45 to develop a bending shear reaction of 20 lb. Thus the final moments are 8.25 times those in Fig. All.43.

The solution as given neglects the effect of axial loads upon the value of the stiffness and carry over factors. Art. All.12 explains how to include these effects.

All.12 Effect of Axial Load on Moment Distribution

In the previous articles the effect of axial loads upon the member end moments of a continuous structure was neglected. The axial loads produce bending in the members of a continuous structure in two ways. (1) The shortening or lengthening of the various members due to axial loads produces translation of the joints of the structure. Since the angles between the members at a joint remain the same due to continuity, this translation of joints bends the members between joints. (2) The bending moments on the members due to external joint or lateral loads or those due to joint rotation produce lateral deflection of the members between joints. The member axial loads times these lateral deflections produce secondary moments. These secondary moments can be handled by the general method of moment distribution however the stiffness and carry over factors and the fixed end moments are not constant but become functions of the axial loads.

Fig. All.45 shows a prismatic beam simply supported at A and fixed at B, with a moment \( M_A \) applied at A and carrying an axial compressive load P. Sub. Figure a, b and c show the 3 parts which make up the moment diagram on the beam. Without the axial load P the portion (c) would be omitted.
fixed end moments and also the carry over factor.

**A11.13 Fixed End Moments, Stiffness and Carry over Factors for Beam Columns of Constant Section**

In deriving expressions for fixed end moments, stiffness and carry over factors the beam columns formulas of chapter A10, must be used in finding slopes, deflection and moments. Mr. S. W. James in an excellent thesis at Stanford University and later published by N.A. C.A. as Technical Note #534 has derived expressions for these factors and has provided graphs of these factors for use in routine design. Figures A11.46 to A11.56 inclusive are taken from this thesis. The use of these Figures can best be illustrated by the solution of several problems.

**A11.14 Illustrative Problems**

**Example Problem #12**

Fig. A11.57 shows the same continuous beam as used in Example Problem #4 (and solution #2). For this example it has been assumed that such axial compression loads exist in spans BC and CD as to make the term L/j = 2.5 and 2.0 for these spans respectively, where j = \(\sqrt{EI/G}\). Due to the axial loads, new values of stiffness and carry over moments as well as fixed end moments must be determined as follows.

**Span BC**

\[
K = \frac{I}{L} = \frac{1}{96} = .0104
\]

From Fig. A11.47 when L/j = 2.5, correction factor for stiffness factor = .775 when far end is fixed and .36 when far end is pinned. Hence,

\[
K_{BC} = .0104 \times .775 = .00806
\]

\[
K_{CB} = .0104 \times .36 = .00374
\]

From Fig. A11.46, when L/j = 2.5 the carry over factors are C_{BC} = .73 and C_{CB} = 0 if B is considered freely supported.

From Fig. A11.48 when L/j = 2.5 fixed end moments for uniform load = \(WL^2/10.57 = 1 \times 96^2/10.57 = 365\) in. lb.

**Span CD**

\[
K = \frac{L}{L} = 1/72 = .0139, L/j = 2.0
\]

From Fig. A11.47 correction factor = .86, hence

\[
K_{CD} = .86 \times .0139 = .01218.\] Carry over factor from Fig. A11.46 = .82. Fixed end moments from Fig. A11.48 = \(WL^2/11.2 = 1 \times 72^2/11.2 = 462\)

The distribution factor at joint C equals (.00374/.00374 + .01186) = .238 to CB and the remainder .762 to CD.

The balancing and carry over process is similar to that in Example Problem #4.
THE MOMENT DISTRIBUTION METHOD

Fig. A11.48

Fixed-end moment coefficient. Uniform Load.

Fig. A11.49


Fig. A11.50


Fig. A11.51

Fixed-end moment coefficient. Concentrated load at mid-span.
Fig. A11.52 Fixed-end moment coefficient. Concentrated load.

Fig. A11.53

Fig. A11.54

Fig. A11.55

Fig. A11.56

Fig. A11.57
Comparing these results with those of Problem #3, we find moment at C is increased 15.7 percent and that at D is decreased 8.5 percent. For larger values of L/j the difference would be greater.

Example Problem #13

Fig. 11.58 shows the upper wing of a bi-plane. The wing beams are continuous over 3 spans. The distributed air loads on front beam are shown in the figure, also the axial loads on front beam induced by the lift and drag truss. The bay sections of the spuce are shown in Fig. 11.58. The moment of inertia in each span will be assumed constant, neglecting the influence of tapered filler blocks at strut points.
Fixed end moment = \( \frac{wL^2}{11.35} = 31 \times \frac{54^2}{11.35} = 19300\# \)

The moment distribution process is given in Fig. A11.60. If the axial loads were neglected, the bending moment, at support A and A', would be 19480, thus the axial influence increases the moment at A approximately 7.5 percent.

Example Problem #14

Fig. A11.61 shows a triangular truss composed of two members fixed at A and B and rigidly joined at C to the axle bar. Let it be required to determine the end moments on the two members considering the effect of axial loads on joint rotation and translation.

Solution:

The magnitude of the axial loads in the members is influenced by the unknown restraining moments at A and B. To obtain a close approximation of the axial loads, the end moments in the two members will be determined without consideration of axial loads. Thus the external joint moment of \( 4 \times 18000 = 72000 \) in. lb. at C is distributed between the two members as shown in Fig. A11.62. With the member end moments known the axial loads and shear reactions at A and B can be found by statics. The resulting axial loads are

\[ P_{AC} = 10400\# \text{ and } P_{BC} = -27150\# \]

The shear reactions, \( S_A = 550\# \) and \( S_B = 2000\# \).

With the axial loads known the modified beam factors can be determined.

For member BC

\[ P = -27150, \ I/L = .0023, I = .332 \]

\[ j = \left( \frac{29,000,000 \times .332}{2} \right) = 18.22, L/j = 40 \]

\[ = 2.12 \]

From Fig. A11.47, stiffness factor = \( .82 \times .0023 = .0068 \)

From Fig. A11.46 C.O. factor = .66

For Member AC

\[ P = 10400\#, \ I/L = .0029, I = .137 \]

\[ j = \sqrt{\frac{29,000,000 \times .137}{3}} = 19.58, L/j = 47.20 \]

\[ = 2.42 \]

Stiffness factor = \( .0029 \times 1.18 = .0034 \)

C.O. factor = .39

Fig. A11.63 shows the moment distribution solution which includes the effect of axial loads on joint rotation. Comparing the results in Figs. A11.62 and 63, the moment \( M_{AC} \) of 24050 is 29 percent larger than that in Fig. 62, and the moment at B is 10 percent larger. The effect on the axial loads of these new final moments will be quite small, and thus further revision is unnecessary.

Effect on End Moments Due to Translation of Joint C Due to Axial Loads

The movement of joint C normal to each member will be calculated by virtual work.

(Reference Chapter A7). Fig. A11.64 shows the virtual loading of \( 1\# \) normal to each member at C. The Table shows the calculation of the normal deflections at C.

<table>
<thead>
<tr>
<th>Member</th>
<th>( L/A )</th>
<th>( S )</th>
<th>( S_{min} L/AX )</th>
<th>( S_{max} L/AX )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>.00000049</td>
<td>10400</td>
<td>1.26</td>
<td>.0962</td>
</tr>
<tr>
<td>BC</td>
<td>.000002344</td>
<td>-27150</td>
<td>-1.60</td>
<td>-1.068</td>
</tr>
</tbody>
</table>

Thus the deflection of joint C normal to BC equals \( -.026" \) in the direction assumed for the unit load, and the deflection of C normal to AC equals \( .207" \).
The fixed end moments due to support deflection equals $M = \frac{6EI'd}{L^3}$ as derived in Art. All.7. However, the secondary moments due to axial loads times lateral deflection modify this equation. "James" has shown that the modified equation is,

$$M_{\text{fixed end}} = \frac{6EI'd}{L^3(2\beta - \alpha)}, \text{ if } K = \frac{1}{L} \text{ and } R = \frac{c}{L}, \text{ than } M = \frac{5KSE}{2\beta - \alpha},$$

Fig. All.66 shows a plot of $2\beta - \alpha$ against $L/J$.

For Member BC:

$$R = 0.222/40 = 0.00555, \quad K = 1/L = 0.0083, \quad 2\beta - \alpha = 1.08 \text{ for } L/J = 2.12.$$ 

Hence, fixed end moments due to translation of joint C equals:

$$M_{BC} = M_{CB} = \frac{5 \times 0.0053 \times 0.00555 \times 29,000,000}{1.08} = -675^\circ\#$$

For Member AC:

$$R = 0.207/47.2 = 0.00438, \quad K = 0.0029, \quad 2\beta - \alpha = .92$$

when $L/J = 2.42$ and member is in tension.

hence

$$M_{AC} = M_{CA} = \frac{5 \times 0.00438 \times 0.0029 \times 29,000,000}{.92} = -2400^\circ\#$$

The signs are minus because inspection shows that the moments tend to rotate ends of members counter clockwise. (Ref. Art. All.2)

Fig. All.65 shows the moment distribution for these moments. The magnitude of the moment at B is 6.7 percent of that in Fig. All.63 and 10.4 percent at Joint C, however, it is relieving in this example.

All.15 Secondary Bending Moments in Trusses with Rigid Joints.

Often in airplane structural design trusses with rigid joints are used. Rigid joints are produced because of welding the members together at the truss joints or by the use of gusset plates bolted or riveted to each member at a truss joint. When a truss bends under loading the truss joints undergo different amounts of movement or deflection. Since the truss members are held rigidly at the joints, any joint displacement will tend to bend the truss members. The bending moments produced in the truss members due to the truss joint deflections are generally referred to as secondary moments and the stresses produced by these moments as secondary stresses.

Since fatigue strength is becoming more important in aircraft structural design, the question of secondary stresses becomes of more importance than in the past. The moment distribution method provides a simple and rapid method for determining these secondary moments in truss members due to truss deflection. The general procedure would be as follows:

1. Find the horizontal and vertical displacements of each truss joint due to the critical design condition. (See Chapter A7 for methods of finding truss deflections).

2. From these displacements, the transverse deflection of one end of a member with respect to the other can be found for each truss member.

3. Compute the fixed end moments on each member due to these transverse displacements.

4. Calculate stiffness and carry-over factors for each truss member.

5. Calculate distribution factors for each truss joint.

6. Carry out the moment distribution process to find the secondary moments at the ends of each member.

7. Calculate the stresses due to these secondary moments and combine with the primary axial stresses in the truss members due to truss action with pinned truss joints.

All.16 Structures with Curved Members.

The moment distribution method can be applied to continuous structures which include curved as well as straight members. The equations for finding the stiffness factors, carry over factors and fixed end moments for straight members cannot be used for curved members. The elastic center method as presented.
in Chapter A9 provides a rapid and simple
method for determining these values for curved
members. The use of the elastic center method
in determining the value of stiffness and
carry-over factor will now be explained.

**A11.7 Structures with Curved Members.**

Before considering a curved member a
straight member of constant EI will be con-
sidered. Fig. A11.56 shows a beam freely

Fig. A11.56

![Diagram showing M/A relationship](image)

A moment \( M_A \) is applied at end A of such magni-
tude as to turn end A through a unit angle of
one radian as illustrated in Fig. A11.56. By de-
definition, the necessary moment \( M_A \) to cause
this unit rotation at A is referred to as the
stiffness of the beam AB. In Art. A11.4 it
was shown that this required moment was equal
in magnitude to \(-2EI/L\). It was also proved
that this moment at A produced a moment at
the fixed end B of \( 2EI/L \) or a moment of one
half the magnitude and of opposite sign to
that at A. Fig. A11.56 shows the bending
moment diagram which causes one radian rota-
tion of end A. Fig. A11.56 shows the M/EI
diagram, which equals the moment diagram
divided by EI which has been assumed constant.

The total moment weight \( \theta \) as explained
in Chapter A9 equals the area of the M/EI
diagram. Thus \( \theta \) for the M/EI diagram in
Fig. A11.56 equals,

\[ \frac{4}{L} \left( \frac{1}{2} \right) + \frac{2}{L} \left( \frac{M_A}{2} \right) = -2 + 1 = -1 \]

In other words the total elastic moment
weight \( \theta \) equals one or unity.

The location of this total moment weight
\( \theta \) will coincide with the centroid of the M/EI
diagram. Thus to find the distance \( x \) from B
to this centroid we take moments of the M/EI
diagram about B and divide by \( \theta \) the total area
of the M/EI diagram. Thus

\[ x = \frac{-2(0.667L) + 1(0.333L)}{-2 + 1} = \frac{-L}{-1} = L \]

Thus the centroid of the total moment weight \( \theta \)
which equals one lies at point A or a distance
L to left of B.

In using the elastic center method to de-
termine the stiffness and carry-over factor for
a straight beam, we assume that the bending
moment curve due to a moment applied at A is of
such magnitude as to turn the end A through an
angle of 1 radian. As shown above, the moment
weight \( \theta \) for this loading is unity or 1 and its
centroid location is at A. Then by the elastic
center method we find the moment required to
turn end A back to zero rotation. The value of
this moment at A will then equal the stiffness
factor of the beam AB. In order to simplify
the equations for the redundant forces the
elastic center method refers them to the elastic
center. From Chapter A9 the equations for the
redundant forces at the elastic center for a
structure symmetrical about one axis are:

\[ M_0 = \frac{-2}{3} \frac{\theta}{EI} \]
\[ Y_0 = \frac{-2}{3} \frac{\theta}{EI} \]
\[ X_0 = \frac{2}{3} \frac{\theta}{EI} \]

Fig. A11.59 shows beam of Fig. A11.56 re-
placed by a beam with the reaction at end A
replaced by a rigid bracket terminating at
point (O) the elastic center of the beam, which
due to symmetry of the beam lies at the mid-
point of the beam. The elastic moment loading
is \( \theta_e = 1 \) and its location is at A as shown
in Fig. A11.59.

Fig. A11.59

![Diagram showing reaction and moment](image)

Solving for redundants at (O) by equations (1),
(2) and (3).

\[ M_0 = \frac{-\theta_e}{2} \frac{3\theta}{EI} = -\frac{1}{2} = -\frac{EI}{L} \]
THE MOMENT DISTRIBUTION METHOD

\[ Y_0 = -\frac{\sum \Delta M_y}{I_y} = -\frac{1(-\frac{1}{2})}{\frac{1}{12}EI} \frac{6EI}{L^2} \]

\[ X_0 = \frac{\sum \Delta M_x}{I_x} \]

Because \( y \) is the vertical distance of \( \Delta M_y \) load to \( x \) axis through elastic center (0) is zero.

Fig. All.70 shows these forces acting at the elastic center.

Fig. All.70

The bending moment at end \( A \) equals

\[ -\frac{EI}{L} \frac{6EI}{L^2} \left( \frac{L}{2} \right) = \frac{6EI}{L} \]

By definition the stiffness factor is the moment at \( A \) which is required to turn \( A \) through an angle of \( \frac{L}{2} \) radian. Thus \( \frac{6EI}{L} \) is the stiffness factor and this result checks the value as previously derived in Art. All.4.

The bending moment at \( B \) in Fig. All.70 equals,

\[ -\frac{EI}{L} \frac{6EI}{L^2} \left( \frac{L}{2} \right) = \frac{6EI}{L} \]

Hence the carry-over factor from \( A \) to \( B \) is \( \frac{1}{2} \) and the carry-over moment is of opposite sign to that of the moment at \( A \).

All.18 Stiffness and Carry-Over Factors For Curved Members.

Fig. All.71 shows a curved member, namely, a half circular arc of constant \( EI \) cross-section. The end \( B \) is fixed and the end \( A \) is freely supported. A moment \( M_A \) is applied at \( A \) of such magnitude as to cause a rotation at \( A \) of \( \frac{L}{2} \) radian as illustrated in the Fig. Fig. All.72 shows the general shape of the bending moment curve which is statically indeterminate. In Fig. All.73 the support at \( A \)

\[ \Delta_{AB} \]

\[ \Delta_{BC} \]

\[ \Delta_{CD} \]

\[ \Delta_{DE} \]

is removed and end \( A \) is connected by a rigid bracket terminating at the elastic center of the structure. We will now find the required forces at (0) to cancel the unit rotation at \( A \) which was assumed in Fig. All.71.

The total area of the \( M/EI \) curve for the curve in Fig. All.72 if calculated would equal one or unity as explained in detail for a straight member. The centroid of this \( M/EI \) diagram would if calculated fall at point \( A \). Thus in Fig. All.73 we apply a unit \( \Delta M \) load at \( A \) and find the redundant force at \( 0 \). Due to symmetry of structure about a vertical or \( y \) axis the elastic center lies on this symmetrical axis. The vertical distance from base line \( AB \) to elastic center equals \( \bar{y} = \frac{L}{3} \). (See page A3.4 of Chapter A3).

The elastic moments of inertia \( I_x \) and \( I_y \) can be calculated or taken from reference sources such as the table on page A3.4.

Whence,

\[ I_x = \frac{.2978r^2}{t}, \text{but } t = \frac{1}{EI} \]

Hence \( I_x = \frac{.2978r^2}{EI} \)

\[ I_y = \frac{m^2t}{2} = \frac{m^2}{2EI} \]

Solving the equations for the redundants at \( 0 \), remembering \( \Delta M = 1 \) and located at point \( A \).

\[ M_0 = -\frac{\sum \Delta M_A}{\sum 38/EI} = -\frac{(1.5)}{\frac{78}{EI}} = \frac{-EI}{78} \]

\[ Y_0 = \frac{-98x}{12EI} = \frac{-1(-\bar{y})}{\frac{m^2}{2EI}} = \frac{28EI}{2EI} \]

\[ X_0 = \frac{28x}{12EI} = \frac{1(-\bar{y})}{\frac{m^2}{2EI}} = 1.14 \frac{28EI}{2EI} \]

Fig. All.74 shows these forces acting at the elastic center. The bending moment at \( A \) equals the stiffness factor for the curved beam fixed at \( A \) and freely supported at near
end A. The ratio of the bending moment at point B to that at point A will give the carry-over factor.

\[ M_B = \frac{2EI}{r^2} - 2.14 \frac{2EI}{r^2} (0.6366r) - \frac{(2EI)(r)}{r^2} \]

\[ = -0.316EI \frac{r}{r} - 1.36EI \frac{r}{r} - 0.636EI \frac{r}{r} = -2.31EI \frac{r}{r} \]

Bending moment at B:

\[ M_B = -0.316EI \frac{r}{r} - 1.36EI \frac{r}{r} - 0.636EI \frac{r}{r} = -1.042 EI \frac{r}{r} \]

Therefore the stiffness factor for a half-circular arch of constant EI is 2.314 EI/r.

The carry-over factor equals the ratio of $M_B$ to $M_A$ or \((-1.042 EI/r) / (-2.314 EI/r) = 0.452\). It should be noticed that the carry-over factor has the same sign as the applied moment at A as compared to the opposite sign for straight members. In other words, there are two points of inflection in the elastic curve for the curved arch as compared to one for the straight member.

**FIXED END MOMENTS**

The fixed end moments on a curved member for any external loading can be determined quite rapidly by the elastic center method as illustrated in Chapter A9 and thus the explanation will not be repeated here.

The student should realize or understand that when the end moments on a straight member in a continuous structure are found from the moment distribution process, the remaining end forces are statically determinate, whereas for a curved member in a continuous structure, knowing the end moments does not make the curved member statically determinate, since we have six unknowns at the two supports as illustrated in Fig. A11.75 and only 3 equations of static equilibrium. Even when the end moments are determined from the moment distribution process there still remains one unknown, namely the horizontal reaction at one of the beam ends.

The method of how to handle this remaining redundant force can best be explained by presenting some example problem solutions.

**A11.19 Example Problems. Continuous Structures Involving Curved Members.**

**Example Problem 1**

Fig. A11.76 shows a frame consisting of both straight and curved members. Although simplified relative to shape, this frame is somewhat representative of a fuselage frame with two cross members, one between A and C to support installations above cabin ceiling and the other between F and D to support the cabin floor loads. The frame supporting forces are assumed provided by the fuselage skin as shown by the arrows on the side members. Eccentricity of these skin supporting forces relative to neutral axis of frame member is neglected in this simplified example problem, since the main purpose of this example problem is to illustrate the application of the moment distribution method to solving continuous structures involving curved members.

**SOLUTION**

Due to symmetry of structure and loading, no translation of the frame joints takes place due to frame sideway.

The frame cross members AC and FD prevent horizontal movement of joints A, C, F, and D due to bending of the two arches. Any horizontal movement of these joints due to axial deformation is usually of minor importance relative to causing bending of frame members. Therefore it can be assumed that the frame joints suffer rotation only and therefore the moment distribution method is directly applicable.

Calculation of stiffness (K) values for each member of frame:

Upper curved member ABC:

\[ K_{ABC} = K_{BCA} = 2.314 \frac{EI}{r} \]

This value was derived in the previous Art. A11.18.
Substituting:

\[ K_{ABC} = K_{CBA} = \frac{2.314 \times 1}{30} = 0.0770 \] (2 is constant and therefore omitted since only relative values are needed for the K values.)

The relative moment of inertia of the cross-section of each frame member is given on Fig. All.76.

\[ K_{ped} = \frac{K_{DEF}}{30} = 0.1540 \]

For all straight members, the stiffness factor equals 4EI/L. Hence,

\[ K_{AC} = K_{CA} = 4 \times 2/60 = 0.1333 \]
\[ K_{PD} = K_{DF} = 4 \times 6/60 = 0.4000 \]
\[ K_{AF} = K_{FA} = 4 \times 1/60 = 0.0666 \]
\[ K_{CD} = K_{DC} = 4 \times 1/60 = 0.0666 \]

**DISTRIBUTION FACTORS AT EACH JOINT:**

**JOINT A:**

\[ z_k = 0.1333 + 0.0770 + 0.0500 = 0.2603 \]

Let D equal symbol for distribution factor.

\[ D_{ABC} = 0.0770/0.2603 = 0.293 \]
\[ D_{AC} = 0.1333/0.2603 = 0.512 \]
\[ D_{AF} = 0.0500/0.2603 = 0.192 \]

**JOINT F:**

\[ z_k = 0.0500 + 0.400 + 0.1540 = 0.6040 \]
\[ D_{FA} = 0.6000/0.6040 = 0.988 \]
\[ D_{PD} = 0.4000/0.6040 = 0.663 \]
\[ D_{PED} = 0.1540/0.6040 = 0.255 \]

**Carry-Over Factors:**

For the straight members the carry-over factor is 0.5 and the moment sign is the same as the distributed moment when the sign convention adopted in this chapter is used.

For a half circular arch of constant I, the carry-over factor was derived in the previous article and was found to be 0.452. The sign of this carry-over moment was the same as the distributed moment at the other end of the beam. However, using the sign convention as adopted for the moment distribution in this book, the carry-over factor would be minus or of opposite sign to the distributed moment.

**Calculation of Fixed End Moments:**

The curved members of the frame have no applied external loadings hence the fixed end moments on the curved members is zero.

For member AC, fixed end moment equals W/L = 10 x 60/12 = 3000 in. lb. and for member FD = 50 x 60/12 = 15000 in. lb.

Since the supporting skin forces on the side members have been assumed acting along centerline of frame members, the fixed end moments on members AP and CD are zero.

**Moment Distribution Process:**

Fig. All.77 shows the calculations in carrying out the successive cycles of the moment distribution process. Due to symmetry of frame and loading, the process need only be carried out for one-half of frame, thus in Figure All.77, only joints A and F are considered since the numerical results at D and C would be the same as at A and F respectively. In the figure the distribution factors are shown in
the \( \sigma \) at each joint. The process is started by placing the fixed and moments with due regard to sign at the ends of members AC and PD, namely, -3000 at AC, 3000 at CA, -15000 at PD and 15000 at DF. We now unlock joint A and find an unbalanced moment of -3000 which means a plus 3000 is needed for static balance. Joint A is therefore balanced by distributing 
\( \frac{1}{2} \times 3000 = 1500 \) to AC, \( \frac{1}{2} \times 3000 = 1500 \) to ABC, and \( \frac{1}{2} \times 3000 = 1500 \) to AP. Short horizontal lines are then drawn under each of these distributed values to indicate that these are balancing moments. Carry-over moments are immediately taken care of by carrying over to joint D, \( \frac{1}{2} \times 3000 = 1500 \). From A to C the carry-over moment would be \( \frac{1}{2} \times 1500 = 750 \) and therefore the carry-over from C to A would be \( \frac{1}{2} \times 1500 = 750 \) which is recorded at A as shown. For the arch member ABC, the carry-over moment from C to A would be \( -0.452 \times \frac{1}{2} = -0.452 \) (not shown) and therefore from C to A = \( -0.452 \times \frac{1}{2} = -0.452 \) as shown at joint A in the figure for arch member CBA. Joint C in the figure has been balanced once for the purpose of helping the student understand the sign of the carry-over moments which flow to the left side from the right side of the frame.

After balancing joint A and taking care of the carry-over moments, we imagine A as fixed again and proceed to joint F where we find an unbalanced moment of -15000 + 288 = -14712, thus plus 14712 is necessary for balancing. The balancing distribution is \( 0.255 \times 14712 = 3750 \) to FED, \( 0.667 \times 14712 = 9750 \) to FD and \( 0.082 \times 14712 = 1212 \) to CF. The carry-over moments are \( \frac{1}{2} \times 1212 = 606 \) to A, \( \frac{1}{2} \times 3750 = 1875 \) from D to F and \( \frac{1}{2} \times 3750 = 1875 \) from D to F. We now go back to joint A which has been unbalanced by the carry-over moments and repeat the balancing and carrying-over cycle. In the complete solution as given in Figure A11.77 each joint A and F was balanced five times. The final bending moments at the ends of the members at each joint are shown below the double short lines.

The arch member ABC has 3 unknown forces at each end A and C or a total of 6 unknowns. With 3 equations of static equilibrium available plus the known values of the end moments at A and C as found from the moment distribution process, the arch member is still statically indeterminate to one degree. Thus the horizontal reaction at A or C as provided by the axial load in member AC must be found before the bending moments on arch ABC can be calculated.

The first step in this problem is to find the elastic center of the frame portion composed of members ABC and AC, as shown in Fig. A11.78, and then find the elastic moments of inertia about x and y axes through the elastic center.

\[ y = \frac{19.1 \times \pi \times (30/1)}{1} = \frac{14.49}{2} \]

(Note: 19.1 equals distance from line AC to centroid of arch member ABC.)

Calculation of moment of inertia \( I_x \):

<table>
<thead>
<tr>
<th>Member</th>
<th>( I_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>( \frac{3r^3}{4} + \frac{\pi r^2}{2} \times 1.449 \times 10^4 )</td>
</tr>
</tbody>
</table>

Total elastic weight of structure equals

\[ 2 x 10^4 = \left( \frac{\pi \times 30}{1} \right) + 60/2 = 124.24. \]

The next step in the solution is to draw the bending moment curves on this frame portion due to the given load on member AC and the end moments as found by the moment distribution process in Fig. A11.77. It is composed of three parts labeled (1) to (3) in Fig. A11.79. Portions (1) and (2) are due to the end moments and portion (3) due to the distributed lateral load on member AC.

The next step is to find the \( \theta_\alpha \) (area of M/I curve) for each portion and its centroid location.
A11.34

THE MOMENT DISTRIBUTION METHOD

\[ \theta_{S_1} = 1094 \times \pi \times 30/1 = 103000 \]
\[ \theta_{S_a} = 60 \times 2353/2 = 70590 \]
\[ \theta_{S_b} = (-0.667 \times 60 \times 4500)/2 = -90000 \]
\[ \sum \theta_{S} = 83590 \]

Fig. A11.30 shows the \( \theta_{S} \) values concentrated at their centroid locations and referred to the x and y axes through the elastic center.

The frame has been imagined cut at \( A \) and the arch end at \( A \) has been connected by a rigid bracket to the elastic center. The redundants at the elastic center which will cancel any relative movement of the cut faces at \( A \) can now be computed.

\[ M_0 = \frac{\sum \theta_{S_a}}{2 \, ds/1} = \frac{-33590}{124.24} = -673 \text{ in} \cdot \text{lb.} \]
\[ X_0 = \frac{\sum \theta_{S_a}x}{I_x} = \frac{(103000 \times (4.61) + 70590 \times (-14.49) - 90000 \times (-14.49))}{16410} = 46 \text{ lb.} \]
\[ Y_0 = \frac{\sum \theta_{S_a}y}{I_y} = 0 \text{ because } x = \text{ zero.} \]

The bending moment at any point on \( ABC \) or \( AC \) equals that due to \( M_0 \) and \( X_0 \) plus the moments in Fig. A11.79.

For example,

At point \( A \) on member \( ABC \),
\[ M_A = 1094 - 670 + 46 \times 14.49 = 1091 \text{ (should be 1094 since moment as found in Fig. A11.77 is correct one. Small error due to slide rule accuracy.)} \]

At point \( B \) on member \( ABC \):
\[ M_B = 1094 - 670 - 46 \times 15.51 = -259 \text{ in} \cdot \text{lb.} \]

The horizontal reaction at \( A \) will not produce any bending on member \( AC \), thus the values in Fig. A11.79 are the true moments.

The axial load in member \( AC \) by statics from Fig. A11.80 equals \( X_0 \) or a tension load of 46 lb. The member \( AC \) also suffers an axial load due to the shear reactions at the top of members \( FA \) and \( DC \). Fig. A11.81 shows free bodies of the side member \( FA \) and \( DC \) with the end moments as found in Fig. A11.77.

The shear reaction at \( A \) and \( C \) can be found by taking moments about lower ends. Thus for \( FA \),
\[ R_A = (1259 + 1791)/30 = 38.1 \text{ lb.} \]
Likewise \( R_C = 38.1 \text{ lb.} \) These reactions react on cross member \( AC \) in the opposite directions thus giving a compression load of 35.1 lb. in member \( AC \), which must be added to the tension load of 46 lb. from the arch reaction to obtain the final load in the cross-member.

Bending Moment in Lower Arch Member \( FED \)

The horizontal reaction on arch \( FED \) must be found before the true bending moments on the arch can be found. The procedure is the same as for the upper arch. Figs. A11.81, 82, 83 show the results for the lower frame portion.

\[ y = 19.1 \times \pi \times (30/2) = 900 \times \frac{57.1}{2} = 15.76'' \]
\[ \Sigma \frac{ds}{t} = 57.1 \]
\[ I_x = \frac{3 \times 30^3}{2} \times 6 \times \frac{15 \times 3.34^3}{2} = 4578 \]
\[ \frac{2}{dx} = 7056 \]

From Fig. A11.82,
\[ \theta_{S_x} = -8753 \times 60/6 = -57300 \]
\[ \theta_{S_a} = 0.667 \times 22500 \times 60/6 = 150000 \]
\[ \theta_{S_b} = -5862 \times (30/2) = -528800 \]
\[ \Sigma \theta_{S} = -265500 \]

Solving for Redundant forces \( M_0 \) and \( X_0 \) (See Fig. A11.33)
\[ M_0 = \frac{\Sigma \theta_{S}}{2 \, ds/1} = -\frac{(-265500)}{57.1} = 4650 \text{ in} \cdot \text{lb.} \]
Example Problem 2. External Loads on Curved Members.

Fig. A11.85 shows a frame which has external loads applied to the curved member as well as to the straight members. The frame supporting forces have been assumed as acting uniformly along the side members AB and CE. The problem will be to determine the bending moments at frame points ABCDE.

Figs. A11.94. Final Bending Moment Curves on Each Member of Frame.
Curved member ABC.

The fixed end moments on this curved member due to the external loads will be determined by the elastic center method. The assumed static frame condition will be an arch pinned at A and supported on rollers at B. (See Fig. All.86)

Fig. All.86 shows the general shape of the static moment curve. For the frame portion between the reactions and the load points, the bending moment equation is \( M = P(r - r \cos \alpha) \).

For the beam portion between the two 1000 lb. loads, the bending moment is constant and equals, \( M = r - r \cos 30^\circ \).

Calculation of \( \phi_s \) Values.

The \( \phi_s \) values equal the area of the M/I curves. The moment curve in Fig. All.86 has been broken down into three parts labeled (1) (1') and (2).

\[
\phi_{s1} + \phi_{s2} = 2 \left[ \frac{P \cdot \alpha}{I} (\alpha - \sin \alpha) \right] = 2 \left[ \frac{1000 \times 30 \times (.524 - .5)}{1.5} \right] = 28600
\]

The vertical distance \( \bar{y} \) from line AC to centroid of \( \phi_{s1} \) and \( \phi_{s2} \) values is,

\[
\bar{y} = \frac{r(1 - \cos \alpha - \sin \alpha)}{\alpha - \sin \alpha} = \frac{30(1 - .667 - 0.5/2)}{.524 - .50} = 10 \text{ in.}
\]

\[
\phi_s = \frac{2r \cdot \bar{y}}{I} (1 - \cos \alpha) \theta = 120^\circ, \alpha = 30^\circ.
\]

\[
\phi_s = \frac{1000 \times 30 \times 2.1}{1.5} = 167300
\]

Vertical distance \( \bar{y} \) from line AC to centroid of \( \phi_s \) equals,

\[
\bar{y} = \frac{2r \cdot \sin \theta/2}{\theta} = \frac{2 \times 30 \times 0.667}{2.1} = 24.3 \text{ in.}
\]

Fig. All.87 shows the \( \phi_s \) values and their location with respect to \( X \) and \( y \) axes through the arch elastic center.

The elastic center method requires the total elastic weight of member and its elastic moments of inertia about axes through the elastic center.

\[
\text{TOTAL ELASTIC WEIGHT} = \sum ds/I = \frac{Fr}{1.5} = \frac{r \times 30}{1.5} = 62.63
\]

Distance \( \bar{y} \) from line AC to elastic center of arch ABC equals 0.6566 \times .6566 \times 30 = 19.1 \text{ in.}

\[
I_x = 0.2978r^3/I = 2978 \times 30^3/1.5 = 5550
\]

Calculation of redundant forces at elastic center.

\[
M_o = -\sum \phi_s \frac{E}{I} = \frac{(-28800 + 167300)}{52.63} = -3122 \text{ in.lb.}
\]

\[
X_o = \sum \phi_s \frac{E}{I} \frac{x}{I_x} = \frac{167300 \times 5.7 + 28600(-9.1)}{5550} = 129.3 \text{ lb.}
\]

The fixed end moments at ends A and C will equal the moment due to the redundant forces \( M_o \) and \( X_o \) since the static moment assumed was zero at A and C.

\[
M_A = M_C = M_0 + X_0 = -3122 + 129.3 (-19.1) = -652 \text{ in.lb.}
\]

Moment Distribution Process

Having determined the fixed end moments, distribution and carry-over factors, the moment distribution can now be carried out. Fig. All.88 shows the solution. The first cycle will be explained. Starting at joint (A) the unbalanced moment is -3000 - 652 = -3652. The joint is balanced by distributing .182 × 3652 = 1534 to AC, .364 x 1534 = 553 to ABC and .312 x 3652 = 776 to AD. The carry-over moment from C to A is .5 x (-1534) = -772; from C on member CSA to A = -.452(.1332) = 601; and from A to C = .5 x 776 = 388. Now proceeding to joint D the unbalanced moment is -7500 + 388 = -7112. The joint is balanced by distributing .75 x 7112 = 5340 to DE and the remainder 25 per cent = 1772 to DA.
The carry-over moments are:

From E to D = .5 x (-5540) = -2770

To A from D = .5 (1772) = 886

The first cycle is now completed. Five more cycles are carried out in Fig. A11.99 in order to obtain reasonable accuracy of results. The final end moments are listed below the double short lines.

On arch member ABC the end moments of 584 are correct. However before the end moments at any other point on the arch can be found the horizontal reaction on the arch at A must be determined. This reaction will be determined by the elastic center method.

Fig. A11.90 shows the bending moment curves for members ABC and AC as made up of 5 parts labeled 1 to 5.

Calculation of \( \phi_B \) values which equal area of each moment curve divided by 1 of member.

\[ 2\phi_{A1} = 23850 \]
\[ \phi_{A2} = 167300 \]
\[ \phi_{A3} = 584 \times \pi \times 30/1.5 = 36800 \]
\[ \phi_{A5} = -(.567 \times 4500 \times 60/2) = -90000 \]
The bending moment at any point on member AC or ABC equals the values in Fig. A11.20 plus those due to $M_0$ and $X_0$ in Fig. A11.91.

Thus at point A on member ABC,

$$M_{ABC} = 584 - 2355 + 191 x 12.37 = 584 \text{ in. lb.}$$

which is the same as found by the moment distribution process.

At point B on arch:

$$M_B \text{ from Fig. A11.90} = 584 + 4000 = 4584$$

$$M_B \text{ from Fig. A11.91} = -2355 - 191 x (30 - 12.37) = -5478$$

Hence $M_B = 4584 - 5478 = -892 \text{ in. lb.}$

**STRUCTURES WITH JOINT DISPLACEMENTS**

For unsymmetrical frames or for frames loaded unsymmetrically, the assumption that only joint rotation occurs may give resultant moments considerably in error. Joint displacement can be handled in a manner as previously explained and illustrated for frames composed of straight members.

The distribution of the shear supporting forces on the frame boundary are usually taken as following the shear flow distribution for the shell in bending as explained in a later chapter.

**A11.20 Problems.**

(1) Determine the bending moment diagram for the load shown in Figs. 1 to 6.

(2) Find the bending moments at all joints and support points of the loaded structures in Figs. 7 to 10.

(3) In the loaded beam of Fig. 11, the supports B and E of the elevator beam deflect 0.1 inch more than supports C and D. Compute the resultant bending moments at the supports and find reactions. $EI = 320,000 \text{ lb-in.}$

(4) Fig. 12 illustrates a continuous 3 span wing beam, carrying a uniform air load of 20 lb/in. Determine the beam bending moments at a strut points A and B. Take $l_{AB} = 17 \text{ in.}$, $l_{AA'} = 20 \text{ in.}$ and $E = 1.3 \times 10^6 \text{ in.}$

Neglect effect of support deflection due to strut axial deformation.

(5) Figs. 13 to 15 are loaded structures that suffer joint translation. Determine bending moment diagram.
A12.1 General. The slope-deflection method is another widely used special method for analyzing all types of statically indeterminate beams and rigid frames. In this method as in the moment distribution method all joints are considered rigid from the standpoint that all angles between members meeting at a joint are assumed not to change in magnitude as loads are applied to the structure. In the slope deflection method the rotations of the joints are treated as the unknowns. For a member bounded by two end joints, the end moments can be expressed in terms of the end rotations. Furthermore for static equilibrium the sum of the end moments on the members meeting at a joint must be equal to zero. These equations of static equilibrium provide the necessary conditions to handle the unknown joint rotations and when these unknown joint rotations are found, the end moments can be computed from the slope-deflection equations. The advantages of the slope deflections method will be discussed at the end of the chapter after the method has been explained and applied to problem solution.

A12.2 Derivation of Slope Deflection Equation.

The problem is to determine the relationships between the displacements of the end supports of a beam and the resulting end moments on the beam.

Fig. A12.1a shows a beam restrained at ends 1 and 2. It is assumed unloaded and of constant cross-section or moment of inertia. This beam is now displaced as shown in Fig. b, namely, that the ends are rotated through the angles $\theta_1$ and $\theta_2$, plus a vertical displacement $d_1$ and $d_2$ of its ends from its original position, which produces a relative deflection $\Delta$ of its two ends. The angle $\theta$ representing the swing of the member equals $\Delta/L$.

The problem is to derive equations for $M_1$ and $M_2$, in terms of the end slopes $\theta_1$ and $\theta_2$, the length of the beam and $EI$. Figs. c, d and e illustrate how the total beam deflection in Fig. b is broken down into three separate deflections. In Fig. c, the end (2) is considered fixed and a moment $M_2$ is applied to rotate end (1) through an angle $\theta_1$. In Fig. d, end (1) is considered fixed and a moment $M_2$ is applied at (2) to rotate end (2) through an angle $\theta_2$. In Fig. e, both ends are considered fixed and end (2) is displaced downward a distance $\Delta$ which causes end moments $M''_1$ and $M''_2$. In deriving the slope deflection equations each of the three beam deflections will be considered separately and the results are then added to give the final equations. Fig. A12.2 shows Fig. c repeated.

Fig. A12.2

Fig. A12.3

Fig. A12.4

The applied moment $M_1$, is positive (tension in bottom fibers). Since end (2) is assumed fixed an unknown moment $M''_2$ is produced at (2). Fig. A12.3 shows the bending moment diagram made up of two triangles. $M''_2$ being unknown will be
assumed also positive, as the algebraic solution will determine the true sign of $M'$.  

Two moment area theorems will be used in deriving the slope-deflection equations, namely (1) that the change in slope of the beam elastic curve between two points on the beam is equal in magnitude to the area of the M/EI diagram between the two points, and (2) the deflection of a point (A) on the elastic curve away from a tangent to the elastic curve at (B) is equal in magnitude to the first moment about A of the M/EI diagram between (A) and (B) acting as a load.

In Fig. A12.2 since $\phi_a = 0$, then $\phi_1$ will equal the area of the M curves divided by EI.

$$\phi_1 = \frac{(MIL/2)}{EI} + \frac{(MIL/2)}{EI} = \frac{(M_1 + M_2)L}{2EI}$$  \hspace{1cm} (1)

The deflection of end (1) away from a tangent at (2) equals zero, thus we take moments of the moment diagram about (1) and equate to zero.

$$\left(\frac{MIL}{2EI}\right) + \frac{(MIL)\theta_1}{2EI} = \frac{M_1 L}{2EI} + \frac{M_2 L}{2EI} = 0$$

whence $M'_L = M_1/2$  \hspace{1cm} (2)

Fig. A12.4 shows the true shape of moment diagram.

Substituting results in equation (2) in equation (1) we obtain,

$$\phi_L = \frac{MIL}{4EI} \text{ or } M_1 = \frac{4EI\theta_L}{L}$$  \hspace{1cm} (3)

Then from equation (2)

$$M'_L = \frac{2EI\theta_L}{L}$$  \hspace{1cm} (4)

However, according to the moment area theorem, the area of M/EI diagram equals zero.

Hence, $\frac{(M + M')L}{2EI} = 0$

whence $M_1 = -M_2$  \hspace{1cm} (9)

From the deflection theorem,

$$\frac{M_1}{2EI} \left(\frac{L}{2}\right) + \frac{M_2}{2EI} \left(\frac{L}{2}\right) = \Delta$$  \hspace{1cm} (10)

Subt. (9) in (10)

$$M_1 = -\frac{2EI\theta_L}{L}$$  \hspace{1cm} (11)
The combined effect of deflection \( C_1, C_2 \) and \( A \) can now be obtained by adding the separate results.

\[
M_{z=1} = M_1 + M_2 + M_f = \frac{EI}{L} (2\phi_1 + \phi_2 - 3A/L)
\]

Let \( K = \frac{EI}{L} \) and \( \phi = A/L \)

Then,

\[
M_{z=1} = 2K (2\phi_1 + \phi_2 - 3\phi) \quad \text{--- (13)}
\]

and

\[
M_{z=1} = -2K (2\phi_1 + \phi_2 - 3\phi) \quad \text{--- (14)}
\]

These end moments due to the distortion of the supports must be added to the moments due to any applied loads on the beam when considered as fixed ended. Let these fixed end moments be labeled \( M_{P_1} \) and \( M_{P_2} \). Then equations (13) and (14) can be written including these moments.

\[
M_{z=1} = 2K (2\phi_1 + \phi_2 - 3\phi) - M_{P_1} \quad \text{--- (15)}
\]

\[
M_{z=1} = -2K (2\phi_1 + \phi_2 - 3\phi) - M_{P_2} \quad \text{--- (16)}
\]

**MODIFIED SIGN CONVENTION**

Equations (15) and (16) have been developed using the conventional signs for bending moments. In general there are advantages of using a statical sign convention as was used in the moment distribution method in Chapter 11. Therefore, the following sign convention will be used in this chapter:

1. The rotation of a joint or member is positive if it turns in a clockwise direction.

2. An end moment is considered positive if it tends to rotate the end of the member clockwise or the joint counterclockwise.

This adopted sign convention is the same as adopted for the moment distribution method (See Art. A12.2).

When equations (15) and (16) are revised for this new sign convention they become:

\[
M_{z=1} = 2K (2\phi_1 + \phi_2 - 3\phi) + M_{P_1} \quad \text{--- (17)}
\]

\[
M_{z=1} = 2K (2\phi_1 + \phi_2 - 3\phi) + M_{P_2} \quad \text{--- (18)}
\]

Equations (17) and (18) are referred to as the slope-deflection equations where \( K = EI/L \) and \( \phi = A/L \).

For no yielding or transverse movement of supports, \( A = 0 \) and equations (17) and (18) become

\[
M_{z=1} = 2K (2\phi_1 + \phi_2) + M_{P_1} \quad \text{--- (18a)}
\]

\[
M_{z=1} = 2K (2\phi_1 + \phi_2) + M_{P_2} \quad \text{--- (18b)}
\]

**A12.3 Hinged End: Slope Deflection Equation.**

Consider that end (2) of beam 1-2 is freely supported or in other words hinged. This means that \( M_{z=1} \) is zero. Thus equation (18) can be equated to zero.

\[
M_{z=1} = 2K (2\phi_1 + \phi_2 - 3\phi) + M_{P_2} = 0
\]

whence

\[
2K\phi_2 = -K\phi_1 + 3K\phi - .5M_{P_2}
\]

Substituting this value in equation (17)

\[
M_{z=1} = 3K(\phi_1 - \phi) + M_{P_{1-2}} - 0.5M_{P_{2-1}} \quad \text{--- (19)}
\]

**A12.4 Example Problems.**

**Problem 1**

Fig. A12.1 shows a two span continuous beam fixed at ends (1) and (3) and carrying lateral loads as shown. The bending moments at points 1, 2 and 3 will be determined. Relative values of I are shown for each member.

\[
\begin{array}{c|c|c|c|c|c|c}
& 1 & 2 & 3 \\
\hline
w & 12" & - & - & w = 24 lb/in. \\
\hline
L & 24" & 24" & 24" \\
I & 1 & 48 & 48 \\
\end{array}
\]

**Fig. A12.1**

**SOLUTION:**

Calculation of fixed end moments due to applied loads -

\[
M_{P_{1-2}} = -PL/6 = 100 \times 24/6 = -300 \text{ in. lb.}
\]

\[
M_{P_{2-1}} = PL/8 = 300
\]

The signs as shown are determined from the fact that the end moment at (1) tends to rotate the end of the member counterclockwise which is a positive moment according to our adopted sign convention. By similar reasoning the fixed moment in moment at (2) is positive because the moment tends to rotate end of member clockwise. For span 2-3:

\[
M_{P_{3-2}} = wL^2/12 = 24 \times 24^2/12 = -1152 \text{ in lb.}
\]

\[
M_{P_{3-2}} = 1152
\]

**K Values**

Since \( E \) is constant it will be considered unity.

FOR BEAM 1-2, \( K = 1/L = 24/24 = 1 \)

FOR BEAM 2-3, \( K = 48/24 = 2 \)
Substituting in slope-deflection equations 12a and 12b. \( \theta = 0 \) since supports do not translate.

**Beam 1-2.** \( \theta_1 = 0 \) because of fixity at (1).

\[
\begin{align*}
M_{1-2} &= 2K(2\theta_2 + \theta_2) + M_F_1 \\
M_{1-2} &= 2 \times 1 (0 + \theta_2) - 300 \\
M_{1-2} &= 2\theta_2 - 300 \quad \text{(a)} \\
M_{2-1} &= 2K(2\theta_2 + \theta_2) + M_F_1 \\
M_{2-1} &= 2 \times 1 (2\theta_2 + 0) + 300 \\
M_{2-1} &= 4\theta_2 + 300 \quad \text{(b)}
\end{align*}
\]

**Beam 2-3.** \( \theta_3 = 0 \), because of fixity at (3).

\[
\begin{align*}
M_{2-3} &= 2K(2\theta_3 + \theta_3) + M_F_2 \\
M_{2-3} &= 2 \times 2 (2\theta_3 + 0) - 1152 \\
M_{2-3} &= 8\theta_3 - 1152 \quad \text{(c)} \\
M_{3-2} &= 2K(2\theta_3 + \theta_3) + M_F_3 \\
M_{3-2} &= 2 \times 2 (0 + \theta_3) + 1152 \\
M_{3-2} &= 4\theta_3 + 1152 \quad \text{(d)}
\end{align*}
\]

At joint (2) \( EM = 0 \) for static equilibrium.

Hence

\[
M_{2-1} + M_{2-3} = 0
\]

Substituting from (b) and (c)

\[
4\theta_2 + 300 + 8\theta_3 - 1152 = 0
\]

where \( \theta_3 = 71 \)

With \( \theta_3 \) known the final end moments can be found by substituting in equations (a), (b), (c) and (d) as follows:

\[
\begin{align*}
M_{1-2} &= 2 \times 71 - 300 = -158 \text{ in.lb.} \\
M_{2-3} &= 4 \times 71 + 300 = 584 \\
M_{2-3} &= 3 \times 71 - 1152 = -584 \\
M_{3-2} &= 4 \times 71 + 1152 = 1456
\end{align*}
\]

Changing the resulting moment signs to the conventional sign practice, gives

\[
\begin{align*}
M_{1-2} &= -158, \ M_{2-3} = M_{2-3} = -584, \\
M_{3-2} &= -1456
\end{align*}
\]

**Example Problem 2**

\[
\begin{align*}
M_F_1 &= -PL/8 = 100 \times 24/3 = -300, \\
M_F_2 &= 300 \\
M_F_3 &= (100 \times 8 \times 15^2/24) + (100 \times 15 \times 5^2/24^2) = -534 \\
M_F &= 534 \\
M_F &= wL^2/12 = 12 \times 20 \times 12 = -400, \\
M &= 400
\end{align*}
\]

**SLOPE DEFORMATION EQUATIONS** \( \theta = 0 \)

\[
\begin{align*}
M_{1-2} &= 2 \times 1 (0 + \theta_2) - 300 = 2\theta_2 - 300 \quad \text{(a)} \\
M_{2-3} &= 2 \times 2 (2\theta_2 + 0) + 300 = 4\theta_2 + 300 \quad \text{(b)} \\
M_{2-3} &= 2 \times 2 (2\theta_2 + \theta_3) - 534 = 8\theta_2 + 4\theta_3 \\
&\quad - 534 \quad \text{---} \quad \text{(c)} \\
M_{3-2} &= 2 \times 2 (2\theta_3 + \theta_3) + 534 = 8\theta_2 + 4\theta_3 \\
&\quad + 534 \quad \text{---} \quad \text{(d)} \\
M_{3-2} &= 2 \times 1 (2\theta_3 + 0) - 400 = 4\theta_2 - 400 \quad \text{(e)} \\
M_{3-2} &= 2 \times 1 (0 + \theta_3) + 400 = 2\theta_2 + 400 \quad \text{(f)}
\end{align*}
\]

Joint-Moment equilibrium equations:

**JOINT (2)** \( M_{2-1} + M_{2-3} = 0 \)

whence

\[
\begin{align*}
4\theta_2 + 300 + 8\theta_2 + 4\theta_3 - 534 &= 0 \quad \text{or} \\
12\theta_2 + 4\theta_3 - 234 &= 0 \quad \text{---} \quad \text{(g)}
\end{align*}
\]

**JOINT (3)** \( M_{3-2} + M_{3-4} = 0 \)

\[
\begin{align*}
30\theta_2 + 4\theta_3 + 534 + 4\theta_2 - 400 &= 0 \quad \text{or} \\
12\theta_2 + 4\theta_3 + 134 &= 0 \quad \text{---} \quad \text{(h)}
\end{align*}
\]
Solving equations (g) and (h) for $\theta_a$ and $\theta_b$ gives,

$$\theta_a = 26.15, \quad \theta_b = -19.9$$

Substituting in equations (a) to (f) inclusive to find final end moments.

$$M_{1-a} = 28_a + 300 = 2 \times 26.15 - 300 = -247.7 \text{ in.lb}.$$  
$$M_{2-a} = 40_a + 300 = 4 \times 26.15 + 300 = 404.6$$  
$$M_{3-a} = 8 \theta_a + 4 \theta_b - 534 = 8 \times 26.15 + 4 \times (-19.9)$$  
$$- 534 = -404.6$$  
$$M_{4-a} = 4 \theta_a + 4 \theta_b + 534 = 8 \times (-19.9) + 4 \times 26.15$$  
$$+ 534 = 479.4$$  
$$M_{5-a} = 40_a - 400 = 4 \times (-19.9) - 400 = -479.4$$  
$$M_{6-a} = 28_b + 400 = 2 \times (-19.9) + 400 = 360.2$$

(Note: Student should convert to conventional moment signs and draw complete bending moment diagram).


Fig. A12.13 shows a loaded 3 span continuous beam with hinged supports at points (1) and (4), which means that $M_{1-a}$ and $M_{4-a} = 0$. The moments at supports (2) and (3) will be determined.

\[
\begin{array}{ccc}
100 \text{ lb.} & 200 \text{ lb.} & 10^3 \text{ in.} \\
\hline \\
(1) & L = 20 & (2) & L = 20 & (3) & L = 20 \\
I = 20 & K = \frac{L}{2} & K = 0.5 & K = 1 \\
K = \frac{L}{2} & K = 1 \\
\end{array}
\]

Fig. A12.13

SOLUTION:

Fixed end moments:

$$M_{F_{1-a}} = -PL/8 = 100 \times 20/8 = -250,$$

$$M_{F_{2-a}} = 250$$

$$M_{F_{3-a}} = M_{F_{4-a}} = 0 \text{ (No load on spans 2-3)}$$

$$M_{F_{5-a}} = PL/8 = 200 \times 20/8 = -500,$$

$$M_{F_{6-a}} = 500$$

Slope-deflection equations:

Since supports at (1) and (4) are hinged or free to rotate we use equation (19) in writing equations for $M_{1-a}$ and $M_{2-a}$.

$$M_{1-a} = 3 \theta_a + M_{F_{1-a}} - 0.5 M_{F_{2-a}}, \quad (\theta \text{ is zero})$$

$$M_{2-a} = 3 x 1 x \theta_a + 250 - 0.5 (-250) =$$

$$3 \theta_a + 375$$

$$M_{3-a} = 3 \theta_a + M_{F_{3-a}} - 0.5 M_{F_{4-a}}$$

$$M_{4-a} = 3 x 1 x \theta_a - 500 - 0.5 (500) = 3 \theta_a - 750$$

Using equations 18a and 18b and substituting,

$$M_{2-a} = 2k(2 \theta_a + \theta_b) + M_{F_{2-a}}$$

$$M_{2-a} = 2 x 0.5(2 \theta_a + \theta_b) + 0 = 2 \theta_a + \theta_b = (c)$$

$$M_{5-a} = 2k(2 \theta_a + \theta_b) + M_{F_{5-a}}$$

$$M_{5-a} = 2 x 0.5(2 \theta_a + \theta_b) + 0 = 2 \theta_a + \theta_b = (d)$$

For static equilibrium of joints:

\[
\begin{align*}
\text{JOINT (2)} & : M_{1-a} + M_{2-a} = 0 \\
\text{Subt.} & : 3 \theta_a + 325 + 2 \theta_a + \theta_b = 0 \\
\text{Solving (e) and (f) for $\theta_a$ and $\theta_b$ gives,} & : \theta_a = 109.4, \quad \theta_b = 172
\end{align*}
\]

The end moments at (2) and (3) can now be found from equation (a) to (d) inclusive.

$$M_{2-a} = 3 \theta_a + 375 = 3 (-109.4) + 375 = 45.8 \text{ in.lb}.$$  
$$M_{3-a} = 2 \theta_a + \theta_b = 2 (-109.4) + 172 = -46.8$$  
$$M_{4-a} = 2 \theta_a + \theta_b = 2 x 172 + (-109.4) = 234$$  
$$M_{5-a} = 3 \theta_a - 750 = 3 x 172 - 750 = -234$$

A12.5 Loaded Continuous Beam with Yielding Supports.

The movable control surfaces of an airplane, namely the elevator, rudder and ailerons are attached at several points to the stabilizer, fin and wing respectively. These supporting surfaces are usually cantilever structures and thus the supporting points for the movable control surfaces suffer a displacement thus promoting a continuous beam on yielding or deflected
supports. The slope deflection equations can take care of this support displacement as illustrated in the following example problem.

Example Problem.

\[ M_{p_{-3}} = \frac{40^2}{60} (5 \times 3 + 1.5) = 507 \]

The beam has a constant section, hence, \( K \) for all spans equals,

\[ K = EI/L = (10,000,000 \times 0.02539) \times 40 = 8347 \]

Calculation of \( \phi \) values.

\[ \Delta = \phi/L = 0.0075 \text{ inches} \]

\[ \phi = -0.0075 \times 40 = -0.00301 \text{ rad.} \]

Since \( \phi \) is negative.\n
\[ \phi = -\Delta/L = -0.0075 \times 40 = -0.00301 \text{ rad.} \]

Since the beam, external loading and support settlement is symmetrical about support \( (4) \), the slope of the beam elastic curve at \( (4) \) is horizontal or zero and therefore \( \theta_4 = 0 \). Thus only one-half of the structure need be considered in the solution. Due to the fact that \( (2) \) is a simple support with a cantilever overhang, the moment \( M_{u-3} \) is statically determinate and equals \( 5 \times 3 \times 3.5 = 50 \) in.lb. Then for static equilibrium of joint \( (2) \), \( M_{u-3} \) must equal \( 50 \).

Substitution in slope deflection equations (17) and (18):

\[ M_{u-3} = 2K(2\theta_a + \theta_3 - 3\theta_4) + M_{p_{-3}} \]

\[ -50 = 2 \times 8347 \left[ 2\theta_a + \theta_3 - 3(-0.0075) \right] \]

or \( 333889 + 166949 + 15 = 0 \)

\[ M_{3-4} = 2K(2\theta_a + \theta_3 - 3\theta_4) + M_{p_{-3}} = 2 \times 8347 \left[ 2\theta_a + \theta_3 - 3(-0.0075) \right] + 440 \]

whence \( M_{3-4} = 333889 + 166949 + 331 = 0 \) (1)

\[ M_{u-4} = 2K(2\theta_a + \theta_3 - 3\theta_4) + M_{p_{-3}} \]

\[ 2 \times 8347 \left[ 2\theta_a + 0 - 3(-0.00468) \right] = 484 \]

whence \( M_{3-4} = 333889 - 259 = 0 \)

\[ M_{u-4} = 2K(2\theta_a + \theta_3 - 3\theta_4) + M_{p_{-3}} \]

\[ 2 \times 8347 \left[ 2\theta_a + 0 - 3(-0.00468) \right] = 507 \]

whence \( M_{u-4} = 166949 + 742 = 0 \)

For static equilibrium of joint \( (3) \),

\[ M_{3-4} + M_{u-4} = 0 \]

\[ 333889 + 166949 + 331 + 333889 = 259 = 0 \]
whence, $567776 + 166946 + 372 = 0 \quad \quad - \quad - \quad - \quad (5)$

Solving equations (1) and (5) for $\theta_a$ and $\theta_b$ we obtain $\theta_a = 0.00059$ and $\theta_b = -0.00067$.

Substituting in equations (2), (3) and (4) gives the moments at these points.

$M_{a-3} = 33268 (-.00067) + 16694 \times .00439$
$\quad \quad \quad \quad \quad \quad + 331 = 582 \text{ in.lb.}$

$M_{a-3} = 33883 (-.00067) - 259 = -582$

$M_{a-3} = 16694 (-.00067) + 742 = 580$

Converting these signs to the conventional signs would give $M_a = -50$, $M_b = -582$ and $M_c = -580$.

**A12.6 Statically Indeterminate Frames.**

Joint Rotation Only.

The slope deflection equations can be used in solving all types of framed structures. In frames as compared to straight continuous beams, the axial loads in the members are usually more important, however the influence on joint displacement due to axial deformation of members is relatively small and is therefore usually neglected in most simple framed structures. To illustrate the slope-deflection method as applied to frames, the structure shown in Fig. A12.16 will be solved.

**Example Problem 1.**

![Diagram](image)

Fig. A12.16 shows a closed rectangular frame subjected to loadings on its four sides which hold the frame in equilibrium. The bending moments at the frame joints will be determined.

**SOLUTION:**

**FIXED END MOMENTS:**

$M_{a-3} = \frac{(100 \times 3 \times 12^2) \div 20^2} + \frac{(100 \times 12 \times 8^2)}{20^2} = -480$

$M_{a-1} = 480$

$M_{a-2} = 100 \times 24 = 300 \quad M_{a-4} = -300$

$M_{a-3} = 10 \times 20^2 \div 12 = 334 \quad M_{a-5} = -334$

Due to symmetry of loading we know that $\theta_a = -\theta_a$ and $\theta_b = -\theta_b$, which fact will shorten the solution.

**Slope deflection equations:**

$M_{1-a} = 2K (\theta_a + \theta_b) + M_{1-a}$

$M_{1-a} = 2x1 (2\theta_a - \theta_b) - 480$

$M_{2-a} = 2K (\theta_a + \theta_b) + M_{2-a}$

$M_{2-a} = 2x1.67 (2\theta_a + \theta_b) + 300$

$M_{3-a} = 6.6781 + 3.3382 + 300$  

$M_{3-a} = 2K (2\theta_a + \theta_b) + M_{3-a}$

$M_{3-a} = 2x1.67 (2\theta_b + \theta_a) - 300$

$M_{4-a} = 6.6783 + 3.3382 - 300$  

$M_{4-a} = 2K (2\theta_b + \theta_a) + M_{4-a}$

$M_{4-a} = 2x1 (2\theta_a - \theta_b) + 334$

$M_{4-a} = 2\theta_a + 334$  

**Static Joint Equations:**

**JOINT (1):**

$M_{1-a} = M_{1-a} = 0$  

which gives,

$2\theta_a - 480 + 6.6781 + 3.3382 + 300 = 0$

whence $6.6781 + 3.3382 - 180 = 0$  

**JOINT (3):**

$M_{a-3} + M_{a-4} = 0$  

which gives,

$6.6782 + 3.3382 + 300 + 2\theta_a + 334 = 0$

whence $6.6782 + 3.3382 + 34 = 0$  

Solving equations (e) and (f) for $\theta_a$ and $\theta_b$ gives,

$\theta_a = 26.3, \quad \theta_b = -13.94$
SPECIAL METHODS - SLOPE DEFLECTION METHOD

A12.6

Subt. in equation (a)

\[ M_{a-5} = 2 \times 25.3 - 480 = -428 \text{ in. lb.} \]

Subt. in equation (d)

\[ M_{a-4} = 2(-13.94) + 334 = 306 \text{ in. lb.} \]

Fig. A12.17 shows the resulting bending moment diagram.

A12.7 Frames with Joint Displacements.

In the previous example problems, the conditions were such that only joint rotations took place, or if any transverse joint displacement took place as in the example problem of Art. A12.5, these displacements were known, or in other words the value of \( \phi \) in the slope-deflection was known.

In many practical frames however, the joints suffer unknown displacements as for example in the frame of Fig. A12.18. The term \( \phi = \Delta/L \) in the slope-deflection equation was derived on the basis of transverse displacement of the member ends when both ends were fixed. Thus in Fig. A12.18 under the action of the external loads, will sway to the right as magnified by the dashed lines. Neglecting any joint displacement due to axial deformation, the upper end of each member will move through the same displacement \( \Delta \). Then we can write,

\[ \phi_{a-1} = \Delta/L_{a-1}, \quad \phi_{a-3} = \Delta/L_{a-3}, \quad \phi_{a-5} = \Delta/L_{a-5} \]

Due to the fixity at joints (1), (3) and (5), \( \phi_{a-1} = \phi_{a-3} = \phi_{a-5} = 0 \). Therefore, the unknowns are \( \phi_{a-1}, \phi_{a-3}, \phi_{a-5} \) and \( \Delta \).

There are three statical joint equations of equilibrium available, namely,

\[ M_{a-1} + M_{a-3} = 0, \quad M_{a-3} + M_{a-5} = 0, \quad \text{and} \quad M_{a-5} = 0 \]

Another equation is necessary because there are four unknowns. This additional equation is obtained by applying the equations of statics to the free bodies of the vertical members. Fig. A12.19 shows the free bodies.

A12.8 Example Problems of Frames with Unknown Joint Displacement.

Example Problem 1

Fig. A12.20 shows an unsymmetrical frame with unsymmetrical loading. The bending moment curve will be determined.

SOLUTION:

The relative moment of inertia of each
The relative $K = 1/L$ values are also indicated adjacent to each member.

Relative values of $\phi$ due to sideway of frame:

The angles $\phi$ are proportional to the $A/L$ for each member where $A$ is same for members (1-2) and (3-4) and zero for member (2-4). (See Fig. A12.10).

Hence, $\phi_{A-1} = \frac{1}{16} = 0.0667\phi$

$\phi_{A-2} = \frac{1}{10} = 0.10\phi$

$\phi_{A-3} = \frac{1}{12} = 0$

FIXED END MOMENTS:

Member 1-2:

$M_{A-2} = -(48 \times 6^2 \times 9)/15^2 = -69.12$ in.lb.

$M_{A-1} = (48 \times 2^2 \times 6)/15^2 = 103.68$

Member 2-4:

$M_{A-4} = -(96 \times 12)/8 = -144$

$M_{A-3} = 144$

Member 3-4 has no lateral loads and thus fixed end moments are zero.

The slope-deflection equations are:

$M_{A-3} = 2K(2\theta_a + \theta_e - 3\theta) + M_{A-3} = -144$ (17)

$M_{A-1} = 2K(2\theta_a + \theta_e - 3\theta) + M_{A-2} = -103.68$ (18)

Writing above equations for each member and noting that $\theta_e$ and $\theta$ are zero because frame is fixed at support points, gives:

$M_{A-3} = 2 \times 0.667(0 + \theta_a - 3 \times 0.667\theta) - 69.12$

$M_{A-1} = 1.3333\theta_a - 2.667\theta - 69.12 = -103.68$ (a)

$M_{A-4} = 2 \times 0.667(2\theta_a + 0 - 3 \times 0.667\theta) + 103.68$

$M_{A-1} = 2.667\theta_a - 0.2667\theta + 103.68 = -103.68$ (b)

$M_{A-4} = 2 \times 1.667(2\theta_a + \theta_e - 0) - 144$

$M_{A-3} = 6.667\theta_a + 3.333\theta - 144 = -144$ (c)

$M_{A-2} = 2 \times 1.667(2\theta_a + \theta_e - 0) + 144$

$M_{A-1} = 6.667\theta_a + 3.333\theta + 144 = -144$ (d)

$M_{A-3} = 2 \times (1 + \theta_a - 3 \times 0.1\theta) + 0$

$M_{A-2} = 2 \times (1 + \theta_a - 3 \times 0.1\theta) + 0$

$M_{A-1} = 2 \times 0.667 + 0 - 3 \times 0.1\theta = 0$

$M_{A-4} = 0.667 \theta - 0.667 \theta = -144$ (e)

Equilibrium equations:

$\text{JOINT (2). } M_{A-1} + M_{A-4} = 0$

Substituting:

$2.667\theta_a - 0.2667\theta + 103.68 + 6.667\theta_a + 3.333\theta - 144 = 0$

whence, $9.333\theta_a - 0.2667\theta + 3.333\theta - 40.32 = 0$

$\theta_a = -144 = 0$

$\text{JOINT (4). } M_{A-3} + M_{A-4} = 0$

$6.667\theta_a + 3.333\theta + 144 - 48 - 0.6\theta = 0$

whence, $10.667\theta_a + 3.333\theta - 0.6\theta + 144 = 0$

Writing the bent equation (See Eq. 20):

$H_1 + H_2 - 48 = 0$

In order to get $H_1$ and $H_2$ in terms of end moments see Fig. A12.21.

For free body of member 1-2 take moments about (2) and equate to zero.

$M_{A-1} + M_{A-2} - 6 \times 48 + 15H_1 = 0$

hence, $H_1 = \frac{288 - (M_{A-1} + M_{A-2})}{15}$

Subt. values of $M_{A-1}$ and $M_{A-2}$ in the equation,

$H_1 = -0.2667\theta_a + 0.3333\theta + 16.38 = 0$

For free body of member 3-4 take moments about (4) and equate to zero.
\[ M_{v\rightarrow} + M_{v\leftarrow} + 10H_v = 0 \]

whence, \[ H_v = - \left( \frac{M_{v\rightarrow} + M_{v\leftarrow}}{10} \right) \]

Substituting values of \( M_{v\rightarrow} \) and \( M_{v\leftarrow} \) in above equations, gives,

\[ H_v = -0.68\theta_a + 0.12\theta_v - 31.12 = 0 \quad (1) \]

Substituting values of \( H_v \) and \( H_v \) in equation (1), gives,

\[-0.2667\theta_v - 0.6\theta_a + 1.1555\theta - 31.12 = 0 \quad (1)\]

Solving equations (g), (h), (l) for the unknowns \( \theta_a \), \( \theta_v \) and \( \theta \) gives,

\[ \theta_a = 196.9, \quad \theta_v = 12.17, \quad \theta = 6.22 \]

The final end moments can now be found by substituting these values in equations (a) to (f) inclusive.

\[ M_{1\rightarrow} = 1.333 \times 12.17 - 0.2667 \times 196.9 \]
\[ = 69.12 = 106.4 \text{ in.} \text{lb.} \]

\[ M_{2\leftarrow} = 2.667 \times 12.17 - 0.2667 \times 196.9 \]
\[ + 103.68 = -83.6 \]

\[ M_{3\rightarrow} = 6.667 \times 12.17 + 3.333 \times (-6.22) \]
\[ - 144 = 83.6 \]

\[ M_{4\leftarrow} = 6.667 \times (-6.22) + 3.333 \times 12.17 + 144 \]
\[ = 142.9 \]

\[ M_{5\rightarrow} = 4 \times (-6.22) - 0.6 \times 196.9 = -142.9 \]

\[ M_{5\leftarrow} = 2 \times (-6.22) - 0.6 \times 196.9 = -130.46 \]

Fig. A12.22 shows the bending moment diagram. The student should also draw the shear diagram, find axial load in members since all these loads are needed when the

strength of the frame members are computed and compared against the stresses caused by the frame loading.

Example Problem 2.

Fig. A12.23 shows a gable frame, carry-

ing the symmetrical distributed load as shown.

w = 12 lb/in.

\[ \theta_{a\leftarrow} = \frac{1}{20} = 0.05\theta \]

\[ \theta_{a\rightarrow} = \frac{\sqrt{2}}{22.4} = 0.10 \]

The bending moment diagram under the given loading will be determined.

**SOLUTION:**

**Relative Stiffness (K values)**

Members 1-2 and 4-5, \( K = 10/20 = 0.5 \)

Members 2-3 and 4-3, \( K = 20/22.4 = 0.892 \)

**Fixed end moments:**

\[ M_{P_{2\leftarrow}} = \frac{wL^2}{12} = \frac{(12 \times 20^2)}{12} = 400 \text{ in.} \text{lb.} \]

\[ M_{P_{2\rightarrow}} = 400, \quad M_{P_{4\leftarrow}} = 400, \quad M_{P_{4\rightarrow}} = -400 \]

**Relative values of \( \theta \):**

Due to the sloping members, the relative transverse deflections of each member are not as obvious as when the vertical members are connected to horizontal members as in the previous example. In this example, joints (1) and (2) are fixed. Because of symmetry of frame and loading, joints (2) and (4) will move outward the same distance \( \Delta \) as indicated in Fig. A12.24. Furthermore, due to symmetry, joint (3) will undergo vertical movement only.

In Fig. A12.24, draw a line from point (2') parallel to 2-3 and equal in length to 2-3 to locate the point (3'). Erect a perpendicular to 2'-3' at 3' and where it intersects a vertical through (3) locates the point (3'). The length of 3'-3'' equals \( \Delta \). In the triangle 3'-3''-5'', the side \( 3''-5'' = \frac{\sqrt{2}}{22.4} \). The relative values of \( \theta \) which are measured by \( \Delta/L \) for each member can now be calculated.

\[ \theta_{a\leftarrow} = \frac{1}{20} = 0.05 \theta \]

\[ \theta_{a\rightarrow} = \frac{\sqrt{2}}{22.4} = 0.10 \]
Substitution in Slope Deflection Equations:

We know that $\theta_0 = \theta_n = 0$, due to fixity at joints (1) and (5). Also $\theta_0 = 0$ due to symmetry or only vertical movement of joint (3). Furthermore due to symmetry $\theta_a = -\theta_a$.

\[
\begin{align*}
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.5(0 + \theta_a + 0.05 \times 3\theta) + 0 \\
M_{a-1} &= \theta_a + 0.15\theta + \ldots - \ldots - \ldots - \ldots \ldots \ldots \ldots (a) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.5(\theta_a + 0.05 \times 3\theta) + 0 \\
M_{a-1} &= 2\theta_a + 0.15\theta + \ldots - \ldots - \ldots - \ldots - (b) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.5(2\theta_a + 0 - 3 \times 0.1\theta) - 400 \\
M_{a-1} &= 3.566\theta_a - 0.5352\theta - 400 - \ldots - \ldots (c) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.862(2\theta_a + 0 - 3 \times 0.1\theta) + 400 \\
M_{a-1} &= 1.734\theta_a - 0.5352\theta - 400 - \ldots - \ldots (d) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.862(0 - \theta_a + 3 \times 0.1\theta) - 400 \\
M_{a-1} &= -1.734\theta_a + 0.5352\theta - 400 - \ldots - \ldots (e) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
M_{a-1} &= 2 \times 0.862(-2\theta_a + 0 + 3 \times 0.1\theta) + 400 \\
M_{a-1} &= -3.566\theta_a + 0.5352\theta + 400 - \ldots - (f) \\
M_{a-1} &= 2K(2\theta_a + \theta_a - 3\theta) + M_{F_{a-1}} \\
\end{align*}
\]

Joist (2). Equilibrium equation.

\[
M_{a-1} + M_{a-2} = 0, \text{ substituting -} \\
2\theta_a + 0.15\theta + 3.566\theta_a - 0.5352\theta - 400 = 0 \\
\text{whence, } 5.566\theta_a - 0.3652\theta - 400 = 0 - \ldots \ldots (1)
\]

The joint equilibrium equations at (3) and (4) will not provide independent equations because in the previous substitutions in the slope-deflection equations $\theta_a$ was made equal to zero and $\theta_a$ was equal to $-\theta_a$.

Shear Equations:

Due to symmetry the horizontal reactions at points (1) and (5) are equal and opposite and therefore in static balance. Since we have two unknowns $\theta_a$ and $\theta$, we need another equation to use with equation (1). This equation can be obtained by stating that the horizontal reaction $H_{a-1}$ on member $1\rightarrow 2$ at end (2) must be equal and opposite to $H_{a-2}$, the horizontal reaction at end (2) of member $2\rightarrow 3$.

Fig. A12.25 and 26 show free body sketches of members 1-2 and 2-3.

![Free body diagram](image)

Fig. A12.26

In Fig. A12.25 take moments about (1) and equate to zero.

\[
M_{1-2} + M_{a-1} - 20H_{a-1} = 0
\]

whence, $H_{a-1} = \frac{M_{1-2} + M_{a-1}}{20}$

and in Fig. A12.26, taking moments about (3),

\[
M_{a-3} + M_{a-2} - 240 \times 20 - 20 \times 12 X 10 - 10H_{a-3} = 0
\]

whence, $H_{a-3} = \frac{M_{a-3} + M_{a-2}}{20} + 2400$
SPECIAL METHODS - SLOPE DEFLECTION METHOD

Shear equation, \( M_{x-1} = M_{x-3} \)

whence,
\[
\frac{M_{x-3} + M_{x-2}}{20} = \frac{M_{x-2} + M_{x-1}}{10} + 2400
\]

Substituting values for the end moments:
\[
\begin{align*}
\Theta_1 &= 0.15\bar{\theta} + 29 + 0.15\bar{\theta} = \frac{3.568\bar{\theta} - 0.5352\bar{\theta}}{\Theta_1} \\
400 + 1.784\bar{\theta} &= 0.5352\bar{\theta} + 400 + 2400
\end{align*}
\]

whence,
\[
0.3552\bar{\theta}, -0.122\bar{\theta} + 240 = 0
\]

Solving equations i and j for \( \Theta_1 \) and \( \bar{\theta} \), gives
\[
\bar{\theta} = 2905, \quad \Theta_1 = 265.8
\]

Substituting these values in equations (a) to (h), the end moments are obtained as follows:
\[
\begin{align*}
M_{x-1} &= \Theta_1 + 0.15\bar{\theta} = 265.8 + 0.15 \times 2905 = 652.5 \text{ in.lb}.
M_{x-2} &= 29 + 0.15\bar{\theta} = 531.6 + 420.7 = 952
M_{x-3} &= 3.568\bar{\theta} - 0.5352\bar{\theta} - 400 = 3.568 \times 265.8 - 0.5352 \times 2905 - 400 = -952
M_{x-4} &= 1.784\bar{\theta} - 0.5352\bar{\theta} + 400 = 1.784 \times 265.8 - 0.5352 \times 2905 + 400 = 628
M_{x-5} &= -1.784\bar{\theta} + 0.5352\bar{\theta} - 400 = -628
\end{align*}
\]

Fig. A12.27 shows the resulting bending moment diagram, first drawn in parts for each member of the frame and then added to form the composite diagram plotted directly on the frame. The shears and axial loads follow as a matter of statics. With the moment diagram known the frame deflected shape is readily calculated.

A12.9 Comments on Slope-Deflection Method.

The example problems solutions indicate that the method of slope-deflection has two advantages, namely, (1) it reduces the number of equations to be solved simultaneously and (2) it presents equations that are easily and rapidly formulated.

Thus, for structures with a high degree of redundancy, the slope-deflection method should be considered as possibly the best method of solution. The solving of the equations in this method are readily programmed for solution by high speed computing machinery.

A12.10 Problems.

(1) Determine the bending moments at support points A, B, C, D, for the continuous beam shown in Fig. A12.28.

(2) Same as problem (1) but consider support A as freely supported instead of fixed.

(3) Determine the bending moment diagram for the various loaded structures in Fig. A12.29.

![Fig. A12.27](image-url)

![Fig. A12.28](image-url)

![Fig. A12.29](image-url)
CHAPTER A13
BENDING STRESSES

A13.0 Introduction.

The bar AB in Fig. a is subjected to an axial compressive load P. If the compressive stresses are such that no buckling of the bar takes place, then bar sections such as 1-1 and 2-2 move parallel to each other as the bar shortens under the compressive stress.

![Fig. a]

In Fig. b the same bar is used as a simply supported beam with two applied loads P as shown. The shear and bending moment diagrams for the given beam loading are also shown. The portion of the beam between sections 1-1 and 2-2 under the given loading are subjected to pure bending since the shear is zero in this region.

Experimental evidence for a beam segment Ax taken in this beam region under pure bending shows that plane sections remain plane after bending but that the plane sections rotate with respect to each other as illustrated in Fig. c, where the dashed line represents the unstressed beam segment and the heavy section the shape after pure bending takes place. Thus the top fibers are shortened and subjected to compressive stresses and the lower fibers are elongated and subjected to tensile stresses. Therefore at some plane n-n on the cross-section, the fibers suffer no deformation and thus have zero stress. This location of zero stress under pure bending is referred to as the neutral axis.

A13.1 Location of Neutral Axis.

Fig. d shows a cantilever beam subjected to a pure moment at its free end, and under this applied moment the beam takes the exaggerated deflected shape as shown.

The applied bending moment vector acts parallel to the Z axis, or in other words the applied bending moment acts in a plane perpendicular to the Z axis. Consider a beam segment of length L. Fig. d shows the distortion of this segment when plane sections remain plane after bending of the beam.

It will be assumed that the beam section is homogeneous, that is, made of the same material, and that the beam stresses are below the proportional limit stress of the material or in other words that Hook's Law holds.

From the geometry of similar triangles,

\[
\frac{e}{y} = \frac{\varepsilon_0}{c} \text{ or } e = \frac{y}{c} \varepsilon_0
\]

(A)

We have from Young's Modulus E, that

\[
\varepsilon = \frac{\text{unit stress}}{\text{unit strain}} = -\frac{\sigma_y}{E/L}
\]

where \(\sigma_y\) is the bending stress under a deformation \(e\) and since it is compression it will be given a minus sign.

Solving for \(\sigma_y\),

\[
\sigma_y = -E \frac{\varepsilon_0}{c} \frac{y}{L}
\]

The most remote fiber is at a distance \(y = c\).

Hence

\[
\sigma_y = -E \frac{\varepsilon_0}{L}
\]

whence

\[
\sigma_y = -\frac{\varepsilon_0}{c} y
\]

(B)

For equilibrium of the bending stress perpendicular to the beam cross-section or in the X direction, we can write \(\Sigma F_X = 0\), or

\[
\Sigma F_X = -\frac{\varepsilon_0}{c} \int yda = 0.
\]

However in this expression, the term \(\frac{\varepsilon_0}{c}\) is not zero, hence the term \(\int yda\) must equal zero and this can only be true if the neutral axis coincides with the centroidal axis of the beam cross-section.

A13.1
A13.2 Equations for Bending Stress, Homogeneous Beams, Stresses Below Proportional Limit Stress.

In the following derivations, it will be assumed that the plane of the external loads contain the flexural axis of the beam and hence, the beam is not subjected to torsional forces which, if present, would produce bending stresses if free warping of the beam sections was restrained, as occurs at points of support. The questions of flexural axes and torsional effects are taken up in later chapters.

![Fig. A13.1](image_url)

Fig. A13.1 represents a cross-section of a straight cantilever beam with a constant cross-section, subjected to external loads which lie in a plane making an angle \( \Theta \) with axis Y-Y through the centroid C. To simplify the figure, the flexural axis has been assumed to coincide with the centroidal axis, which in general is not true.

Let NN represent the neutral axis under the given loading, and let \( \Theta \) be the angle between the neutral axis and the axis X-X. The problem is to find the direction of the neutral axis and the bending stress \( \sigma \) at any point on the section. In the fundamental beam theory, it is assumed that the unit stress varies directly as the distance from the neutral axis, within the proportional limit of the material. Thus, Fig. A13.2 illustrates how the stress varies along a line such as \( mm \) perpendicular to the neutral axis N-N.

Let \( \sigma \) represent unit bending stress at any point a distance \( y_n \) from the neutral axis. Then the stress \( \sigma \) on \( da \) is

\[
\sigma = ky_n
\]

where \( k \) is a constant. Since the position of the neutral axis is unknown, \( y_n \) will be expressed for convenience in terms of rectangular coordinates with respect to the axes X-X and Y-Y.

Thus, \( y_n = (y - x \tan \Theta) \cos \Theta - \sin \Theta \)

\[
y \cos \Theta - x \sin \Theta
\]

\[
(2)
\]

Then Eq. (1) becomes

\[
\sigma = k(y \cos \Theta - x \sin \Theta)
\]

\[
(3)
\]

This equation contains three unknowns, \( \sigma \), \( k \), and \( \Theta \). For solution, two additional equations are furnished by conditions of equilibrium namely, that the sum of the moments of the external forces that lie on one side of the section ABCD about each of the rectangular axes X-X and Y-Y must be equal and opposite, respectively, to the sum of the moments of the internal stresses on the section about the same axes.

Let \( M \) represent the bending moment in the plane of the loads; then the moment about axis X-X and Y-Y is \( M_x = M \cos \Theta \) and \( M_y = M \sin \Theta \). The moment of the stresses on the beam section about axis X-X is \( \int \sigma \, da \). Hence, taking moments about axis X-X, we obtain for equilibrium,

\[
M \cos \Theta = \int \sigma \, da \, y
\]

\[
= k \cos \Theta \int (y^2 - x^2 \sin \Theta \cos \Theta) \, dy \, dx
\]

\[
(4)
\]

In similar manner, taking moments about the Y-Y axis

\[
M \sin \Theta = \int \sigma \, da \, x
\]

whence

\[
M \sin \Theta = -k \sin \Theta \int x^2 \, dx + k \cos \Theta \int xy \, dx
\]

A13.3 Method I. Stresses for Moments About the Principal Axes.

In equation (4), the term \( \int y^2 \, dx \) is the moment of inertia of the cross-sectional area about axis X-X, which we will denote by \( I_x \), and the term \( \int xy \, dx \) represents the product of inertia about axes X-X and Y-Y. We know, however, that the product of inertia with respect to the principal axes is zero. Therefore, if we select XX and YY in such a way as to make them coincide with the principal axes, we can write equation (4):

\[
M \cos \Theta = k \cos \Theta \int_{yp} x^2 \, dx
\]

\[
(5)
\]

In like manner, from equation (4a)

\[
M \sin \Theta = -k \sin \Theta \int_{yp} x^2 \, dx
\]

\[
(6)
\]

To find the unit stress \( \sigma \) at any point on the cross-section, we solve equation (5) for \( \cos \Theta \) and equation (6) for \( \sin \Theta \), and then substituting these values in (3), we obtain the following expressing giving \( \sigma \) the subscript \( b \) to represent bending stress:
Let the resolved bending moment \( M \cos \theta \) and \( M \sin \theta \) about the principal axes be given by the symbols \( M_{xp} \) and \( M_{yp} \). Then we can write

\[
\sigma_b = \frac{M \cos \theta \cdot y_p}{I_{xp}} + \frac{M \sin \theta \cdot x_p}{I_{yp}}
\]

The minus signs have been placed before each term in order to give a negative value for \( \sigma_b \) when we have a positive bending moment, or if \( M_{xp} \) is the moment of a couple acting about \( x_p-y_p \), positive when it produces compression in the upper right hand quadrant. \( M_{yp} \) is the moment about the \( y_p-y_p \) axis, and is also positive when it produces compression in the upper right hand quadrant.

**BENDING STRESS EQUATION FOR SYMMETRICAL BEAM SECTIONS**

Since symmetrical axes are principal axes (\( tan / xyd = 0 \)), the bending stress equation for bending about the symmetrical \( XX \) and \( YY \) axes is obviously,

\[
\sigma_b = -\frac{M_{xy}}{I_{x}} - \frac{M_{yx}}{I_{y}}
\]

### A13.4 Method 2. Stresses by use of Neutral Axis for Given Plane of Loading.

The direction of the neutral axis NN, measured from the \( XX_p \) principal axis is given by dividing equation (6) by (5).

\[
\tan \phi = -\frac{I_{xp} \tan \theta}{I_{yp}}
\]

### A13.5 Method 3. Stresses from Moments, Section Properties and Distances Referred to any Pair of Rectangular Axes through the Centroid of the Section.

The fiber stresses can be found without resort to principal axes or to the neutral axis.

Equation (4) can be written:

\[
M_x = k \cos \phi I_x - k \sin \phi I_{xy} = - - - - - - - (11)
\]

where \( I_x = J \cdot y d a \) and \( I_{xy} = J \cdot x y d a \), and \( M_x = M \cos \phi \).

In like manner,

\[
M_y = -k \sin \phi I_y + k \cos \phi I_{xy} = - - - - - - - (12)
\]

Solving equations (11) and (12) for \( \sin \phi \) and \( \cos \phi \) and substituting their values in equation (3), we obtain the following expression for the fiber stress \( \sigma_b \):

\[
\sigma_b = \left( \frac{M_{Ix} - M_{I_{xy}}}{I_{xy} - I_{I_{xy}}} \right) x - \left( \frac{M_{Iy} - M_{I_{xy}}}{I_{xy} - I_{I_{xy}}} \right) y
\]

For simplification, let

\[
K_1 = I_{xy}/(I_{Ixy} - I_{I_{xy}}) \\
K_2 = I_{y}/(I_{Ixy} - I_{I_{xy}}) \\
K_3 = I_{x}/(I_{Ixy} - I_{I_{xy}})
\]

Substituting these values in Equation (13):

\[
\sigma_b = -(K_1 M_y - K_1 M_x) x - (K_2 M_y - K_2 M_x) y
\]

In Method 2, equation (8) was used to find the position of the neutral axis for a given plane of loading. The location of the neutral axis can also be found relative to any pair of rectangular centroidal axes \( X \) and \( Y \) as follows: - Since the stress at any point on the neutral axis must be zero, we can write from equation (14) that:

\[
(K_1 M_y - K_1 M_x) x = -(K_2 M_y - K_2 M_x) y
\]

for all points located on the neutral axis. From Fig. A13.1 \( \tan \phi = \frac{y}{x} \),

\[
\tan \phi = \frac{K_2 M_y - K_2 M_x}{K_1 M_y - K_1 M_x} = - - - - - - - - (15)
\]
A13.4 BEAM BENDING STRESSES

It frequently happens that the plane of the bending moment coincides with either the X-X or the Y-Y axis, thus making either $M_x$ or $M_y$ equal to zero. In this case, equation (15) can be simplified. For example, if $M_y = 0$

$$\tan \theta = \frac{I_{xy}}{I_y}$$  \hspace{1cm} (16)

and if $M_x = 0$

$$\tan \theta = \frac{I_{xy}}{I_x}$$  \hspace{1cm} (17)

A13.6 Advantages and Disadvantages of the Three Methods.

Method 2 (bending about the neutral axis for a given plane of loading) no doubt gives a better picture of the true action of the beam relative to its bending as a whole. The point of maximum fiber stress is easily determined by placing a scale perpendicular to the neutral axis and moving it along the neutral axis to find the point on the beam section furthest away from the neutral axis. In airplane design, there are many design conditions, which change the direction of the plane of loading, thus, several neutral axes must be computed for each beam section, which is a disadvantage as compared to the other two methods.

In determining the shears and moments on airplane structures, it is common practice to resolve air and landing forces parallel to the airplane XYZ axes and these results can be used directly in method 3, whereas method 1 requires a further resolution with respect to the principal axes. Methods 1 and 3 are more widely used than method 2.

Since bending moments about one principal axis produces no bending about the other principal axis, the principal axes are convenient axes to use when calculating internal shear flow distribution.

A13.7 Deflections.

The deflection can be found by using the beam section properties about the neutral axis for the given plane of loading and the bending moment resolved in a plane normal to the neutral axis. The deflection can also be found by resolving the bending moment into the two principal planes and then using the properties about the principal axes. The resultant deflection is the vector sum of the deflections in the direction of the principal planes.

A13.8 Illustrative Problems. Example Problem 1.

Fig. A13.3 shows a unsymmetrical one cell box beam with four corner flange members a, b, c and d. Let it be required to determine the bending axial stress in the four corner members due to the loads $P_x$ and $P_y$ acting $60^\circ$ from the section abcd.

In this example the solution the stresses connecting the corner members will be considered ineffective in bending. The stresses will be determined by each of the three methods as presented in this chapter.

**SOLUTION**

The first step common to all three methods is the calculation of the moments of inertia about the centroidal X and Y axes. Table A13.1 gives the detailed calculations. The properties are first calculated about the reference axes $x'x'$ and $y'y'$ and then transferred to the parallel centroidal axes.

<table>
<thead>
<tr>
<th>Mem.</th>
<th>Area &quot;A&quot;</th>
<th>$y'$</th>
<th>$x'$</th>
<th>$Ay'$</th>
<th>$Ax'$</th>
<th>$Ay'^2$</th>
<th>$Ax'^2$</th>
<th>$Ax'y'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.0</td>
<td>12</td>
<td>-16</td>
<td>12</td>
<td>-16</td>
<td>144.0</td>
<td>256.0</td>
<td>-192</td>
</tr>
<tr>
<td>b</td>
<td>0.5</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>32.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0.8</td>
<td>0</td>
<td>-16</td>
<td>0</td>
<td>-12.8</td>
<td>0</td>
<td>204.8</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>2.7</td>
<td>18</td>
<td>28.8</td>
<td>176.0</td>
<td>460.8</td>
<td>-192</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Calculation of Centroid of Section:

$$x = \frac{\sum Ax'}{2A} = \frac{-28.8}{2.7} = -10.667''$$

$$y = \frac{\sum Ay'}{2A} = \frac{16}{2.7} = 5.926''$$

$$I_x = 176 - 2.7 \times 5.926^2 = 81.18$$

$$I_y = 460.8 - 2.7 \times 10.667^2 = 153.58$$

$$I_{xy} = -192 - (2.7 \times -10.667 \times 5.926) = -21.33$$
Solution by Method 1 (bending about principal axes)

The angle $\phi$ between the $x$ axis and the principal axes is given by the equation,

$$\tan 2\phi = \frac{2 I_{xy}}{I_y - I_x} = \frac{2 (-21.33)}{153.55 - 81.18} = \frac{-42.66}{72.40} = -0.589$$

$2\phi = 30^\circ - 30' = 30^\circ - 15^\circ = 15'$

$I_{x_p} = I_x \cos^2 \phi + I_y \sin^2 \phi - 2 I_{xy} \sin \phi \cos \phi$

$$= 81.18 \times 0.866^2 + 153.58 \times 0.2636^2 - (-21.33 \times 0.2636 \times 0.866)$$

$$= 75.56 + 10.67 - 10.85 = 75.36 \text{ in.}$$

$I_{y_p} = I_x \sin^2 \phi + I_y \cos^2 \phi - 2 I_{xy} \sin \phi \cos \phi$

$$= 81.18 \times 0.2636^2 + 153.58 \times 0.866^2 + 10.85 = 159.34 \text{ in.}$$

(Stringer a)

$$x_p = -6.74, \quad y_p = 4.45$$

hence

$$\sigma_b = -\frac{268700 \times 4.45}{75.38} - \frac{156200 (-6.74)}{159.34} = -15900 - 6600 = -22500 \text{#/in.}$$

(Stringer b)

$$x_p = 9.75', \quad y_p = 4.32$$

$$\sigma_b = -\frac{268700 \times 4.32}{75.38} - \frac{156200 \times 9.75}{159.34} = -17180 - 3570 = -7610 \text{#/in.}$$

(Stringer c)

$$x_p = -3.68, \quad y_p = -7.12$$

$$\sigma_b = -\frac{268700 \times -7.12}{75.38} - \frac{156200 \times (-3.68)}{159.34} = 25400 - 3520 = 21880$$

and similarly for stringer d, $\sigma_b = 21900$.

Solution by Method 2 (Neutral Axis Method)

Let $\phi'$ = angle between $yy$ axis and plane of loading.

$$\tan \phi' = \frac{50000}{300000} = -0.2667$$

hence

$\phi' = 14^\circ - 56'$ and $\phi = 14^\circ - 56' + 15^\circ - 15^\circ = 30^\circ - 11$

Let $a = \angle$ between neutral axis NN and the $x_p$ axis,

$$\tan a = \frac{I_{x_p} \tan \phi'}{I_{y_p}} = \frac{75.38 \times (-0.5816)}{159.34} = -0.275$$

hence $a = 15^\circ - 24'$ (see Fig. A13.3e).

Since the angle between the $x$ axis and the neutral axis is only $\phi'$, we can say

$I_N = I_x = 91.18$.

Resolving the external bending moments normal to neutral axis, we obtain
The bending stress at any point is given by:

\[ \sigma_b = \frac{M_n y_n}{I_n} \]

### Stringer a

\[ y_n = 6.074 + 5.33 \times 0.0026 = 6.087'' \]

\[ \sigma_b = \frac{-300 \times 200 \times 6.087}{81.13} = -22500/\text{in.}^2 \]

### Stringer b

\[ y_n = 2.065'' \]

\[ \sigma_b = \frac{-300 \times 200 \times 2.05}{81.13} = -7570 \]

### Stringer c

\[ y_n = -5.33 \]

\[ \sigma_b = \frac{-300 \times 200 \times (-5.92)}{81.13} = 21650 \]

### Stringer d

\[ y = -5.95 \]

\[ \sigma_b = \frac{-300 \times 200 \times (-5.95)}{81.13} = 22000 \]

---

**Solution by Method 3 (Method Using Properties About X and Y Axes)**

\[ M_x = 300000 \times 0.9999 + 80000 \times \sin 0^\circ - 9' = 300200'' \]

\[ I_x = 81.13, \ I_y = 153.58, \ I_{xy} = -21.33 \]

\[ K_x = I_{xy}/(I_x I_y - I_{xy}^2) = \frac{-21.33}{81.18 \times 153.58 - 21.33^2} = -0.00177 \]

\[ \sigma_b = I_y/(I_x I_y - I_{xy}^2) = 153.58/12016 = 0.012794 \]

\[ K_y = I_{xy}/(I_x I_y - I_{xy}^2) = 81.18/12016 = 0.00674 \]

\[ \sigma_b = -[K_x Y - K_x M_x] x - [K_x M_y - K_y M_x] y \]

### Stringer a

\[ x = -5.333'', \ y = 6.074'' \]

\[ \sigma_b = \left[ -0.00177 \times 80000 + 0.00177 \times 300000 \right] x - \left[ 0.012794 \times 300000 - 0.00177 \times (-80000) \right] y \]

hence \( \sigma_b = (8) x - (3697)y; \)

hence \( \sigma_b = 8 x - 5.333 \times 3697 \times 6.074 = -22450 \)

### Stringer b

\[ x = 10.667, \ y = 2.074 \]

\[ \sigma_b = 8 \times 10.667 - (3697)2.074 = 85 - 7680 = -7575 \]

### Stringer c

\[ x = -5.333, \ y = -5.926 \]

\[ \sigma_b = 8 \times 5.333 - (3697) - 5.926 = -42 + 21900 = 21858 \]

### Stringer d

\[ x = 10.667, \ y = -5.926 \]

\[ \sigma_b = 8 \times 10.667 - (3697) - 5.926 = 85 + 21900 = 21985 \]

**Note:** The stresses \( \sigma_b \) by the three methods were calculated by 10th slide rule, hence the small discrepancy between the results for the three methods.

**Error in Stresses Due to Assumption that Section Bends About X and Y Centroidal Axes Due to \( M_x \) and \( M_y \).**

\[ \sigma_b = -\frac{M_x y}{I_x} - \frac{M_y x}{I_y} \quad \text{(Equation 7a)} \]

### Stringer a

\[ y = 6.074, \ x = -5.33 \]

\[ \sigma_b = -\frac{300000 \times 6.074 - 80000 \times (-5.33)}{81.18} = -25180 \]

### Stringer b

\[ y = 2.07, \ x = 10.667 \]
Stringer b - cont’d.

\[ \sigma_b = \frac{-300000 \times 2.07}{81.18} = -300000 x 10.667 = -2110 \]

In like manner for stringers c and d

\[ \sigma_{bc} = 19130 \quad \text{and} \quad \sigma_{bd} = 27440 \]

Comparing these results with the previous results it is noticed that considerable error exists. Under these erroneous stresses the internal resisting moment does not equal the external bending moment about the X and Y axis.

Example Problem 2.

Fig. A13.4 shows a portion of a cantilever 2-cell stressed skin wing box beam. In this example, the beam section is considered constant, and the section is identical to that used in Fig. A3.13 of Chapter A3 for which the section properties were computed and are as follows:

\[ I_x = 186.5 \text{ in.}^2, \quad I_y = 431.7 \text{ in.}^2, \quad I_{xy} = 36.41 \text{ in.}^2 \]

\[ \theta = 90^\circ - 16.25^\circ = 181.2 \quad I_{Xp} = 437 \]

The resultant air load on the wing outboard of section ABC is 14260# acting up in the Y direction and 760# acting forward in the X direction, and the location of these resultant loads is 50" from section ABC (Fig. A13.4).

The bending stress intensity at the centroids or stringers number (1), (9) and (12) will be calculated using all three methods.

Solution by Method 1 (Bending about Principal Axes)
The bending moment at section ABC about the X and Y axes is,

\[ M_x = 14260 \times 50 = 713,000" \]

\[ M_y = -760 \times 50 = -38000" \]

These moments are resolved into bending moments about the principal axes, as follows:

\[ M_{xP} = 713000 \times \cos \theta - 38000 \times \sin \theta = 700,000" \]

\[ M_{yP} = -713000 \times \sin \theta - 38000 \times \cos \theta = -140000" \]

From equation (7), the general formula for \( \sigma_b \) is:

\[ \sigma_b = \frac{M_{xP} y_p}{I_{xP}} - \frac{M_{yP} x_p}{I_{yP}} \]

Stress on Stringer 1:

\[ y_p = 1.56" \quad \text{and} \quad x_p = -17.86" \text{ (scaled from full size drawing)} \]

\[ \sigma_b = \frac{-(700000 \times 1.85)}{181.17} - \frac{-140000 \times (-17.85)}{437} \]

\[ = -7150 - 5700 = -12850#/\text{sq. in.} \]

Stress on Stringer 9:

\[ y_p = 9.04", \quad x_p = 14.24" \]

\[ \sigma_b = \frac{-(700000 \times 3.04)}{181.17} - \frac{-140000 \times 14.24}{437} \]

\[ = -34900 + 4560 = -30340#/\text{sq. in.} \]

Stress on Stringer 12:

\[ y_p = -6.80", \quad x_p = -3.22" \]

\[ \sigma_b = \frac{-700000 \times (-6.80)}{181.17} - \frac{-140000 \times (-8.22)}{437} \]

\[ = 26200 + 2630 = 23570#/\text{sq. in.} \]

Solution by Method 2 (Neutral Axis Method)
In Fig. A13.5 let \( \theta' \) be the angle between the Y-Y axis and the plane of the resultant bending moment. Resultant bending moment,

\[ M = \sqrt{713000^2 + 38000^2} = 714600" \text{ lb.} \]

\[ \tan \theta' = \frac{38000}{713000} = -0.0533, \quad \text{hence} \quad \theta' = -3^\circ - 3' \]

Let \( \theta \) equal the angle between the plane of the resultant moment and the \( Y_p \) axis.

Then \( \theta = \theta' + \beta = 3^\circ - 3' + 8^\circ - 15' = 11^\circ - 19' \)

From equation (8), the angle between the \( X_p \) axis and the neutral axis = \( \alpha \), and

\[ \tan \alpha \frac{I_{xP}}{I_{yP}} = \frac{181.17 \times (-0.200)}{437} \]
Fig. A13.5 shows the relative positions of the neutral axis, principal axes, and plane of loading.

The component of the external resultant moment about the neutral axis \( N \) equals:

\[
M_N = 714080 \times \sin 63^\circ\ = 26^\prime = 709350 \text{ in. lb.}
\]

\[
I_N = I_{x_p} \cos^2 a + I_{y_p} \sin^2 a = 151.17 \times \frac{.9966^2}{2} + 437 \times .0873^2 = 183.37 \text{ in.}^4
\]

**Stress on Stringer 1:***

\[
\sigma_{b_1} = \frac{M_N \times x_p}{I_N} = \frac{-709350 \times 3.32}{183.37} = -12850\text{#/sq. in.}
\]

**Stress on Stringer 9:**

\[
\sigma_{b_9} = \frac{M_N \times x_p}{I_N} = \frac{-709350 \times 7.34}{183.37} = -30300\text{#/sq. in.}
\]

**Stress on Stringer 12:**

\[
\sigma_{b_{12}} = \frac{M_N \times x_p}{I_N} = \frac{-709350 \times -610}{183.37} = 23620\text{#/sq. in.}
\]

**Solution by Method 3**

\[
M_X = 713000\text{#}, \quad M_Y = -38000\text{#}
\]

\[
I_X = 186.46 \quad I_Y = 431.7 \quad I_{XY} = -36.41
\]

The constants \( K_1, K_2, \) and \( K_3 \) are first determined:

\[
K_1 = \frac{I_{XY}}{I_X I_Y - I_{XY}^2} = \frac{-36.41}{186.46 \times 431.7 - 36.41^2}
\]

\[
K_2 = \frac{I_Y}{I_X I_Y - I_{XY}^2} = \frac{431.7}{186.46 \times 431.7 - 36.41^2}
\]

\[
K_3 = \frac{I_X}{I_X I_Y - I_{XY}^2} = \frac{-36.41}{186.46 \times 431.7 - 36.41^2}
\]

**Stress on Stringer 1:**

\[
y_1 = 4.39\text{", } x_1 = -17.41\text{"}
\]

\[
\sigma_{b_1} = -\left( K_3 M_Y - K_1 M_X \right) x - \left( K_2 M_X - K_1 M_Y \right) y
\]

\[
\sigma_{b_{12}} = \left[ \frac{0.002355 \times (-38000)}{23620} - \left( \frac{-0.00046 \times 713000}{23620} \right) \right] (-17.41)
\]

\[
-\left[ \frac{0.00545 \times 713000}{23620} - \left( \frac{-0.00046 \times 38000}{23620} \right) \right] 4.39
\]

\[
= - (236.5)(-17.41) - (3868) 4.39
\]

\[
= 1450 - 17000 = -12850\text{#/in.}^2
\]

**Stress on Stringer 9:**

\[
y_9 = 6.89\text{", } x_9 = 15.39\text{"}
\]

\[
\sigma_{b_9} = \left[ \frac{[236.5] 15.39}{3868} \right] 6.89 = -30320\text{#/in.}^2
\]

**Stress on Stringer 12:**

\[
y_{12} = -5.55\text{", } x_{12} = -9.11\text{"}
\]

\[
\sigma_{b_{12}} = -\left[ \frac{[236.5] (-9.11)}{3868} \right] (-5.55) = 23620\text{#/in.}^2
\]

**NOTE:** In the three solutions, the distance from the axis in question to the stringers 1, 9, and 12 have been taken to the centroid of each stringer unit. Thus, the stresses obtained are average axial stresses on the stringers. If the maximum stress is desired, the arm should refer to the most remote part of the stringer or the skin surface.

**Approximate Method.**

It is sometimes erroneously assumed that the external bending moments \( M_X \) and \( M_Y \) produce bending about the X and Y axes as though they were neutral axes. To show the error of this assumption, the stresses will be computed for stringers 1 and 9.
Stringer 1:

\[ y_1 = 4 + .39 = 4.39 \]
\[ x_1 = -33.15 + 15.74 = -17.41 \]

\[ \sigma_b = \frac{-M_y x_1}{I_y} = \frac{-38000 \times -17.41}{431.7} = 18330 \text{#/in}^2. \]

Stringer 9:

\[ y_s = 6.89, \; x_s = 15.39 \]
\[ \sigma_b = \frac{-713000 \times 6.89}{185.46} = \frac{-38000 \times 15.39}{431.7} = \frac{-250000}{185.46} \text{#/in}^2. \]

Example Problem 3.

The previous example problems were solved by substituting in the bending stress equations. The student should solve bending stress problems by equating the internal resisting moment at a beam section to the external bending moment at the same section. To illustrate Fig. A13.6 shows a simply supported loaded beam. The shear and bending moment diagram for the given beam loading is also shown. Fig. A13.7 shows the beam section which is constant along the span.

The maximum bending moment occurs over the left support and equals 24000 in. lbs. Due to symmetry of the beam cross-section the centroidal horizontal axis is the center line of the beam and thus the neutral axis is at the midpoint of the beam.

Fig. A13.6 shows a free body of that portion of the beam from the left end to a section over the left support where the bending moment is maximum. The triangular bending stress intensity diagram is shown acting on the cut section, with a value of \( \sigma_b \) at the most remo e fiber. The forces \( T_A, T_B \) etc. represent the total load on the beam cross-sectional areas labeled A and B respectively in Fig. A13.8. The

\[ \sigma_b = \frac{M \times c}{I} \]

where \( M = -24000 \)

\[ \sigma_b = \frac{24000}{3} = 8000 \text{ psi}. \]

Solution using Bending Stress Formula.

\[ \sigma_b = \frac{M \times c}{I} \]

where \( M = -24000 \)

\[ \sigma_b = \frac{24000}{3} = 8000 \text{ psi}. \]

Calculation of I.

For Portion A and A'

\[ I = \frac{1}{12} \times 1 \times 75 \left( 6^2 - 4.75^2 \right) = 18.26 \text{ in}^4. \]

For Web Portions B, B',

\[ I = \frac{1}{12} \times 0.25 \times 6^3 = 4.50 \text{ in}^4. \]

\[ I_{\text{total}} = 22.76 \text{ in}^4. \]
hence

\[ \sigma_b = \frac{-24000 \times 3}{22.76} = -3170 \text{ psi,} \]

which checks first solution.

Example Problem 4.

Fig. A13.10 shows a loaded beam and Fig. A13.11 shows the cross-section of the beam at section a-a’. Determine the magnitude of the maximum bending stress at section a-a’ under the given beam loading. The beam section is symmetrical about the horizontal centroidal axis. Simple calculations locate the neutral axis as shown in Fig. A13.11.

This method of solution involves more calculations than that required in substituting in the bending stress formula, however, the student should obtain a better understanding of the internal force action from this method of solution. In solving non-uniform beams and beams stressed above the elastic limit stress, this method of solution often proves necessary or advantageous because no simple beam bending stress formula can be derived.

In Fig. A13.12, let \( \sigma_b \) be the intensity on the most remote fiber, or 3.09" above the neutral axis. Table A shows the calculations of the total stress on each of the portions of the cross-section and their moment about the neutral axis, all in terms of the unknown stress, \( \sigma_b \).

### TABLE A

<table>
<thead>
<tr>
<th>Portion</th>
<th>Area A</th>
<th>Average Stress in terms of ( \sigma_b )</th>
<th>Total Load ( x \times (2) )</th>
<th>( y ) = arm to N.A.</th>
<th>Resisting moment ( (4) \times (5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.045</td>
<td>-3.37( \sigma_b )</td>
<td>-353( \sigma_b )</td>
<td>1.39</td>
<td>-490( \sigma_b )</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>-757</td>
<td>-757</td>
<td>2.35</td>
<td>-1.780</td>
</tr>
<tr>
<td>3 – 3’</td>
<td>1.00</td>
<td>-837</td>
<td>-837</td>
<td>2.62</td>
<td>-2.190</td>
</tr>
<tr>
<td>4</td>
<td>1.08</td>
<td>.350</td>
<td>.387</td>
<td>-1.44</td>
<td>-0.557</td>
</tr>
<tr>
<td>5 – 5’</td>
<td>0.56</td>
<td>.578</td>
<td>.323</td>
<td>-1.93</td>
<td>-0.622</td>
</tr>
<tr>
<td>6</td>
<td>1.50</td>
<td>.821( \sigma_b )</td>
<td>1.237( \sigma_b )</td>
<td>-2.57</td>
<td>-3.180( \sigma_b )</td>
</tr>
</tbody>
</table>

Totals 6.19 \hspace{1cm} 0.000\( \sigma_b \) \hspace{1cm} -8.82\( \sigma_b \)

Explanation of Table A.

Column 3 gives the average stress on each of the 6 portions. For example, the stress on portion (1) varies from zero at the neutral axis to 2.09 \( \sigma_b \) at the upper edge of 2.09. Thus, the average stress \( -0.674 \sigma_b + \sigma_b = \frac{2}{2} \). On portions (2), (3), (5), and (6) the bending stress distribution is triangular, and the arm is to the centroid of this trapezoid.

The total internal resisting moment of \(-8.82 \sigma_b\) from Table A equals the external bending moment of 36670\( \sigma_b \).

Thus,

\[ \sigma_b = -\frac{36670}{8.82} = -4160\text{#}./\text{in.}^2. \]

For equilibrium, the total compressive stresses on the cross-section of the beam must equal to the total tensile stresses, or \( ZH \) must equal zero. Column 4 of Table A gives the
total load on each portion of the cross-section and the total of this column is zero.

The bending stress on the lower fiber of the cross-section is directly proportional to the distance from the neutral axis or, \( \sigma_{\text{lower}} = \frac{2.91 \times 4150}{3.09} = 3630 \text{ psi} \).

**Solution Using Bending Stress Formula.**

\[ \sigma_{b} = \frac{M \cdot Y}{I_{N.A.}} \]

\[ I_{N.A.} = 27.2 \text{ in}^{4} \]

\[ \sigma_{b} (\text{upper fiber}) = \frac{-36670 \times 3.09}{27.2} = -4150 \text{ psi} \]

which checks the above solution.

A13.9 Bending Stresses in Beams with Non-Homogeneous Sections, Stresses within the Elastic Ranges.

In general, beams are usually made of one material, but special cases often arise where two different materials are used to form a beam section. For example, a commercial airplane in its lifetime often undergoes a number of changes or modifications such as the addition of additional installations, fixed equipment, larger engines, etc. This increase in airplane weight increases the structural stresses and it often becomes necessary to strengthen various structural members at the critical stress points. Since space limitations are usually critical, it often necessitates that the reinforcing material be stiffer than the original material. Years ago, when spruce wood was a common material for wing beams, it was normal practice to use reinforcements at critical stress points of stiffer wood material such as maple in order to cut down the size of the reinforcements. In aluminum alloy beams, the use of heat-treated alloy steel reinforcements are often used because steel is 25000/105000,000 or 2.75 times stiffer than aluminum alloy.

**Solution by Means of Transformed Beam Section.**

To illustrate how the stresses in a composite beam can be determined, two example problems will be presented.

**Example Problem 5.**

Fig. A13.14 shows the original beam section which is entirely spruce wood. Fig. A13.15 shows the reinforced or modified beam section. Two ears of maple wood have been added to the upper part as shown and a steel strap has been added to the bottom face of the beam. The problem is to find the stress at the top and bottom points on the beam section when the beam section is subjected to an external bending moment of 60000 in. lbs. about a horizontal axis which causes compression in the upper beam portion. The modulus of elasticity (E) for the 3 different materials is:

- \( E_{\text{spruce}} = 1,300,000 \text{ psi} \)
- \( E_{\text{maple}} = 1,600,000 \text{ psi} \)
- \( E_{\text{steel}} = 29,000,000 \text{ psi} \)

**Solution:**

The first step in the solution is to transform the reinforced beam section of Fig. A13.15 into an equivalent beam section composed of the same material throughout. This is possible, because the modulus of elasticity of each material gives us the measure of stiffness for that material. In this solution the reinforced beam will be transformed into a spruce beam section as illustrated in Fig. A13.16.

\[ \frac{E_{\text{maple}}}{E_{\text{spruce}}} = \frac{1,600,000}{1,300,000} = 1.23 \]

\[ \frac{E_{\text{steel}}}{E_{\text{spruce}}} = \frac{29,000,000}{1,300,000} = 22.3 \]

Thus to transform the maple reinforcing strips in Fig. A13.15 into spruce we increase the width of each strip to 1.23 x 0.5 = 0.615 inches as shown in Fig. A13.16. Likewise, to transform the steel reinforcing strip into spruce, we make the width equal to 22.3 x 1.5 = 33.4 inches as shown in Fig. A13.16.

This transformed equivalent section is now handled like any homogeneous beam section which is stressed within the elastic limit of the materials. The usual calculation would locate the neutral axis as shown and the moment of inertia of the transformed section about the neutral axis would give a value of 51.5 in. lbs. Bending stress at upper edge of beam section:

\[ \sigma_{bc} = \frac{-M_{x} \cdot Y}{I_{x}} = \frac{-60000 \times 3.33}{51.5} = -3900 \text{ psi} \]

Referring to Fig. A13.15, this stress would be
the stress in the spruce section. Since the reinforcing strips are maple, the stress at the top edge of these maple strips would be 1.23 times (- 3900) = - 4680 psi.

The bending stress at the lower edge of the transformed beam section of Fig. A13.15 would be:

$$\sigma_{bt} = \frac{60000 \times (-2.73)}{31.30} = 3200 \text{ psi.}$$

The stress in the steel reinforcing strap thus equals 22.3 times 3200 = 71500 psi.

Since all these stresses are below the elastic limit stress for the 3 materials the beam bending stress formula as used is applicable.

**Example Problem 6.**

Fig. A13.17 shows an unsymmetrical beam section composed of four stringers, a, b, c and d of equal area each and connected by a thin web. The web will be neglected in this example problem. Each of the stringers is made from different material as indicated on Fig. A13.17. The beam section is subjected to the bending moment $M_X$ and $M_Y$ as indicated. Let it be required to determine the stress and total load on each stringer in resisting these applied external bending moments.

![Figure A13.17](image)

**SOLUTION:**

Since the 4 stringers are made of different materials we will transform all the materials into an equivalent beam section with all 4 stringers being magnesium alloy.

$$E_{\text{mag.}} = 6,500,000$$

$$E_{\text{steel}} = 29,000,000$$

$$E_{\text{stain. steel}} = 28,000,000$$

$$E_{\text{alum. alloy}} = 10,500,000$$

Using the ratio of stiffness values as indicated above gives the transformed beam section of Fig. A13.18 where all material is now magnesium alloy. The original area of 0.1 sq. in. each have been multiplied by these stiffness ratio values.

![Figure A13.18](image)

The solution for the beam section of Fig. A13.18 is the same as for any other unsymmetrical homogeneous beam section.

The first step is to locate the centroid of this section and determine the moments of inertia of this section about centroidal X and Y axes.

$$y = \frac{ \sum A_y}{\sum A} = \frac{0.446 \times 10 + 0.1615 \times 10}{1.1365} = 5.35$$

$$x = \frac{ \sum A_x}{\sum A} = \frac{0.1615 \times 6 + 0.431 \times 4}{1.1365} = 2.365$$

$$I_X = 0.6075 \times 4.55 + 0.531 \times 5.35 = 28.27 \text{ in.}^4$$

$$I_Y = 0.546 \times 2.355 + 0.165 \times 3.838 + 0.431 \times 1.635 = 6.34 \text{ in.}^4$$

$$I_{xy} = 0.446 (- 2.365)(4.55) = - 4.90$$

$$0.1615 \times 3.65 \times 4.65 = 2.73$$

$$0.10 (- 2.365)(- 5.35) = 1.27$$

$$0.431 \times 1.635 (- 5.35) = - 3.77$$

**TOTAL** = - 4.67 in.\(^4\) = I_{xy}

The bending stresses will be calculated by using method 3 of Art. A13.5.

From Equation (14) Art. A13.5

$$\sigma = \left( K_{M_X} - K_{M_Y} \right) \times \left( K_{W_X} - K_{W_Y} \right)$$

$$M_X = - 10,000 \text{ in. lb.}$$

$$M_Y = 5000 \text{ in. lb.}$$
Consider Stringer (c):
\[ x = -2.365, \ y = -5.35 \]

Substituting in \( \sigma_b \) equation
\[
\sigma_b = -\left[0.1797 \times 5000 - (-0.0296)(-1000)\right] x
- \left[(0.0403)(-1000,00) - (-0.0296 \times 5000)\right] y
= (888 - 296) x - \left[(403) + 148\right] y
\]
\[ \sigma_c = -602 \ x + 255 \ y \]  
For stringer (c) \( x = -2.365, \ y = -5.35 \)
\[
\sigma_c = -602 \ (-2.365) + 255 \ (-5.35)
= 1423 - 1360 = 63 \text{ psi}
\]

The total load \( P_c \) in stringer (c) thus equals
\[ \sigma_c \text{ (Area)} = 63 \times 0.1 = 6 \text{ lbs. tension.} \]

Stringer (a):
\[ x = -2.365, \ y = 4.65 \]
from equation (18)
\[ \sigma_a = -602 \ (-2.365) + 255 \times 4.65 = 2613 \text{ psi.} \]
\[ P_a = 2613 \times 0.446 = 1167 \text{ lb. tension.} \]

Since the true area of this stringer is 0.1 square the stress in this steel stringer equals 1167 /0.1 = 11670 psi. tension.

Stringer (b):
\[ x = 3.635, \ y = 4.65 \]
\[ \sigma_b = -602 \times 3.635 + 255 \times 4.65 = -1005 \text{ psi.} \]
\[ P_b = -1005 \times 0.1515 = -162 \text{ lbs.} \]

True stress in stainless steel stringer = \( -162 / 0.1 = -1622 \text{ psi.} \)

Stringer (d):
\[ x = 1.635, \ y = -5.35 \]
\[ \sigma_d = -602 \times 1.635 + 255 \ (-5.35) = -2345 \text{ psi.} \]
\[ P_d = -2345 \times 0.431 = -1010 \text{ lbs.} \]

True stress = \( \frac{1010}{0.1} = -10100 \text{ psi.} \)

To check the results, check total stringer loads to see if they equal zero.
\[ \Sigma P = 6 + 1167 - 162 - 1010 = 1 \text{ check.} \]

Furthermore the moments of the stringer loads about the X and Y axes must equal the applied external bending moments.
Take moments about a Y axis through stringers (a) and (c).
\[ \Sigma M_y = -6 \times 162 - 4 \times 1010 + 5000 = -12 \text{ in. lb.} \]

Take moments about X axis through stringers (c) and (d),
\[ \Sigma M_x = 1162 \times 10 - 162 \times 10 - 10000 = 0 \text{ in. lb.} \]

(The calculations in this example being done on a slide rule can not provide exact checks).

A13.10 Bending Stresses of Homogeneous Beams Stressed above the Elastic Limit Stress Range.
In structural airplane design, the applied loads on the airplane must be taken by the structure without suffering permanent strain which means the stresses should fall within the elastic range. The airplane structural design loads which in general equal the applied loads times a factor of safety of 1.5 must be taken by the structure without collapse or rupture with no restriction on permanent strain. Many airplane structural beams will not fail until the stresses are considerably above the elastic stress range for the beam material.

Since the stress-strain relationship in the inelastic range is not linear and also since the stress-strain curve for a material in the inelastic range is not the same under tensile and compressive stresses (See Fig. A13.10), the beam bending stress formulas as previously derived do not apply since they were based on a linear variation of stress to strain. Experimental tests however, have shown that even when stressed in the inelastic range, that plane sections before bending remain planar after after bending, thus strain deformation is still linear which fact simplifies the problem since the stress corresponding to a given strain can be found from a stress-strain curve for the beam material.

A general method of approach to solving beams that are stressed above the elastic range can best be explained by the solution of a problem.

Example Problem 7.
Portion (a) of Fig. A13.2C shows a solid round bar made from 24ST aluminum alloy material. Fig. A13.19 shows a stress-strain curve for this material. Let it be assumed
that the maximum failing compressive stress occurs at a strain of 0.01 in. per inch. The problem is to determine the ultimate resisting moment developed by this round bar and then compare the result with that obtained by using the beam bending stress formula based on linear variation of stress to strain.

The problem as stated assumed that a compressive unit strain of 0.01 caused failure. Fig. b thus shows the strain picture on the beam just before failure since plane sections remain plane after bending in the inelastic range. Table A13.2 gives the detailed calculations for determining the internal resisting moment developed under the given strain condition.

---

**TABLE A13.2**

<table>
<thead>
<tr>
<th>Strip No.</th>
<th>Strip Area &quot;A&quot;</th>
<th>ε</th>
<th>Unit Stress σ</th>
<th>F = σA</th>
<th>Res. Moment M = Fr</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>.035</td>
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</tr>
<tr>
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<td>.130</td>
<td>.00016</td>
<td>42500</td>
<td>2075</td>
</tr>
</tbody>
</table>

| Total     | 3.140          |                      | 740  | 58735 |

Col. 1 Rod divided into 20 strips .1" thick.
Col. 2 y = distance from centerline to strip c.g.
Col. 3 ε = strain at midpoint = (y - .0375)/103.75
Col. 4 σ = unit stress for 1 strain from Figs. A13.19
Col. 5 Total stress on strip.
Col. 6 Total strain on strip.
Col. 7 Moment about neutral axis. r = (y - .0375)

The summation of column (ε) should be zero. Since the discrepancy is 740 lbs., it means that the assumed position for the neutral axis is a little too high, however the discrepancy is negligible. The total internal resisting moment is 58735 in. lbs. (Col. 7).

If the maximum unit compressive strain of 0.01 is not exceeded, we find the corresponding stress from Fig. A13.19 to be 46500 psi. If this stress is used as the failing stress in the beam formula

\[
M = \frac{9F}{c}
\]

we obtain

\[
M = 46500 \times 0.785 = 36000 \text{ in. lbs.}
\]

\[
(0.785 = \frac{2}{3} \text{ of round bar } = \pi r^2/4)
\]
Thus using the same failure stress at the far extreme fiber the beam formula based on linear stress-strain relationship gives an ultimate bending strength of 38000 as compared to a true strength of 56735 or only 67 percent as much.

Fig. c of Fig. A13.10 shows the true stress distribution on the cross-section, which explains why the resisting moment is higher than when a triangular distribution is used.

The problem of the ultimate bending strength of structural shapes is discussed in detail in Volume II.

A13.11 Curved Beams. Stresses Within the Elastic Range.

The equations derived in the previous articles of this chapter were for beams that were straight. Thus in Fig. A13.21, the element of length (L) used in the derivation was constant over the depth of the beam. The strain (dL/L) was therefore directly proportional to dL which had a linear variation.

In a curved beam, the assumption that plane sections remain plane after bending still applies, however the beam segment of a curved beam cannot have equal width over the depth of the beam because of the curvature as illustrated in Fig. A13.22, or in other words the length of the segment is greater on the outside edge (L₁) than on the inside edge (L₂). Thus in calculating the strain distribution over the beam depth the change in length dL at a point must be divided by the segment width at that point. Thus even though plane sections remain plane the strain distribution over the section will not be linear. The width of the segment at any point is directly proportional to the radius of curvature of the segment and thus the strain at a point on the segment is inversely to the radius of curvature. This gives a hyperbolic type of strain distribution as illustrated in Fig. A13.22, and if the strains are within the elastic limit the stress distribution will be similar. The development of a beam formula based on a hyperbolic stress distribution is given in most books on advanced engineering mechanics and will not be repeated here.

It is convenient however to express the influence of the beam curvature in the form of a correction factor K by which the stresses obtained by the beam formula for straight beams can be multiplied to obtain the true stresses for the curved beams. Thus for a curved beam the maximum stress can be calculated from the equation

\[ \sigma = K \frac{M}{I} \]

Table A13.3 gives the value of K for various beam section shapes and beam radius of curvatures. The table shows that for only rather sharp curvatures is the correction appreciable. In general for airplane fuselage rings on frames the curvature influence can be neglected. However there are often fittings and mechanical structural units in airplane construction whose parts involve enough curvature to make the influence on the stress of primary importance. The concentration of stress on the inside edge of a curved unit in bending may influence the fatigue strength of unit considerably, thus a consideration of the possible influence of curvature should be a regular part of design procedure.

In the inelastic or plastic stress range, the influence of beam curvature should be considerably less since the stiffness of a material in the inelastic range is much less than in the elastic stress range and changes rather slowly as the stress increases.

A13.12 Problems.

1. Fig. A13.22 shows the cross-section of a single cell beam with 12 stringers. Assume the walls and webs are ineffective in bending. Calculate load in each stringer by use of beam formula. Also calculate stringer loads by equating internal resisting moment to external bending moment. Each stringer area is same and equals 0.1 sq. in. applied bending moment Mₓ = -100,000 in.lb.

2. Same as problem (1) but change external bending moment to Mᵧ = 200,000 in.lb.

3. Fig. A13.24 shows a beam section with 4 stringers. Assume web end walls ineffective in bending. Stringer areas
**TABLE A13.3**

VALUES OF K FOR USE IN THE BEAM FORMULA $\sigma = \frac{Mc}{I}$

<table>
<thead>
<tr>
<th>SECTION</th>
<th>R/c</th>
<th>FACTOR K INSIDE FIBER</th>
<th>FACTOR K OUTSIDE FIBER</th>
<th>$e^*/R$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.2</td>
<td>3.41</td>
<td>0.54</td>
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</tr>
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<td>1.4</td>
<td>2.40</td>
<td>0.60</td>
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</tr>
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<td>1.6</td>
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<table>
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<tr>
<th>SECTION</th>
<th>R/c</th>
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<th>OUTSIDE FIBER</th>
<th>$e^*/R$</th>
</tr>
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<tr>
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<td>1.2</td>
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<td>0.57</td>
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<td>2.43</td>
<td>0.76</td>
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<tr>
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<td>1.8</td>
<td>2.23</td>
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<tr>
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<td>0.071</td>
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<tr>
<td>3.0</td>
<td>3.0</td>
<td>1.59</td>
<td>1.11</td>
<td>0.034</td>
</tr>
</tbody>
</table>

* $e^*$ equals distance from centroidal axis to neutral axis.

shown in (1) on figure. External applied bending moments are:

\[ M_x = -500,000 \text{ in}.lb. \text{ and } M_y = 200,000 \text{ in}.lb. \]

Find stress on all four stringers by all three methods which were explained in this chapter.

4) The Zee section shown in Fig. A13.25 is subjected to bending moments of \( M_x = 500 \text{ in}.lb. \text{ and } M_y = 2000 \text{ in}.lb. \) Find bending stresses at points a, b, c, d.

5) Fig. A13.26 shows a beam section composed of three different materials. Find the stress at top and bottom points on beam due to a bending moment \( M_y = 50,000 \text{ in}.lb. \)

\[ E_{	ext{Magnesium}} = 6.5 \times 10^6 \]
\[ E_{	ext{Aluminum}} = 29 \times 10^6 \]
\[ E_{	ext{Steel}} = 10.5 \times 10^6 \]

6) Fig. A13.26 shows a cross-section of a wood beam composed of 3 kinds of wood labeled A, B and C, glued together to form a composite beam. If the beam is subjected to a bending moment \( M_y = 7500 \text{ in}.lb. \text{, find intensity of bending stress at top edge of beam. Also find total end load on portions B and C.} \)

\[ A = \text{Spruce. } E = 1,300,000\text{ psi} \]
\[ B = \text{Maple. } E = 1,500,000\text{ psi} \]
\[ C = \text{Fir. } E = 1,600,000\text{ psi} \]

7) Fig. A13.27 shows 3 different beam sections. They are made of aluminum alloy whose stress-strain diagram is the same as that plotted in Fig. A13.19. Determine the ultimate internal resisting moment if the maximum compressive strain is limited to 0.01 in./in. Consider that upper portion is in compression. Compare the results obtained with formula \( M = \frac{\sigma_0}{2}, \) where \( \sigma_0 \) = compressive stress when unit strain is 0.01.

8) Fig. A13.28 shows a curved beam, carrying two 500 lb. loads. Find bending stresses at points C and C', when beam cross-section is made 3 different ways as indicated by sections 1, 2 and 3. Use Table A13.3.
DOUGLAS DC-3 AIRPLANE. Over-all view of the test wing section representing center wing section of DC-3.

A close-up view of wing test section showing details of wing ribs, stringers, etc.
CHAP. A. 14
BENDING SHEAR STRESSES - SOLID AND OPEN SECTIONS

A14.1 Introduction.

In Chap. A6, the shear stresses in a member subjected to pure torsional forces were considered in detail. In Chap. A13, the subject of bending stresses in a beam subjected to pure bending was considered in considerable detail. In practical structures, however, it seldom happens that pure bending forces (couples) are the loading forces on the beam. The usual case of bending moments on a beam are due to a transfer of external shear forces. Thus bending of a beam usually involves both bending (longitudinal tension and compression stresses) and shear stresses.

The same assumption that were made in Chap. A13 in deriving the bending stress equations are likewise used in deriving the equations for flexural shear stresses. With flexural shear stresses existing, the assumption that plane sections remain plane after bending is not completely true, since the shearing strains cause the beam sections to slightly warp out of their plane when the beam bends. This warping action is usually referred to by the term "shear lag". However, except in cases of beams with wide thin flanges, the error introduced by neglecting shearing strains is quite small and therefore neglected in deriving the basic flexural shear stress formula. The problem of shear lag influence is considered in other chapters.

A14.2 Shear Center.

When a beam bends without twisting, due to some external load system, the shear stresses are set up on the cross sections of the beam. The centroid of this internal shear force system is often referred to as the shear center for the particular section. The resultant external shear load at this section must pass through the shear center of the section if twist of the section is to be prevented. Thus, if the shear center is known, it is possible to represent the external load influence by two systems, one that causes flexure and the other which causes only twist.

A14.3 Derivation of Formula for Flexural Shear Stress.

Fig. A14.1 shows a loaded simply supported beam. When the beam bends downward due to the given loading, the beam portion above the neutral axis is in compression and that below the neutral axis in tension. Consider a short portion dx of the beam at points DF on the beam and treat it as a free body as shown in Fig. A14.3. The variation of tensile and compressive stress on each face of the beam portion is as indicated. The stress \( \sigma_t \) is greater than \( \sigma_c \) because the bending moment due to the given beam loading is greater at beam section DD' than at FF'. Now consider that this beam portion dx is further cut as indicated by the notch DCEF in Fig. A14.1, and this segment is shown in Fig. A14.4 as a free body with the forces as indicated. Let \( \sigma_t \) = maximum tensile stress at a distance c from the neutral axis

Then the stress at a distance y from neutral axis is \( \sigma_y = \sigma_t \frac{y}{c} \).

The total load on an element of area dy of the beam cross-section (see Fig. A14.2) thus equals \( \frac{dy}{y} \) dy.

Now, referring to Fig. A14.4, the total tensile load on each face of this segment will be calculated.

\[
\text{Total load on face CD} = \int_{y}^{c} y \, dy \quad \text{(1)}
\]

\[
\text{Total load on face FE} = \int_{y}^{c} y \, dy \quad \text{(2)}
\]

From Chap. A13, the equation for flexural stress \( \sigma \) was derived, namely \( \sigma = Mx/I \). Let M equal the bending moment at beam section DD' and M' that at beam section FF' and let I and I' the moment of inertia of the cross-sectional area about the neutral axis at these same beam sections respectively. Then substituting value of \( \sigma_t \) in equations (1) and (2) we can write,

\[
\text{Total load on face CD} = \frac{M}{I} \int_{y}^{c} y \, dy \quad \text{(3)}
\]
Total load on face FE = \( \frac{M}{I} \int y \, dA \)  

Now let \( \tau \) b dx equal the shearing force on face CE of the segment in Fig. A14.4 where \( \tau \) equals the shearing stress and b dx the shearing area. For equilibrium of the segment, the total forces parallel to x-x axis must be zero. If the beam is of uniform cross-section, which is the case in our problem, then \( I = I' \) and c and \( y_o \) are the same in both equations (3) and (4).

Then the resultant horizontal force on the sides of the segment equals the difference of the values in (3) and (4), or

Resultant horizontal load = \( \frac{M - M'}{I} \int y \, dA \).

For equilibrium of segment in x-x direction,

\[ \sum F_x = \frac{M - M'}{I} \int y \, dA + \tau \, b \, dx = 0 \]

hence \( \tau = \frac{M - M'}{I \, b \, dx} \int y \, dA \)  

\[ \text{(5)} \]

However \( \frac{M - M'}{dx} = \frac{dM}{dx} = V = \text{the external shear on the beam section.} \)

hence \( \tau = \frac{V}{I \, b} \int y \, dA \)  

\[ \text{(6)} \]

It is important to note that equation (6) applies only to beams of uniform section (constant moment of inertia). In airplane wing structures the common case is for beams to vary in cross-section or moment of inertia, and if this variation is considerable, equation (5) should not be used and resort should be made to equations (3) and (4). This fact is illustrated in example problem 2. This matter of variable cross-sections is discussed later in this chapter.

A14.4 Example Problems. Symmetrical Sections.

External Shear Loads Act Thru Shear Center.

Example Problem 1.

Fig. A14.5 shows the cross section of a beam symmetrical about the Y-Y axis. Assume that a beam with this cross-section is subjected to a loading which produces a shear load in the Y direction = to \( V_y = 850 \) lb. Its location is through the shear center of the section which lies on the centroidal Y axis of the beam section due to the symmetry of the section about this axis. Let it be required to determine the shearing stress at the neutral axis x-x and at points 1-1 and 2-2 of the cross-section as shown in Fig. A14.5. The neutral axis location and moment of inertia of the section about the neutral axis are given on the figure.

SOLUTION:

Shear stress at neutral axis x-x

\[ \tau_{x-x} = \frac{V_y}{I_x} \int y \, dA \]

Table A shows the calculation of the term

\[ \int y \, dA \].

<table>
<thead>
<tr>
<th>PORTION</th>
<th>Area dA</th>
<th>y</th>
<th>y dA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.09 \times 0.5 = 1.045</td>
<td>1.045</td>
<td>1.096</td>
</tr>
<tr>
<td>2</td>
<td>3 \times 0.5 = 1.50</td>
<td>2.34</td>
<td>3.510</td>
</tr>
<tr>
<td>3</td>
<td>0.5 \times 0.5 = 0.25</td>
<td>2.84</td>
<td>0.710</td>
</tr>
<tr>
<td>1</td>
<td>0.5 \times 0.5 = 0.25</td>
<td>2.84</td>
<td>0.710</td>
</tr>
<tr>
<td>SUM</td>
<td>6.026</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \tau = \frac{850 \times 6.026}{27.2 \times 0.5} = 377 \text{ psi.} \]

Calculation of shear stress at point 1-1:

\[ \tau_{1-1} = \frac{V_y}{I_x} \int y \, dA \]

\[ \tau_{1-1} = \frac{850 \times 4.93}{27.2 \times 0.5} = 308 \text{ psi.} \]

Table A. Refer to Fig. a.

\[ \text{Fig. a} \quad 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3 \]

\[ \text{Fig. b} \quad 1 \quad \text{Fig. c} \quad 1 \quad \text{Fig. d} \quad 1 \]

hence \( \tau = \frac{850 \times 6.026}{27.2 \times 0.5} = 377 \text{ psi.} \)

FIG. 14.5
This shearing stress is at a section just adjacent to portion (2). If it was taken just adjacent to portion (1), then the width \( b \) in the equation would be 3 inches instead of 0.8 inch and the shearing stress would be 0.5/3 times that shown above, or in other words the shearing stress changes abruptly when the shear area changes abruptly.

Shearing stress at point 2-2 on cross-section: 
\[
\tau_{22} = \frac{350 \times 1.42}{27.2 \times 1.0} = 44.4 \text{ psi}.
\]

Substituting, \( \tau_{2-2} = 44.4 \text{ psi} \).

(The width \( b = 0.5 \times 0.5 \) for the two portions 3, 3').

The shearing stresses as calculated act in the plane of the beam cross-section in the Y direction and also with the same intensity parallel to the Z axis which is normal to the beam section.

Example Problem 2. VARIABLE MOMENT OF INERTIA.

Fig. A14.6 shows a cantilever beam loaded with a single load of 600 lb. at the end and acting through the centroid of the beam cross-section. The beam section is constant between stations 0 and 132, then it tapers uniformly to the sections shown for stations 175 and 218. The shear stress distribution on the beam cross-section at stations 175 and 218 will be determined.

\[
\text{Bending Moments:}
\]
\[
M_{132} = -600 \times 132 = -79,200^* \]
\[
M_{175} = -600 \times 175 = -105,000^* \]
\[
M_{218} = -600 \times 218 = -130,600^* \]

Table A14.1 shows the results of calculating the bending stresses at 3 points on each side of the neutral axis for the 3 stations. For example for station 132

\[
\sigma_{\text{MAX.}} = \pm \left( \frac{M_0}{I} \cdot \frac{1}{3} \right) = \pm \left( 79200 \times \frac{4}{33.67} \right) = \pm 6180 \text{ psi}.
\]

(tension at top edge and compression at lower edges).

For a point 1 inch from either edge of the beam

\[
\sigma = \pm \left( \frac{79200 \times 3}{33.67} \right) = \pm 6135 \text{ psi}.
\]

### Table A14.1

<table>
<thead>
<tr>
<th>Station</th>
<th>Bending Moment M</th>
<th>On Top or Bottom Fiber y = 4&quot;</th>
<th>Point 1&quot; From Top or Bottom y = 3&quot;</th>
<th>Point 2&quot; From Top or Bottom y = 2&quot;</th>
<th>Portion &quot;A&quot;</th>
<th>Portion &quot;B&quot;</th>
<th>Portion &quot;C&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>132</td>
<td>79,200*</td>
<td>8180</td>
<td>6135</td>
<td>4090</td>
<td>7157</td>
<td>5112</td>
<td>1023</td>
</tr>
<tr>
<td>175</td>
<td>105,000*</td>
<td>8180</td>
<td>6135</td>
<td>4090</td>
<td>7157</td>
<td>1023</td>
<td>5124</td>
</tr>
<tr>
<td>218</td>
<td>130,600*</td>
<td>8180</td>
<td>6135</td>
<td>4090</td>
<td>7157</td>
<td>5112</td>
<td>1023</td>
</tr>
</tbody>
</table>

From the results in Table A14.1, it should be noticed that the change in moment of inertia between the three stations is directly proportional to the change in bending moment, hence the same value for the bending stresses for all three stations. Columns 6, 7, and 8 give the total bending stress load on portions A, B, and C of the three cross-sections (see Fig. A14.6). These values equal the average stress on the portions times the area of the portion; for example, for station 132, the load on portion A:

\[
\text{load} = \frac{6180 + 6135}{2} \times 1 \times 1 = 7157^*, \text{ and for portion C:}
\]

\[
\text{load} = \frac{4090 + 0}{2} \times 2 \times .25 = 1023^*
\]

Fig. A14.7 shows the tension and compressive stresses acting on a portion of the beam between
stations 175 and 218. Fig. A14.8 shows the resulting horizontal shear stress pattern resulting from the loads in Fig. A14.7. For example, if we take a section along the beam 1" from the top or bottom edge of the beam and treat this portion as a free body as shown in Fig. A14.12 applying \( ZH = 0 \),

\[
ZH = -7157 + 7157 + \tau \times 43 \times 1.0 = 0, \text{ hence } \tau = 0.
\]

\[
\begin{align*}
&8180 \quad 7157\# \quad A \quad B \quad 10224\# \quad \text{Fig. A14.7} \\
&6135 \quad 7157\#
\end{align*}
\]

Fig. A14.8 Fig. A14.10 Fig. A14.11

Similarly, treating the portion between the edge of the beam and a point 2" from the edge as a free body diagram as shown in Fig. A14.13,

\[
ZH = -7157 + 7157 - 15336 + 10224 + \tau \times 43 \times 0.25 = 0, \text{ therefore } \tau = 477 \text{ psi}.
\]

Obviously, the shear stress on portion C is constant, since the end load on this portion at both stations is the same, or 10224#.

Fig. A14.8 shows the general shape of the shear stress distribution on the beam section at any point between stations 175 and 218. The shear stresses between stations 132 and 175 would be the same, since the change in bending moment and moment of inertia have been made the same as between stations 175 and 218.

Figs. A14.9 and A14.10 show the shear stress patterns if the formula \( \tau = \frac{V}{I_b} \int y \, dA \) be used for each station. The discrepancy is considerable as the equation does not apply to beams of varying section.

To illustrate the calculation by the shear stress formula, the shear stress will be calculated at the neutral axis for the beam section at station 175.

\[
\tau = \frac{V}{I_b} \int_0^4 y \, dA
\]

where \( \int_0^4 y \, dA = 1 \times 1 \times 3.5 + 2 \times 1 \times 2.5 + 2 \times 0.25 \times 1 = 9.0 \)

hence, \( \tau = \frac{500 \times 9.0}{51.55 \times 9.0} = 420 \text{ psi} \) as compared to the true shear stress of 477 in Fig. A14.8.

**TABLE A14.2**

<table>
<thead>
<tr>
<th>Maximum Shear Stress for Simple Sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross Section</td>
</tr>
<tr>
<td>---------------</td>
</tr>
<tr>
<td>N</td>
</tr>
<tr>
<td>O</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>Ellipse</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
A14.5 Maximum Shear Stresses for Simple Cross-Sections.

Table A14.2 gives the value of the maximum shear stress on a few simple sections and where it occurs on the cross section, (e is distance from neutral axis to point of maximum shear stress). $V$ equals the shear load normal to the neutral axis and it acts along the centerline axis, thus no twisting on the section. $A$ is the total cross-sectional area. The maximum shear stress is given in terms of the average shear stress which equals $V/A$.

A14.6 Derivation of Flexural Shear Flow Equation.

Symmetrical Beam Section.

To emphasize further the fundamental relationships, a second derivation of the equation for shear stress distribution will be presented.

Fig. A14.14 shows a portion of a cantilever beam carrying a load $P$ at the free end as shown. This load is so located as to cause the beam to bend in the XZ plane without twist about a Y axis. The problem is to derive relationships which will give the magnitude and sense of the shear flow distribution on the cross-section of the beam.

Fig. A14.15 shows a free body of a small portion of the beam cut out from the upper flange of the beam at points (a b) in Fig. A14.14. Under the given external load $P$ it is obvious that the upper half of the beam is subjected to compressive stresses. In Fig. A14.15, $C_Y$ is larger than $C_y$ since the cantilever bending moment is greater at station Y'.

The free edge of beam flange forms the right side face of the element in Fig. A14.15 and thus the shear flow force on this face is zero as indicated. The shear flow forces on the internal or cut faces in lbs. per inch are $q_y$ and $q_x$ as indicated. Since the sense of these shear forces is unknown, they will be assumed as acting in the positive direction. (See Fig. A14.16 for positive sense of forces acting parallel to each of the coordinate axes X Y Z.

Now consider the equilibrium of forces in the Y direction for the element in Fig. A14.15 $2F_y = 0$, or

$$q_y \, dy + (C_y' - C_y) = 0,$$

hence

$$q_y = -\left(\frac{C_y' - C_y}{dy}\right).$$

But

$$\left(C_y' - C_y\right) = \frac{M_x - M_x}{I_x} \int_a^b z \, da \quad \text{(See Art. A14.3)}.$$

hence

$$q_y = -\frac{M_x}{I_x} \int_a^b z \, da.$$

however $\frac{M_x}{I_x} = V_z$, the external shear in the $Z$ direction.

hence $q_y = -\frac{V_z}{I_x} \int_a^b z \, da$.

Equation (7) gives the change in shear flow force $q_y$ between points (a) and (b) and since in figure A14.15 the value of $q_y$ at (a) is zero because of a free surface, the value of $q_y$ in equation (7) is the true shear flow force in lbs. per inch at point (b). The student should realize that equation (7) gives the shear flow $q_y$ in the $Y$ direction. The minus sign in equation (7) means that the positive sense as assumed by the arrowhead on $q_y$ in Fig. A14.15 is incorrect or should be reversed.

The initial problem was to determine the shear flow force system in the plane of the beam cross-section or the XZ plane. From elementary engineering mechanics, we know that if a shearing stress occurs on one plane at a point in a body, a shearing stress of the same magnitude exists on planes at right angles to the first plane, or in general at a point,

$$q_y = q_x = q_z.$$

Before the shear flow in other planes is completely defined its sense (positive or negative) must be known. Equation (7) gives the magnitude and sense for $q_y$ at any desired point on the cross-section. The question of the sense of the associated $q_y$ and $q_x$ is easily determined from an observation involving equilibrium of moments. This fact will be explained by referring to a number of free body diagrams.

Fig. A14.17 shows a free body of a small element cut from the beam in Fig. A14.14 at point (a) on the cross-section. The forces on this free body are the compressive forces $C'$ and $C$ on the front and rear faces and the shear forces on the various faces as indicated. The right side face of the element is a free surface and thus $q$ on this face is zero.

The shear flow $q_y$ on the left side face is calculated from equation (7) namely...
For the given beam loading in Fig. A14.14 the load \( P \) is up, therefore \( V_y \) has a positive sign. For any portion of area \( A \) of the cross-section the distance \( z \) in the above equation is therefore positive. Therefore in substituting in the above equation \( q_y \) comes out negative for any point above neutral axis, and likewise for any point on beam section below the neutral axis the distance \( z \) would have a negative sign and \( q_y \) would come out positive. Therefore the sense of the calculated shear flow \( q_y \) on the left side of the element in Fig. A14.17 is negative or as indicated by the arrow on the force vector. Now to find the sense of the shear flow \( q_x \) on the front face take moments about a \( z \) axis acting in the plane of the rear face and through point (0) which is on the line of action of the forces \( C' \) and \( C \). Only two forces have moments, namely the side shear force \( q_y \) dy and the front face shear force \( (q_y/2)dx \). It is obvious by observation that \( q_y \) on front face must act to the left as shown if the moment is to be equal zero. The total shear force on front face is \( (q_y/2)dx \) because shear flow at right edge is zero and it varies linearly to \( q_x \) at left edge of front face.

Figs. A14.18, 19 and 20 show free bodies of elements taken at the other three corners of the beam section which are labeled \( c, d, e \) in Fig. A14.14. For all free surfaces \( q \) is zero. The sense of \( q_y \) as before is given by the equation as explained before, hence \( q_y \) is negative in Fig. A14.18 and positive in figures 19 and 20. A simple consideration of moment equilibrium as explained for Fig. A14.17 gives the sense of the shear flow \( q_x \) as shown in the 3 figures.

Fig. A14.21 shows a free body of the entire upper beam flange and a short portion of the beam web. \( q_y \) from the equation is negative and this acts as shown in the figure. Now if we take moments about a \( X \) axis in the plane of the back side and through a point on the line of action of \( C' \) and \( C \), the shear flow \( q_x \) on the front face must act downward in order to balance the moment due to \( q_y \).

Fig. A14.22 shows the results for a free body of the lower flange plus a small web portion. \( q_x \) is positive from equation and thus \( q_x \) must be downward for moment equilibrium.

From the results obtained for these 6 different locations, a simple rule can be stated relative to sense of shear flows in the plane of the cross-section, namely:

- If the calculated shear flow is directed toward the boundary line between the two intersecting planes of the particular free body, then the shear flow on the other plane is also directed toward the common boundary line, and conversely directed away if the calculated shear flow is directed away.

Fig. A14.14 shows the sense of the shear flow pattern on the beam section as determined for the given external loading.

**A14.7 Shear Stresses and Shear Center for Beam Sections with One Axis of Symmetry.**

**Example Problem. CHANNEL SECTION.**

Fig. A14.23 shows a cantilever beam with channel shaped cross-section carrying a 100 lb. downward load as shown. The problem is to determine the lateral position of this load so that the beam will bend without twist. This position will coincide with the lateral position of the centroid of the shear flow system on the beam cross-section which holds the external load in equilibrium without twisting of the beam section. The cantilever beam has been cut at a section \( abc \) (Fig. A14.23) which is far enough from the fixed end of the beam (not shown) so that the effects of beam end restraint against section warping can be neglected. In Fig. A14.23, the internal forces holding the beam in equilibrium is sketched in. They consist of a longitudinal stress system of tension and compression and variable shear flow system in the plane of the cross-section. In this problem we are only considered with the internal resisting shear flow system.

For solution of this problem the moment of inertia \( I_X \) must be known. If calculated it would be \( I_X = 0.2667 \) in.².

**SOLUTION:** From equation (7)

\[
q_y = -\frac{V_y}{I_x} \begin{array}{c}
\frac{Z}{A} \\
\frac{Z}{A}
\end{array}
\]  

we know that the shear stress is zero at a free edge, thus the solution of equation (6) is started at either points (a) or (d).

The shear \( V_y \) is = 100 lb. Thus equation (8) for our problem reduces to,

\[
q_y = -\frac{100}{0.2667} \begin{array}{c}
\frac{Z}{A} \\
\frac{Z}{A}
\end{array} = 375 \frac{Z}{A}
\]
We will start at point (a) in solving equation (9) and proceed around the section.

Point (a) \( q_y = 0 \) (free surface)

Point (b) \( z = -1, A = \text{area between (a) and (b)} \)

\[
q_y = q_{y_a} + 375 z_a^2 A
\]

\[
q_y = 0 + 375 \times (-1) \times (1 \times 0.1) = -37.5 \text{ lb/in.}
\]

Point (0) on X axis.

\[
q_y = -37.5 + 375 z_b^2 A
\]

\[
= -37.5 + 375 \times (-0.5) \times (1 \times 0.1) = -56.3 \text{ lb/in.}
\]

Point (c).

\[
q_y = -56.3 + 375 (0.5) \times (1 \times 0.1) = -37.5 \text{ lb/in.}
\]

Point (d).

\[
q_y = -37.5 + 375 (1) \times (1 \times 0.1) = 0 \text{ (free surface)}
\]

We know that the intensity of shear flow in the ZX plane at any point equals that in the Y direction at the same point. Thus \( q_x \) or \( q_z \) equal the \( q_y \) values above. The sense of the \( q_x \) and \( q_z \) shear flows must be known before they are completely defined or known. Fig. A14.24 shows a free body of a small element at point (a) on the end of the lower beam flange. For any point below the X centroidal axis equation (9) will give a minus sign for \( q_y \). Thus in Fig. A14.24 \( q_y \) acts as shown, or directed toward the boundary line between the side face and the front face. Then by the simple rule as given in the previous article \( q_x \) is also directed toward this common boundary line as shown in Fig. A14.25.

Common sense tells us that the resisting shear flow \( q_z \) on the channel web must be directed upward because it is the only force system that can balance the 100 lb. load as far as \( EF_z = 0 \) is concerned.

In general the shear flow is continuous around the section and only reverses when it passes through zero which only happens in closed tubular sections. In general, it is possible in most cases by observation only, to determine the sense of the shear flow at some one point on the beam cross-section. The shear flow being like a flow or liquid will continue in the same general direction along the center line of the parts that make up the beam section.

The small arrows on the beam section of Fig. A14.23 show the sense of the shear flow pattern over the beam section. In Fig. A14.25, the shear flow values as calculated at the various points are plotted to form a shear flow diagram for the beam section. Between points a and b or d and c, the arm \( z \) in equation (9) is constant and thus \( q \) varies linearly as plotted. Between b and 0 or c and d the arm \( z \) changes and the area is also a function of \( z \), thus \( q \) varies as \( z^2 \) or parabolic as plotted.

The initial problem was to locate the centroid of this final shear flow system which is generally referred to as the shear center. In Fig. A14.25, \( q_{ab} \), \( q_{bc} \), and \( q_{cd} \) represent the resultant of the shear force system on these three portions of the beam cross-section. Each force is equal in magnitude to the area of the shear flow diagram for the particular beam section portion. Hence,

\[
q_{ab} = 1 \times 37.5/2 = 18.75 \text{ lb.}
\]

\[
q_{bc} = 2 \times 37.5 + (56.3 - 37.5) \times 2 \times 2/2 = 100 \text{ lb.}
\]

\[
q_{cd} = 1 \times 37.5/2 = 18.75 \text{ lb.}
\]

The resultant \( R \) of these three shear forces will now be determined.

\[
\Sigma F_z = 100 \text{ lb.}, \quad \Sigma F_x = 18.75 - 18.75 = 0
\]

Hence

\[
R = (\Sigma F_z^2 + \Sigma F_x^2)^{1/2} = (100^2 + 0)^{1/2} = 100 \text{ lb.}
\]

The moment of the resultant about any point such as (b) in Fig. A14.17 must equal the moment of the shear flow force system about point (b). Let \( e \) be distance from point b to line of action of the resultant \( R \). Hence

\[
Re = \Sigma M_b \text{ (of shear flow system)}
\]

\[
100 e = 18.75 \times 2, \quad e = 0.375 \text{ inches}
\]

Thus the centroid of the internal shear resisting force system lies on a vertical line.
BENDING SHEAR STRESSES. SOUND AND OPEN SECTIONS. SHEAR CENTER.

0.375 inches to left of point b as shown in Fig. A14.17. For bending about the centroidal Z axis without twist the resultant of the internal shear flow system would obviously, due to symmetry of section about X centroidal axis, lie on the X axis, hence shear center for the given channel section is at point C in Fig. A14.17.
The external load of 100 lb. would have to be located 0.375 inches to the left of the centerline of the channel web if bending of the channel without twist is to occur.

A14.8 Shear Stresses for Unsymmetrical Beam Sections.

In chapter A13, which dealt with bending stresses in beams, three methods were presented for determining the bending stresses in beams with unsymmetrical beam sections. The bending stress equations for these three methods will be repeated here:

Method 1. The Principal Axis Method.

\[ \sigma = -\frac{M_p}{I_{xy}} z_p - \frac{M_z}{I_{xy}} x_p \]  

\[ \sigma = -\frac{M_n}{I_n} z_n \]  

Method 3. The Method using section properties about centroidal Z and X axes. For brevity this method will be called the k method.

\[ \sigma = -(k_x M_x - k_y M_y) x - (k_x M_x - k_y M_y) z \]  

where,

\[ k_x = \frac{I_{xz}}{I_x - I_{xz}} \]  

\[ k_y = \frac{I_{zy}}{I_y - I_{zy}} \]

In referring back to the derivations of equations (5), (6) and (7) the above equations (9), (10) and (11) can be written in terms of beam external shears instead of external bending moments as follows:

Method 1. The Principal Axis Method.

\[ q_y = -\frac{V_{2p}}{I_{xy}} \Sigma z_p A \]  

Method 2. The Neutral Axis Method.

\[ q_y = -\frac{V_{2n}}{I_n} \Sigma z_n A \]  

Method 3. The k Method.

\[ q_y = -(k_x V_x - k_y V_y) \xi A - (k_x V_z - k_y V_z) \eta A \]  

\[ \xi = \frac{z}{A} \]  

EXAMPLE PROBLEM USING THE THREE DIFFERENT METHODS.

Fig. A14.26 shows a Zee Section subjected to a 10,000 lb. shear load acting through the shear center of the section and in the direction as shown. The problem will be to calculate the shear flow qy at two points on the beam section, namely points b and c as indicated on the figure. The shear flow at these two places will be calculated by all 3 methods.

Since all 3 methods require the use of beam section properties and since the direction of either the principal axes or the neutral axes are unknown the first step in the solution regardless of which method is used is to calculate the section properties about centroidal X and Z axes. Table A14.3 gives the calculations. The section has been divided into 4 portions labeled 1, 2, 3 and 4.

\[ V = 10000 \text{ lb.} \]

![Fig. A14.26](image-url)

<table>
<thead>
<tr>
<th>Portion</th>
<th>Area (A)</th>
<th>Arm about X (x)</th>
<th>Arm about Y (y)</th>
<th>Ax</th>
<th>Ay</th>
<th>Ix</th>
<th>Iy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>1.45</td>
<td>-0.45</td>
<td>0.0825</td>
<td>0.02025</td>
<td>0.00838</td>
<td>0.000017</td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
<td>0.70</td>
<td>0</td>
<td>0</td>
<td>0.08660</td>
<td>0</td>
<td>0.00117</td>
</tr>
<tr>
<td>3</td>
<td>0.14</td>
<td>-0.70</td>
<td>0</td>
<td>0</td>
<td>0.08660</td>
<td>0</td>
<td>0.00117</td>
</tr>
<tr>
<td>4</td>
<td>0.10</td>
<td>-1.45</td>
<td>0.45</td>
<td>-0.0625</td>
<td>-0.2025</td>
<td>-0.0838</td>
<td>0.000017</td>
</tr>
<tr>
<td>Σ</td>
<td>-1.305</td>
<td>0.53770</td>
<td>0.04090</td>
<td>0.01690</td>
<td>0.04577</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ I_x = \Sigma A z^2 + \Sigma I_x = 0.6035 \text{ in}^4 \]

\[ I_z = \Sigma A x^2 + \Sigma I_z = 0.0574 \]

\[ I_{xz} = 2 A x z = -1.305 \]

(Note: In Table A14.3 Ix and Iy are the moments of inertia of each portion about its own centroidal axis).

SOLUTION BY PRINCIPAL AXES METHOD. (Method 1)

Let \( \phi \) be angle between Principal axes and the X and Z axes. From chapter A13,

\[ \tan 2\phi = \frac{2 I_{xz}}{I_z - I_x} \]

\[ = \frac{2(-0.1305)}{0.0574 - 0.6035} = 0.4778 \]
hence $2 \phi = 25^\circ - 32.2^\prime$ or $\phi = 12^\circ - 46.1^\prime$.

\[
\sin \phi = 0.2210 \text{ and } \cos \phi = 0.97527.
\]

The moments of inertia about the principal axes can now be calculated.

\[
I_x = I_x \cos^2 \phi + I_z \sin^2 \phi - 2 I_{xz} \sin \phi \cos \phi = 0.6035 \times 0.97527 + 0.0574 \times 0.2210 - 2 (-0.1305) \\
\quad \times 0.97527 \times 0.2210 = 0.63316 \text{ in}^4.
\]

\[
I_z = I_x \sin^2 \phi + I_z \cos^2 \phi + 2 I_{xz} \sin \phi \cos \phi = 0.6035 \times 0.2210 + 0.0574 \times 0.97527 + 2 (-0.1305) \\
\quad \times 0.97527 \times 0.2210 = 0.02782 \text{ in}^4.
\]

The equation for shear flow q is,

\[
q = -\frac{V_{z_p}}{I_{z_p}} \sum z_p - \frac{V_{x_p}}{I_{x_p}} \sum x_p - - - - - (15)
\]

In Fig. A14.26 the external shear load is 10000 lbs. acting in a direction as shown. Resolving this shear load into z and x components, we obtain,

\[
V_z = 10000 \times \cos 30^\circ = 8667 \text{ lb.}
\]
\[
V_x = 10000 \times \sin 30^\circ = 5000 \text{ lb.}
\]

Resolving these z and x components further into components along the principal axes we obtain,

\[
V_{z_p} = 8667 \times 0.37527 - 50000 \times 0.2210 = 7348 \text{ lb.}
\]
\[
V_{x_p} = 8667 \times 0.2210 + 5000 \times 0.97527 = 6792 \text{ lb.}
\]

Calculation of shear flow at point (b). See Fig. A14.26.

Fig. A14.27 shows the position of the principal axes as calculated. The shear flow at the free edge of the upper portion (1) is zero. For the shear flow at point (b), the area to be used in the summations $\sum z_p$ and $\sum x_p$ is the area of element (1). The arms $z_p$ and $x_p$ can be calculated by simple trigonometry. Fig. A14.27 shows the value of these distances, namely $x_p = -0.1184$ and $z_p = 1.5136$.

![Fig. A14.27](image)

Now substituting in equation (15)

\[
q_b = -\frac{7348}{0.63316} (1 \times 0.1 \times 1.5136) - 6792(1 \times 0.1) (-0.1184) = -1756 + 2890 = 1134 \text{ lb/in.}
\]

Calculation of shear flow at point (c).

For portion (2) area $A = 1.4 \times 0.1 = 0.14$

\[
z_p = 0.70 \times 0.97527 = 0.6825 \text{ in.}
\]
\[
x_p = 0.70 \times 0.2210 = 0.1547
\]

The shear flow at point (c) equals the shear flow at point (b) plus the effect of the portion (2) between points (b) and (c), hence

\[
q_c = 1134 - \frac{7348}{0.63316} (0.14 \times 0.6825) - 6792(0.14 \times 0.1547) = 1134 - 1109 - 5285 = -5260 \text{ lb/in.}
\]

The shear stresses at these two points (b) and (c) would equal $q_b/2 = 1134/0.1$ and $-5260/0.1$ or 11340 psi and -52600 psi respectively, respectively.

**SOLUTION BY NEUTRAL AXIS METHOD. (Method 2)**

In this solution it is necessary to find the neutral axis for the given external loading. In Fig. A14.28, the angle $\phi$ is the angle between the plane of loading and the $z_p$ principal axis, and this angle $\phi$ equals $30^\circ + 12^\circ - 46^\circ = 42^\circ - 46^\circ$.

Let a equal angle between $x_p$ principal axis and neutral axis R.M.

From chapter A13, we find,

\[
\tan \alpha = \frac{I_{z_p} \tan \theta}{I_{x_p}}
\]

\[
= \frac{0.53316 \times 0.5245}{0.02782} = -21.052
\]

whence, $\alpha = 87^\circ 17'$ (See Fig. A14.28 for location of neutral axis).

\[
\sin \alpha = 0.9989, \quad \cos \alpha = 0.04742
\]

\[
I_n = I_{x_p} \cos^2 \alpha + I_{z_p} \sin^2 \alpha, \quad \text{substituting}
\]
The component of the given external shear load normal to the neutral axis n-n equals

\[ V_n = 10000 \times \sin 45° - 25° = 7130 \text{ lb.} \]

From equation (13)

\[ q = \frac{V_n}{In} \cdot z_n \cdot A \]

Shear flow at point (b):
The distance from the neutral axis to the centroid of portion (l) equal \( z_n = -0.0466 \text{ in.} \) hence

\[ q_b = -\frac{7130}{0.02819} \cdot (1 \times 0.1) \cdot (-0.0466) = 1135 \text{ lb/in.} \]

Shear flow at point (c):

\[ q_c = 1135 - \frac{7130}{0.02819} (1.4 \times 0.1) \cdot 0.1868 \]

\[ = 1135 - 6390 = -5255 \text{ lb/in.} \]

**SOLUTION BY METHOD 3 - (The k Method).**

In this method, only the section properties about the centroidal X and Z axes are needed. These properties as previously calculated in Table A14.3 are,

\[ I_x = 0.6035, \quad I_z = 0.0574, \quad I_{xz} = -0.1305 \]

The shear flow equation from eq. (14) is,

\[ q = -(k_x V_x - k_z V_z) \cdot z \cdot A - (k_x V_x - k_z V_z) \]

\[ k_i = \frac{I_{xz}}{I_x I_z - I_{xz}^2} = \frac{0.6035 \times 0.0574 - (-0.1305)}{0.6035 \times 0.0574 - (-0.1305)} = -0.1305 \]

\[ = \frac{0.1305}{0.01762} = -7.406 \]

\[ k_x = \frac{I_z}{I_x I_z - I_{xz}^2} = \frac{0.0574}{0.6035 \times 0.01762} = 3.257 \]

\[ k_z = \frac{I_x}{I_x I_z - I_{xz}^2} = \frac{0.6035}{0.01762} = 34.25 \]

Resolving the external shear load of 10,000 into x and z components, we obtain,

\[ V_x = 10000 \sin 30° = 5000 \text{ lb.} \]

\[ V_z = 10000 \cos 30° = 8667 \text{ lb.} \]

Substituting values of \( V_x, V_z \) and \( k \) values in equation (16) we obtain -

\[ q = [34.25 \times 5000 - (-7.406 \times 8667)] \cdot \Sigma x A - \]

\[ [3.257 \times 8667 - (-7.406 \times 5000)] \cdot \Sigma z A, \text{ whence} \]

\[ q = -235438 \cdot \Sigma x A - 65258 \cdot \Sigma z A \]

Shear flow at point (b)
For portion (1), \( x = -0.45 \text{ in.}, \ z = 1.45 \text{ in.} \)

\[ A = 1 \times 0.1 = 0.1 \]

Substituting,

\[ q_b = -235438 (-0.45) 0.1 - 65258 1.45 \times 0.1 \]

\[ = 10597 - 9462 = 1135 \text{ lb/in.} \]

Point (c)
For portion (2), \( x = 0, \ z = 0.7, \ A = 1.4 \times 0.1 = 0.14 \)

\[ q_c = 1135 - 235438 (0) \cdot 0.14 - 65258 0.7 \times 0.1 \]

\[ = 1135 - 0 - 5255 = -5255 \text{ lb/in.} \]

**General Comments.**

The author prefers solution method number 3 since it avoids the calculation of additional angles and section properties as required in methods 1 and 2. Furthermore, in calculating the shears and moments on the airplane wing, fuselage and other major structural units it is convenient to refer these shears and moments to the conventional X Y Z axes, and thus these values can be used in method 3 without further resolution. From an investigation of many airplane stress analysis reports, it appears that the engineers of most airplane companies prefer to use method 3.

A14.9 Beams with Constant Shear Flow Webs.

Fig. A14.29 shows a beam composed of heavy flange members and a curved thin web. For bending about the X-X axis, the web on the compressive side of the beam absorbs very little compressive stress, since buckling of the web will take place under low stresses, particularly when the curvature of the web is small. On the tension side, the web will be more effective, but if the flange areas are relatively large, the proportion of the total bending tensile stress carried by the web is small as compared to that carried by the tension flange. Thus for beams composed of individual flange members connected by thin webs it is often assumed that the flanges develop the entire longitudinal bending resistance which therefore means that the shear flow is constant.
over a particular web. In other words in the shear flow equation \( q = \frac{V_x}{I_x} \), if the area of the web is neglected then q is constant between flange members.

**RESULTANT OF CONSTANT SHEAR FLOW FORCE SYSTEM**

Fig. A14.29 shows a beam assumed to be carrying a downward shear load (not shown) and to cause bending about axis-x-x without twist. Assuming the two flanges develop the entire bending resistance, the shear flow q is constant on the web and acts upward along the web to balance the assumed external downward load. The resultant of this resisting shear flow force system will give the lateral position of the shear center for this beam section. The problem then is to find the resultant of the shear flow system.

Let \( q = \) load per inch along web (constant).

Let \( R = \) resultant of the q force system.

From elementary mechanics,

\[
R = \sqrt{\sum q_x^2 + \sum q_y^2},
\]

where \( q_x \) and \( q_y \) are the x and y components of the q forces along the web. Since q is constant, \( \Sigma q_x = 0 \), hence,

\[
R = \Sigma q_x = q_h \quad (17)
\]

Equation (17) states that the magnitude of the resultant of a constant flow force system is equal to the shear flow q times the straight line distance between the two ends of the shear flow system.

Since \( \Sigma q_x = 0 \), the direction of the resultant is parallel to the straight line joining the ends of the web.

The location of the resultant force is found by using the principle of moments, namely, that the moment of the resultant about any point must equal the moment of the original force system about the same point. In Fig. A14.29 assume point (0) as a moment center.

Then \( R \cdot e = q \cdot L \cdot r \)

but \( R = q \cdot h \)

hence \( e = \frac{q \cdot L \cdot r}{q \cdot h} = \frac{L \cdot r}{h} \quad (18) \)

In equation (18) the term \( L \cdot r \) is equal to twice the area (A), where area (A) is the enclosed area formed by drawing straight lines from moment center (0) to the ends of the shear flow force system. Thus

\[
e = \frac{2 \cdot A}{h} \quad (19)
\]

The shear center thus lies at a distance e to the left of point (0), and the external shear load would have to act through this point if twisting were to be eliminated.

**EXAMPLE PROBLEM – RESULTANT OF A CONSTANT FLOW FORCE SYSTEM**

Fig. A14.30 shows a constant flow force system thru points A B C D E with \( q = 10 \) lb. per inch. The resultant of this force system is required.

\[
\begin{align*}
R &= 200 \text{lb} \\
q &= 10 \text{ lb/in.} \\
h &= 20' \\
A &= 5 \times 10^5 + 5 \times 10^5 + 0.5 \times 5^8 + 10 \times 50 = 189.3 \text{ sq in.}
\end{align*}
\]

From eq. (19)

\[
e = \frac{2 \cdot A}{h} = \frac{2 \times 189.3}{20} = 18.93 \text{ in.}
\]

Fig. A14.30 shows the resultant of 200 lb. acting at a distance e from (0) and parallel to line AE.

A14.10 Example Problems for Beams with Constant Shear Flows Between Flange Members.

**EXAMPLE PROBLEM 1. Beam Section Symmetrical About One Axis**

Fig. A14.31 shows an open beam section composed of 8 flange members connected by thin
sheet to form the webs and walls. The flange members are numbered a to h and the areas of each are given on the figure. It will be assumed that the webs and walls develop no bending resistance and thus the shear flow between adjacent flange members will be constant. The problem is to determine the shear center for the beam section.

**SOLUTION:**

Since the beam section is symmetrical about the X axis, the centroidal X and Z axes are also principal axes, since the product of inertia Ixz is zero.

The vertical position of the beam section centroid due to symmetry is midway between the upper and lower flanges.

To find the horizontal position of the centroid, take moments of the flange areas about the left end or line bc:

\[
\bar{x} = \frac{\sum A_x x}{\sum A} = \frac{0.4 \times 15 + 0.2 \times 10 + 0.2 \times 5}{15} = 5.625 \text{ in.}
\]

The moments of inertia for the section about the centroidal x and y axes are:

\[
I_x = \sum A z^2 = (0.8 \times 5^2) 2 = 40 \text{ in}^4.
\]

\[
I_z = 0.8 \times 5.625^2 + 0.2 \times 0.625^2 + 0.2 \times 4.375^2 + 0.4 \times 9.376^2 = 64.4 \text{ in}^4.
\]

**HORIZONTAL POSITION OF SHEAR CENTER:**

The horizontal position of the shear center will coincide with the centroid of the shear flow system due to bending about axis xx without twist. For simplicity, to eliminate large decimal values for shear flow values an external shear load \(V_2 = 100 \text{ lb.}\) will be assumed and the internal resisting shear flow system will be calculated for this external loading.

From equation (8)

\[
q_y = -\frac{V_2}{I_x} \sum z A , \text{ substituting values of } V_2 \text{ and } I_x
\]

\[
q_y = -\frac{100}{40} \sum z A = -2.5 \sum z A
\]

We could start the solution at either of two points (a) or (h) since these points are free edges and thus \(q_y\) is zero. In this solution, we will start at the free edge at point (a) and go counterclockwise around the beam section. The area of each flange member has been concentrated at a point coinciding with the centroid of each flange area. In solving for the \(q\) values the subscript \(y\) will be omitted, and subscripts using the flange letters will be used in order to indicate at what point the shear flow \(q\) is being calculated.

\[
q_{ab} = -2.5 \sum z A = -2.5 \times 5 \times 0.1 = 1.25 \text{ lb./in.}
\]

The first letter of the subscript refers to the flange member where the shear flow \(q\) is being calculated and the second letter indicates on which adjacent side of the particular flange member. Hence \(q_{ab}\) means the shear flow at flange (a) but on the side toward (b).

\[
q_{bc} = q_{ba} = -1.25 \text{ (since no additional flange area is added, and thus shear flow is constant on sheet ab.)}
\]

\[
q_{db} = q_{dc} = -2.5 \sum z A = -2.5 \times 0.4 = -6.25 \text{ lb./in.}
\]

\[
q_{cb} = -6.25
\]

\[
q_{cd} = q_{dc} = -2.5 \sum z A = -6.25 - 2.5 x (-5 x 0.4) = -1.25
\]

\[
q_{da} = q_{dc} = -2.5 \sum z A = -1.25 - 2.5 (-5 x 0.1) = 0
\]

\[
q_{ed} = q_{de} = 0
\]

\[
q_{ef} = 0 - 2.5 \sum z A = 0 - 2.5 (-5 x 0.1) = 1.25
\]

\[
q_{fe} = q_{ef} = 1.25
\]

\[
q_{fg} = 1.25 - 2.5 \sum z A = 1.25 - 2.5 (-5 x 0.2) = 3.75
\]

\[
q_{gf} = q_{fg} = 3.75
\]

\[
q_{gh} = q_{hg} = 3.75 - 2.5 \sum z A = 3.75 - 2.5 \times 5 \times 0.2 = 1.25
\]

\[
q_{gh} = q_{hg} = 1.25
\]

\[
q_{ha} = 1.25 - 2.5 \sum z A = 1.25 - 2.5 \times 5 \times 0.1 = 0 \text{ (checks free edge at h).}
\]

The sign or sense of each shear flow is for the shear flow in the y direction as explained in the derivations of the shear flow equations. The procedure now is to determine the sense of the shear flow in the plane of the cross-section or in the xy plane. It is only necessary to determine this sense at the beginning point, that is in sheet panel ab. The surest way to determine this sense is to draw a simple free body sketch of flange member (a) as illustrated in Fig. A14.31. The shear flow on the cut face is \(q_{y(ab)} = -1.25\) and this value is shown on the free body. By simple rule given at the end of Art. A14.5, the shear flow in the plane of the cross-
section is also directed toward the common boundary line and thus $q_x(ab)$ has a sense as shown in Fig. A14.31. The sense of the shear flow on the cross-section will now continue in this direction until the sign changes in the original calculation, which means therefore the shear flow sense will reverse. Fig. A14.32 shows a plot of the shear flow pattern with the sense indicated by the arrow heads.

![Diagram](image)

**Fig. A14.32**

The results will be checked to see if static equilibrium exists relative to $\Sigma F_x$ and $\Sigma F_z = 0$.

$$\begin{align*}
\Sigma F_z &= 100 \text{ (ext. load)} - 10 x 6.25 - 10 x 3.75 + 1.25 x 0.5 x 4 - 1.25 x 0.5 x 4 = 0 \text{ (check)}. \\
\Sigma F_x &= -5 x 1.25 + 5 x 1.25 - 5 x 1.25 + 5 x 1.25 = 0 \text{ (check)}. 
\end{align*}$$

The shear flow force system in Fig. A14.32 causes the section to bend about axis $xx$ without twist. The resultant of this system is 100 lb. acting down in the $z$ direction. The position of this resultant will thus locate the lateral position of the shear center.

Equating the moments of the shear flow system about some point such as (c) to the moment of the resultant about the same point we obtain:

$$e = 10 x 3.75 x 15 - 1.25 x 0.5 x 2 x 5 - 1.25 x 0.5 x 2 x 10 + 1.25 x 0.5 x 2 x 15$$

hence $e = 562.5/100 = 5.625$ inches.

Thus the shear center lies on a vertical line 5.625 inches to right of line $bc$.

**CALCULATION OF VERTICAL POSITION OF SHEAR CENTER.**

For convenience as before, we will assume a shear load $V_x = 100$ lb. and compute the resisting shear flow system to resist this load in bending about axis $zz$ without twist. The resultant of this shear flow system will give the vertical location of the shear center. The shear flow equation is,

$$q_y = -\frac{V_x}{I_z} \Sigma x A = -\frac{100}{64.4} \Sigma x A = -1.55 \Sigma x A$$

We will again start at the free edge adjacent to flange (a) where $q_y = 0$.

$q_{ab} = -1.55 \Sigma_a x A = -1.55 (-0.625 \times 0.1) = 0.0971 \text{ lb./in.}$

$q_{ba} = q_{ab} = 0.0971$

$q_{bc} = 0.0971 - 1.55 \Sigma_b x A = 0.0971 - 1.55 (-5.625 \times 0.4) = 3.592$

$q_{cb} = q_{bc} = 3.592$

$q_{cd} = 3.592 - 1.55 \Sigma_c x A = 3.592 - 1.55 x 5.625 \times 0.4 = 7.087$

$q_{dc} = 7.087$

$q_{de} = 7.087 - 1.55 (-0.625 \times 0.1) = 7.184$

$q_{ed} = 7.184$

$q_{ef} = 7.184 - 1.55 \times 4.375 \times 0.1 = 6.504$

$q_{fe} = 6.504$

$q_{fg} = 6.504 - 1.55 \times 9.375 \times 0.2 = 3.589$

$q_{gf} = 3.589$

$q_{gh} = 3.589 - 1.55 \times 9.375 \times 0.2 = 0.674$

Fig. A14.33 shows the plotted shear flow results. The signs of the calculated shear flows are for shear flows in the $y$ direction. Simple consideration of a free body of flange member (a) will give the sign or sense of the shear flow in the plane of the beam section. Thus in Fig. A14.34 $q_y$ must act as shown when $q_y$ is positive.

![Diagram](image)

**Fig. A14.33**

**Fig. A14.34**

**Fig. A14.35**
Checking to see if \( \Sigma F_x = 0 \) and \( \Sigma F_y = 0 \):

\[
\begin{align*}
\Sigma F_x & = 3.593 \times 10 - 3.539 \times 10 + 0.5 \times 0.971 - 0.5 \\
& \times 0.971 + 5 \times 0.674 - 0.5 \times 0.674 + 0.5 \times 0.971 - 0.5 \\
& \times 0.674 - 0.5 \times 6.504 - 0.5 \times 7.087 = 0 \text{ (check)}
\end{align*}
\]

\[
\Sigma F_y = 100 + 5 \times 0.971 + 5 \times 0.674 - 5 \times (7.087 + \\
7.184 + 6.504) = 0 \text{ (check)}
\]

The resultant \( R \) of the internal shear flow system is a horizontal force of 100 lb. acting toward the left. To find the location of the resultant take moments about a point 0.5 inch below point (a).

\[
Re = \Sigma M
\]

\[
100 \times 5 (0.971 + 0.674) 11 + 10 \times 3.569 \times 15 + \\
0.5 \times 0.971 \times 5 - (0.5 \times 0.674 \times 10) + 0.5 \times 0.674 \times 15 + \\
0.5 \times 6.504 \times 15 + 0.5 \times 0.674 \times 10 - (0.5 \times \\
0.971 \times 5)
\]

\[
100 \times e = 643
\]

\[
e = \frac{643}{100} = 6.43 \text{ inches}
\]

Fig. A14.35 shows the resulting shear center location for the given beam section.

EXAMPLE PROBLEM 2. Unsymmetrical Beam Section.

Fig. A14.36 shows a four flange beam section. The areas of each flange are shown adjacent to flange. The external shear load \( V \) equals 141.14 lb. and acts in a direction as shown. The problem is to find the line of action of \( V \) so that section will bend without twisting.

\[
\text{Load} = 141.14
\]

\[
\begin{align*}
\Sigma F_x & = 141.14 \sin 45^\circ = 100 \text{ lb.} \\
\Sigma F_y & = 141.14 \cos 45^\circ = 100 \text{ lb.}
\end{align*}
\]

From equation (14)

\[
q_y = - (k_e V_x - k_i V_z) \Sigma z A - (k_e V_x - k_i V_z) \Sigma z A
\]

Substituting

\[
q_y = - [-0.006037 \times 100] - [(-0.00142 \times 100)] \Sigma z A
\]

whence

\[
q_y = -0.7457 \Sigma z A - 1.278 \Sigma z A
\]

We will start at flange member (a) where \( q_y \) is zero on the free edge side of the member.

\[
q_{ab} = -0.7457 \times 1 \times (-5.333) - 1.278 \times 1 \times 6.667 = -4.544
\]

\[
q_{ba} = q_{ab} = -4.544
\]

\[
q_{bc} = -4.544 - 0.7457 \times 1 \times (-5.333) - 1.278 \times 1 \times (-5.333) = 6.249 \text{ lb./in.}
\]

\[
q_{cb} = q_{bc} = 6.249
\]
\[
\begin{align*}
q_{cd} &= 6.249 - 0.7457 \times 0.5 \times 10.667 - 1.278 \times 0.5 = 5.880 \\
q_{dc} &= q_{cd} = 5.880 \\
q_{da} &= 5.880 - 0.7457 \times 0.5 \times 10.667 - 1.278 \times 0.5 \\
&\quad \times 2.567 = 5.880 - 3.977 - 1.704 = 0 \\
&\quad \text{(checks free edge at d where } q_y \text{ must be zero.)}
\end{align*}
\]

Fig. A14.37 shows the resulting shear flow resisting pattern. The sense of the shear flow in

\[
\begin{align*}
\alpha &= 4.544 \\
V &= 141.14 \\
R &= 141.14 \\
q &= 5.880
\end{align*}
\]

Fig. A14.37

the plane of the cross-section is determined in web at flange member (a) by the simple free body diagram of stringer (a) in Fig. A14.37a. Check \( \Sigma F_x \) and \( \Sigma F_z \) to see if each equals 100.

\[
\begin{align*}
\Sigma F_x &= -6.249 \times 16 = -99.99 \quad \text{(checks } V_x = 100) \\
\Sigma F_z &= -12 \times 4.544 - 8 \times 5.88 = -99.94 \quad \text{(checks } V_z = 100).
\end{align*}
\]

The resultant of the internal resisting shear flow system equals \( \sqrt{100^2 + 100^2} = 141.14 \) lb.

To locate this resultant we use the principle of moments. Taking point (b) as a moment center,

\[
141.14 \times e = 8 \times 5.88 \times 16 - 4.544 \times 36 \pi
\]

hence \( e = \frac{212}{141.14} = 1.50 \) inch.

Therefore external load must act at a distance \( e = 1.50" \) from (b) as shown in Fig. A14.37. The load so located will pass thru shear center of section. To obtain the shear center location, another loading on the beam can be assumed, and where the line of action of the resultant of the resisting shear flow system intersects the resultant as found above would locate the shear center as a single point. If the shear center location is desired it is convenient to assume a unit \( V_x \) and \( V_z \) acting separately and find the horizontal and vertical locations of the shear center from the 2 separate shear flow force systems.

A14.11 Shear Center Location By Using Neutral Axis Method.

In a beam subjected to bending there is a definite neutral axis position for each different external plane of loading on the beam. The shear flow equation with respect to the neutral axis is,

\[
q_y = -\frac{V_x}{I_x} \sum Z \Delta A \quad \text{(20)}
\]

where, \( V_x = \) Shear resolved normal to neutral axis
\( I_x = \) Moment of inertia about neutral axis
\( Z = \) Distance to neutral axis

In finding the shear center location of an unsymmetrical section, it is convenient to assume that the \( Z \) and \( X \) axes are neutral axis and find the shear flow system for bending about each axis by equation (20). The resultant of each of these shear flow force systems will pass through the shear center, thus the intersection of these two resultant forces will locate the shear center.

Example Problem

The same beam section as used in the previous article (see Fig. A14.26) will be used to illustrate the neutral axis method.

Fig. A14.38 shows the section with the centroidal axis drawn in. The \( X \) axis will now

\[
\begin{align*}
&\frac{5.33}{5.33} = \frac{100}{100} - 6.667 \\
&\frac{7.35}{1.47} = \frac{1.47}{1.47}
\end{align*}
\]

Fig. A14.38
Fig. A14.39

be considered as the neutral axis for an external plane of loading as yet unknown. We will further assume that when this unknown external loading is resolved normal to the \( X \) neutral axis, that it will give a value of 100 lb., or \( V_x = 100. \) From the previous article \( I_x = 90.667. \)

Since the \( X \) axis has been assumed as the neutral axis, equation (20) can be written

\[
q_y = -\frac{V_x}{I_x} \sum Z \Delta A, \quad \text{hence,}
\]

\[
q_{ab} = -\frac{100}{90.667} \times 6.667 \times 1.0 = -7.35 \text{ lb./in.}
\]

\[
q_{bc} = -7.35 - \frac{100}{90.667} \times (-5.333)1 = -1.47
\]

\[
q_{cd} = -1.47 - \frac{100}{90.667} (-5.333)0.5 = 1.47
\]

Fig. A14.39 shows the resulting shear flow values.
BENDING SHEAR STRESSES. SOUND AND OPEN SECTIONS. SHEAR CENTER.

\[ \Sigma F_x = 1.47 \times 16 = 23.52 \text{ lb.} \]
\[ \Sigma F_z = 7.35 \times 12 + 8 \times 1.47 = 99.96 \text{ lb. (check)} \]
\[ R = \sqrt{100^2 + 23.52^2} = 103 \text{ lb.} \]
\[ \tan \theta = 23.52/100 = .2352 \]
\[ \text{hence } \theta = 13^\circ - 16' \]

Let \( e \) = distance from resultant \( R \) to point \( b \).

Equating moments of resultant about \( b \) to that of shear flow system about \( b \),

\[ 103e^2 = -7.35 \times 6^2 + 1.47 \times 8 \times 16 \]
\[ e = \frac{54.4}{103} = -6.25 \text{ in.} \]

Fig. A14.40 shows the location of the resultant. We know the shear center lies on the line of action of this resultant. Thus we must obtain another resultant force which passes through the shear center before we can definitely locate the shear center. Therefore we will now assume that the \( Z \) centroidal axis is a neutral axis and that a resolution of the external load system gives a shear \( V_x = 100 \text{ lb.} \)

\[ Q_y = \frac{V_x}{I_z} \Sigma XA , \quad I_z = 170.667 \]
\[ Q_{ab} = -\frac{100}{170.667} \times (-5.333)1 = 3.125 \text{ lb./in.} \]
\[ Q_{bc} = 3.125 - \frac{100}{170.667} \times (-5.333)1 = 6.25 \text{ lb./in.} \]
\[ Q_{cd} = 6.25 - \frac{100}{170.667} (0.5)(10.667) = 3.13 \]

Fig. A14.41 shows the shear flow results.

\[ \Sigma F_x = -6.25 \times 16 = -100 \text{ lb.} \]
\[ \Sigma F_z = -8 \times 3.13 + 12 \times 3.125 = 12.5 \text{ lb.} \]
\[ R = \sqrt{100^2 + 12.5^2} = 100.8 \]
\[ \tan \theta = \frac{12.5}{100} = .125 \]
some flange members develop entire bending stress resistance.

(5) Determine the shear center for the beam section of Fig. A14.49. Assume only the 8 stringers as being effective in bending. Area of stringers (a) and (b) = 2 sq. in. each. All other stringers 1 sq. in. each.

(6) Determine the shear center for the unsymmetrical beam section of Fig. A14.50. Assume sheet connecting the four stringers as ineffective. Areas of stringers shown on Fig.

(7) In Fig. A14.51, the shell structure is subjected to a torsional moment M = 50,000 in. lb. The shell skin shown dashed is cut out, thus the torsional moment is resisted by the constant shear flow on the two curved sheet elements ac and bd. Determine the value of the shear flow.

(8) Determine the moment of the constant flow force system in Fig. A14.52 about point (O). Also find the resultant of this force system.

(9) In Fig. A14.53, the four stringers a, b, c and d have the same area. Assume the webs ineffective in resisting bending stresses. Determine the distance (a) to product bending about the horizontal axis without twist.

(10) For the wing cell beam section in Fig. A14.54, determine the location of the shear center. Assume webs and walls ineffective in bending.
STRUCTURAL TESTING IS AN IMPORTANT PHASE OF STRUCTURAL DESIGN.
A15.1 Introduction. The wing, fuselage and empennage structure of modern aircraft is essentially a single or multiple cellular beam with thin webs and walls. The design of such structures involves the consideration of the distribution of the internal resisting shear stresses. This chapter introduces the student to the general problems of shear flow distribution. Chapter A14 should be covered before taking up this chapter.

A15.2 Single Cell Beam. Symmetrical About One Axis. All Material Effective in Resisting Bending Stresses.

Fig. A15.1 shows a single cell rectangular beam carrying the load of 100 lb. as shown. The problem is to find the internal resisting shear flow pattern at section abcd.

![Fig. A15-1](image)

Solution 1

Due to symmetry of material the X centroidal axis lies at the mid-height of the beam. The shear flow equation requires the value of \( I_x \), the moment of inertia of the section about the X axis.

\[
I_x = \frac{1}{12} x 15 x 10^5 + 2 \left[ 20 x 0.05 x 5 \right] = 62.5 \text{ in}^4
\]

From Chapter A14, the equation for shear flow is,

\[
q_y = -\frac{V_x}{I_x} z \Delta A
\]

(1)

This equation gives the change in shear flow between the limits of the summation. In open sections we could start the summation at a free surface where \( q \) would be zero, thus the summation to any other point would give the true shear flow \( q_y \). In a closed cell there is no free end, therefore the value of \( q_y \) is unknown for any point.

Equation (1) gives the shear distribution for bending about the X axis without twist. The general procedure is to assume a value of the shear flow \( q_y \) at some point and then find the shear flow pattern for bending without twist under the given external load. The centroid of this internal shear flow system will be the location where the external shear load should act for bending without twist. Since the given external shear would have a moment about this centroid, this unbalanced moment must be made zero by adding a constant shear flow system to the cell.

To illustrate we will assume \( q_y \) to be zero at point 0 on the web ad.

\[
q_0 = 0. \text{ The term } \frac{V_x}{I_x} = \frac{100}{62.5} = 1.6
\]

\[
q_{0a} = -1.6 z_0^a \Delta A = -1.6 x 2.5 x 5 x 0.1 = -2 \text{ lb/in.}
\]

\[
q_{ba} = -2 - 1.6 z_0^b \Delta A = -2 - 1.6 x 5 x 20 x 0.05 = -10
\]

\[
q_{0'b} = -10 - 1.6 z_0^{0'} \Delta A = -10 - 1.6 x 2.5 x 5 x 0.05 = -11
\]

\[
q_{0co} = -11 - 1.6 z_0^{c} \Delta A = -11 - 1.6 x (-2.5) x 5 x 0.05 = -10
\]

\[
q_{0d} = -10 - 1.6 z_0^{d} \Delta A = -2 - 1.6 (-2.5) x 50 x 0.1 = 0
\]

Fig. A15.2 shows a plot of the shear flow results. On the vertical web the increase in shear is parabolic since the area varies directly with distance z.

The intensity of \( q_x \) and \( q_z \) in the plane of the cross-section is equal to the values of \( q \) found above which are in the y direction. The sense of \( q_x \) and \( q_z \) is determined as explained in detail in Art. A14.6 of Chapter A14.
A15.2 SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

Fig. A15-2 also shows the resultant shear flow force on each of the four walls of the cell. The resultant shear force on each equals the area of the shear flow diagram on each position.

For example,

\[ \text{Qab} = \frac{2 + 10}{2} \times 20 = 120 \text{ lb.} \]
\[ \text{Qda} = \frac{1}{3} \times 2 \times 10 = 6.67 \text{ lb.} \]
\[ \text{Qbc} = 10 \times 10 + 0.667 \times 1 \times 10 = 106.7 \]

The internal shear flow force system as given in Fig. A15.2 will now be checked for equilibrium with given external shear loading of 100 lb. as shown in Fig. A15.1.

\[ \Sigma F_x = 100 \text{ (external)} + 6.67 - 106.67 = 0 \text{ (check)} \]
\[ \Sigma F_y = 120 - 120 = 0 \text{ (check)} \]

Equilibrium of moments must also be satisfied. Take moments of all forces about point d. The external load has no moment about d.

\[ \Sigma M_d = 120 \times 10 + 106.67 \times 20 = 3344 \text{ in. lb. clockwise} \]

Thus we have an unbalanced moment which must be made zero if we are to have equilibrium.

The unbalanced shear flow of 3344 in. lb. can be balanced by adding a constant shear in a counterclockwise direction around the cell. The value of this balancing shear flow would equal,

\[ q = -\frac{M}{2A} = -\frac{-3344}{2 \times 200} = -8.34 \text{ lb/in.} \]
(A equals area of cell = 10 \times 20)

Adding this constant shear flow to that of Fig. A15.2 we obtain the final shear flow pattern of A15.3.

Fig. A15-3

If this constant shear flow of -8.34 was not added then the external load of 100 lb. would have to be re-located so that it passed through the centroid of the shear flow pattern in A15.2. To find this centroid location, we can equate the moment of the internal shear flow pattern about some point to the moment of the resultant of the system about the same point.

\[ \text{Resultant } R = \sqrt{2F_x^2 + 2F_y^2} = \sqrt{100^2 + 0} = 100 \text{ acting down.} \]

Take moments about point d. Let \( \bar{x} \) equal distance to resultant.

\[ R\bar{x} = \Sigma M_d \text{ (internal system)} \]
\[ 100 \bar{x} = 120 \times 10 + 106.67 \times 20 \]

hence \( \bar{x} = 33.33 \text{ in.} \) Thus the external load would have to be moved 33.33 inches to right if the pattern of A15.2 would hold it in equilibrium. Since we assume q = 0 at point 0, this means an open cell with the free end at 0 would bend without twist if the external load was moved the distance \( \bar{x} \).

The student should work this same problem by assuming \( q_x \) at some other point is zero instead of point 0 as assumed in the above solution.

Solution No. 2. Shear Center Method.

In this solution, we determine the centroid of the internal shear flow system for bending of the closed section about axis X without twist. This point is called the shear center. The external shear load can then be resolved into a shear force acting through the shear center plus a torsional moment about the shear center.

We start the solution, exactly as in solution 1, by assuming the shear flow \( q = 0 \) at point 0. In a sense we are cutting the cell at 0 and making it an open section. The resulting shear flow is as given in A15.2. This open
section will bend without twist if the external shear load acts through the shear center of the open section.

The closed section will be assumed to bend without twist, and the resulting shear flow pattern will be determined.

The equation for angular twist \( \theta \) (See Chapter A6) per inch length of beam is,

\[
\theta = \frac{1}{2AG} \sum \frac{qL}{t}
\]

where \( L \) equals the length of a web or wall.

or \( 2AG = \frac{1}{\theta} \cdot \frac{qL}{t} \). The right hand side of this equation represents the total shearing strain around the cell which must be zero for no twist of cell. Since \( G \) is constant, we can assume it as unity as only relative values of strain are needed in the solution. Thus the total shearing strain \( \delta \) around cell is proportional to,

\[
\delta = \sum \frac{qL}{t} = \frac{2 x 5}{3 x .1} + \frac{(10 + 2) 20}{2} \cdot 05 + \frac{10 \times 5}{2} \cdot 05
\]

or \( \delta = \frac{11 - 10 + 667 x 5}{2} \cdot 05 = 7000 \)

Since the section and shear flow pattern is symmetrical about the X axis, the substitution above is written for one-half of the cell and the results multiplied by 2. When the shear flow \( q \) varies over a portion the average shear flow is used in the above substitution.

If the cell is not to twist the relative twist of 7000 must be cancelled by adding a constant shear flow \( q \) around the cell to give a total shear strain of 7000.

The shearing strain for a constant \( q \) equals,

\[
\delta = 2 \frac{qL}{t} = q \left[ \frac{5}{0.1} + \frac{20}{0.05} + \frac{5}{0.05} \right] 2 = -7000
\]

hence \( q = \frac{7000}{11.05} = -6.36 \text{ lb/in.} \)

Fig. A15.4 shows the resultant shear flow pattern if the constant shear flow of -6.36 is added to the shear flow pattern of A15.2.

The location of the resultant of the shear flow force system of Fig. A15.4 will locate the horizontal position of the shear center. Due to symmetry of the section about the X axis, the vertical position of the shear center will be on the X axis, because for bending about the Z axis, the shear flow would be symmetrical and thus, the resultant would coincide with the X axis.

Fig. A15.4 also shows the resultant shear load \( Q \) on each portion of the cell wall which equals the area of the shear flow diagram for various portions as shown.

\( ZF_x = -56.9 - 43.1 = -100 \), which balances the external load of 100 lb.

\( ZF_y = 0 \) by observation. Take moments about point \( O \), the intersection of axis XX and side ad.

\[ \frac{Z}{F_x} = \frac{43.1 x 20 + (23.8 + 16.58)10}{100} = \]

7.90 in.

Hence the shear center lies on the X axis, 7.9" from side ad.

The moment of the external load of 100 lb. about shear center equals 100 x 7.9 = 790 in. lb. clockwise. The moment of the internal shear flow of Fig. A15.4 is zero thus we have an unbalanced moment of 790. Therefore for equilibrium of moments we must add a constant shear flow \( q \) around cell to develop -790 in.lb. or

\[ q = \frac{M}{2A} = -\frac{790}{2 \times 200} = -1.98 \text{ lb/in.} \]

Adding this constant shear flow to that of Fig. A15.4, we obtain the final shear flow pattern which will be identical to that in Fig. A15.3 or the results of solution 1.

A15.3 Single Cell - 2 Flange Beam. Constant Shear Flow Webs.

As discussed in Art. A14.9, the common aircraft cellular beam is made up of thin sheet
walls and webs which are stiffened by members usually referred to as flange members. A simplifying common assumption is to assume that the flange members alone develop the resistance to the bending moment. This assumption therefore means that the shear flow is constant between flange members.

Fig. A15.5 shows a single cell beam with two flanges at A and B. Find the internal shear flow force system when the beam carries the external load of 100 lb, as shown.

![Diagram of a single cell beam with flanges and loads](image)

It is assumed that the two flanges develop the entire bending stress resistance. This means that shear flow is constant on each web. Let \( q_1 \) and \( q_2 \) be the constant shear flow as shown in Fig. A15.6. The sense or direction of these shear flows will be assumed as indicated by the arrow heads in the figure.

The force system in the plane of the cross-section has 2 unknowns, namely \( q_1 \) and \( q_2 \), and thus we can solve for \( q_1 \) and \( q_2 \) by simple statics.

To find \( q_1 \) take moments about point A.

\[
M_A = 100 \times 10 - q_1 \times (2 \times 80.52) = 0
\]

hence \( q_1 = 1000/2.81 = 6.21 \text{ lb/in.} \)

The sign of \( q_1 \) comes out plus, thus the assumed sense of \( q_1 \) as shown in Fig. A15.6 is correct.

In the above equation of moments about point A, the moment of the constant flow system \( q_1 \) about A, equals \( q_1 \) times double the enclosed area formed by lines running from A to the ends of the web which carries the shear flow \( q_1 \). In this case the area is the area of the cell, or 80.52 sq. in.

To find the remaining unknown \( q_2 \) we use the equilibrium equation,

\[
\Sigma F_y = 0 = -100 - 10 \times 6.21 + 10q_2 = 0
\]

hence \( q_2 = 16.21 \text{ lb/in.} \) (sign comes out positive, hence assumed sense of \( q_2 \) is correct).

Fig. A15.7 shows a plot of the internal shear flow resisting system.

A single cell beam having only two flanges can resist only external loads which are parallel to the line AB, and thus a two flange box beam is not used very often in aircraft structure. When the bending moment in a plane at right angles to line AB is small, the resistance of the curved panel to compressive bending stresses may be sufficient to resist such external bending moments and thus be satisfactory.

A15.4 Shear Center of Single Cell - Two Flange Beam.

Let it be required to find the shear center of the beam as given in Fig. A15.8. In other words where would the external load have to be placed so that the beam would bend without twist.

![Diagram of a single cell beam with shear center](image)

Fig. A15.8 shows the resulting shear flow system in resisting the 100 lb. external load acting as shown in Fig. A15.5. This shear flow system will cause the cell to twist. Therefore we add a constant shear flow \( q \) to the cell to produce zero twist (Fig. A15.8). The centroid of the combined shear flow system will then locate the lateral location of the shear center.

To find \( q \) we must write an expression which measures the twist when subjected to the shear flows of Figs. A15.8 and 9 and equate the result to zero, and then solve for the one unknown \( q \).

\[
\delta = \frac{3qL^2}{t} = 0. \text{ (Clockwise } q \text{ is positive)}
\]

Substituting, using the shear flows of Figs. A15.8 and A15.9,
\[ 5 = \frac{16.21 \times 10}{.04} - \frac{6.21 \times 24.28 + 10q}{.025} + 24.28q \frac{24.28q}{.025} = 0 \]

hence \( q = 8.26 \) lb/in.

Adding this constant shear flow to that of Fig. A15.8, we obtain the shear flow of Fig. A15.10.

The lateral position of the shear center is given by the location of the resultant of the shear flow system of Fig. A15.10.

The resultant \( R = \sqrt{\frac{ZF_x}{ZF_z} + \frac{ZF_z}{ZF_x}} \)

\( ZF_x = 0 \)

\( ZF_z = 10 \times 7.35 + 2.05 + 10 = 100 \) lb.

Therefore \( R = 100 \) lb.

Equate moments of \( R \) about point \( A \) to moment of shear flow system about \( A \).

\[ Re = ZM_A \]

\[ 100e = 2.05 \times (2 \times 80.52) \]

\[ e = 3.30 \text{ in.} \]

Thus shear center lies 3.30 inches to left of line \( AB \). (See Fig. A15.10).

Thus if the given external load of 100 lb. acts through the shear center, it will produce the shear flow system of Fig. A15.10. However it acts 13.3 to right of shear center, hence it produces a clockwise moment of 100 x 13.3 = 1330 in.lb. on the cell. For equilibrium, this moment must be balanced by a constant resisting shear flow around cell which will produce a moment of -1330.

The required \( q = \frac{-1330}{2 \times 80.52} = -8.26 \text{ lb/in.} \)

which if added to the shear flow system of A15.10 will give the true shear flow system of A15.8.

Thus having the shear center location, the external load system can be broken down into a load through the shear center plus a moment about the shear center. The shear flow due to each is then added to give the true resisting shear flow.

It should be noticed that the web or skin thickness does not influence the magnitude of the shear flow system in a single cell beam. A change in thickness, however, effects the unit shearing stress and therefore the shearing strain and thus in computing angular twist of the cell, the web and wall thickness does influence the amount of twist for a given torsional load. In the shear center solution, it is known what portion of the shear flow is due to torque or pure twist, and also that due to bending without twist, which fact is sometimes of importance.

**TORSIONAL DEFLECTION OF CELL**

The angular twist as given by the final shear flow pattern of Fig. A15.8 equals

\[ 2\theta_{AG} = \frac{ZqL}{t}, \text{ whence} \]

\[ 2\theta_{AG} = \frac{-5.21 \times 24.28 - 16.21 \times 10}{.025 \times .04} = -10082 \]  \( \text{-- (3)} \)

After finding the shear center location, we found that the external load had a moment of 1330 in. lb. about shear center, which was resisted by a constant shear flow of - 8.26 lb/in. The angular twist under this pure torque shear flow should therefore give the same result as equation (3) above.

\[ 2\theta_{AG} = \frac{-8.26 \times 24.28 - 8.26 \times 10}{.025 \times .04} = -10082 \]

which checks the result of equation (3).

**A15.5 Single Cell-Three Flange Beam. Constant Shear Flow Webs.**

Fig. A15.12 shows a single cell beam with three flange members, A, B, and C, carrying the external load as shown. A three flange box if the flanges are not located in a straight line can take bending in any direction and therefore is often used in design because of its simplicity.

For such a structure, there are six unknowns, namely, the axial load in each stringer and the shear flow \( q \) in each of the three sheet panels that make up the cell. For a space structure, we have six static equations of equilibrium, thus a three flange single cell
beam can be solved by statics if we assume that the three flange members develop all the bending stress resistance, thus producing constant shear flow webs.

Fig. A15.12 shows the cross section ABC. The three unknown resisting shear flows have been assumed with a positive sign. (Clockwise flow is positive shear flow). These three unknown shear flows can be determined by statics.

To find \( q_{ca} \) take moments about point B and equate to zero.
\[
ZM_B = 100 \times 5 - 25 \times 10 + q_{ca} (128.54 \times 2) = 0
\]

hence \( q_{ca} = - \frac{250}{257.08} = -0.972 \) lb/in.

To find \( q_{ab} \) take \( EF_Z = 0 \)
\[
ZF_Z = 100 - 10 \times 0.972 - 10q_{ab} = 0
\]

\( q_{ab} = 9.13 \) lb/in.

To find \( q_{bc} \) take \( EF_X = 0 \)
\[
ZF_X = 15 \times 0.972 - 25 - 15q_{bc} = 0
\]

hence \( q_{bc} = -2.639 \) lb/in.

The signs of \( q_{ca} \) and \( q_{bc} \) came out negative, hence the sense of the shear flow on these cell wall portions is opposite to that assumed in Fig. A15.12. The resulting shear flow pattern is plotted in Fig. A15.13.

The student should realize the thickness of the wall elements does not influence the shear flow distribution if we assume the three flanges develop the entire resistance to the bending moment.

A15.6 Shear Center of Single Cell-Three Flange Beam. Constant Shear Flow Webs.

Let it be required to determine the shear center location for the beam in Fig. A15.11. The shear center is a point on the beam cross-section through which the resultant external shear must act if the cell is to bend without twist.

The shear center location will be determined in two steps, first its horizontal location and then its vertical location.

Calculation of horizontal location:-

We will assume any vertical shear load, as the example, the same vertical shear as used in the problem of Art. A15.4, namely, a 100 lb. load acting five inches from A, as illustrated in the following Fig. A15.14.

The three unknown resisting shear flows will be assumed with the sense as indicated by the arrow heads.

To find \( q_{ac} \) take moments about B
\[
ZM_B = 100 \times 5 - q_{ac} (128.54 \times 2) = 0
\]

\( q_{ac} = 1.945 \) lb/in.

\[
ZF_X = 15 \times 1.945 + 15q_{bc} = 0
\]

\( q_{bc} = 1.945 \)

\[
ZF_Z = 100 - 10 \times 1.945 - 10q_{ab} = 0
\]

\( q_{ab} = 8.055 \) lb/in.

The algebraic signs of the unknown \( q \) value all come out positive, thus the assumed direction of shear flows in Fig. A15.14 is correct.

To make the cell twist zero, we must add a constant shear flow \( q \) to the cell (see Fig. A15.15). The relative twist under the shear flow of Figs. 14 and 15 will be equated to zero.

\[
\frac{128.54 \times 20.71}{0.03} - 1.945 \times 15 \times 0.025 \times 0.04
\]

\[
+ \frac{20.71q}{0.03} + 10q + \frac{15q}{0.025} = 0
\]

Whence, \( q = 0.322 \) lb/in. with sense as assumed in Fig. A15.15. Adding this constant shear flow to that of Fig. A15.14 we obtain the shear flow system of Fig. A15.16. The resultant R of this shear flow system is obviously - 100 lb., since the external load was 100 lb. The location of this resultant R will therefore locate the horizontal position of the shear center. Equate moment of resultant R about point B to the moment of the shear flow system about B, whence,

\[
100e = 1.623 (128.54 \times 2)
\]

or \( e = 41.7/100 = 4.17 \) in. from line AB. (Fig. A15.16)

Calculation of Vertical Position of Shear Center

A convenient horizontal shear load will be
assumed acting on the cell. Since we used a 25 lb. load in the example problem of Art. 15.5, we will assume the same load in this solution. Fig. A15.17 shows the loading and the assumed directions of the three unknown shear flows.

\[
\begin{align*}
\Sigma M_B &= -25 \times 10 + q_{ca} (128.54 \times 2) = 0 \\
q_{ca} &= 0.972 \text{ lb/in.} \\
\Sigma F_X &= -25 + 0.972 x 15 + 15q_{cb} = 0 \\
q_{cb} &= 0.695 \\
\Sigma F_Z &= 10 \times 0.972 - 10q_{ab} = 0 \\
q_{ab} &= 0.972
\end{align*}
\]

A constant shear flow \( q \) is now added to cell to make twist zero (Fig. A15.18).

Writing \( ZqL/t \) for both loadings and equating to zero:

\[
ZqL/t = \frac{0.972 \times 20.71 - 0.695 \times 15 + 0.972 \times 10}{.03} \times \frac{.023}{.04} + \frac{20.71q_{ca} + 10q_{cb} + 15q_{ab}}{.03} = 0
\]

whence, \( q = -0.324 \text{ lb/in.} \)

Adding this constant shear flow to that of Fig. A15.17 we obtain the values in Fig. A15.19.

\( R \) (the resultant) = 25 lb.

Equating moment of resultant about B to moment of shear flow system about B,

\[
Re = ZM_B
\]

\[
25e = 0.648 (128.54 \times 2)
\]

Therefore \( e \) = 6.65 inches.

Thus shear center lies 6.65 inches above B, and 4.17 inches to left of B as previously found.

---

A15.7 Single Cell-Multiple Flange-One Axis of Symmetry.

Fig. A15.20 shows a single cell beam with 8 flange members, carrying a 100 lb. shear load. The resisting shear flow system will be calculated.

The beam section which is symmetrical about the X axis is identical to the beam section relative to flange material which was used in example problem 1 of Art. A14.10.

**SOLUTION:**

Assuming the 8 flanges develop all the bending stress resistance, the shear flow will therefore be constant between flanges. Since the beam section is a closed one the value of the shear flow \( q \) at any point is unknown. Thus we will imagine the top cover cut between flange members a and h, thus making \( q \) zero in this panel due to the free end at the cut. We now find the internal resisting shear flow system for bending of this open section about axis \( x-x \) under an external shear load \( V_x = 100 \text{ lb.} \)

The calculations would be exactly like those in example problem 1 of Art. A14.10 and will not be repeated here. Fig. A15.21 shows the plotted results as recopied from Fig. A14.22.
If $\Sigma F_x$ and $\Sigma F_z$ are considered for equilibrium of external and internal loads, they will be found to equal zero.

To check $\Sigma M_y = 0$, take moments about some point such as C.

$$\Sigma M_C = -100 \times 7.5 + 10 \times 3.75 \times 15 - 1.25 \times 5$$
$$-1.25 \times 10 + 1.25 \times 15 = -187.5 \text{ in} \cdot \text{lb.}$$

Therefore to make $\Sigma M_y = 0$, a constant shear flow equal to $M/2A = (187.5/2 \times 11 \times 15)$ $\approx 0.57 \text{ lb./in.}$ is required. Adding this constant shear flow to that in Fig. A15.21, we obtain the final shear flow pattern of Fig. A15.22. This final pattern is not much different from that of Fig. A15.21, the reason being that the location of the imaginary cut to make $q$ equal zero, was not far from the true fact, since the final $q$ in this panel was only 0.57. If we had started the solution by assuming the web $bc$ cut or $Q_{bc} = 0$, then the correction constant flow that would be needed to satisfy $\Sigma M_y = 0$ would have come out $q = -5.88$, since this is the final $q$ in web $bc$. Since the shear flow which is a load on the cell wall influences the required thickness of sheet required, it is good practice to try to place the imaginary cut at a point where the shear is near zero, so that preliminary estimates in routine design relative to shell thickness required will be based on shear flow values that are near the final values.

Solution 1. Using Section Properties and External Shears with Reference to Centroidal Axes (The K Method).

In Art. A13.8 the calculations for this beam section gave:

$$I_x = 81.18, \quad I_z = 153.58, \quad I_{xz} = -21.33$$

Fig. A15.23a shows the location of $x$ and $z$ centroidal axes.

$$q_y = -(k_x V_x - k_y V_y) \Sigma A = -(k_x V_x - k_y V_y) \Sigma A$$

$$k_x = I_{xz}/I_x I_z - I_{xx}^* = \frac{-21.33}{81.18 \times 153.58 - 21.33^2} = -0.001775$$

$$k_y = I_z/I_x I_z - I_{yy}^* = 153.58/12016 = 0.01279$$

$$k_x = I_x/I_x I_z - I_{xx}^* = 81.18/12016 = 0.00674$$

For the given beam loading the external shear loads at section abcd are:

$$V_y = 6000 \text{ lb., } V_x = -1600 \text{ lb.}$$
Substituting in the equation for \( q_y \) as given above,

\[
q_y = \frac{-0.0674 \cdot (-1600) - (-0.0177 \cdot 6000)}{0.1279 \cdot 6000 - (-0.00177) \cdot (-1600)} Z\Sigma A
\]

whence

\[
q_y = 0.16 \cdot Z\Sigma A - 73.91 \cdot Z\Sigma A
\]  \( - - - - - - (4) \)

In using equation (4) to compute the shear flow pattern we will imagine top panel ab cut, thus making the shear flow \( q_y = 0 \) in this panel. Subst. in (4) \(- - - - - -

\[
q_{ac} = 0 + 0.16 \cdot 1 \cdot (-5.333) - 73.91 \cdot 1 \cdot 1 \cdot 6.074 = -449.78 \text{ lb/in.}
\]

\[
q_{cd} = -449.78 + 0.16 \cdot 0.8 \cdot (-5.333) - 73.91 \cdot 0.8 \cdot (-5.926) = 100.05
\]

\[
q_{db} = -100.05 + 0.16 \cdot 0.4 \cdot 10.667 - 73.91 \cdot 0.4 \cdot (-5.926) = 75.80
\]

\[
q_{bd} = 75.80 + 0.16 \cdot 0.5 \cdot 10.667 - 73.91 \cdot 0.5 \cdot 2.074 = 0
\]

Fig. A15.24 shows the plotted shear flow results. This pattern satisfies \( \Sigma F_x = 0 \) and \( \Sigma F_y = 0 \). To check equilibrium of moments

\[
6000 = V_x
\]

Fig. A15-24

about a y axis. Assume a y axis going through point d.

\[
X_d = 6000 \cdot 8 - 1600 \cdot 8 - 449.78 \cdot 12 \cdot 0.16 = 51160 \text{ in.lb.}
\]

Thus for equilibrium a moment of plus 51160 in.lb. is required. This is produced by adding a constant shear flow q around the cell walls, where

\[
q = \frac{M}{Z\Sigma A} = \frac{51160}{2 \cdot 160} = 159.88 \text{ lb/in.}
\]

(A = area of cell = 160)

Adding this volume of q to those in Fig. A15.24 we obtain the final shear flow resisting pattern in Fig. A15.25.

Solution 2. Principal Axes Method

The shear flow system can of course be found by referring section properties and external shear loads to the principal axes of the beam section. The equation for shear flow is (see Eq. 15 of Chapter A14),

\[
q_y = -\frac{\Sigma x P A}{I_{xp}} x z P A - \frac{\Sigma x P A}{I_{zp}} z x P A
\]

(The subscript p refers to principal axes.)

The section properties about the principal axes were computed for this same beam section on page A13.5 of Chapter A13. The values are:

\[
I_{xp} = 75.38, \ I_{zp} = 159.34
\]

Fig. A15.26 which was also taken from page A13.5 shows the location of the principal axes and the distances from the four flange members to the principal axes.

Before substitution in equation (5) can be made, the given shear loads \( V_x = 6000 \), and \( V_y = -1600 \) must be resolved normal to the principal axes.

\[
V_{zp} = 6000 \cos 15^\circ - 15^\circ - 1600 \sin 15^\circ - 15^\circ = 5367.9 \text{ lb.}
\]

\[
V_{xp} = -6000 \sin 15^\circ - 15^\circ - 1600 \cos 15^\circ - 15^\circ = -5121.9 \text{ lb.}
\]

Hence \( \frac{V_{zp}}{I_{zp}} = 71.21 \) and \( \frac{V_{xp}}{I_{xp}} = -19.59 \)

Subst. in equation (5),

\[
q_y = -71.21 \Sigma z P A + 19.59 \Sigma x P A
\]  \(- - - - - - (6)\)
A15.10 SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

Assume \( q_y = 0 \) in top panel ab.

\[
\begin{align*}
q_{ac} &= -71.21 \times 4.45 \times 1 + 19.59 \times (-6.74) \\
&= -448.92 \\
q_{dd} &= -448.92 - 71.21 (-7.12) \times 8 + 19.59 \\
&= -3.58, 8 = -99.41 \\
q_{db} &= -99.41 - 71.21 (-2.90) \times 4 + 19.59 \\
&= 11.80 \times \frac{4}{3} = 75.55 \text{ lb/in.} \\
q_{ba} &= 75.55 - 71.21 (4.82) \times 5 + 19.59 \\
&= 9.75 \times \frac{5}{3} = 0 \text{ (check)}
\end{align*}
\]

These shear flows are practically the same as obtained in solution no. 1 as recorded in Fig. A15.24. Discrepancies are due to slide rule accuracy.

For equilibrium of moments, take moments about (b).

\[
\begin{align*}
2M_b &= 6000 \times 8 - 1600 \times 3 - 448.92 \times 12 \\
&= -50993 \text{ in.lb.}
\end{align*}
\]

A constant shear flow \( q \) around the cell must be added to produce 50993 in.lb. for equilibrium. This balancing shear flow is,

\[
q = \frac{M}{2a} = \frac{-50993}{2 \times 160} = 159.35 \text{ lb/in.}
\]

which is the same as in solution no. 1.

Example Problem 2

Fig. A15.27 illustrates a typical single cell wing beam with multiple flange members. The external shear load on this beam section is \( V_z = 1000 \) and \( V_x = 400 \) located as shown. The internal shear flow resisting pattern will be calculated.

This beam section is the same as that used in example problem 5 of Chapter A3, where the calculations of the section properties were made.

The results were:

\[
\begin{align*}
I_x &= 186.5, \quad I_z = 431.7, \quad I_{zz} = 36.41 \\
V_z &= 1000 \text{ lb} \\
V_x &= 400 \text{ lb} \\
\text{Fig. A15.27}
\end{align*}
\]

Solution. The K method of solution will be used.

\[
\begin{align*}
k_1 &= \frac{I_{xz}}{I_x - I_{zz}} = \frac{36.41}{186.5 \times 431.7 - 36.41^2} \\
&= \frac{36.41}{79185} = 0.0004598 \\
k_x &= -\frac{I_z}{I_{xz}} = \frac{431.7}{79185} = 0.005452 \\
k_2 &= -\frac{I_x}{I_{xz}} = \frac{186.5}{79185} = 0.002355 \\
q_y &= (k_x V_x - k_z V_z) E A = (k_x V_x - k_z V_z) E A \\
\text{Substituting in above equation,} \\
q_y &= (-0.002355 \times 400 - 0.0004598 \times 1000) E A \\
&= (-0.005452 \times 1000 - 0.0004598 \times 400) E A \\
q_y &= -0.4277 E A - 5.268 E A \\
\text{Substituting in equilibrium,} \\
q_y &= 0.4277 E A + 5.268 E A \\
\text{Since the value of the shear flow is unknown at any point on the cell wall, it will be assumed that the cell wall is cut between flange members 1 and 10, thus making q zero on the sheet panel numbered (1-10). Then using equation (7) the shear flow is calculated in going clockwise around the panel. Columns 1 to 10 of Table A15.1 show the calculations in solving equation (7). For explanation on how to determine sense of shear flows \( q_y \) in Columns 9, 10, and 11, review Art. A14.8 of Chapter A14.}
\]

The shear flow values in column 11 would be the results if the external loads as given were so located as to act through the centroid of this shear flow force system. Since they do not we will solve for the unbalanced moment on the beam section about point (0) the centroid of the beam cross-section. The moment of the shear flow force on a sheet panel between any two adjacent flange members is equal in magnitude to \( q \times \) times double the enclosed area formed by drawing lines from the moment center (0) and the ends of the particular sheet panel. Fig. A15.28 illustrates this explanation. The value \( q \) in column 12 of the table lists the double areas of these various triangular areas.

Taking moments of all forces both external and internal about point (0),

\[
\begin{align*}
E M_0 &= 1000 \times 2 + 400 \times 3 + 17123 = 20323 \text{ in.lb.} \\
(17123 \text{ equals summation of column 13)}
\end{align*}
\]

Thus for equilibrium a negative moment of 20323 is needed. This moment is provided by adding a
constant negative shear flow around cell where magnitude equals

\[ q = \frac{M}{2A} = \frac{20323}{2 \times 493} = -20.6 \text{ lb./in.} \]

(493 = area of cell)

Adding this constant shear flow to that in column 11, we obtain the final shear flow in column 14. Fig. A15.29 shows true shear flow pattern.

**Table A15.1**

<table>
<thead>
<tr>
<th>Member</th>
<th>Area A</th>
<th>Arm Z</th>
<th>ZA</th>
<th>Arm X</th>
<th>XA</th>
<th>ZZA</th>
<th>ΣXA</th>
<th>qy = -0.4822 (ΣXA)</th>
<th>qxz = -0.4822 (ΣZXA)</th>
<th>qy = -5.269 (Σcol 9 + col 10)</th>
<th>m</th>
<th>sq. in.</th>
<th>qzzma</th>
<th>Final ( q )</th>
<th>( q_{xz} ) -20.6</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.14</td>
<td>4.396</td>
<td>0.615</td>
<td>17.41</td>
<td>-2.437</td>
<td>0.615</td>
<td>2.437</td>
<td>1.175</td>
<td>-3.240</td>
<td>2.056</td>
<td>55.2</td>
<td>114</td>
<td>146</td>
<td>-18.53</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.14</td>
<td>5.464</td>
<td>0.602</td>
<td>15.34</td>
<td>-1.896</td>
<td>1.517</td>
<td>4.333</td>
<td>2.059</td>
<td>-7.991</td>
<td>5.902</td>
<td>44.2</td>
<td>261</td>
<td>146</td>
<td>-18.53</td>
<td></td>
</tr>
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<td>8.111</td>
<td>-3.453</td>
<td>4.327</td>
<td>7.795</td>
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<td>19.038</td>
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<td>1.308</td>
<td>7.14</td>
<td>1.214</td>
<td>9.548</td>
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<td>10.888</td>
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<td>15.39</td>
<td>5.386</td>
<td>12.888</td>
<td>6.659</td>
<td>0.318</td>
<td>-67.894</td>
<td>67.976</td>
<td>261.3</td>
<td>1702</td>
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**A15.9 Two Cell-Multiple Flange Beam. Symmetrical About One Axis.**

Fig. A15.30 shows a two cell cantilever beam with 10 flange stringers. The cross-section is constant. Let it be required to determine the internal shear flow in resisting the 1000 lb. load acting as shown. For simplification, the top and bottom sheet covering and the three vertical webs will be considered ineffective in taking bending flexural loads. Since the beam section is symmetrical about the X axis, the beam will bend about this axis in resisting the given external load. The moment of inertia of the section about the X axis equals 250 in.².

**Solution 1 (Without use of shear center)**

The internal shear flow is statically indeterminate to the second degree, since the
shear flow at any point in each cell is unknown. Therefore, to make the flexural shear flow statically determinate, a value for the shear flow $q$ in each cell will be assumed at some point, and the flexural shear flow for each cell will then be calculated, consistent with the assumed conditions. These resulting static shear flow systems will, in general, produce a different total shearing strain around the perimeter of each cell, or in other words, produce a different cell twist. Since full continuity exists between cells, this condition cannot exist, and therefore an unknown constant shear flow of $q$ in cell (1) and $q_s$ in cell (2) must be added to make the twist of both cells identical. This fact gives us the basis for one equation and the other equation necessary for the solution of the two unknowns $q$ and $q_s$ is given by the requirement of equilibrium, namely, that the moment of the external and the internal shear forces about any point in the plane of the cross section must equal zero. In Fig. A15.31 the flexural shear flow has been assumed as zero just to the left of stringer c in cell (1) and just right of stringer c in cell (2). The balance of the flexural shear system consistent with this assumption is calculated as follows:

Fig. A15-31

The general shear flow equation is,

$$q_y = \frac{V_2}{Z_y A} = \frac{-1000}{250} = -4Z_y A$$

Cell (1). Starting in panel cb where the shear has been assumed zero and proceeding counter-clockwise around cell

$$q_{cb} = \text{zero (assumed)}$$

$$q_{ba} = -4Z_y A = -0.4 \times 5 \times 0.5 = -10 \text{ lb./in.}$$

$$q_{ad'} = -10 - 4 \times 5 \times 2 = -50 \text{ lb./in.}$$

$$q_{a'd'} = -50 - 4 \times (-5) \times 2 = -10 \text{ lb./in.}$$

$$q_{b'o'} = -10 - 4 \times (-5) \times 0.5 = 0 \text{ lb./in.}$$

We cannot proceed beyond stringer c' because there are two connecting webs with unknown shear flows. We can get around this difficulty by going back to stringer c, where the shear flow on each side of c was assumed zero. Thus the shear flow in the vertical web $c'o'$ is determined by the stringer $c'$ alone, namely

$$q_{oc'} = -4Z_y A = -0.4 \times 1 \times 5 = -20 \text{ lb./in.}$$

We can now continue around cell (2) starting with stringer $c''$ where we were previously stopped.

$$q_{c'd'} = q_{b'o'} + q_{oc'} = -20 \text{ lb./in.}$$

$$q_{d'e'} = 0 - 4 \times (-5) 0.5 = 10 \text{ lb./in.}$$

$$q_{e'a} = 10 - 4 \times (-5) 0.5 = 30 \text{ lb./in.}$$

$$q_{e'd} = 30 - 4 \times 5 \times 1 = 10 \text{ lb./in.}$$

$$q_{d'c'} = 0 - 4 \times 5 \times 0.5 = 0 \text{ lb./in.}$$

The shear flows in cell (2) could of course been found by starting in panel cd where the shear has been assumed zero and proceeding clockwise around cell as for example

$$q_{de} = 0 - 4 \times 5 \times 0.5 = -10 \text{ lb./in.}$$

$$q_{ee'} = -10 - 4 \times 5 \times 1 = -30 \text{ lb./in.}$$

$$q_{e'd'} = -30 - 4 \times (-5) 1 = -10 \text{ lb./in.}$$

$$q_{d'c'} = -10 - 4 \times (-5) 0.5 = 0 \text{ lb./in.}$$

The magnitude of the results are the same as previously calculated but the signs are opposite. As emphasized previously the shear flow calculated together with its sign is in the $y$ direction or $q_y$. The direction of the shear flow along the cell walls in the $xy$ plane can be determined by drawing simple free body diagrams as illustrated in Chapter A14 but it is simpler to use the automatic rule of Art. A14.11. To illustrate, the solution started in panel cb and proceeded counterclockwise around cell (1). By the rule this means that the shear flows in the $xz$ plane will have the same sign as $q_y$. Since the signs of $q_y$ were negative, the direction of the shear flows in the $xz$ plane will be counter-clockwise around cell (1).

To obtain the shear flow in the vertical web $c'o'$ it was necessary to start at c and go toward c'. This direction is counter-clockwise with respect to cell (2) or clockwise with respect to cell (1). $q_y$ for this panel from the equation was found to be -20. If we consider web $c'o'$ as part of cell (1) then the direction in solving the summation $Z_2 A$ was clockwise with respect to cell (1) and by our rule the shear flow in the $z$ direction will have the opposite sign to $q_y$ or plus 20, which means clockwise
Since there is continuity between cells, \( q_1 = q_2 \). Also since area of each cell is the same, \( A_1 = A_2 \). Equating (1) and (2),

\[
1533q_1 - 1533q_a - 10840 = 0
\]

One other equation is necessary to solve for unknowns \( q_1 \) and \( q_a \), and it is given by the moments of the external and internal shear forces about any point in the plane of the cross section, which must equal zero for equilibrium.

Take moments about point (b) of the shear flow system of Figs. A15.31 and A15.32 and also the external shear load of 1000 lb., which in this case has no moment about our assumed moment center.

\[
2M_b = -50 \times 10 \times 5 + 20 \times 10 \times 5 + 10 \times 30 \times 15 + 200q_1 + 200q_a = 0
\]

hence, \( 200q_1 + 200q_a + 3000 = 0 \)  

Solving equations (3) and (4) for \( q_1 \) and \( q_a \), we obtain

\[
q_1 = -4.07 \text{ lb./in.} \quad q_a = -10.80 \text{ lb./in.}
\]

The final or true internal shear flow system then equals that of Fig. A15.31 plus that of Fig. A15.32 when \( q_1 = -4.07 \) and \( q_a = -10.80 \) lb./in., which gives the shear flow diagram of Fig. A15.33.

**Solution 2 (By use of shear center)**

In this solution, we find the flexural shear flow for bending about axis X-X without twist. The centroid of this internal shear system locates the shear center. The moment of the external shear load about the shear center produces pure torsion on the 2 cell beam. Thus, adding the shear due to this pure torsion to that of pure bending, we obtain the final resisting internal shear flow.

In bending about axis X-X without twist, the shearing strain for each cell as given by equations (1) and (2) must equal zero. Hence:

\[
1200q_1 - 333q_a - 6670 = 0
\]

\[
-333q_1 + 1250q_a + 4170 = 0
\]
Solving equations (5) and (6) for $q_1$ and $q_2$, we obtain $q_1 = -2.0$ lb./in., $q_2 = 5.00$ lb./in. Therefore, taking these values of $q_1$ and $q_2$ as shown in Fig. A15.32 and adding the results to that of Fig. A15.31, we obtain the shear flow pattern of Fig. A15.34 which is the shear flow system for bending without twist about X axis. The centroid of this shear system locates the shear center.

In Fig. A15.34,

$$
\Sigma V = 0 = -10 \times 45 - 10 \times 28 - 10 \times 27 = -1000 \text{ lb., which checks the external shear of 1000 lb.} \quad \Sigma H = 0 \text{ by observation of Fig. A15.34.}
$$

![Fig. A15-34](image)

To find the horizontal position of the centroid of the shear flow in Fig. A15.34 take moments about point a:

$$
ZM_a = 10 \times 27 \times 10 + 10 \times 28 \times 20
+ 5 \times 8 \times 10 - 5 \times 2 \times 10 = 8600 \text{ in.lb.}
$$

hence, $\bar{x} = \frac{8600}{1000} = 8.6"$ to the right of web aa'.

The external shear load of 1000 lb. acts 6" to the right of aa', and therefore causes a moment about the shear center equal to $(9.6 - 8.0) 1000 = 3600 \text{ in.lb.}$ To resist this torsional moment, a constant torsional shear flow $q_t(1)$ and $q_t(2)$ must act on cells (1) and (2) respectively.

The values of $q_t(1)$ and $q_t(2)$ can be found by using equations (16) and (17) of Art. A5.11 of Chapter A5. Thus

$$
q_t(1) = -\frac{1}{2} \left[ \frac{a_{11}A_{11} + a_{12}A_{12}}{a_{11}A_{11}^* + a_{12}A_{12}^* + a_{13}A_{13}^*} \right] T
$$

$$
q_t(2) = -\frac{1}{2} \left[ \frac{(10/0.03) \times 100 + 10/0.03 \times 100}{(10/0.03) \times 100^* + 10/0.03 \times 100^* *}
\right] T = \frac{1}{2} \left[ \frac{158200}{31190000} \right] T = 0.00264 T
$$

Since the external torque equals 3600 in.lb.,

the resisting internal torque must therefore equal -3600. Therefore,

$$
q_t(1) = 0.00264(-3600) = -9.17 \text{ lb./in.}
$$

Solving for $q_t(2)$

$$
q_t(2) = -\frac{1}{2} \left[ \frac{10/0.03 + 10/0.03 + 10/0.03 + 10/0.03 \times 200}{31190000} \right] T = 0.00264 T
$$

hence: $q_t(2) = 0.00264 \times -3600 = -9.17 \text{ lb./in.}

Therefore, if we add to the shear flow system of Fig. A15.34, a constant shear flow of $-9.17 \text{ lb./in.}$ to cell (1) and $-8.85 \text{ lb./in.}$ to cell (2), we will obtain the true internal resisting shear flow of Fig. A15.35, which checks solution 1, any discrepancy being due to slide rule accuracy.

Torsional Deflections

The angular twist of each cell is the same. The value of the angular twist $\theta$ per unit length of the beam can be found using the shear flow pattern of Fig. A15.35 which is the true resultant shear flow, or the pure torsional shear flows of $q_t(1) = -9.17 \text{ lb./in.}$ and $q_t(2) = -8.85 \text{ lb./in.}$ may be used if desired.

The results will, of course, be the same. For example:

For cell (1) due to $q_t(1) = -9.17 \text{ lb./in.}$ and $q_t(2) = -8.85 \text{ lb./in.}$

$$
2GGA_1 = \frac{2G}{t} \left[ \frac{(9.17 \times 10)}{0.03} \right] 3 + \frac{9.17 \times 10}{0.03} - \frac{8.85 \times 10}{0.03} = 8000
$$

Cell (2)

$$
2GGA_2 = \frac{2G}{t} \left[ \frac{(8.85 \times 10)}{0.03} \right] 3 + \frac{8.85 \times 10}{0.04} - \frac{9.17 \times 10}{0.03} = 8000
$$
Cell (1) Final stresses - Fig. A15.35

\[ \text{2GA} = \frac{E t}{0.03} \left( 14.17 \times 10^5 \right) + 0.03 \times 54.17 
\]

\[ = \frac{26.65 \times 10^5}{0.03} = 8000 \]

Cell (2) Final stresses

\[ \text{2GA} = \frac{E t}{0.03} \left( 10.85 \times 10^5 \right) + 0.03 \times 26.68 
\]

\[ = \frac{-19.15 \times 10^5}{0.04} = 8000 \]

A15.10 Three Cell - Multiple Flange Beam. Symmetrical

About One Axis.

Fig. A15.36 shows a 3-cell box beam subjected to an external sheaf load of 1000 lbs. as shown. The section is symmetrical about axis XX. The area of each stringer is shown in parenthesis at each stringer point. The internal sheaf flow system which resists the external load of 1000 lbs. will be calculated assuming that the webs and walls take no bending loads, or, the stringers are the only effective material in bending. The moment of inertia about the XX axis of effective material equals 250 in^4 (Note: this beam section is identical to the two cell beam of Fig. A15.30 plus the leading edge cell (3).

![Fig. A15-36](image-url)

Solution No. 1 (Without use of shear center)

The system is statically indeterminate, to the third degree, since the value of the shear flow q at any point in each cell is unknown.

The value of the shear flow will be assumed at a point in each cell and the flexural sheaf flow for bending about the XX axis will be determined consistent with this assumption. A constant unknown shear flow \(q_A\), \(q_B\), and \(q_C\) for cells (1) and (2) and (3) respectively will be added to the static flexural sheaf flow so as to make the angular twist \(\phi\) of each cell the same, since if any twisting takes place, all cells must suffer the same amount. Furthermore, for equilibrium, the moment of the internal sheaf flow system plus the moment of the external sheaf load must equal zero.

For bending about axis XX, the flexural sheaf flow will be assumed as zero at a point just to the left of stringer \(a\) in cell (3) and just to the left and right of stringer \(c\) in cells (1) and (2) respectively. One might consider the cells as cut at these three points. Fig. A15.37 shows the flexural sheaf flow under these assumptions. Since the leading edge cell (3) has no stringers and the covering is considered ineffective in bending, the sheaf flow will be zero on the leading edge portion since the sheaf flow was assumed zero just to the left of stringer \(a\). The resulting flexural sheaf flow for the 3-cell section will therefore be identical to Fig. A15.31 and the calculations for the flexural sheaf flow will be identical to those in Art. A15.7.

![Fig. A15-37](image-url)

Fig. A15.38 shows the unknown constant sheaf flows \(q_1\), \(q_2\), and \(q_3\) which must be added to the flexural sheaf flow of Fig. A15.37 to make the twist \(\phi\) of each cell the same. The sense of each has been assumed positive in each cell.

![Fig. A15-38](image-url)

The angular twist \(\phi\) for each cell equals

\[ \phi = \frac{1}{2GA} \sum q L/t \]

Using the values of \(q\) in Figs. A15.37 and A15.38, the value \(\phi\) will be computed for each cell.
A15.16 SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

\[ 2Q_a d = \frac{2qL}{t} \]

\[ 2q \times 39.7G = 10 \times 50 \cdot \frac{15.71}{.05} \cdot q_s + \frac{10q_s}{.05} \cdot \frac{-10q_s}{.05} \]

or \( Q_G = 10.5q_s - 2.55q_s + 127 \)

(1)

Cell (1)

\[ 2G_a G = \frac{2qL}{t} \]

\[ 2q \times 100G = \frac{-50 \times 10}{.05} \cdot 2(10 \times 5) \cdot \frac{10 \times 20}{.03} + \frac{3(10q_s)}{.05} \]

\[ + \frac{10q_s}{.05} \cdot \frac{10q_s}{.05} \cdot \frac{-10q_s}{.05} \]

hence \( G = 6q_s - q_s - 1.67q_s \)

\[ - 33.34 \]

(2)

Cell (2)

\[ 2G_a G = \frac{2qL}{t} \]

\[ 2q \times 100 \times G = \frac{-20 \times 10}{.03} \cdot 2(10 \times 5) \cdot \frac{10 \times 30}{.03} + \frac{10q_s}{.03} \cdot \frac{-10q_s}{.03} \cdot \frac{-10q_s}{.03} \]

\[ + 3 \cdot 100 \cdot q_s \]

hence \( G = 6.25q_s - 1.67q_s + 20.83 \)

(3)

Taking moments of the internal shear flow systems of Fig. A15.37 and A15.38 and the external load of 1000 lbs. about stringer \( a \) and equating to zero:

\[ 2M_a = 10 \times 20 \times 10 + 10 \times 30 \times 20 - 5 \times 1000 \]

\[ + 78.6q_s - 200q_s + 200q_s = 0 \]

\[ = 3000 + 78.6q_s - 200q_s + 200q_s = 0 \]

(4)

Solving equations (1) (2) (3) and (4) for the unknown \( q_1, q_s, q_a \), and \( G_G \) we obtain:

\[ q_1 = -2.12 \text{ lb./in.} \]

\[ q_s = -7.09 \text{ lb./in.} \]

\[ q_a = -14.5 \text{ lb./in.} \]

\[ G_G = -19.9 \]

Adding these constant shear flows to the flexural shear flow of Fig. A15.37, we obtain the true internal resisting shear flow as shown in Fig. A15.39.

A15.11 Shear Flow in Beam with Multiple Cells. Method of Successive Approximations.

The general trend in airplane structural design appears to be to the use of a relatively large number of cells. There are various reasons for this trend some of which are: (1) using multiple interior webs, the detrimental effect of shear deformation on bending stress distribution is decreased; (2) the fail safe characteristic of the wing is increased because the wing is made statically indeterminate to a high degree and thus failure of individual units due to fatigue or shell fire can take place without greatly decreasing the over-all ultimate strength of the wing; (3) the ultimate compressive strength of wing flange units is usually increased because column action is prevented by the multiple webs which attach to flange members.

In Chapter A9, Art. A9.13, the method of successive approximation was presented by determining the resisting shear flow system when a multiple cell beam was subjected to a pure torsional moment. This method of approach has now been extended to determine the resisting shear flow when the beam is subject to flexural bending without twist*. Using these two methods the shear flow in a beam with a relatively large number of cells can be determined rather rapidly as compared to the usual method of solving a number of equations.

PHYSICAL EXPLANATION OF THE METHOD

Fig. A15.40 shows a 3-cell beam carrying and external shear load \( V \) acting through the shear center of the beam section but as yet unknown in location. In other words, the beam bends about the symmetrical axis \( X-X \) without twist. The problem is to determine the internal resisting shear flow system for bending without

---

twist. In this example, it is assumed that the bending moment is resisted entirely by the flange members as represented by the small circles on the figure, which means that the shear flow will be constant between the flange members.

The first step in the solution is to make the structure statically determinate relative to shear flow stresses for bending without twist. In Fig. A15.41 imagine each cell cut at points a, b and c as shown. For the given shear load V, the static shear flow q_s can be calculated, assuming the modified section bends about axis X with no twist. Fig. A15.41 shows the general shape of this static shear flow pattern.

Now consider each cell as a separate cell. The static shear flow q_s acting on each cell will cause each cell to twist. Since zero twist is necessary a constant shear flow q'_s to cell (1), q'_s to cell (2), and q'_s to cell (3) must be added as shown in Fig. A15.42, and the magnitudes of such value as to make the twist of each cell zero. However, the cells are actually not separate but have a common web between adjacent cells, thus the shear flow q'_s acts on web 2-1 which is part of cell (1), and thus causes cell (1) to twist. Likewise cell (3) is twisted by q'_s and cell (2) by both q'_s and q'_s. Therefore to cancel this additional cell twist, we must add additional constant shear flows q''_s, q''_s and q''_s as shown in Fig. A15.43, and considering each cell separate again. However, since the cells are not separate these additional shear flows effect the twist of adjacent cells through the common web. As before this disturbance in cell twist is again cancelled or made zero by adding further closing shear flows q''''_s, q''''_s, q''''_s as shown in Fig. A15.44. This procedure is repeated until the closing shear flows become negligible. In general the converging of this system is quite rapid and only a few cycles are necessary to give the desired accuracy of results.

The total closing shear flows q_s and q_s are then equal to

\[ q_s = q'_s + q''_s + q''''_s + \ldots \]

\[ q_s = q'_s + q''_s + q''''_s + \ldots \]

The final shear flow on any panel then equals, (See Fig. A15.45)

\[ q = q_s + q_s + q_s + \ldots \] \hspace{1cm} (1)

The centroid of this final shear flow system locates the shear center of the section, relative to bending about the X axis.

**DERIVATION OF EQUATIONS FOR USE IN SUCCESSIVE APPROXIMATION METHOD**

Fig. A15.46 shows cell (2) of the 3-cell beam shown in Fig. A15.45. q_s is the static shear flow and q_s and q_s are the redundant or unknown shear flows. Since cell (2) does not twist under these shear flows we can write in general,

\[ \sum_{i=1}^{n} q_s = 0 \] \hspace{1cm} (1)

Substituting the various shear flows on cell (2) in Fig. A15.46 into eq. (1),
The subscript (a) on the summation symbol implies summation completely around cell (2) whereas the subscripts "w" and "s" implies summation only along webs 1-2 or 3-2 respectively. L is the length of a sheet panel and t its thickness.

Solving equation (2) for q

\[ q_a = \frac{\sum_{a} q_a L}{L} + \left( \frac{\sum_{w} L}{L} \right) q_w + \left( \frac{\sum_{s} L}{L} \right) q_s \]  

The first term in equation (3) represents the proportion of the static shear flow \( q_s \) which must act as a constant shear flow around cell (2) to cancel the twist due to \( q_a \). The resulting value of this first term will be given the term \( q_w \).

The second and third terms in (3) represent the constant closing shear flows required in cell (2) to cancel the twist of cell (2) due to the influence of \( q_w \) and \( q_s \) in the adjacent cells acting on the common webs between the cells. The ratio in equation (3) before \( q_s \) will be referred to as the carry over influence factor from cell (1) on cell (2) and will be given the symbol \( C_{-w} \), and the ratio before \( q_w \) in equation (3), the carry over influence factor from cell (3) to cell (2) and it will be given the symbol \( C_{s} \). Thus equation (3) can now be written as,

\[ q_a = q_w + C_{-w} q_s + C_{s} q_a \]  

As explained above, \( q_w \) is the value of the necessary closing shear flow for zero twist when the adjacent cell shear flows are zero. Hence first approximations to the final shear flows in each cell can be taken as neglecting the effect of adjacent cells, or in other words each cell is considered separate. Hence the first approximations are,

\[ q_a \approx q_i \]  

By substituting (5) in (4) a second approximation for \( q_a \) is obtained, namely,

\[ q_a \approx q_i + q_w^* + q_s^* \]  

or, \[ q_a \approx q_i + q_w^* + C_{s} q_s^* \]  

where \( q_w^* \) and \( q_s^* \) are made to the approximations for \( q_w \) and \( q_s \). Therefore as a third estimate for \( q_a \), these further corrections should be added and thus equation (7) becomes,

\[ q_a \approx q_i + q_w^* + q_s^* + C_{s} (q_w^* + q_s^*) \]  

Thus by repeating the above procedure, a power series of the carry over influence factor is obtained. In general the convergence is rapid and only a relatively few cycles or operations are needed for sufficient accuracy for final shear flows. A solution of a problem will now be given to show how the necessary operations form a very simple routine.

A15.12 Example Problem Solution. Problem No. 1.

Fig. A15.47 shows a cellular beam with five cells. The flange areas and the web and wall thicknesses are labeled on the figure. The problem will be to determine the internal shear flow pattern when resisting an external shear load of \( V_s = 1000 \) lbs. without twist of the beam. Having determined this shear flow system the shear center location follows as a simple matter.

Fig. A15.48 shows the assumed static condition for determining the shear flow system in carrying a \( V_s \) load of 1000 lb. without twist. The static condition is that all webs except the right end web have been imagined cut as indicated thus making the shear flow \( q_s \) at these points zero.

In this example problem it will be assumed that the flange members develop all the bending stress resistance, which assumption makes the shear flow constant between adjacent flange members.

The total top flange area equals 5.5 in.*, and also the total bottom flange area. Due to symmetry the centroidal X axis lies at the mid-depth point.

Hence, \[ I_X = (5.5 \times 5)*2 = 275 \text{ in.}^4 \]

\[ q_s = \frac{V_s E_Z A}{I_X} = \frac{1000}{275} E_Z A = 3.636 E_Z A \]

Starting at the lower left hand corner, the static shear flow \( q_s \) will be computed going counter-clockwise around beam.

\[ q_a = -3.636*(-5)*2 = 36.36 \text{ lb./in.} \]
\[ q_b = 36.36 + 3.636*(-3) = 54.55 \text{ lb./in.} \]
\[ q_c = 54.55 - 3.636*(-5)*0.5 = 63.64 \]

Continuing in like manner around the beam, the values of \( q_s \) as shown in Fig. A15.48 would be obtained.

The solution from this point onward is made in table form as shown in Table A15.2 which should be located below a drawing of the
cellular beam as illustrated, and the numbers in the Table should be lined up with respect to the cells as indicated.

The solution as presented in Table A15.2 is carried out in 17 simple steps. The first step as given in row 1 of the Table is to compute for each cell the value for $2 \times q_8 \cdot \frac{L}{t}$, where $q_8$ is the static shear flow on each shear panel of a cell; $L$ the length of the panel and $t$ its thickness. Values for $q_8$ are taken from Fig. A15.48.

For example, for Cell 1

$2 \times q_8 \cdot \frac{L}{t} = 2(36.36 \times 10) \cdot \frac{1}{.04} = 18180$

The sign is positive because $q_8$ is positive. (Clockwise shear flow on a cell is positive.) Row 1 in the Table shows the values as calculated for the 5 cells.

The second step as indicated in row 2 of the Table is to calculate the value of the expression $2L/t$ for each cell.

For example, for Cell 1,

$2 \cdot \frac{L}{t} = \frac{10}{.06} + (10) \cdot \frac{1}{.06} + \frac{10}{.06} = 856$

For Cell 2,

$2 \cdot \frac{L}{t} = \frac{10}{.04} + 2(10) \cdot \frac{10}{.04} = 950$

The third step as indicated in row 3 is to calculate the value for the $L/t$ of the common web between two adjacent cells.

For example, for web bb' between cells (1) and (2),

$\frac{(L)}{(t)}_{bb'} = \frac{10}{.08} = 200$

The fourth step is to determine the carry over factor from one cell to the adjacent cell. The results are recorded in row 4 of the Table.

Referring to equation (3) for general explanation, the carry over factor from cell (1) to cell (2) is,

$C_{1-2} = \frac{(L)}{(t)}_{bb'} \times \frac{200}{856} = .2105$

and the carry over factor from cell (2) to cell (1) equals,

$C_{2-1} = \frac{(L)}{(t)}_{bb'} \times \frac{200}{950} = .2105$

We are now ready to start the solution proper by successive approximations. In row 5 of the Table, the first approximation is to assume a value $q'$ added to each cell which will cancel the twist due to the static shear flow when the cell webs are not cut, but each cell is considered separate or independent of the other. This constant closing shear flow $q'$ equals,

$q' = -\frac{2q_8}{\frac{L}{t}}$. The minus sign is necessary because the twist under the static shear flow must be canceled. The values for $q'$ are recorded in row 5 of the Table.

For example for Cell (1),

$q' = -\frac{18180}{856} = -21.238$

For Cell (2),

$q' = -\frac{27275}{950} = -28.71$

Steps 6 to 13 as recorded in rows 6 to 13 of the Table are identical in operation, namely, the carry over influence from one cell to the adjacent cell because of the common web between the cells. As a closing shear flow is added to each cell to make the cell twist zero when they are considered separate, this result is continually disturbed because of the common web. Gradually these corrective shear flows become smaller and smaller until the cells reach their true state and possess zero twist. In the Table, arrows have been used for two cycles to help clarify the operations.

For example in row 6, the carry over shear flow from cell (1) to cell (2) is,

$-5.700 \times .2105 = -1.414$

From cell (2) to cell (1), the carry over value is

$(-4.480 - 8.330) \times .2336 = -3.000$

From cell (2) to cell (3),

$(-4.480 - 8.330) \times .250 = -3.215$
SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

**Fig. A15-47**
Flange and Web Data

<table>
<thead>
<tr>
<th>10&quot; X</th>
<th>1000 lb. + Vc</th>
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</thead>
<tbody>
<tr>
<td>.04</td>
<td></td>
</tr>
<tr>
<td>.04</td>
<td></td>
</tr>
<tr>
<td>.04</td>
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<table>
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<tr>
<td>.03</td>
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5 Cells at 10" = 50°

**Fig. A15-48**
Assumed Static Condition for Shear Flow q_a

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<th>100°</th>
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<tbody>
<tr>
<td>a'</td>
<td>36.36</td>
</tr>
<tr>
<td>b'</td>
<td>54.55</td>
</tr>
<tr>
<td>c'</td>
<td>62.64</td>
</tr>
<tr>
<td>d'</td>
<td>72.73</td>
</tr>
<tr>
<td>e'</td>
<td>81.82</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cut Cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>o (1)</td>
</tr>
<tr>
<td>o (2)</td>
</tr>
<tr>
<td>o (3)</td>
</tr>
<tr>
<td>o (4)</td>
</tr>
<tr>
<td>o (5)</td>
</tr>
</tbody>
</table>

Table A15.2

<table>
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<tr>
<th>Row</th>
<th>OPERATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$Zq_a$ L/t for each cell</td>
</tr>
<tr>
<td>2.</td>
<td>$Z L/t$ for each cell</td>
</tr>
<tr>
<td>3.</td>
<td>L/t of cell web</td>
</tr>
<tr>
<td>4.</td>
<td>Carry Over Factor (C)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>18180</th>
<th>27275</th>
<th>31820</th>
<th>48487</th>
<th>87890</th>
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<td>950</td>
<td>1000</td>
<td>1250</td>
<td>1333</td>
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<td>5.</td>
<td>200</td>
<td>250</td>
<td>250</td>
<td>333</td>
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<td>2633</td>
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<td>250</td>
<td>333</td>
<td>266</td>
</tr>
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<td>250</td>
<td>2633</td>
</tr>
<tr>
<td>9.</td>
<td>250</td>
<td>250</td>
<td>333</td>
<td>266</td>
</tr>
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<td>250</td>
<td>2633</td>
</tr>
<tr>
<td>11.</td>
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<td>250</td>
<td>333</td>
<td>266</td>
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<td>250</td>
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<tr>
<td>13.</td>
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<td>333</td>
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<td>250</td>
<td>2633</td>
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<tr>
<td>17.</td>
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<td>250</td>
<td>333</td>
<td>266</td>
</tr>
<tr>
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<td>2336</td>
<td>250</td>
<td>2633</td>
</tr>
<tr>
<td>19.</td>
<td>250</td>
<td>250</td>
<td>333</td>
<td>266</td>
</tr>
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<td>2336</td>
<td>250</td>
<td>2633</td>
</tr>
<tr>
<td></td>
<td>250</td>
<td>250</td>
<td>333</td>
<td>266</td>
</tr>
</tbody>
</table>

| 1.  | 21288 | 18283 |
| 2.  | 13958 | 10958 |
| 3.  | 9514  | 7514  |
| 4.  | 6173  | 4173  |
| 5.  | 3832  | 1832  |
| 6.  | 2492  | 1492  |
| 7.  | 1152  | 752   |
| 8.  | 8182  | 6182  |
| 9.  | 5842  | 4842  |
| 10. | 3502  | 2502  |
| 11. | 2162  | 1162  |
| 12. | 1122  | 722   |
| 13. | 7882  | 5882  |
| 14. | 5542  | 3542  |
| 15. | 3202  | 2202  |
| 16. | 1862  | 1062  |
| 17. | 1122  | 722   |
| 18. | 7882  | 5882  |
| 19. | 5542  | 3542  |
| 20. | 3202  | 2202  |

**Fig. A15-49**
Closing Shear Flows to Make Twist of Each Cell Equal Zero

\[ q_1 = 33.51 \]
\[ q_2 = 52.45 \]
\[ q_3 = 63.43 \]
\[ q_4 = 73.91 \]
\[ q_5 = 84.39 \]

**Fig. A15-50**
Final Shear Flows.
(Fig. A15-48 plus Fig. A15-49)

<table>
<thead>
<tr>
<th>2.85</th>
<th>2.10</th>
<th>0.27</th>
<th>1.18</th>
<th>2.57</th>
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<td>33.51</td>
<td>18.94</td>
<td>10.98</td>
<td>10.48</td>
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<tr>
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<td>2.10</td>
<td>0.27</td>
<td>1.18</td>
<td>2.57</td>
</tr>
</tbody>
</table>

CALCULATION OF SHEAR CENTER LOCATION

In Fig. A15.47 let $X$ = distance from left end of beam to shear center. Taking moments about upper left corner of the shear flow forces in Figs. A15.48 and A15.49 and equating to 1000 $X$, $1000X = 10(36.36+54.55+63.64+72.73+81.82)10 + 100 x 10 x 50 - 2 x 100(33.51 + 52.45 + 63.43 + 73.91 + 84.39)$, hence $1000X = 19472$ or $X = 19.47$ inches.
In row 13 of the Table, the carry over values are so small that the process is terminated. The final constant shear flow that must be added to each cell to cancel the twist due to the static shear flow equals the algebraic sum of the values from the beginning row 5 to row 14. The results are shown in row 14 of the Table.

The results in row 14 are obtained after a considerable number of multiplications and additions of numbers, thus it is easy to make a numerical mistake. To check whether any appreciable mistakes have been made, we take the values in row 14 and consider these values of constant shear flow in each cell as that causing zero twist if cells are separate. Then bringing the cells together, through the common webs causes a disturbance in twist and this is made zero by the carry over values. This step in the Table is referred to as a reiteration and is indicated in row 15. Then adding the values in row 15 to the initial approximation q' in row 5, which value is repeated in row 16, we obtain the final value of q in row 17. The values in row 17 are practically the same magnitude as in row 14, thus no appreciable mistakes have been made. If the difference was appreciable, then a second, and if needed, even a greater number of reiterations should be carried out. In the Table a second reiteration is shown in rows 18, 19, 20 and the results in row 20 are practically the same as in row 17.

It will be assumed that the solution was stopped after first iteration, and thus the values in row 17 are the constant shear flows that must be added to the static shear flows to produce bending without twist. Fig. A15.49 shows these final closing constant shear flows. Adding these values to those in Fig. A15.48 we obtain the final shear flows in Fig. A15.49.

The lateral location of the shear center for this given 5 cell beam coincides with the centroid of the shear force system in Fig. A15.50. The calculations for locating the shear center are given below Fig. A15.50.

Solution 2 of Problem 1

In solution 1, the assumed static condition involved cutting all vertical webs except the right end web. Thus the static beam section became an open channel section and the resulting static shear flows must obviously be far different than the final true shear flow values, since the webs always carry the greater shear flows in bending without twist. This fact is indicated by the relatively large number of steps required in Table A15.2 to reach a state where successive corrections were small enough to give a desired accuracy of final result. Thus it is logical to assume a static condition where the static shear flows in the webs should be much closer to the final values and thus hasten the convergence in the successive approximation procedure.

Thus in Fig. A15.51, we have assumed the top panel in each cell as cut to give the static condition. The static shear flow is now confined to the vertical webs and zero values for top and bottom sheet. Table A15.3 shows the calculations for carrying out the successive approximations and needs no further explanation. It should be noticed that after the first approximation was made in row 5, only three carry over cycles were needed in rows 6, 7 and 8 to obtain the same degree of accuracy as required in 9 cycles in Table A15.2 for solution 1. Fig. A15.52 shows the final shear flows which equal the constant shear flows in each cell from row 9 of Table added to the static shear flows in Fig. A15.51. These values check the results of solution 1 as given in Fig. A15.50, within slide rule accuracy. In Table A15.3 no reiteration steps were given. The student should make it a practice to use such checks.

### A15.13 Example Problem 2

All Material Effective in Bending Resistance

The general trend in supersonic wing structural design is toward a large number of cells and relatively thick skins, thus in general, all cross-sectional material of the wing is effective in resisting bending stresses and thus the sheaf flow varies in intensity along the walls and webs of the beam cells. Fig. A15.53 shows a ten cell beam with web and wall thicknesses as shown. It will be assumed that all beam material is effective in bending. The shear flow resisting system for bending about the horizontal axis without twist will be determined. The centroid of this system will then locate the shear center.

Fig. A15.54 shows the static condition that has been assumed, namely, that the upper sheet panel in each cell has been imagined cut at its midpoint, thus making the static shear flow zero at these points. The static shear flow values are shown on Fig. A15.54. To explain how they were calculated, a sample calculation will be given.

The moment of inertia of the entire cross-section about the horizontal centroidal axis is, 

\[ I_x = (50 \times 0.125 \times 2.5)^2 = 78.0 \text{ in}^4 \]

\[ I_x \text{ of all webs} = (0.912 \times 5)^2 = 9.5 \text{ in}^3 \]

Total \[ I_x = 87.5 \text{ in}^4 \]

For convenience an external shear load \[ V_z = 8750 \text{ lb.} \] will be assumed acting on this beam section.

Hence, \[ q = \frac{V_z \cdot 2 z A}{I_x} = \frac{-8750 \cdot 22}{87.5} = -100 \text{ lb} \]
A15.22 SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

Now consider Fig. A15.55 which shows a sketch of cell (1) plus half of cell (2). As previously explained the upper cell panels were assumed cut at their midpoints (a) and (a).

Solution II

Starting at point (a) in cell (1) where the shear flow is zero and going counter-clockwise around the cell, the static shear flows are as follows:

- 2045
- 1362
  0
  758
  3031

Table A15.3

<table>
<thead>
<tr>
<th>Row</th>
<th>OPERATION</th>
<th>L/t for each cell</th>
<th>L/t of cell web</th>
<th>Carry over factor C</th>
<th>1st approx. ( q = \frac{q'' \cdot q'''}{L'/L} )</th>
<th>( q'' + q''' ) (carry over)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( L/g )</td>
<td>-2045</td>
<td>856</td>
<td>200</td>
<td>2105.0</td>
<td>1.334</td>
</tr>
<tr>
<td>2</td>
<td>( L'/L )</td>
<td>-1362</td>
<td>956</td>
<td>250</td>
<td>2350.0</td>
<td>3.545</td>
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<tr>
<td>3</td>
<td>( L'/L )</td>
<td>0</td>
<td>1000</td>
<td>250</td>
<td>2500.0</td>
<td>11.33, 1333</td>
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<tr>
<td>4</td>
<td>( L'/L )</td>
<td>0</td>
<td>1250</td>
<td>250</td>
<td>2500.0</td>
<td>1.333</td>
</tr>
<tr>
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<td>Carry over factor C</td>
<td>( q'' )</td>
<td>0.534</td>
<td>0.535</td>
<td>( q''' ) (carry over)</td>
<td>0.005</td>
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<tr>
<td>6</td>
<td>( q'' + q''' ) (carry over)</td>
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<td>0.005</td>
<td>0.041</td>
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</tr>
<tr>
<td>7</td>
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<tr>
<td>8</td>
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<td></td>
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<tr>
<td>9</td>
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<td>0.026</td>
<td>0.026</td>
<td>0.041</td>
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Table A15.4

Fig. A15-53

Final Shear Flows
(Row 9 plus Fig. A15-51)
Compare Results with
Solution I (See Fig. A15-50)

Example Problem 2: Ten Cell Beam - All Material Effective in Bending.
Top Skin = .125 Inches Thick

Table A15.4

<table>
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<tr>
<th>( L'/L )</th>
<th>(1)</th>
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<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
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<th>(11)</th>
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<td>0</td>
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</tr>
<tr>
<td>211.4</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
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<td>0</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>-4760</td>
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</tr>
</tbody>
</table>

Fig. A15-55

Final Shear Flow Values
(Note: Shear Flow at Ends of Webs Equal Sum of Shear Flows in Adjacent Skin Panels.)
Fig. A15-55

At point (f) there are two other connecting sheet panels so we cannot proceed past this joint in calculating the shear flow in the two connecting sheets at (f).

Thus we go back to point (a) and go clockwise,

\[ q_a = 0 \]

\[ q_{ba} = 0 + 1000^bZA = 0 + 100 (2.5 \times 2.5 \times 0.125) = 75 \text{ lb./in.} \]

With two other webs intersecting at joint (f) the shear flow summation cannot continue past (f), hence we go to point (m) in cell (2) where shear flow is zero due to the assumed cut at point (m).

\[ q_m = 0 \]

\[ q_{nm} = 0 - 1000^hZA = -100(2.5 \times 2.5 \times 0.125) = -75 \]

Now at joint (b) we have the shear flow of 75 magnitude on each top panel, thus the shear flow in the vertical web at (h) equals the sum of these two shear flows or 156.

Proceeding to (g)

\[ q_g = q_{bg} + 1000^fZA = 156 + 100 (1.25 \times 2.5 \times 0.084) = 156 + 29.4 = 185.4 \text{ lb./in.} \]

\[ q_{fr} = 185.4 + 1000^gZA = 185.4 + 100 (-1.25) = 156 \text{ lb./in.} \]

\[ q_{rk} = q_{rg} - q_{fe} = -156 + 78 = -78 \]

\[ q_{kf} = -78 - 1000^kZA = -78 - 100 (-2.5)(2.5 \times 0.125) = -78 + 78 = 0 \]

Fig. A15.56 shows a plot of these calculated values. The arrows give the sense of the shear flows.

Fig. A15.54 shows the calculated static shear flow values for the entire 10 cells. The values are recorded at the ends of each sheet panel and at the midpoints of each sheet panel. Clockwise shear flow in a cell is positive shear flow. Since an interior web is part of two adjacent cells, the sign of the shear flow on vertical webs is referred to the left hand cell in order to determine whether sense is positive or negative.

Having determined the static shear flows which will be referred to as \( q_g \), we can now start the operations Table A15.4. The first horizontal row gives the calculations of the twist of each cell under the static shear flows, which is relatively measured by the term \( \frac{E}{I} \frac{L}{t} \) for each cell.

With all material effective in bending the shear flow varies along each sheet. Fig. A15.56 shows this variation on the sheet panels of cell (1). The term \( \frac{E}{I} L/t \) is nothing more than the area of the shear flow diagram on each sheet divided by the sheet thickness. To illustrate, consider cell (1) in Fig. A15.56.

Upper sheet panel:

\[ \frac{E}{I} \frac{L}{t} = \frac{-(0 + 78)}{2} \frac{2.5}{0.125} + (0 + 78) \frac{2.5}{0.125} = 0 \]
A15.24  SHEAR FLOW IN CLOSED THIN-WALLED SECTIONS. SHEAR CENTER.

For lower sheet panel:

\[ \frac{Q_{ls}}{t} = 0 \text{ (Same figure as for upper sheet)} \]

For left hand vertical web:

Treating the shear flow diagram as a rectangle with height 78 and a parabola with height 98 - 78 = 20,

\[ \frac{Q_{ls}}{t} = \frac{-78 \times 5}{0.064} \cdot \frac{(20 \times 5 \times 0.667)}{0.064} = -7142 \]

For right hand web of cell (1),

The shear flow diagram is likewise made up of a rectangle and a parabola.

\[ \frac{Q_{ls}}{t} = \frac{156 \times 5}{0.094} \cdot \frac{(186.5 - 156)5 \times 0.667}{0.094} = 9342 \]

Therefore for entire cell (1):

\[ \frac{Q_{ls}}{t} = -7142 + 9342 = 2200, \text{ which is the value in row (1) of Table A15.4 under cell (1).} \]

The results for the other nine cells as calculated in a similar manner are recorded in row 1 of the Table. The procedure as followed in the remaining rows of Table 4 is the same as explained in detail for example problem 1. In Table A15.4 only one reiteration is carried through as the values in the bottom or last row are practically the same as arrived at after the fifth carry over cycle. Adding the constant shear flow values in the last row in the Table to the static shear flow values in Fig. A15.54 we obtain the final shear flow values of Fig. A15.57. The resultant of this shear flow pattern is a force of 9750 acting down in the Z direction. Its location would be through the centroid of the shear flow force system. Let \( \bar{x} \) equal distance from upper left hand corner of beam to line of action of shear flow resultant force.

Taking moments of shear flow force system of Fig. A15.54 plus the constant shear closing values in each cell as given in the last row of Table A15.4 and equating to 8750 \( \bar{x} \), we obtain:

Due to uniform static shear flow on each web:

\[ M = (156 \times 5)(5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45) + 78 \times 5 \times 50 = 175000 \]

Due to parabolic static shear flow in each web:

\[ M = (29.4 \times 5 \times 0.667)(5 + 10 + 15 + 20 + 25) + 25.3 \times 5 \times 0.667(30 + 35) + 22.5 \times 5 \times 5 \times 0.667 + 40 + 45) + 20 \times 5 \times 0.667 \times 50 = 22520 \]

Due to constant closing shear flows as listed in last row in Table A15.4:

\[ M = -(11.31 - 3.57 - 1.33 - 1.17 - 2.88 - 0.97 - 4.27 - 5.04 + 5.55 + 22.61) \times 25 = -820 \]

Total Moment = 175000 + 22520 - 820 = 179000 hence \( \bar{x} = 179000/8750 = 22.5 \) in., which equals the distance from left end of beam to shear center location.

Referring to the final shear flow values in Table A15.57, it will be noticed that the final results are not much different from the assumed static shear flows with the possible exception of the two end webs. If we had assumed all the webs cut except one to form the static condition, then Table A15.4 would have required several times as many carry-over cycles to obtain the same accuracy of final results.

A15.14  Use of Successive Approximation Method for Multiple Cell Beams when Subjected to Combined Bending and Torsional Loads.

The internal shear flow resisting force system for a beam subjected to bending and twisting loads at the same time is carried out in two distinct steps and the results are added to given the true final shear flow system. First, the shear flow resisting system is found for being without twist as was explained in this chapter. The results of this first step locates the shear center. The external load system is then transferred to the shear center, which normally would produce a torsional moment about the shear center. The internal resisting shear flow system to balance this torsional moment is then handled by the successive approximation method as explained and illustrated in detail in Art. A6.13 of Chapter A6.

A15.15  Shear Flow in Cellular Beams with Variable Moment of Inertia.

The previous part of this chapter dealt entirely with beams of constant moment of inertia along the flange direction. In airplane wing and fuselage structures, the common case is a beam of non-uniform section in the flange direction. In cases where the change of the cross-sectional areas is fairly well distributed between the various flange members which make up the beam cross section, the shear flow results as given by the solution for beams of constant moment of inertia are not much in error. For beams where this is not the case, the shear flow results may be considerably different from the actual shear flows. This fact will be illustrated later by the solution of a few example problems.

A15.16  The Determination of the Flexural Shear Flow Distribution by Considering the Changes in Flange Loads. (The 2P Method.)

Fig. A15.58 shows a single cell distributed
flange beam. Consider the beam acts as a cantilever beam with the bending moment existing at section (A) being greater than that existing at section (B) and that the bending moment produces compression on the upper surface. By the use of the flexural stress equations, the bending stress on each stringer can be found, which if multiplied by the stringer area gives the stringer axial load. Thus at beam section (B), let $P_1, P_2, P_3, P_4, P_5$ etc. represent the axial loads due to a bending moment $M$. The external bending moment at section (A) is $M + \Delta M$, hence the stringer axial loads at section (A) will equal $P_1 + \Delta P_1, P_2 + \Delta P_2, P_3 + \Delta P_3, \ldots$. These stringer axial loads are shown on Fig. A15.58.

Imagine the upper sheet panel 2, 2', 3, 3' is cut along line (a-a). Furthermore consider stringer number (3) cut out and shown as a free body in Fig. A15.59. Let $q_y$ be the average shear flow per inch over the distance d on the sheet edge bb. It has been assumed positive relative to sense along y axis.

For equilibrium of this free body, $\Sigma F_y = 0$, hence $\Delta P_3 + q_yd = 0$

whence $q_y = -\frac{\Delta P_3}{d}$

For a free body including two stringers or flange members, see Fig. A15.60. Again $\Sigma F_y = 0$, whence $\Delta P_2 + \Delta P_3 + q_yd = 0$

or $q_y = -\frac{\Delta P_2 + \Delta P_3}{d}$

Therefore starting at any place where the value of $q_y$ is known, the change in the average shear flow to some other section equals $q_y = \Sigma \frac{\Delta P}{d}$

(1)

If the summation is started where $q_y$ is zero then equation (1) will give the true average shear flow $q_y$.

Fig. A15.61 shows sheet panel (3,3',4,4') isolated as a free body. Taking moments about corner 4 and equating to zero for equilibrium, $\Sigma M_x = d(\Delta P_4b) - q_xbd = 0$

whence, $q_x = \frac{\Delta P_4}{d}$

(2)

Thus for rectangular sheet panels between flange members, the shear flow $q_x$ or $q_z$ equals the average shear flow $q_y$.

The same rules as previously presented to determine the sense of $q_x$ or $q_z$ after having $q_y$ can be used and will not be repeated here.

To show that equation (1) reduces to the shear flow equation previously derived and used, consider a beam with constant cross-section and take a beam length $d = 1$ inch. Then, $\Delta M = V_2d = V_2(1) = V_2$

$\Delta P = \frac{M}{I_x}ZA = \frac{V_2}{I_x}ZA$ (where $A$ = area of stringer)

From equation (1) $q_y = -\Delta P$

Substituting value of $\Delta P$ found above, $q_y = -\frac{V_2}{I_x}ZA$

(3) which is the shear flow equation previously derived for beams with constant moment of inertia.

A15.17 Example Problem to Compare Results in Using Equations (1) and (3).

Fig. A15.62 shows a square single cell beam with six flange stringers. Between points B and C, the beam has a constant flange section which is shown in Section B-B. The numerals beside each stringer represent the area of the stringer. Between points B and D, the flange material tapers uniformly with the flange material at point A as indicated in Section A-A. It should be noticed that the increase in flange area is
with the shear flow system of Fig. A15.53, which therefore is the final shear flow system for this method of solution.

Solution No. 2. Considering ΔP Loads in Flange Stringers. (Equation 1)

Bending moment at section AA = 1000 x 50 = 50000 in.lb.
Bending moment at section BB = 1000 x 30 = 30000 in.lb.

Considering Section B-B:
Bending stress intensity at midpoint of stringers by the flexural formula:

\[ \sigma_b = \frac{M_2}{I_x} \times \frac{50000 \times 5}{150} = 1000 \text{ psi.} \]

Axial load in each of the stringers a, b, and c = 1000 x 1 = 1000 lb.

Considering Section A-A:

\[ \sigma_a = \frac{M_2}{I_x} \times \frac{50000 \times 5}{250} = 1000 \text{ psi.} \]

Axial load in stringer (a) = 1000 x 1 = 1000 lb.
Axial load in stringer (b) or (c) = 1000 x 2 = 2000 lb.

These resulting axial loads are shown acting on the portion between points A and B in Fig. A15.65, which equals the results as shown in Fig. A15.66.

Having found the ΔP flange loads over a length (d) of 20°, the shear flow can be computed by equation (1).

It will be assumed that one-half of the ΔP load in stringer (a) will flow to each adjacent web. However, there is no ΔP load in stringer (a) hence \( q_{ab} = q_{ac} = 0 \). Then from equation (1),

\[ q_{bb'} = 0 - \frac{q_{bb}}{d} - 0 = \frac{1000}{20} = -50 \text{ lb./in.} \]
\[ q_b' a' = -50 + \frac{3}{2} \cdot \frac{\Delta P_b'}{d} = -50 - \frac{1000}{20} = 0 \]

Due to symmetry the left side of cell would give the same results. The results are plotted in Fig. A15.64. Since the increase in section moment of inertia between beam points B and A is increasing at the same rate as the external bending moment, the average shear of 50 lb./in. is constant between the two beam section A and B. Comparing Figs. 63 and 64, we find the first method gives a shear flow of 10 lb./in. in the top and bottom webs, whereas actually it is zero. This seems reasonable since the entire increase of flange area was placed in stringers b and c.

Example Problem No. 2

The same beam as in Problem 1 will be used except that the cross-sections at beam points B and A are as shown in Fig. A15.67. The increase in flange area between beam points B and A has been placed entirely in stringer (a) which changes from 1 sq. in. at B to 3 sq. in. at A.

\[
\begin{array}{ccc|ccc}
\text{Section} & \text{A-A} & \text{B-B} & \text{Fig. A15-67} \\
\hline
l_x = & 150'' & 150'' & \\
\gamma' = & 150'' & 150'' & \\
1 & 1 & 1 & \\
\end{array}
\]

The results of using equations (3) and (1) relative to the shear flow pattern are given in Figs. A15.68 and A15.69 respectively. (The student should check these results.) It should be noticed that the true shear flow is greater in the top and bottom skin than that given by equation (1) which applies only for beams of constant cross-section.

![Fig. A15-68](image)

![Fig. A15-69](image)

A15.18 Shear Flow in Tapered Sheet Panel.

Major aircraft structural units such as the wing, fuselage, etc., are tapered in both plan form and depth and therefore the sheet panels between flange members usually are tapered in width. Fig. A15.70 shows a cantilever beam tapered in depth and carrying a load \( V \) at its end. The flange reactions at the left end have been found by statics. A free body diagram of the web is shown in Fig. A15.71. Take moments about point (0) and equate to zero.

\[ M_0 = \left( \frac{V_d}{b_1} \right) b_a = q_a b_a d = 0 \]

where \( q_a = \frac{V_d}{b_1} \).

But \( V = q_a b_a \), whence

\[ q_a = \left( \frac{b_a}{b_1} \right) q_a \]

From Fig. A15.71

\[ q_a = \frac{V}{b_a} = \frac{q_a b_a}{b_1} \]

Substituting value of \( q \) in (4)

\[ q_a = \left( \frac{b_a}{b_1} \right) q_a \]

Thus having the shear flow on the stringer edge of the sheet panel, the shear flow on the large end of the tapered panel can be found by equation (5).

A15.19 Example Problem of Shear Flow in Tapered Multiple Flange Single Cell Beam.

Fig. A15.72 shows a tapered single cell beam with 6 spanwise stringers or flange members. The beam is loaded by a 1000 lb. load located as shown. Assuming the webs ineffective in bending the internal resisting shear flow pattern will be determined.

In this solution the shear flow at Station 120 will be determined by considering the AP flange loads over a length of 30° or between Stations 90 and 120.

Consider section at Station 120:

Bending stress \( \sigma_b = \frac{M_0}{I} = \frac{1000 \times 120 \times 5}{450} = 1333.33 \) psi.

The horizontal component of the axial load in a stringer equals \( \sigma_A \) (where \( A \) = area of the stringer).

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**ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES** A15.27
hence, the horizontal components, of stringer loads are,

\[ P_a = 4 \times 1333.33 = 5333.33 \text{ lb.} \]
\[ P_b = 3 \times 1333.33 = 4000 \text{ lb.} \]
\[ P_c = 2 \times 1333.33 = 2666.7 \text{ lb.} \]

Consider beam section at Station 90:-
\[ \sigma_0 = \frac{1000 \times 90 \times 4.25}{271} = 1411.5 \text{ psi.} \]

hence stringer loads are,
\[ P_a = 3.25 \times 1411.5 = 4597.4 \]
\[ P_b = 2.50 \times 1411.5 = 3528.7 \]
\[ P_c = 1.75 \times 1411.5 = 2470.1 \]

The change in axial load \( \Delta P \) in the stringers between stations 90 and 120 equals the difference between the above loads, whence
\[ \Delta P_a = 746, \quad \Delta P_b = 471, \quad \Delta P_c = 196.6 \]

Fig. A15.73 shows these \( \Delta P \) loads acting as wing portion. Since shear flow is unknown at any point, assume \( q \) equals zero in web aa'. The average shear flow in each sheet panel over a length \( d = 30 \) inches can now be calculated by using equation (1).
\[ q_y = q_{a'a} - \frac{\Delta P}{d}, \quad q_{a'a} = 0 \text{ (assumed)} \]
\[ q_y(ab) = 0 - \frac{746}{30} = - 24.87 \text{ lb./in.} \]

Since panel ab is tapered in width from 6" at station 120 to 5.5" at station 90, the shear flow \( q_x(ab) \) at station 120 can be found from equation (5).
\[ q_x(ab) = - 24.87 \times 5.5/6 = - 23.2 \text{ lb./in.} \]
\[ q_y(bc) = - 24.87 - 471/30 = - 40.57 \]
\[ q_x(bc) = - 40.57 \times 5.5/6 = - 37.2 \]
\[ q_y(cc') = - 47.12 \times 8.5/10 = - 40.0 \text{ lb./in.} \]

Since the \( \Delta P \) loads are the same on the lower stringers but tension the shear flow calculations if continued would give the same values as found on the top surface. Fig. A15.74 shows the shear flow pattern on station 120.

**IN PLANE FORCES PRODUCED BY INCLINATION OF FLANGE MEMBERS**

Since the box tapers in depth and width, the flange stringers are not normal to section 120, thus X and Z force components are produced on section by the stringer loads.

These in plane force components are:

For stringers a and a',
\[ P_x = 5333.33 \times 2/120 = 88.9 \text{ lb.} \]
\[ P_z = 5333.33 \times 3/120 = 133.3 \text{ lb.} \]

For stringers b and b',
\[ P_x = 4000 \times 0/120 = 0 \]
\[ P_z = 4000 \times 3/120 = 100 \text{ lb.} \]

For stringers c and c',
\[ P_x = 2666.7 \times 2/120 = 44.4 \text{ lb.} \]
\[ P_z = 2666.7 \times 3/120 = 66.7 \text{ lb.} \]

Fig. A15.74 shows these in plane force components due to the flange axial loads.
The forces in Fig. A15.74 will be checked for equilibrium.

\[ Z_F = 1000 - 266.6 - 200 - 133.4 - 10 \times 40 = 0 \]
\[ Z_F = 0 \text{ by observation.} \]

Take moments about stringer (a),

\[ Z_M = -3 \times 1000 + 200 \times 6 + 133.4 \times 12 + 10 \times 40 \times 12 + (-38.9 - 44.4) \times 23.2 \times 6 + 10 + 37.2 \times 6 \times 10 = 8670 \text{ in.lb.} \]

For equilibrium a moment of -8670 is needed, which is provided by a constant shear flow \( q = M/2A = -8670/240 = -36.1 \text{ lb./in.} \)
Adding this constant shear flow to that of Fig. A15.74 gives the final shear flow as shown in Fig. A15.75.

Solution No. 2

This same beam and loading will now be solved using the shear flow equation derived for beams of constant cross-section.

Since the stringers are not perpendicular to the beam cross-section they have a \( z \) force component which thus assists in carrying the external shear load in the \( z \) direction. These \( P_z \) components at station 120 have been calculated in the other solution.

\[ Z_F = -(2 \times 133.3 + 2 \times 100 + 2 \times 66.7) = -600 \text{ lb} \]
Total \( V_z \) (external) = 1000 lb.

Let \( V_z \) (net) be shear load to be taken by cell walls,

\[ V_z \text{(net)} = V_z - Z_F = 1000 - 600 = 400 \text{ lb.} \]
Calculation of static shear flow assuming \( q \) in shear panel aa' is zero.

\[ q_{y(ab)} = \frac{V_z \text{(net)} \times 2A}{I_x} = \frac{400 \times 5 \times 4}{450} = 17.8 \text{ lb.in.} \]
This corresponds to value of 23.2 in previous solution (see Fig. A15.74).

\[ q_{y(bc)} = 17.8 + \frac{400}{450} \times 5 \times 3 = 31.1 \text{ lb./in.} \]
As compared to 37.2 in Fig. A15.74.

\[ q_{y(c')} = 31.1 + \frac{400}{450} \times 5 \times 2 = 40.0 \text{ lb.in.} \]
which is the same as in Fig. A15.74.

Taking moments about point a of the forces in Fig. A15.74 but replacing the shear flows in the top and bottom panels by the values found above, we would obtain an unbalanced shear moment of 7970 in.lb. instead of 8670 previously found. The correcting shear flow would then be \( q = -7970/240 = -33.2 \) instead of -36.1 as previously found. The final shear flow pattern would be as shown in Fig. A15.75, which values should be compared to those in Fig. A15.75.

A15.20 Problems

(1) Determine the resisting shear flow pattern for the loaded single cell beam as shown in Fig. A15.77. Assume load \( P \) = zero in this problem. Assume all material effective in bending. Make two solutions, one of them involving the use of the shear center.

(2) Same as problem (1) but add load \( P = 1000 \text{ lb.} \)

(3) Fig. A15.78 shows an unsymmetrical single cell beam loaded as shown. Assume all material effective in bending. Determine resisting shear flow diagram.

(4) Figs. A15.79 and A15.80 show two loaded single cell - 2 flange beams. Assume the flanges develop all the bending resistance. Determine the shear flow resisting pattern by two solutions, namely, without and with use of shear center.

(5) Fig. A15.81 shows a single cell - 3 flange beam subjected to loads as shown. Assume the 3
flanges develop the entire bending resistance. Determine the internal shear flow resisting system.

(6) Find shear center location for beam in Fig. A15.81 if the 3 flanges provide the entire bending resistance.

(7) Find the internal resisting shear flow pattern for the 3 flange-single cell beam of Fig. A15.82. Assume webs or walls ineffective in bending.

(8) Determine shear center location for beam of Fig. A15.82. Webs and walls are ineffective in bending.

(9) Fig. A15.83 shows a multiple flange-single cell beam section. Find resisting shear flow system if webs and skin are ineffective in resisting bending stresses. All skin flange members have area of 0.1 sq. in. each.

(10) Find shear center location for beam section in Fig. A15.83, if webs and skin are ineffective in bending.

(11) Fig. A15.84 shows a multiple flange-circular beam section. Find the resisting shear flow pattern when carrying the external shear load of 5000 lb. as located in figure. Assume cell skin ineffective in resisting beam bending stresses.

(12) Determine the shear flow resisting system for the beam section of Fig. A15.85. The 6 flanges have areas of 0.2 sq. in. each. Skin is 0.035 thickness. Assume skin ineffective in bending.

(13) Find the shear flow resisting system for the unsymmetrical beam section in Fig. A15.86. Flange areas and skin thicknesses are given on figure. Assume skin ineffective in bending.

(14) Determine shear center location for beam section in Fig. A15.86.

(15) Fig. A15.87 shows a 2 cell beam section. Consider all material effective in resisting bending stresses. For the given beam loading determine the internal resisting shear flow system.

(16) Find shear center location for beam section in Fig. A15.87. All material effective in bending.

(17) For the 2 flange-2 cell beam in Fig. A15.88, determine the resisting shear flow pattern when beam section is loaded as shown. Webs are ineffective in bending.

(18) Fig. A15.89 shows a 4 cell beam section with 6 flange members. Assume walls and webs ineffective in bending.

(a) As a first problem assume that the left and right curved sheet panels are removed,
leaving a 2 cell beam section. Find the resisting shear flow system under the given loading.

(b) Now add the left curved sheet to form cell (1) thus giving a 3 cell beam. Find shear flow system.

(c) Now add the right curved sheet to form cell (4), thus giving a 4 cell beam section. Find the shear flow system.

19) Fig. A15.89 shows a 10 cell multiple flange beam section. Area of each of the 22 flange members equal 0.3 sq. in. Assume webs and skin ineffective in resisting bending stresses. Find internal resisting shear flow system for 2000 lb. shear load acting through shear center of beam section. Find location of shear center. For solution use method of successive approximation.

(20) Find the shear center for the 7 cell beam section of Fig. A15.91 for bending about xx axis without twist. All beam material effective in bending. For solution use method of successive approximation.

(21) Fig. A15.92 shows a single cell-6 flange tapered beam carrying a 1000 lb. load as shown. Calculate resisting shear flow pattern at section A-A by two methods. (1) By ΔP method over a distance of 20 inches between sections A-A and B-B, and (2) By using general shear flow equation for beams of constant section. Compare the results. All six flange members have 0.2 sq. in. area each at section A-A and taken uniformly to 0.1 sq. in. at end C-C.

(22) Add two interior webs to the beam of Fig. A15.92, connecting flange members a-a" and b-b", thus making it a 3 cell beam. Find the shear flow resisting pattern at section A-A by the ΔP method.

(23) Fig. A15.93 shows a circular single cell beam with 8 flange members. The area of each flange member is 0.1 sq. in. throughout the beam length. For the given 400 lb. external loading determine the resisting shear flow pattern at section A-A using the ΔP method over a distance of 25 inches between sections A-A and B-B. Assume cell wall ineffective in resisting bending stresses.
CHAPTER A-16
MEMBRANE STRESSES IN PRESSURE VESSELS
ALFRED F. SCHMITT

16.1 Introduction.

The structural designer is often called upon to develop a vessel which is to contain a fluid under pressure. Occasionally the design of such a vessel is not critical from either a weight or shape standpoint and almost any suitably strong sealed vessel will suffice. More often, the strength, weight and form of such a unit are closely prescribed and rigidly controlled. Thus, the pressurized cabin of a modern aircraft is a sealed pressure vessel containing an atmosphere at near sea level pressures and whose functional requirements include:

i) the transmittal of heavy loads from the tail surfaces and from internal dead loads,

ii) the necessity for nonstructural cut-outs for doors and windows,

iii) an efficient shape from both the aerodynamic and space utilization points of view,

iv) a minimum of weight.

Structurally, the most efficient form of pressure vessel is one in which the lateral pressures are supported by tensile stresses alone in the curved walls of the vessel. Examples of such shapes are those assumed by pressurized rubber balloons and canvas fire hoses and by the free surface of a drop of water (in which the surface tension forces provide the support). The walls of these vessels have zero bending stiffneses and hence have the properties of a membrane. The stresses developed, lying wholly in tangential directions at each point, are called membrane stresses.

In shells of technical importance, the walls do, of course, have some bending stiffness and hence may carry some transverse loadings by flexural stresses. Indeed, the boundary conditions imposed on the shell may be such as to necessitate some localized bending near edges and seams. An efficient pressure vessel design is one in which the configuration minimizes these departures from a true membrane stress system, i.e., minimizes the degree of local bending stresses induced.


Consider the equilibrium of a differential element cut from the shell of revolution of Fig. 16.1. (The figure is drawn to resemble a familiar folding paper Christmas ball, since such an object may aid in visualization.) The element is cut out by the intersection of a pair of adjacent meridian curves and a pair of adjacent parallels.

The radii \( R_m \) and \( R_t \) shown on the figure are found by erecting local normals to the surface of the element at its corners. \( R_m \) is the radius of curvature of the meridian curve; it may be either positive (inward pointing), negative (outward pointing as in Fig. 16.1), or infinite (at inflection points or straight-line meridian segments). \( R_t \) is the radius of curvature of the section normal to the meridian curve. For simple forms of pressure vessel \( R_t \) is always positive; all radii \( R_t \) point inward and intersect the axis 0-0, although not generally normal to 0-0 (see radius \( R_t \) erected from point C of Fig. 16.1).

Fig. 16.2 is a detail of the surface element. The forces per unit length* in the meridional and tangential directions are denoted by \( N_m \) and \( N_t \), respectively. Shear stresses are

\[ \text{Fig. 16.2} \]

* Hereafter referred to as "stresses" although their units are pounds per inch rather than psi.

16.1
absent due to the symmetry of the problem. The included angles between the pairs of meridional and tangential forces are $ds_m/R_m$ and $ds_t/R_t$, respectively.

Summing forces normal to the differential element, one has

$$ds_m N_m = ds_t N_t = p ds_m ds_t$$

or,

$$N_m + N_t = \frac{p}{R_m}$$

(1)

Here $p$ is the internal pressure, positive outward. Note that the shell wall thickness does not enter eq. (1). The pressure $p$ may vary in the meridional direction but is constant in the tangential direction by hypothesis (rotational symmetry assumed).

Eq. (1) is one equation containing two unknowns. Another equation may be obtained by the condition of equilibrium of a portion of the shell above or below a parallel circle. Thus in Fig. A16-3, the pressures acting downward on the lower portion of the shell are equilibrated by the upward vertical components of the meridional stresses, $N_m$.

Summing forces vertically

$$p \pi (R_t \sin \phi)^2 = 2 \pi (R_e \sin \phi) N_m \sin \phi$$

Solving,

$$N_m = \frac{p R_t}{2}$$

(2)

Eqs. (1) and (2) determine completely the membrane stress state in the rotationally symmetric shell problem: the problem is thus seen to be statically determinate.

We note that eq. (2) should not be used in cases of hydrostatic pressure loadings. The basic concept is that of shell equilibrium, and consequently for this class of problems the manner of shell support must be considered. Thus, in the tank of Fig. A16-4a, the upper cylindrical portion requires no meridional stresses since the load is reacted at the supporting ring $O$. (In these analyses the structural weight, which always requires some stresses for its support, is neglected.) In the lower hemispherical portion meridional stresses are required as shown in Fig. A16-4b. Hence, in this class of problems it is best to derive the necessary second equilibrium equation (corresponding to eq. 2) by considering the individual characteristics of the structure.

A16.3 Applications to Simple Pressure Vessels.

Example Problem 1

Determine the membrane stresses in a cylindrical pressure vessel of circular cross section (radius $R_0$), having hemispherical ends, if the internal gas pressure is $p$. Also find the greatest combined normal stress.

SOLUTION:

In the hemispherical ends $N_m = N_t$ by symmetry and, of course, $R_m = R_t = R_0$. Hence eq. (1) is sufficient to determine the stresses in this portion of the structure. One has

$$2 \frac{N_m}{R_0} = 2 \frac{N_t}{R_0} = p$$

$$N_m = N_t = \frac{p R_0}{2}$$

$$\text{Stress} = \frac{N_m}{t} = \frac{N_t}{t} = \frac{p R_0}{2t}$$

In the cylindrical portion the radii are $R_m = \infty$ (the curve of Fig. A16-5 is the meridian curve and this is a straight line for the cylindrical portion).

Eq. (1) becomes

$$\frac{N_m}{R_0} + \frac{N_t}{R_0} = p$$
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[ N_t = p \frac{R_o}{t} \]

Stress \[ = \frac{N_t}{t} = \frac{p R_o}{t} \]

The meridional stress in the cylindrical portion is found from eq. (2): \[ N_m = \frac{p R_o}{2} \]

Stress \[ = \frac{p R_o}{2t} \]

Since shear stresses are absent everywhere in the meridional and tangential directions, these are the directions of principle stresses. Hence the greatest combined normal stress is identically equal to the greatest meridional or tangential stress as just computed. It is seen to be \[ s_{\text{max}} = \frac{p R_o}{t} \].

Example Problem 2.

An important problem in pressurized cabin design concerns the shape of the end bulkheads. While hemispherical bulkheads (such as used in Example Problem 1) are highly desirable from a stress standpoint, such forms are uneconomical as regards space utilization. On the other extremes, a flat bulkhead, while providing far more useful volume, cannot resist the pressure loading by membrane stresses and hence is structurally inefficient. A compromise configuration is that shown in Fig. A16-6, in which the bulkhead is a spherical surface of less curvature ("dished head") supporting the pressure loading by membrane stresses. A reinforcing ring, placed at the seam, resists the radial component of these stresses. Problem: find the compressive load acting in the reinforcing ring.

Example Problem 3.

Another form of bulkhead used to close a circular pressure cylinder is elliptical in section as shown in Fig. A16-7. Such a bulkhead shape provides tangential meridional forces at the seam (requiring no reinforcing ring as in the last example) and yet is reasonably efficient as regards space utilization. Problem: determine the membrane stresses in such a bulkhead.

Solution: In cartesian coordinates the equation of the bulkhead meridian is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

From calculus, the radius of curvature of this meridian curve is

\[ R_m = \frac{1}{\frac{dy}{dx}} = \frac{(a^2 y^2 + b^2 x^2)^{3/2}}{a^2 b^2} \]

The radius \( R_m \) is found most readily by observing that it is normal to a tangent to the meridian curve (see figure). After finding the
slopes of the tangent (angle $\phi$), one computes $R_t = x / \sin \phi$. The results are

$$\sin \phi = \sqrt{\frac{a y^2 + b x^2}{c^2}}$$

$$R_t = \frac{a y^2 + b x^2}{c^2}$$

The meridional stress is found from eq. (2). Thus,

$$N_m = \frac{p R_t}{2} = \frac{p}{2} \left( \frac{a y^2 + b x^2}{c^2} \right)^{1/2}$$

Substituting the expressions for $R_m$, $R_t$ and $N_m$ into eq. (1) one finds

$$N_t = p \left( \frac{a y^2 + b x^2}{c^2} \right)^{1/2} \left[ 1 - \frac{a^2 b^2}{3 (a y^2 + b x^2)} \right]$$

Of particular interest are the stresses at the seam. Here $y = 0$ and $x = a$. One finds

$$N_m = \frac{p a^2}{2}$$

$$N_t = p a \left( 1 - \frac{a^2}{2b^2} \right)$$

This last result is important since it indicates that compressive tangential stresses are possible if $a > b \sqrt{2}$. As will be seen below, such a situation is undesirable because of the large resultant difference in radial expansions between the cylinder and bulkhead (the bulkhead actually contracts radially if $N_t$ is negative) producing high secondary bending stresses in the vessel walls.

**Example Problem 4.**

Determine the weight per unit length of the double-cylinder fuselage cross section as a function of the internal pressure, allowable stress and the geometric parameters of Fig. A16-8. For structural efficiency it is desired to maintain equal membrane stresses in the skin and floor.

**Solution:**

From eq. (1) the "hoop" loadings in the upper and lower cylindrical lobes are

$$N_U = p R_U$$

$$N_L = p R_L$$

Summing forces horizontally at the floor joint:

$$N_f = p (R_U \cos a_1 + R_L \cos a_2)$$

Assume all stresses are equal and are given by $s$.

$$s_U = s = p R_U / t_U$$

$$s_L = s = p R_L / t_L$$

$$s_f = s = \frac{p R_U}{t_f} (R_U \cos a_1 + R_L \cos a_2)$$

Letting the weight density of the material be $w$, the weight per unit length (axially) along the cabin is ($w$ times the developed length of walls and floor).

$$W = w \left\{ 2 \ t_U R_U \left( \pi - a_1 \right) + 2 \ t_L R_L \left( \pi - a_2 \right) + 2 \ t_f R_U \sin a_1 \right\}$$

Solving for the various thicknesses from above and substituting, one finds (to obtain a result symmetric in appearance use was made of the fact that $R_U \sin a_1 = R_L \sin a_2$)

$$W = \frac{2wp}{s} \left\{ R_U^2 \left( \pi - a_1 + \frac{1}{2} \sin 2 \alpha_1 \right) + R_L^2 \left( \pi - a_2 + \frac{1}{2} \sin 2 \alpha_2 \right) \right\}$$

Since one may show that the cross sectional area of this fuselage is

$$A = R_U^2 \left( \pi - a_1 + \frac{1}{2} \sin 2 \alpha_1 \right) + R_L^2 \left( \pi - a_2 + \frac{1}{2} \sin 2 \alpha_2 \right)$$

an important consequence of this calculation is that the ratio of shell weight to shell volume is

$$\frac{W}{A} = \frac{2wp}{s}$$

and is therefore independent of the combinations of $R_U$, $R_L$, $\alpha_1$ and $\alpha_2$ used. The designer is thus free to choose these shape parameters so as to satisfy other requirements.

**Example Problem 5.**

Determine the membrane stresses in a conical vessel of height $h$ and half apex angle $\alpha$. The cone forms the bottom of a large vessel filled with a liquid of specific weight $w$ and having a head of liquid $H$ above the cone. The complete unit is supported from above.
SOLUTION:

The meridian curve of the cone has a radius of curvature \( R_m = \infty \), and at any point a distance \( y \) down from the top of the cone, the radius \( R_t \) is

\[
R_t = \frac{(h - y) \tan \alpha}{\cos \alpha}
\]

Then from eq. 1, at any level \( y \)

\[
N_t = p R_t = \frac{w (H + y) (h - y) \tan \alpha}{\cos \alpha}
\]

To find the meridional stresses \( N_m \) in the cone, the equilibrium of a segment of height \( h - y \) is considered (Fig. A16-5b). Summing forces vertically,

\[
N_m = 2 \pi \frac{(h - y) \tan \alpha}{\cos \alpha} + \frac{w H (h - y)}{2} + \frac{w H^2}{2} (h - y)^2
\]

Solving

\[
N_m = \frac{w \tan \alpha}{\cos \alpha} (h - y) \left[ H + y + \frac{1}{2} (h - y) \right]
\]

A16.4 Displacements, Boundary Conditions and Local Bending in Thin-Walled Shells.

It is appropriate at this point to examine some of the foregoing illustrative cases to determine whether or not the membrane stresses computed gave satisfactorily accurate measures of the shell stresses. Anticipating the answer, we state that, while the membrane analysis will give the primary stress system in a shell-like pressure vessel, a careful (and often lengthy) analysis of induced bending caused by boundary effects will reveal localized secondary stress peaks. In static strength analyses of properly designed vessels it is the practice to neglect these secondary stress peaks, arguing that local yielding of the material will level them out. However, such stress peaks may prove to be of great importance in cases of repeated loadings wherein fatigue failure is considered likely.

To point up the major weakness in the membrane analysis one need only compute the radial displacements in the two different elements that make up the pressure vessel of Fig. A16-6, viz., the cylinder and the hemispherical bulkhead. By Hooke's law, the tangential strain in the cylinder is

\[
\varepsilon_{t_{\text{CYL}}} = \frac{1}{E_t} \left( \sigma_t - \mu \sigma_m \right)
\]

\[
(\mu = \text{Poisson's ratio}, = 0.3 \text{ for aluminum})
\]

\[
= \frac{p R_o}{E_t} \left( 1 - \mu \right)^2 = 0.85 \frac{p R_o}{E_t}
\]

By integration the radial displacement of any point on the cylinder is seen to be (ref. Fig. A16-10).

\[
\delta_{\text{CYL}} = \int_{0}^{\pi/2} \varepsilon_{t_{\text{CYL}}} \cos \theta \, R_o \, d\theta = 0.85 \frac{p R_o^2}{E_t}
\]

Adjacent to the seam, the tangential strain in the hemispherical bulkhead is

\[
\varepsilon_{t_{\text{BLKHD}}} = \frac{p R_o^2}{2E_t} \left( 1 - \mu \right) = 0.35 \frac{p R_o^2}{E_t}
\]

Hence, by integration

\[
\delta_{\text{BLKHD}} = 0.35 \frac{p R_o^2}{E_t}
\]

Thus the cylinder tends to expand more than the bulkhead - a situation prevented by the seam between these elements. It follows then that the seam experiences a transverse shearing action as indicated in Fig. A16-11. These shear forces in turn produce bending moments in the shell wall as shown on the figure.

---

* Various codes and standards give proportions of common vessels which will correctly limit secondary stresses. See for example reference (1).
While it is not our purpose here to take up shell bending in detail, some indication of the character and magnitude of these bending stresses should be available to place them in proper perspective. The most striking thing about these wall moments is that they are quickly dampened out, becoming negligibly small (down to 1% of their maximum value) at a distance of about $\frac{4}{3}R_0 t$ from the seam. Thus, for an instance, in a circular cylindrical shell of 40" radius and .085" wall thickness, these moments are so damped at 6.5" from the seam.

The next important consideration is an appreciation of the magnitude of these secondary bending stresses. For the case of the pressure vessel of Fig. A16-5, the meridional stresses are increased about 30% at the point of maximum moment, while the tangential stresses are increased only about 3%. Fortunately, in this class of vessel, the tangential stresses are the ones designed by (they are twice as great as the meridional stresses) and hence the secondary stresses have little importance for this case (see Chap. 11, pp. 389-422 of reference 2). In other configurations one is not always so fortunate, and detailed analysis may be required. (see references 3, 4, 5 and 6).

The situation at the seam of the above vessel is typical of many seams or boundaries where elements are joined which would experience different expansions if loaded separately. Among such seams and boundaries are those:

1. where the meridional curvature changes abruptly. It changes from $R_m = R_0$ to $R_m = \infty$ at the seam in Fig. A16-5.

2. where a sudden change in direction of the meridional curve occurs. In Example Problem 5, above, considerable shell wall bending would be induced near the seam. In fact, a reinforcing ring would probably have to be added at the seam as was done in Example Problem 2, above.*

3. at which structural members of different stiffnesses and different loadings join. In Example Problem 2, the cylinder tends to expand the most, the bulkhead quite a bit less, and the reinforcing ring, being loaded in compression, tends to contract. Other seams and/or boundaries of this type are those where an abrupt change in shell wall thickness occurs (addition of a doubler) or where a shell is fastened into a foundation.

Good design tends to minimize the magnitude of the secondary bending stresses by avoiding combinations of elements which would have highly incompatible distortions. Thus, the analysis of Example Problem 3 shows that if one closes a circular pressure cylinder with an elliptical bulkhead in which $a = 2b$, compressive tangential stresses would develop in the bulkhead. In such a case the bulkhead would tend to contract radially while the main cylinder would tend to expand as always. Thus, the shear and induced moment at the seam would be aggravated, producing (as it happens) a tangential maximum stress 13% above the membrane stress (as against only 3% above for the hemispherical bulkhead) (reference 2, p. 410). For this type of bulkhead, boiler codes sometimes permit a ratio of $a/b$ as high as 2.5, however.

A16.4 Special Problems in Pressurized Cabin Stress Analysis.

Because of functional requirements over and above those of a simple pressure vessel, the pressurized cabin shell of an airplane has a number of stress analysis problems peculiar to its configuration. Several of the more general of these will be considered here.

**DISTRIBUTION OF STRESSES BETWEEN SHELL AND STRINGERS.**

To stabilize the shell wall in transmitting heavy tail loads through the fuselage, longitudinal stringers are added. These same stringers will also help to carry the meridional pressure loads. The skin and stringers must of course carry strains in the longitudinal directions but, because the skin is in a two-dimensional state of stress, they cannot have equal longitudinal stresses; hence the following analysis.

Let the meridional (longitudinal) stresses in the skin and stringers be $s_m$ and $s_s$, respectively. $s_t$ will be the tangential (hoop) stress in the skin. From eq. (1) we again have

$$s_t = \frac{D R_o}{t}$$

If $N$ is the total number of stringers, each of cross sectional area $A_L$, then equilibrium longitudinally requires

$$p \pi R_o^2 = 2 \pi R_o t s_m + N A_L s_L.$$  

The condition of equal longitudinal strain in the skin and stringers yields

$$\varepsilon = \varepsilon_L = s_m - \mu s_t$$

where $\mu$ is Poisson's ratio (= .3 for aluminum).

Solving these three equations one finds

$$s_t = \frac{D R_o}{t}$$

$$s_m = \frac{D R_o}{2t} \left( \frac{1 + 2 \mu}{1 + \mu} \right) = \frac{D R_o}{2t} \left( \frac{1 + \mu}{1 + \mu} \right)$$

$$s_L = \frac{D R_o}{2t} \left( \frac{1 - 2 \mu}{1 + \mu} \right) = \frac{D R_o}{2t} \left( \frac{1 - 2 \mu}{1 + \mu} \right)$$

where $\mu = \frac{NA_L}{2nt}$ is the ratio of total stringer area to skin area. A little study will show that $t/\mu$ is a sort of "effective shell wall thickness": it is the result of taking all the cross sectional area (skin plus stringers)

* Certain details of the design of such reinforcing rings are given in the codes and standards.
and distributing it uniformly around the perimeter. On this basis, the results are a little disappointing: the stringers are carrying only 40% of the stress one might expect if the net longitudinal load \((2 \pi R \Delta a)\) were distributed evenly over the entire cross sectional area \((2 \pi R \Delta (1 + a))\).

The meridional skin stresses are reduced by the factor \(1 + 0.6 a / (1 + a)\) from what they would be without the stringers. For structures of usual proportions this decrease may amount to 20 to 30% but clearly can never exceed 40%. Inasmuch as the bending stresses due to tail loads will be superposed on these pressure membrane stresses, the reduction is certainly beneficial.

**Interaction Between Rings and Shell.**

Because of the necessity for transmitting various concentrated loads from within the cabin and from the wings and tail to the main shell and because it is also necessary to provide some lateral restraint which will stabilize the stringers and skin against an overall instability failure, the pressurized fuselage of an airplane contains a considerable number of rings and frames distributed along the length of the shell. These rings are seldom, if ever, spaced closely enough such that they can be considered effective in carrying a part of the hoop stresses (in the way the stringers were effective in carrying part of the meridional stress). Rather, they act more like widely spaced restraining bands having the effect shown exaggerated in Fig. A15-12.

![Fig. A16.12 Restraining rings along a pressurized tank. The action is representative of a fuselage with widely spaced rings inside.](image-url)

It is obvious that the rings in this case will produce secondary bending stresses in the skin and hence may have a detrimental effect on the simple membrane stress system. Equally harmful are the tensile loadings developed in the rivets joining the skin and rings. Detailed analyses which will permit quantitative evalu-

*If one looks at the problem from the point of view of a stiffened shell, loaded primarily by bending and shear loads from the tail, on which the pressure membrane loadings are to be superposed, an interesting effect appears. Because the internal pressure tends to stabilize the curved skin panels on the compression side, the effective width of skin acting with the stringers is increased. The section properties of the cross section may then change in such a way as to produce little or no variation in the maximum fiber stresses. Indeed, the maximum tensile stresses may actually be reduced by the addition of the internal pressure loading. (see reference 7).*

**Variations of this type of "mount" suggests themselves, some of which may have merit for other reasons. For instance, the transmittal of wind and other vibration noise into the cabin of a high speed transport is a problem which might be treated simultaneously by the proper choice of connection between the ring and the shell.**

**Doors and Windows.**

The various cutouts in the shell of a pressurized cabin require special consideration if an excessive weight penalty is to be avoided.

Consider the panel removed from the pressurized cylinder of Fig. A16-14a. Following a common practice in dealing with cutouts, we determine what forces the panel-to-be-removed applies to the main structure around the border of the cutout, and then superpose a set of equal but opposite, self-equilibrating stresses to cancel these. The cutout border is then unstressed and the panel may be removed without disturbing the new stress system in the main structure.
In examining the figure to determine what sort of canceling stress system must be supplied, we see that the tangential hoop stresses bordering the cutout cannot be canceled by a self-equilibrating set since they have a radial component. However, the radial component of these stresses will actually be supplied by the door or window pressing outward against its frame. Hence, it is only the component of the hoop stresses along a chord which need to be canceled (Fig. A16-14b)*.

The immediate problem becomes one of designing a structure to effectively support a set of uniformly distributed self-equilibrating stresses acting in the plane of the chord connecting the upper and lower edges of the opening (Fig. A16-15a).

Fig. A16.15

All that appears necessary to support the stress system is to provide horizontal headers at the top and bottom of the cutout, which, as beams, will carry the loads across to the sides of the frames where the loads cancel (Fig. A16-15b). For cutouts of usual sizes in pressurized fuselages, the stress system to be supported in this manner is quite large and it proves uneconomical to design a single horizontal frame member of sufficient bending stiffness to resist them. Instead, the shell wall itself is employed to help carry these loads across. The skin is used to form a beam of considerable depth, the skin being the web of this beam, with the horizontal frame member and one or more longitudinals forming the beam flanges (Fig. A16-15c).

Because of the heavy shear flows and direct stresses developed, the skin is usually doubled in this region. Additional stringers may also be added to relieve the stresses. The rings bordering the cutout (and forming part of the frame) are extended some distance above and below the cutout proper (unless they coincide with a regular ring location, in which case they carry all the way around).

* Clearly one of the design requirements will be to make the frame sufficiently stiff in bending against radial forces so that the door or window can bear up evenly against the frame.

Fig. A16.16 shows the typical cutout structural arrangement. While analytical approaches have been tried, it is probably safe to say that the true elastic stress distribution in such a configuration cannot be computed. The necessity for avoiding high intensity stress concentrations (with their attendant fatigue likelihood) makes empirical information most useful in such cases. On the other hand, a simple rational analysis, based on principles outlined above, will very likely suffice for a static strength check and for most design purposes. (Also see reference 8, pp. 16-23).

The above discussion has concentrated attention on the problems of carrying the hoop stresses around a cutout. The longitudinal pressure stresses, while being smaller themselves, are intensified by bending stresses from the tail loads. Hence, the longitudinal stresses across the cutout may make this condition (or the combination) most severe.

Fig. A16.16 Structural arrangement around a cutout. Most or all of the shaded skin area would probably be doubled.

LARGE DEFORMATIONS OF PLANE PANELS; "QUILTING".

The use of flat skin panels in a pressurized fuselage cannot always be avoided. Since the thin skin has little bending stiffness, it cannot support the lateral pressure as a beam ("plate", more correctly) and hence must deflect to develop some tensile membrane stresses which will then carry the loading. The resultant bulges of the rectangular skin panels between their bordering stiffeners give a "quilting" appearance to the surface.

Even in the case of curved skin panels quilting will occur: if the internal stiffening framework (transverse rings and frames and longitudinal stringers) is relatively rigid and is everywhere tightly fastened to the skin, then each skin panel is restrained along its
four sides (borders) against the radial expansion normally associated with the shell membrane stresses. The result is a sort of three-dimensional-case of the behavior depicted in Fig. A15-12.

From a structural viewpoint, the unfortunate aspects of quilting lie in the high concentration of stresses occurring near the panel edges and in the tensile loadings on the rivets which join shell to stiffeners. The aerodynamic characteristics of a quilted surface are highly undesirable in a high performance airplane; hence again quilting is to be avoided.

Computations of stresses in quilted panels, inasmuch as they involve large, nonlinear deflections, are difficult. An additional (and quite necessary) complication is that of having to introduce the stiffness properties of the bordering members. The reader is referred to Chapter A15.7 for a further discussion of the problem. A simplified approach, indicative of trends, is given there along with further references to the literature.

A16.3 Shells of Revolution Under Unsymmetrical Loadings.

Problems in which the shell of revolution experiences unsymmetrical loadings are not uncommon in aircraft structural analysis. The nose of a fuselage, the external fuel tank and the protruding radomes are shells of revolution which may be loaded unsymmetrically by external aerodynamic pressures. Again, the same external fuel tank shell receives an unsymmetric internal hydrostatic pressure load from the weight of fuel directed normal to the shell axis.

Because of the unsymmetry of the problem, membrane shear stresses are now present and so the analyst must solve not two, but three equations in three unknowns (N_r, N_o, and M_θ). Moreover, these become differential rather than algebraic equations.

Because the derivation of the differential equations of equilibrium is rather lengthy, and because their general solution cannot be written (rather, only specific solutions for certain cases may be found), no details are reproduced here. The reader is referred to pp. 373-379 of reference 2 for the derivation of the equations and for an example problem.

* One design which reduces quilting, in the curved skin, fastens rings and frames to the inner surface of "hat" section stringers only. Thus the ring is not directly fastened to the skin which is therefore not continuously restrained around each ring circumference. The result is a modified floating skin.

REFERENCES


(2) Timoshenko, S. "Theory of Plates and Shells" McGraw-Hill, N.Y., 1940


Designs by Kraft
Ehrlich of Convair
for Space Travel

Outter Space Vehicles will Present Many New Problems to the Aeronautical Structures Engineer.

* American Petroleum Institute - American Society of Mechanical Engineers.
Douglas DC-8 Under Construction and Assembly. Fuselage is Pressurized to Permit High Altitude Flight.

Section of DC-8 Fuselage being Lowered into Hydrostatic Test Tank.
A17.1 Introduction.

It was seen in the last chapter that thin curved shells can resist lateral loadings by means of tensile-compressive membrane stresses. As will be seen later, thin flat sheets, by deflecting enough to provide both the necessary curvature and stretch, may also develop membrane stresses to support lateral loads. In the analysis of these situations no bending strength is presumed in the sheet (membrane theory).

In contrast to the membrane, the plate is a two-dimensional counterpart of the beam, in which transverse loads are resisted by flexural and shear stresses, with no direct stresses in its middle plane (neutral surface).

The skin may also be classified as either a plate or a membrane depending upon the magnitude of transverse deflections under loads. Transverse deflections of plates are small in comparison with the plates' thicknesses — on the order of a tenth of the thickness. On the other hand, the transverse deflections of a membrane will be on the order of ten times its thickness. *

Unfortunately for the engineers' attempts at an orderly cataloging of problems, most aircraft skins fall between the above two extremes and hence behave as plates having some membrane stresses.

Plate bending investigations have for a longtime been important in aircraft structural analyses in their relation to sheet buckling problems. Recently they have assumed new importance with the introduction of thick skinned construction and still more recently with the use of very thin low aspect ratio wings and control surfaces which behave much like large plates, or even are plates in some cases.

It is the purpose of this chapter to present briefly the classic plate formulas and some applications. Appropriate references are cited in lieu of an exhaustive treatise, which could hardly be presented in one chapter (or even one volume) as witness the voluminous literature on the subject.

A17.2 Plate Bending Equations**.

Technical literature in this field abounds with many excellent and elegant derivations of the plate bending equations (references 1 and 2, for instance). Rather than labor the subject

*As will be seen later, the presence or absence of membrane stresses is not wholly dependent upon the magnitude of deflections, but is also determined by the form of deflection surface assumed by the sheet (in turn dependent upon the shape of boundary and loading).

**The assumptions implicit in the following analysis are spelled out in detail in Art. A17.5, below.

with another such, we write the equations down by a direct appeal to past experience and intuition.

Fig. A17.1 shows the differential element of a thin, initially flat plate, acted upon by bending moments (per unit length) $M_x$ and $M_y$ about axes parallel to the y and x directions respectively. Sets of twisting couples $M_{xy} (= - M_{yx})$ also act on the element.

As in the case of a beam, the curvature in the $x, z$ plane, $\frac{\partial^2 w}{\partial x^2}$, is proportional to the moment $M_x$ applied. The constant of proportionality is $1/EI$, the reciprocal of the bending stiffness. For a unit width of beam $I = t^3/12$. In the case of a plate, due to the Poisson effect, the moment $M_y$ also produces a (negative) curvature in the $x, z$ plane. Thus, altogether, with both moments acting, one has

$$\frac{\partial^2 w}{\partial x^2} = \frac{12}{Et^3} (M_x - \mu M_y)$$

where $\mu$ is Poisson's ratio (about .3 for aluminum). Likewise, the curvature in the $y, z$ plane is

$$\frac{\partial^2 w}{\partial y^2} = \frac{12}{Et^3} (M_y - \mu M_x) .$$

These two equations are usually rearranged to give the moments in terms of the curvature. They are written

$$M_x = D \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = D \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)$$

where $D = Et^3/12(1 - \mu^2)$

The twist of the element, $\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x}$, is the change in $x$-direction slope per unit distance in the $y$-direction (and
BENDING OF PLATES

(and visa versa)***. It is proportional to the twisting couple $M_{xy}$. A careful analysis (see references 1 and 2) gives the relation as

$$M_{xy} = D (1 - \mu) \frac{\partial^2 w}{\partial x \partial y} \quad \quad \quad \quad \quad (3)$$

Equations (1), (2) and (3) relate the applied bending and twisting couples to the distortion of the plate in much the same way as does $M = EI \frac{d^2 y}{dx^2}$ for a beam.

While a few highly instructive problems may be solved with these equations (see reference 1, pp. 45-49 and reference 2, pp. 111-113), they are of little technical importance. Hence we move on to consider bending due to lateral loads.

Fig. A17-2 shows the same plate element as in Fig. A17-1, but with the addition of internal shear forces $Q_x$ and $Q_y$ (corresponding to the "y" of beam theory) and a distributed transverse pressure load $q$ (psf). With the presence of these shears, the bending and twisting moments now vary along the plate as indicated in Fig. A17-2a. (For clarity, the several systems of forces on the plate element were separated into the two figures of Fig. A17-2. They do, of course, all act simultaneously on the single element).

$$\text{Fig. A17.2. The differentials are increments which should be written more precisely as, for instance, } \frac{dQ_y}{dy} = \frac{\partial Q_y}{\partial y} dy.$$  

The next relations are obtained by summing moments in turn about the x and y axes. For example, we visualize the two loading sets of Fig. A17-2 acting simultaneously on the single element, and sum moments about the y axis.

$$M_x dy + (M_{yx} + d M_{yx}) dx + (Q_x + d Q_x) dx dy = (M_x + d M_x) dy + M_{yx} dx$$

Dividing by $dx dy$ and discarding the term of higher order gives

$$Q_x = \frac{\partial M_y}{\partial y} - \frac{\partial M_{yx}}{\partial x}$$

or,

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} \quad \quad \quad \quad \quad \quad \quad \quad (4)$$

In a similar manner, a moment summation about the x axis yields

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad \quad \quad \quad \quad (5)$$

(Equations (4) and (5) correspond to $V = dy/dx$ in beam theory).

One final equation is obtained by summing forces in the z direction on the element:

$$q = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \quad \quad \quad \quad \quad (6)$$

Equations (4), (5) and (6) provide three additional equations in the three additional quantities $Q_x$, $Q_y$ and $q$. The plate problem is thus completely defined.

To summarize, we tabulate below the quantities and equations obtained above. For comparison, the corresponding items from the engineering theory of beams are also listed.

<table>
<thead>
<tr>
<th>CLASS</th>
<th>ITEM</th>
<th>PLATE THEORY</th>
<th>BEAM THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry</td>
<td>Coordinates</td>
<td>x y</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>Deflections</td>
<td>$w$</td>
<td>y</td>
</tr>
<tr>
<td></td>
<td>Distortions</td>
<td>$\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x \partial y}$</td>
<td>$\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$</td>
</tr>
<tr>
<td>Structural</td>
<td>Characteristic</td>
<td>$D = \frac{E t^3}{12 (1 - \mu^2)}$</td>
<td>$EI$</td>
</tr>
<tr>
<td>Loadings</td>
<td>Shear $Q_x$, $Q_y$</td>
<td>$M$</td>
<td>$M$</td>
</tr>
<tr>
<td></td>
<td>Lateral $q$</td>
<td>$q$</td>
<td>$q$</td>
</tr>
<tr>
<td>&quot;Hooke's</td>
<td>Moment-</td>
<td>$M_x = D \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial x}\right)$</td>
<td>$M = EI \frac{d^2 w}{dx^2}$</td>
</tr>
<tr>
<td>Law&quot;</td>
<td>Distortion</td>
<td>$M_y = D \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x^2}\right)$</td>
<td>$M = EI \frac{d^2 w}{dx^2}$</td>
</tr>
<tr>
<td></td>
<td>Relation</td>
<td>$M_{xy} = D (1 - \mu) \frac{\partial^2 w}{\partial x \partial y}$</td>
<td>$M = EI \frac{d^2 w}{dx^2}$</td>
</tr>
<tr>
<td>Equilibrium</td>
<td>Moments</td>
<td>$Q_x = \frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x}$</td>
<td>$V = \frac{d M}{dx}$</td>
</tr>
<tr>
<td></td>
<td>$Q_y = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y}$</td>
<td>$V = \frac{d M}{dx}$</td>
<td></td>
</tr>
<tr>
<td>Forces</td>
<td>$q = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}$</td>
<td>$q = \frac{d q}{dx}$</td>
<td></td>
</tr>
</tbody>
</table>

Finally, the result (which the student should obtain by himself as an exercise) is a relation between the lateral loading $q$ and the deflections $w$**:

$$\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} = \frac{q}{D} \quad \quad \quad (7)$$

*** If $w$, the deflection function, is a continuous function of $x$ and $y$ (as it must be, of course, in any technically important plate problem) then at each point $\partial^2 w/\partial x \partial y = \partial^2 w/\partial y \partial x$, as is proven in the calculus.

** the corresponding equation for a simple beam is $q/ EI \neq dy/dx$. 
The plate bending problem is thus reduced to an integration of eq. (7). For a given lateral loading q(x, y), a deflection function w(x, y) is sought which satisfies both eq. (7) and the specified boundary conditions. Once found, w(x, y) can be entered into eqs. (1) to (5) to determine the internal forces and stresses.

A17.3 An Illustrative Plate Bending Analysis.

Assume a lateral loading applied to a rectangular plate having all edges simply supported (hinged). The coordinates are chosen as in Fig. A17-2. With foreknowledge of the general usefulness of the result, we assume a sinusoidal loading of the form

\[ q = q_{m n} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \quad (8) \]

where \( q_{m n} \) is the unknown deflection amplitude. This trial deflection function is known to satisfy the boundary conditions on the plate since at \( x = 0, \) and at \( y = 0, \) we have

\[ w = 0 \quad \text{(zero deflection at the supported edges)} \]

\[ \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{(zero moment at the hinged edges: see eqs. 1 and 2)} \]

It remains only to find the value of \( q_{m n} \) which will satisfy eq. (7). Substituting (8) and (9) into (7) one obtains

\[ \lambda_{m n} = \frac{q_{m n}}{D} \frac{1}{\pi^4} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \]

Hence the required deflection surface (and the solution to the problem) has the equation

\[ w = \frac{q_{m n}}{D} \frac{1}{\pi^4} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \quad (10) \]

The maximum deflection is seen to occur where the trigonometric functions have values of unity and q is also a maximum.

If eq. (10) is substituted into eqs. (1), (2), and (3) one obtains

\[ M_x = -\pi^4 \frac{m^2}{a^2} \frac{n^2}{b^2} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \]

\[ M_y = -\pi^4 \frac{m^2}{a^2} \frac{n^2}{b^2} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \]

\[ M_{xy} = \pi^4 \frac{m^2}{a^2} \frac{n^2}{b^2} \frac{1}{ab} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \]

In a similar manner the transverse shears may be found from eqs. (4) and (5).

With such results as these the plate's stresses may be determined as desired. For example, the maximum direct bending stresses are seen to occur where the shear stresses (due to \( M_{xy} \)) are zero. Thus

\[ S_x = \frac{M_y}{I} \]

and hence

\[ S_{x \text{MAX}} = \frac{M_{xy}}{I} \frac{1}{t^2} = \frac{6 M_{xy}}{t^2} \quad (11) \]

The reader having a familiarity with Fourier series methods will recognize immediately that the above analysis provides the key to the solution of the problem of any general loading q(x, y) on the same plate. Such an application is made by determining the proper combination of sinusoidal pressure terms (each of the form of eq. 8) such that their sum will closely represent the desired loading. The sum of the corresponding deflection functions (each of the form of eq. 10) gives the desired solution. Details of this type of analysis are to be found in reference 1 on pp. 113-176 and 199-258.

In common with all problems which are formulated in terms of a partial differential

**the uniqueness of solutions to the differential equation of the form of eq. (7) is a classical proof appearing in numerous advanced texts on mathematics and mathematical physics. Since the equation is known to have a unique solution, then any solution found for it is the one and only correct solution.**
A17.4 BENDING OF STRESSES

equation, the solution of the plate bending problem depends strongly upon the boundary conditions (both the shape of the boundary and the types of support provided there). The above example may be said to have been deceptively easy because of both the simple shape of the boundary and the type of support. Plate problems wherein the plate planform is not a simple geometric figure must be solved by numerical means. As to the type of support, a full discussion of boundary conditions for plates is to be found in reference 1, pp. 89-95.

A17.4 Computations of Results for Plate Bending Problems.

Fortunately for the practicing engineer, it is not necessary to perform analytic computations as discussed above for the great majority of practical plate problems. Problems of the type illustrated above, plus the myriad variations possible, became very fashionable exercises amongst mathematicians following the discovery by Lagrange of eq. 7 in the year 1751. The results of many researchers’ labors have been compiled in various forms for handy reference.

A common and important case is that of a uniformly loaded rectangular plate (Fig. A17-4). The major engineering results are the values of the maximum deflections and the maximum stresses developed. These may be put in the form (a is the length of the short side):

\[ w_{\text{MAX}} = \frac{q a^4}{E t^3} \]  \hspace{1cm} (12)

\[ s_{\text{MAX}} = \beta \frac{q a^4}{t^3} \]  \hspace{1cm} (13)

where the coefficients \( a \) and \( \beta \) are given in Table A17.1 for the four most common edge conditions.

![Diagram of a rectangular plate](image)

**Fig. A17.4**

Similar presentations may be made for many dozens of other cases. With the ready availability of comprehensive catalogues of these problems in references devoted to the purpose, there appears to be little virtue in duplication here. Hence the following list of selected references is presented. Additional references are to be found in turn within these works. We note that, because of the linearity of the plate bending problem, superposition of solutions is possible to extend even further the usefulness of these extensive listings.

### Table A17.1

<table>
<thead>
<tr>
<th>( b/a )</th>
<th>All Sides Clamped</th>
<th>Long Sides Clamped, Short Sides Pinned</th>
<th>Short Sides Pinned, Long Sides Pinned</th>
<th>Short Sides Pinned, All Sides Clamped</th>
<th>Long Sides Clamped, All Sides Pinned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0443</td>
<td>0.2874</td>
<td>0.0209</td>
<td>0.420</td>
<td>0.0136</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0616</td>
<td>0.2756</td>
<td>0.0460</td>
<td>0.224</td>
<td>0.462</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0770</td>
<td>0.2126</td>
<td>0.0502</td>
<td>0.262</td>
<td>0.466</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0906</td>
<td>0.1712</td>
<td>0.0686</td>
<td>0.273</td>
<td>0.500</td>
</tr>
<tr>
<td>1.8</td>
<td>0.1017</td>
<td>0.1468</td>
<td>0.0799</td>
<td>0.319</td>
<td>0.502</td>
</tr>
<tr>
<td>2.0</td>
<td>0.1108</td>
<td>0.1202</td>
<td>0.0867</td>
<td>0.314</td>
<td>0.504</td>
</tr>
<tr>
<td>2.4</td>
<td>0.1335</td>
<td>0.1034</td>
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<td>0.509</td>
<td>0.508</td>
</tr>
<tr>
<td>3.0</td>
<td>0.1400</td>
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<td>0.518</td>
</tr>
<tr>
<td>4.0</td>
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<td>0.0746</td>
<td>0.202</td>
<td>0.528</td>
<td>0.524</td>
</tr>
<tr>
<td>5.0</td>
<td>0.1422</td>
<td>0.0700</td>
<td>0.242</td>
<td>0.535</td>
<td>0.538</td>
</tr>
</tbody>
</table>

**Rectangular Plates Under Various Loadings**


**Circular Plates Under Various Loadings**

(same three references, in order)

- pp. 55-64, 257-287.
- pp. 128-132.
- pp. 194-201, 209-211.

A17.5 Deflection Limitations in Plate Analyses.

In the introductory remarks of this chapter it was stated that a plate may be distinguished from a membrane by the small order of its deflections (on the order of a few tenths of its thickness). We will re-examine this statement here to show that this is not so much a definition as it is an accuracy limitation imposed by one of the assumptions made in the plate analysis.

There are several familiar assumptions from beam theory which, of course, carry over here, inasmuch as the plate analysis resembles the beam analysis rather closely. These "beam theory assumptions" are:

1. elastic stresses only are presumed,
2. small slopes (so that \( \theta_x'w' = \theta_y^2' \) and \( \theta_y'w' = \theta_x^2' \) are good approximations to the curvatures),
3. at least one transverse dimension (length or width) be large compared to the thickness so that shear deflections may be neglected.
However, the beam theory assumptions do not place a very severe restriction on the magnitude of deflections permitted. Deflections of several times the plate thickness would be permissible if these were the only restricting assumptions.

In deriving the plate bending equations it was assumed that no stresses acted in the middle (neutral) plane of the plate (no membrane stresses). Thus, in summing forces to derive eq. (5), no membrane stresses were present to help support the lateral load. Now in the solutions to the great majority of all plate bending problems (solved as in Art. A17.3), the deflection surface solution found is a non-developable surface, i.e., a surface which cannot be formed from a flat sheet without some stretching of the sheets' middle surface*. But, if appreciable middle surface strains must occur, then large middle surface stresses will result, invalidating the assumption upon which eq. (6) was derived.

Thus, practically all loaded plates deform into surfaces which induce some middle surface stresses. It is the necessity for holding down the magnitude of these very powerful middle surface stretching forces that results in the more severe rule-of-thumb restriction that plate bending formulas apply accurately only to problems in which deflections are a few tenths of the plates' thickness.

A17.6 Membrane Action in Very Thin Plates.

There is still another source of middle surface strains in plates: this is the restraint against in-plane movements offered by the edge supports. While not important in problems wherein deflections are limited in accordance with the restriction of the last article, such restraint does assume great importance in the case of large deflections of very thin plates which support a major share of the load by membrane action. It is, in fact, useful to consider the limiting case of the flat membrane which cannot support any of the lateral load by bending stresses and hence has to deflect and stretch to develop both the necessary curvatures and membrane stresses.

The two-dimensional membrane problem is a nonlinear one whose solution has proven to be very difficult. Rather than attempt to treat the complete problem, we can study a simplified version whose solution retains the desired general features. The one-dimensional analysis of a narrow (unit width) strip will be treated. This strip is cut from an originally flat membrane whose extent in the y-direction is very great (Fig. A17-5a).

\[ \text{Fig. A17.5} \]

Fig. A17-5b shows the desired one-dimensional problem which now resembles a loaded cable. The differential equation of equilibrium is obtained by summing vertical forces on the element of Fig. A17-5c (draw with all quantities; loads, deflections, slopes and curvatures shown positive). One obtains

\[ s \frac{dw}{dx} - \frac{dw}{dx} \left| \begin{array}{c} x + dx \\ x \end{array} \right| + q \, dx = 0 \]

or

\[ \frac{d^2w}{dx^2} = - \frac{q}{st} \]

where \( s \) is the membrane stress in psi.

Eq. (14) is the differential equation of a parabola. Its solution is

\[ w = \frac{q}{2st} \left( a - x \right) \]

The (as yet) unknown stress in eq. (15) can be found by computing the change in length of the strip as it deflects. This "stretch" is given by the difference between the curved arc length and the original straight length (a). Thus:

\[ \delta = a \int_0^1 ds - a \]

\[ = a \sqrt{\frac{w^2}{a^2} + \frac{dx^2}{a^2}} - a \]

Since the slope \( dw/dx \) is small compared with unity, we use the binomial theorem to write

\[ \left( 1 + \frac{(dw/dx)^2}{a^2} \right)^{1/2} \approx 1 + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \]

*The cone and cylinder are examples of developable surfaces, the sphere is a nondevelopable one. It is a familiar experience that the skin of an orange cannot be developed into a flat sheet without tearing.

*here "ds" is the differential arc length of the calculus and has no kinship with the s which denotes the membrane stress throughout the remainder of the analysis.
A17.6 BENDING OF PLATES

Hence
\[
\delta = \int_0^a \left[ 1 + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] dx - a
\]
\[= \frac{1}{2} \int_0^a \left( \frac{dw}{dx} \right)^2 dx .
\]

Substituting through the use of eq. (15) and integrating we find
\[
\delta = \frac{q a^4}{24 s^2} t^2 .
\]

Now by elementary considerations
\[
\delta = \frac{s a}{E} .
\]

Equating these last and solving we find
\[
s = 0.347 \left[ \frac{E}{\left( \frac{1}{2} \right)^{1/2}} \right]^{1/3} .
\]  

If eq. (16) is substituted into eq. (15) one gets for the maximum deflection \( x = \frac{a}{2} \)
\[
\delta_{\text{MAX}} = 0.360 a \left( \frac{q a}{s^2} \right)^{1/3} .
\]  

Equations (16) and (17) display the essential nonlinearity of the problem, the stress and the deflection both varying as fractional exponents of the lateral pressure q.

Of the complete two-dimensional nonlinear membrane problem have been carried out,* the results being expressed in forms identical with those obtained above for the one-dimensional problem, viz.,
\[
\delta_{\text{MAX}} = n_1 \left( \frac{q a}{s^2} \right)^{1/3} .
\]  

\[
\delta_{\text{MAX}} = n_2 \left[ \frac{E}{\left( \frac{1}{2} \right)^{1/2}} \right]^{1/3} .
\]

Here \( a \) is the length of the long side of the rectangular membrane and \( n_1 \) and \( n_2 \) are given in Table A17.2 as functions of the panel aspect ratio \( a/b \).

The maximum membrane stress \( \delta_{\text{MAX}} \) occurs at the middle of the long side of the panel. We note that the limiting case, \( a/b = 0 \), corresponds to the one-dimensional case analyzed earlier. Unfortunately, an extrapolation of these two-dimensional results to that limit does not show agreement with the one-dimensional result. Presumably the discrepancy may be traced to the excessive influence of inaccuracies in the assumed deflection shape of the membrane as used in the approximate two-dimensional solutions.

Experimental results reported in reference 4 show good agreement with the theory for square panels in the elastic range.

<table>
<thead>
<tr>
<th>TABLE A17.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Membrane Stress and Deflection Coefficients</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>a/b</td>
</tr>
<tr>
<td>n_1</td>
</tr>
<tr>
<td>n_2</td>
</tr>
</tbody>
</table>

A17.7 Large Deflections in Plates**

In the previous articles of this chapter the results of analyses were outlined for the two extreme cases of sheet panels under lateral loads. At one extreme, sheets whose bending stiffness is great relative to the loads applied (and which therefore deflect only slightly) may be analyzed satisfactorily by the plate bending solutions. At the other extreme, very thin sheets, under lateral loads great enough to cause large deflections, may be treated as membranes whose bending stiffness is ignored.

As it happens, the most efficient plating designs generally fall between these two extremes. On the one hand, if the designer is to take advantage of the presence of the interior stiffening structure (rings, bulkheads, stringers, etc.), which is usually present for other reasons anyway, then it is not necessary to make the skin so heavy as to behave like a "pure" plate. On the other hand, if the skin is made so thin as to necessitate supporting all pressure loads by stretching and developing membrane stresses, then permanent deformation results, producing "quilting" or "washboarding".

The exact analysis of the two-dimensional plate which undergoes large deflections and thereby supports the lateral loading partly by its bending resistance and partly by membrane action is very involved. A one-dimensional

---

*The work of Henry and Fopp in summarized in reference 3, pp. 258-259 and in reference 4. The partial differential equation solved is given in reference 1 on p. 344 (eq. 302) and the approximate method of solution usually employed is sketched out on pp. 343, 345 of this same reference. The reader who would compare presentations amongst these references should note the differences in the definitions of the plate dimensioning symbols "a" and "b".

**The discussion to follow will be concerned primarily with problems dealing with the support of a uniform pressure load on a flat skin panel. It may, therefore, help the reader to fix his ideas if he visualizes the discussion as applied to the problems of analysis of a single rectangular skin panel taken between the stringers and bulkheads of a semiplane hull bottom. Equally useful is the picture of the very nearly flat panel between rings and stringers in the slightly curved side of a large pressurized fuselage.
The approximate large-deflection method outlined above has serious shortcomings insofar as the prediction of stresses is concerned. For simply supported edges the maximum combined stresses are known to occur at the panel midpoint. Fig. A17-7 shows plots of these stresses for a square panel as predicted by the approximate method (substituting q' and q" into eqs. (13) and (14) respectively and cross plotting with the aid of Fig. A17-6*). Also shown are the maximum stresses computed by the exact large-deflection theory (reference 5).

Because of the obvious desirability of using the results of the more exact theory, some of these are presented in Table A17.3. The treatment of additional cases (other types of edge support) may be found in reference 6, pp. 221, 222.

### Table A17.3

| a/b | Large Deflection Rectangular Plate Junctions (Uniform Pressure Load (q), Simply Supported Edges) |
|-----|-----------------------------------------------------------------------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|     | Stress Coefficients (k) | kmax | k2 | k3 | k5 | k6 | k7 | k8 | k9 | k10 | k11 | k12 | k13 | k14 | k15 | k16 | k17 | k18 |
|     |                      |       |    |    |    |    |    |    |    |      |      |      |      |      |      |      |      |      |
| 1   |                      | 1.10  | 1.30 | 1.50 | 0.90 | 1.50 | 0.60 | 1.00 | 0.30 | 1.00 | 0.30 | 1.00 | 0.30 | 1.00 | 0.30 | 1.00 | 0.30 | 1.00 | 0.30 |
| 2   |                      | 1.20  | 1.40 | 1.60 | 1.00 | 1.60 | 0.70 | 1.00 | 0.35 | 1.00 | 0.35 | 1.00 | 0.35 | 1.00 | 0.35 | 1.00 | 0.35 | 1.00 | 0.35 |
| 3   |                      | 1.30  | 1.50 | 1.70 | 1.10 | 1.70 | 0.80 | 1.00 | 0.40 | 1.00 | 0.40 | 1.00 | 0.40 | 1.00 | 0.40 | 1.00 | 0.40 | 1.00 | 0.40 |

* but using n2 = .260 in eq. (19). This value gives the stresses at the center of a square panel whereas n2 = .356 in Table A17.2 is for stresses at the panel edge.
large-deflection plate and the membrane analyses were developed for applications where the plate bending analysis appeared inadequate. However, these analyses themselves presumed conditions seldom encountered in practice.

FIRST, the analyses assume unyielding supports on the boundaries of the sheet panel. In practice, the skin is stretched across an elastic framework of stringers and bulkheads. It follows, therefore, that the heavy membrane tensile forces developed during large deflections will cause the supports to deflect toward each other thereby increasing the plate deflection and relieving some of the stresses.

A simple one-dimensional analysis for a membrane strip having elastic edge supports (parallel to the analysis of Art. A17.6), shows errors on the order of 25 per cent are likely if the framework elasticity is neglected (reference 7). At this writing no two-dimensional treatment of this problem is known to the writer.

SECOND, it is seldom that the analyst has to check a panel for lateral pressure loads alone. Most often, the entire “field” of panels on the framework of stringers and bulkheads must simultaneously transmit in-plane loadings from the tail load bending stresses and the cabin pressurization stresses.

Inasmuch as the large-deflection plate and membrane analyses are nonlinear, it follows that correct stresses cannot be found by a straight superposition. The magnitude of the error introduced by such a procedure is difficult to estimate in the absence of an exact analysis. A one-dimensional analysis, parallel to that of Art. A17.6, but with elastic supports and axial load, is given in reference 7. These results, which indicate the effect of the axial load to be quite important, may be used as a guide in lieu of more complete two-dimensional studies. The interested reader is referred to the original work for details.

REFERENCES

5. (selected large-deflection plate references).
   c) Levy, S. Square Plate With Clamped Edges Under Normal Pressure Producing Large Deflections, NACA TR 740, 1942.
   d) Levy, S. Bending of Rectangular Plates With Large Deflections, NACA TR 747, 1942.
   f) Chi-Teh Wang, Bending of Rectangular Plates With Large Deflections, NACA TN 1462, 1948.
CHAPTER A18

THEORY OF THE INSTABILITY OF COLUMNS AND THIN SHEETS
(BY DR. GEORGE LIANIS)

PART I

ELASTIC AND INELASTIC INSTABILITY OF COLUMNS

A18.1 Introduction.

Part I of this chapter will be confined to the theoretical treatment of the instability of a perfect elastic column and an imperfect elastic column. The column is the simplest of the various types of structural elements that are subject to the phenomenon of instability. The theory as developed for columns forms the basis for the study of the instability of thin plates, which subject is treated in Part 2.

A18.2 Combined Bending and Compression of Columns.

Consider a column with one end simply supported and the other end hinged (Fig. A18.1) under the simultaneous action of a compressive load P and a transverse load Q. Without the load P the bending moment due to Q would be:

\[ M = \frac{Qz}{1} \]

Thus the total bending moment at section z will be:

\[ M = Pu + \frac{Qz}{1} \] on upper portion \( a \) \( (d) \)

\[ M = Pu + Q(1-a)(1-z) \] on lower portion \( a \)

From mechanics of simple bending, we have the deflection equation,

\[ \frac{d^2u}{dz^2} = \frac{M}{EI} \]

Thus the deflection \( u(z) \) of the column is,

\[ \frac{d^2u}{dz^2} = Pu - \frac{Qz}{1}, \quad (0 \leq z \leq 1 - a) \]

\[ \frac{d^2u}{dz^2} = Pu - \frac{(1-a)(1-z)}{1}, \quad (1 - a \leq z \leq 1) \]

If we introduce the notation,

\[ \frac{P}{EI} = K \]

The general solution of eq. (2) is:

\[ u = C_1 \cos Kz + C_2 \sin Kz - \frac{Qaz}{Pl}, \quad (0 \leq z \leq 1 - a) \]

\[ u = C_3 \cos Kz + C_4 \sin Kz - \frac{Q(1-a)(1-z)}{1}, \quad (1 - a \leq z \leq 1) \]

where \( C_1, C_2, C_3, \) and \( C_4 \) are constants of integration to be determined from boundary conditions.

For eqs. (4), since \( u = 0 \) for \( z = 0 \) and \( z = l \), it follows that:

\[ C_1 = 0 \] and \( C_2 = - C_4 \tan Kl \)

At \( z = (l - a) \) the two portions of the deflection curve given by (4a) and (4b) respectively must have the same deflection and slope. From these two conditions we determine \( C_3 \) and \( C_4 \).

Fig. A18.1

On the lower portion of the column:

\[ M_1 = \frac{Qa}{1} \]

On the upper portion:

\[ M_1 = \frac{Q(1-a)(1-z)}{1} \]

Due to the deflection \( u(z) \), the axial load \( P \) contributes to the bending moment by the amount:

\[ M_2 = Pu \]
the body would cause only infinitesimal changes in the displacements and the body recovers if the added loads are removed. When the displacements are continuously increased with little or no further increment of loads, the system is unstable. If the body will remain in the displaced position after the removal of the disturbance, the body is said to be in neutral equilibrium. Having these definitions, we will not investigate the behavior of the column before and after the critical load is reached.

**Fig. A18.2**

Assume, as shown in Fig. A18.2, that a simply supported column loaded by an axial load \( P \) is bent by a small disturbance. If the deflection is \( u \), the bending moment due to \( P \) is \( Pz \).

From basic mechanics, we know that,

\[
\frac{EI}{R} = -M, \quad \text{whence} \quad \frac{EI}{R} = -Pu
\]

The exact expression for the curvature of the neutral axis is:

\[
\frac{1}{R} = \frac{\partial^2 \Theta}{\partial s^2}, \quad \text{where} \ s \ \text{is the arc length of the deformed axis, and} \ \theta \ \text{the angle between the tangent to the curve and the z axis. Thus,}
\]

\[
\frac{EI}{R} \frac{\partial^2 \Theta}{\partial s^2} + Pu = 0
\]

Differentiating (8b) with respect to \( s \) and since \( \frac{du}{ds} = \sin \Theta \), we obtain:

\[
\frac{EI}{R} \frac{d^2 \Theta}{ds^2} + P \sin \Theta = 0
\]

Multiplying (8c) by \( d \Theta \) and noting that: 

\[
\frac{d^2 \Theta}{ds^2} d \Theta = \frac{d \Theta}{ds} d (\frac{d \Theta}{ds}), \quad \text{and integrating}
\]

\[
\frac{EI}{R} \frac{d}{ds} \left( \frac{d \Theta}{ds} \right) + P \sin \Theta = C, \quad \text{or}
\]

\[
\frac{EI}{2} \left( \frac{d \Theta}{ds} \right)^2 - P \cos \Theta = C
\]

Since at end \( A, \Theta = a \) and \( M = EI \frac{d \Theta}{ds} \), we find that \( C = -P \cos a \).
Now let \( k = \sqrt{\frac{P}{EI}} \) \(-\text{(8a)}\)

Then eq. (8d) becomes,

\[
\frac{1}{\sqrt{2}} \frac{d\theta}{ds} = -k \sqrt{\cos \theta - \cos a}, \text{ or}
\]

\[
ds = \frac{d\theta}{\sqrt{2k^2 \left( \cos \theta - \cos a \right)}} \quad \text{---(8f)}
\]

The total length of the column in the deflected shape is given by:

\[
l = \int_0^1 ds = \int_0^1 \frac{d\theta}{\sqrt{2k \sqrt{\cos \theta - \cos a}}} \quad \text{---(8h)}
\]

or

\[
l = \int_0^a \frac{d\theta}{\sqrt{2k^2 \sin^2 \frac{a}{2} - \sin^2 \theta/2}} \quad \text{---(8i)}
\]

Denoting \( \sin \frac{a}{2} \) by \( p \) and introducing a new variable \( \phi \):

\[
\sin \frac{\theta}{2} = p \sin \phi
\]

Equation (8i) then becomes,

\[
l = \frac{a}{k} \int_0^p \frac{d\phi}{\sqrt{1 - p^2 \sin^2 \phi}} = \frac{2k}{K} \quad \text{---(8j)}
\]

where \( K = \int_0^p \frac{d\phi}{\sqrt{1 - p^2 \sin^2 \phi}} \) is called the complete elliptic integral of the first kind, and it can be found in tables. If \( a \) and therefore \( p \) is very small, then \( p^2 \sin^2 \phi \) can be neglected in equation (8j) and then

\[
l = \frac{a}{k} \int_0^p \frac{d\phi}{\sqrt{1 - p^2 \sin^2 \phi}} = \frac{2k}{K} \quad \text{---(8k)}
\]

whence \( P = \frac{\pi^2 EI}{1^2} \quad \text{---(8m)} \)

The deflection at the mid-point of column is:- \( \phi = 0, du = ds \sin \theta, \) and from Eq. (8f)

\[
u(z = \frac{1}{2}) = 5 = \int_0^a \frac{\sin \phi d\phi}{2k \sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\phi}{2}}} \quad \text{---(8l)}
\]

or in terms of \( \phi \):

\[
\phi = \frac{2P}{k} \int_0^a \frac{d\phi}{\sin \phi} = \frac{2P}{k} \quad \text{---(8m)}
\]

From equation (8j) and equation (8k), we obtain

\[
\frac{P}{1} = \frac{\pi^2 EI}{1^2} \quad \text{---(8n)}
\]

\[
\delta = \frac{2P}{k} \quad \text{---(8o)}
\]

Let us now write the bending moment \( M = P \delta \) at the middle point in non-dimensional form:

\[
m = \frac{P}{P_{cr}} \quad \delta = \frac{4KP}{\pi^2} \quad \text{---(8p)}
\]

Since \( p = \sin \frac{a}{2} \) is a function of \( a \) so is the elliptic integral \( K \) and the ratios \( \frac{P}{P_{cr}} \) and \( \frac{\delta}{1} \) calculated from equations (8a) and (8o).

Thus \( \frac{P}{P_{cr}} \) is a function of \( \frac{\delta}{1} \) calculated by means of tables giving elliptic integrals. Thus \( m \) can be plotted against \( \delta/1 \) as shown in Fig. A18.3.

![Fig. A18.3](image-url)

Let us now examine the stability of various equilibrium configurations. Assume that a load \( P^* \) is acting on the column and the column has a certain maximum deflection \( \delta \) where \( P^* \) does not correspond to \( \delta \).

The non-dimensional maximum bending moment is:

\[
\delta^* = \frac{P^*}{P_{cr}} \quad \text{---(8g)}
\]

The \( m^* \) versus \( \delta^*/1 \) curves are straight lines. The column is in equilibrium if \( m = m^* \), or in other words, if the \( m(\delta/1) \) curve intersects the \( m(\delta/1) \) curve.

We see from Fig. A18.3 that these curves intersect at the origin only if \( P^* < P_{cr} \). The column, therefore, has only one possible equilibrium form, for example, that for which \( \delta/1 = 0 \), which is the straight form. When \( P^* = P_{cr} \), there are two points of intersection,
one when $\delta/l = 0$ and the other (point A) for which $\delta/l \neq 0$. The column thus has two possible equilibrium forms, one straight and one bent.

Let us now assume that at $\delta = 0$, the column is displaced by a small disturbance and acquires a deflection $\delta$. For $P^* = P_{Cr}$, we see from Fig. 3 that $m^* > m$. Thus $P^*$ is not sufficient to maintain the column in equilibrium in the bent form and it will spring back to its straight form. Thus for $P^* = P_{Cr}$, the straight form is stable.

If $P^* > P_{Cr}$, then $m^* = m$. Thus $m^*$ will bend the column still further. This means that if $P^* > P_{Cr}$, the straight form of equilibrium is unstable. The column will continue to bend until $m^*$ becomes equal to $m$ (point A in Fig. 3). If the column is displaced further from A, the deflection becomes larger than $\delta^*$ and $m^* = m$ at the new position. The column will spring back to point A. Point A is therefore stable.

At $P = P_{Cr}$, the $m^*$ versus $\delta/l$ line is tangent to the $m$ curve at the origin. Therefore, for an infinitesimal disturbance, the column will remain in equilibrium at the displaced position since for such small disturbances $m^*$ remains equal to $m$. The column is therefore in neutral equilibrium.

### A18.4 The Failure of Columns by Compression

In discussing the stability of a column in the previous section, it was shown that below the critical Euler load (Eq. 7), the straight form is stable, above the $P_{Cr}$ the bent form is stable and at $P_{Cr}$ the equilibrium is neutral. By plotting the curve $P/P_{Cr}$ versus $\delta/l$ as shown in Fig. A18.4, we observe the following behavior.

(a) Below $P/P_{Cr} = 1$, there is only one equilibrium position, $\delta/l = 0$.

(b) At $P/P_{Cr} = 1$, or at point (A), a bifurcation of equilibrium occurs and the column starts to acquire two possible neighbor positions of equilibrium, the straight and the bent.

(c) Above $P/P_{Cr} = 1$, the column has two possible equilibrium positions $\delta/l = 0$ and $\delta/l \neq 0$.

Thus as far as initiation of instability is concerned, the Euler load as given by Eq. 7 can be considered as the critical load. The question arises whether this load has a practical use for design purposes. A logical design criterion is obviously the maximum load which a column can sustain. We observe from Fig. A18.4 that the load $P$ increases for increasing displacement $\delta$. This behavior is due to the development of large deflections due to bending. However, over a considerable range of deflections $\delta$, the $P \sim \delta$ curve is practically horizontal (for instance, between points A and B the ratio $\delta/l$ varies from zero to $\approx 0.4$). For such large deflections for which the column load does not change practically, it is obvious that the column ceases to function properly. Therefore, from this point of view, the Euler load can be considered that which characterizes the maximum strength of the column.

The rising part of the curve BD holds as long as the material behaves elastically. At some point B, however, inside the almost flat portion of the curve, the inner fibers of the column acquire maximum stress equal to the yield stress. If we carry out an elastic-plastic analysis of the subsequent behavior, we observe that the curve drops almost immediately. Again this maximum load $P_B$ is very near the Euler load. For design purposes, therefore, the Euler load, which is a buckling load, is a very good approximation to the ultimate load which the column can sustain.

Another argument will confirm the above conclusion. In discussing the buckling of columns in the previous paragraphs, we have assumed that the column is initially straight, centrally loaded and made of homogeneous material. Actual columns, however, are imperfect due to initial crookedness (for instance, due to unavoidable tolerances in their manufacture), due to slight load eccentricities and due to lack of complete homogeneity. Therefore, a certain amount of bending is always present even for small loads.

Let us now examine the behavior of such initially imperfect columns by assuming a certain initial deflection $\delta_0$ of the column axis (see Fig. A18.5). For small deflections,
the change of curvature due to subsequent bending (after loading) is:

\[ \frac{1}{R} = \frac{d^2u}{dz^2} - \frac{d^2u_0}{dz^2} \]

In the differential equation of deflection one can prove that \(1/R\) is the change of curvature which for an initial straight column coincides with the curvature itself. Thus in the present case, where the bending moment is \(P_u\), the equation of deflection becomes:

\[ \frac{d^2u}{dz^2} - \frac{d^2u_0}{dz^2} = - k^2 u \quad (9) \]

Let us express \(u_0\) in Fourier series:

\[ u_0 = \sum_{n=1}^{\infty} \delta_n \sin \frac{n\pi z}{l} \quad (10) \]

Substituting (10) in (9), we find the solution which satisfies the boundary conditions \(u = 0\) for \(z = 0, z = 1\) is:

\[ u = \sum_{n=1}^{\infty} \delta_n \sin \frac{n\pi z}{l} \quad (11a) \]

where \(\delta_n = \frac{\delta_n}{1 - P/P_n} \quad (11b)\)

\[ P_n = \frac{n^2 \pi^2 EI}{l^2} \]

The deflection of the column at the center is:

\[ \delta_{\text{max}} = \delta_1 - \delta_2 + \delta_3 \quad (12) \]

If we plot the deflection versus the load we obtain the curve (Fig. A18.6), which approaches the horizontal line \(P/P_{cr} = 1\) asymptotically. This curve, however, is valid for small deflections for which the approximation:

\[ \delta \left( \frac{1}{R} \right) \approx \frac{d^2u}{dz^2} - \frac{d^2u_0}{dz^2} \quad \text{is valid.} \]

By a treatment similar to that in the previous paragraph, we will find that for large deflections the load deflection curve rises after the point \(I\) (curve \(F\)). Due to the onset of plasticity, the actual curve drops at the point \(I^*\) (curve \(F^{(1)}\)). The failing load at \(I^*\) can be either greater or smaller than \(P_{cr}\), but it is usually very near to it.

In the above discussions we have shown that for all practical purposes the Euler buckling load can be considered as the ultimate load which a real or practical column can sustain. Besides its closeness to the actual ultimate load, the critical load can be easily calculated from equation (7) without the necessity of carrying out a lengthy calculation which will include the initial imperfections and plastic effects.

It should be noted, however, that the buckling load given by equation (7) is valid when the uniform stress due to a compressive load \((\sigma = P/A)\), where \(A\) is cross-sectional area) is below yield stress. If \(\sigma\) is above the yield stress, the theory of plasticity predicts another value for the buckling load. Referring now to equations (11) we find:

\[ P_n = n^2 P_{cr} \quad (P_{cr} \text{ from equation 7}) \]

\[ \delta_n = \frac{\delta_n}{1 - n^2 P_{cr}} \]

Thus as \(P\) approaches \(P_{cr}\), we see that

\[ \frac{\delta_1}{\delta_2} \to \delta_1 \to \frac{4}{3}, \; \delta_2 \to 9/8 \; \text{etc.} \]

Thus \(\delta_1 = \delta_2 = \delta_3 \) and:

\[ \delta_{\text{max}} = \delta_1 = \frac{\delta_n}{1 - P/P_{cr}} \]

In a buckling test we measure \(\delta = \delta_{\text{max}}\)

- \(\delta\), where \(\delta\) is the initial deflection at the middle point. Thus:

\[ \delta = \delta_{\text{max}} = \frac{\delta_1}{P_{cr} - \frac{P}{P_{cr}}} \quad \text{and,} \]

\[ P_{cr} \delta - \delta = \delta_1 \quad (13) \]
If in a buckling test we plot \( \delta / P \) versus \( \delta \), we can obtain the critical load experimentally without knowing the initial deflection \( \delta_0 \). It is simply the inverse of the slope of this curve.

**A18.5 Buckling Loads of Columns with Various End Conditions.**

From the conclusions reached in the previous discussion, we can consider the buckling problem as an instability problem of an initially straight column. Thus we assume a certain deflected position near the straight configuration as another possible equilibrium form and seek the loads under which the non-straight form is possible. Furthermore, only a small deflection analysis is necessary.

The general differential equation of bending-buckling is:

\[
\frac{d^4u}{dz^4} + k^2 \frac{d^2u}{dz^2} = 0
\]

and the general solution is:

\[
u = C_1 \sin kz + C_2 \cos kz + C_3 z + C_4
\]

The coefficients \( C_1, C_2, C_3 \) and \( C_4 \) depend on the conditions of the end supports. The various end conditions are:

**Free end:** \( \frac{d^2u}{dz^2} = 0 \), \( \frac{d^4u}{dz^4} = 0 \)

**Pin end:** \( u = 0 \), \( \frac{d^4u}{dz^4} = 0 \)

**Fixed end:** \( u = 0 \), \( \frac{du}{dz} = 0 \)

Thus we have 4 end conditions. These give systems of four linear homogenous equations. A trivial solution of these is the zero solution. For the buckling state, however, \( C_1, C_2, C_3, C_4 \) are not all zero.

The condition of non-zero solution of the above system is that the determinant of the coefficients of \( C_1, C_2, C_3 \) and \( C_4 \) is equal to zero. From this equation, we calculate the buckling load.

For example, in a simply supported beam, \( (u = 0, \frac{d^2u}{dz^2} = 0 \), at both ends), the end conditions furnish give

\[
C_1 \sin kl + C_2 \cos kl + C_3 l + C_4 = 0
\]

For buckling we must have:

\[
\begin{array}{cccc}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\sin kl & \cos kl & 1 & 1 \\
- \sin kl & - \cos kl & 0 & 0
\end{array}
\]

or \( \sin kl = 0 \) or \( kl = \pi n \) \((n = 1, 2, 3, \ldots)\)

whence \( P_n = \frac{n^2 \pi^2 EI}{l^2} \)

Thus for \( P_n \) equal to the right hand side of equation (16a), we have a possible form of equilibrium of the bent form. The smallest value of \( P_n \) occurs at \( n = 1 \), and this is the buckling load:

\[
Pcr = \frac{\pi^2 EI}{l^2}
\]

The buckling load for other end conditions can be derived in similar manner.

**INELASTIC COLUMN STRENGTH**

**A18.6 Inelastic Buckling. Introduction.**

Euler's theory of buckling is valid as long as the stress in the column nowhere exceeds the elastic limit of the column material. We have seen that the analysis for perfect and imperfect elastic columns leads to the same result, namely, equation (7).

The case of the inelastic buckling, that is, instability under axial load exceeding the elastic limit stress, presents some difficulties. As we will see, the perfect column analysis leads to a different expression for the critical load than for the perfect column. This is due to the fact that in the plastic stress range, the material behaves differently under loading and unloading, as illustrated in Fig. A18.7.

Let us now examine the two cases:

![Diagram](image)

**Fig. A18.7**

**A18.7 Perfect Column. Reduced Modulus Theory.**

Let us assume that the perfect column is first compressed uniformly up to the stress \( \sigma \). To study the critical value, \( \sigma_{cr} \), of \( \sigma \) for which...
the column becomes unstable we assume:—

(1) That the displacements are small so that the relation between the radius of curvature R and the deflection u of the elastic axis is,

\[ \frac{1}{R} = \frac{d^2 u}{dz^2} \]  

\[ \frac{h}{h} = h \frac{d^3 u}{dz^3} \]  

(17a)

(2) Plane sections remain plane, therefore the change of strain due to bending at a distance h on the plane of bending is,

\[ \varepsilon = \frac{h}{h} = h \frac{d^3 u}{dz^3} \]  

(17b)

(3) The stress-strain relation follows the simple tension curve for the material.

(4) The plane of bending is a plane of symmetry of the cross-section.

Assume now a column with the cross-section as shown in Fig. A18.8a be compressed in the plastic stress range and that the compressive stress prior to instability be \( \sigma \). To consider the condition of buckling, let the column be slightly deflected transversely. The stress on one side of the column will then increase due to the bending following the stress-strain curve, while on the other side the stress will decrease and will therefore follow the unloading elastic line shown in Fig. A18.8b.

For small changes of the stress, on the first side, the variation of stress is related to the variation of strain by:

\[ \delta S_1 = E \varepsilon_1 \]  

\[ \delta S_1 = E \varepsilon_1 \]  

where \( E \varepsilon_1 \) is the slope of the stress-strain curve at stress \( \sigma \). On the second side the changes will follow the elastic relation, that is,

\[ \delta S_2 = E \varepsilon_2 \]  

\[ \delta S_2 = E \varepsilon_2 \]  

The distribution of the compressive (-) stress and the tensile (+) stress due to bending is shown in Fig. A18.8a. The stress becomes zero on line \((a \ a')\), which is at a distance \( e \) from the centroid \( c \). For equilibrium of stresses on the cross-section we have,

\[ - \int_0^{a_1} \sigma S_1 dA + \int_0^{a_2} \sigma S_2 dA = 0 \]  

\[ - \int_0^{a_1} \sigma S_1 dA + \int_0^{a_2} \sigma S_2 dA = 0 \]  

(20)

and for equilibrium for moments,

\[ - \int_0^{a_1} \sigma S_1 (h_1 + e) dA + \int_0^{a_2} \sigma S_2 (h_2 - e) dA = Pu \]  

\[ - \int_0^{a_1} \sigma S_1 (h_1 + e) dA + \int_0^{a_2} \sigma S_2 (h_2 - e) dA = Pu \]  

(21)

Due to the linear distribution of stress, we have:

\[ \sigma S_1 = \frac{\partial \sigma}{\partial} h_1 \]  

\[ \sigma S_2 = \frac{\partial \sigma}{\partial} h_2 \]  

(22)

Introducing now (17b), (18), (19) and (22) in equation (20), we obtain,

\[ -E \frac{d^2 u}{dz^2} + E \frac{d^2 \varepsilon}{dz^2} = 0 \]  

\[ -E \frac{d^2 u}{dz^2} + E \frac{d^2 \varepsilon}{dz^2} = 0 \]  

(23)

where,

\[ Q_1 = \int_0^{a_1} h_1 dA, \quad Q_2 = \int_0^{a_2} h_2 dA \]  

\[ Q_1 = \int_0^{a_1} h_1 dA, \quad Q_2 = \int_0^{a_2} h_2 dA \]  

(24)

are the moments of the cross-sectional areas to the right and left of line \( a \ a' \).

From eq. (21) we obtain,

\[ \int_0^{a_1} \frac{d^2 u}{dz^2} = Pu \]  

\[ \int_0^{a_1} \frac{d^2 u}{dz^2} = Pu \]  

(25)

where,

\[ \bar{E} = \frac{E I_1 + E t I_s}{I} \]  

\[ \bar{E} = \frac{E I_1 + E t I_s}{I} \]  

(26)

\( \bar{E} \) is the so-called reduced modulus, and \( I_1 \) and \( I_s \) being the moment of inertia of the two sides.

We observe that the position of the neutral axis in terms of the axial stress is given by eq. (24), while the buckling eq. (25) is similar to the elastic buckling eq. (14). However, the value of \( K \) here is not given by eq. (3), but by,

\[ K = \sqrt{\frac{E}{\bar{E}}} \]  

\[ K = \sqrt{\frac{E}{\bar{E}}} \]  

(27)

Therefore all the results of the previous analysis will be valid for the case of inelastic buckling. For instance, for a simply
supported column according to eq. (7), we will have,

\[ \sigma_{cr} = \frac{n^2 E I}{A^2} \]  

(28)

Since \( E \) is a function of \( \sigma_{cr} \) given by eq. (28) and the value of \( E \) at the unknown \( \sigma_{cr} \), the calculation of the critical stress requires a trial and error simultaneous solution of equations (23), (25) and (26).

A18.8 Imperfect Columns. Tangent-Modulus Theory.

The tangent-modulus theory, originally proposed by Engesser (Ref. 1), is based on the assumption that at the critical state, no stress reversal takes place, and the critical stress, therefore, is determined only by the tangent modulus \( E_t \). This theory was abandoned early since according to the previous discussion with the classical definition of instability (perfect column, bifurcation equilibrium) strain reversal does take place. In recent years, however, this tangent modulus theory has been proved useful.

Under the assumption of no strain reversal both sides of the cross-section in Fig. A18.9a, will be characterized by the same linear stress distribution, corresponding to the tangent-modulus \( E_t \). Thus the buckling equation will be,

\[ E_t I \frac{d^4 u}{dx^4} + P u = 0 \]  

(29)

and the critical stress for simply supported end conditions becomes,

\[ \sigma_t = \frac{n^2 E_t I}{A^2} \]  

(30)

Since \( I_1 + I_2 > I \) and \( B > E_t \), it follows from (26)

\[ E > E_t \]  

and \( \sigma_t = \sigma_{cr} \)

(31)

The critical stress \( \sigma_t \), therefore predicted by the tangent-modulus theory, is smaller than \( \sigma_{cr} \) from the reduced modulus theory. Although for perfect columns, the assumption of no strain reversal is in contradiction to the material behavior in the plastic range, most experiments have given results more closely to the results by the tangent-modulus theory.

To resolve this controversy, Shanley (Ref. 2), proposed the following explanation. For simplicity, let a two-flange buckled column be formed by two rigid legs (see Fig. A18.9) joined in the middle by a plastic hinge. Assume that this column starts to buckle as soon as \( \sigma_t \) is reached. By considering the effect of a load \( P \) above the critical load \( P_t \) corresponding to tangent-modulus, Shanley proves the relation,

\[ P = P_t \left( 1 + \frac{1}{b} \frac{d}{2d + b} \right) \]  

(32)

where \( \tau = E_t / E \)

It must be emphasized that the buckled configuration is a stable one similar to that considered in the refined Euler's theory. Shanley has recognized the fact that such a stable configuration may exist after exceeding the tangent modulus load.

If \( R = P / P_t \), Shanley found that the relation between the variation of stress due to bending and the compressive strain \( \varepsilon_t \) corresponding to \( \sigma_t \) is:

\[ \frac{\delta \varepsilon}{\varepsilon} = \frac{2}{R - 1} \frac{1 - \tau}{1 - \tau} \]  

(33)

Convex side:

\[ \frac{\delta \varepsilon}{\varepsilon} = \frac{1 - \tau}{R - 1} \frac{1 - \tau}{1 - \tau} \]  

(33)

Concave side:

In Fig. A18.10, \( \delta \varepsilon / \varepsilon_t \) and \( \delta \varepsilon / \varepsilon_t \) are plotted against \( R \) for \( \tau = 0.75 \). We observe that while the strain on the concave side increases very rapidly and reaches an infinite value at the reduced-modulus load, the strain on the convex side decreases initially very slowly. Due to this picture we can conceive that in a real column, which has initial imperfections, the compressive strain will increase more rapidly. Furthermore, the rapid increase of \( \varepsilon_t \) will cause a fast reduction of \( E_t \). The column, therefore, loses its usefulness after the tangent-modulus has been slightly increased. Thus the tangent-
modulus, even though it does not actually define an unstable configuration, it represents the lower limit of a spectrum of possible buckled configurations, the upper limit of which is the reduced modulus load which corresponds to infinite deflections.

Thus to summarize, sufficient experimental results are available to show that the falling stress of a column in the inelastic range can be found by replacing $E$ by the tangent modulus $E_T$ in Euler's equation, or,

$$\sigma_{cr} = \frac{\pi^2 E_T}{(L/r)^2}, \quad r = \sqrt{\frac{E_T}{E}} \quad \text{(34)}$$

Figs. A18.11 and 12 show how experimental results check the strength as given by the Euler equation using the tangent-modulus $E_T$.

References:

THEORY OF THE ELASTIC INSTABILITY OF THIN SHEETS

A18.9 Introduction.

Thin sheets represent a very common and important structural element in aerospace structures since the major units of such structures are covered with thin sheet panels. Since compressive stresses cannot be eliminated in aerospace structures, it is important to know what stress intensities will cause thin sheet panels to buckle. Equations for the buckling of thin sheet panels under various load systems and boundary conditions have been derived many years ago and are readily available to design engineers. Part C of this book takes up the use of the many buckling equations in the practical design of thin sheet structural elements. The purpose of this chapter is to introduce the student to the theory of thin plate instability or how these buckling equations so widely in use by design engineers were derived. For a broad comprehensive treatment of the subject of instability of structural elements, the student should refer to some of the references as listed at the end of this chapter.

A18.10 Pure Bending of Thin Plates.

To derive the theory of instability of thin plates, we must first derive the theory of the pure bending of thin plates.

In the following the analysis will be confined to small deformations. Let \( x, y \) be the middle plane of the plate before bending occurs and \( z \) be the axis normal to that plane. Points of the \( x, y \) plane undergo small displacements, \( w \) in the \( z \)-direction, which will be referred as the deflection of the plate. The slope of the middle-surface in the \( x \)- and \( y \)-directions after bending are

\[
\frac{dx}{dz} = \frac{\partial w}{\partial x}, \quad \frac{dy}{dz} = \frac{\partial w}{\partial y}
\]

For small deflections, the curvature of the middle surface can be found approximately by omitting powers of \( \frac{\partial^2 w}{\partial x^2} \), \( \frac{\partial^2 w}{\partial y^2} \), as compared to unity, as it has been done for the curvature of beams. Thus the curvature of the deflected middle surface in planes parallel to \( xz \) and \( yz \) planes respectively are:

\[
\frac{1}{Rx} = -\frac{\partial^2 w}{\partial x^2}, \quad \frac{1}{Ry} = -\frac{\partial^2 w}{\partial y^2}
\]

Another quantity used in the problem of plates is the so-called twist of the middle surface given by:

\[
\frac{1}{R_{xy}} = \frac{\partial^2 w}{\partial x \partial y}
\]

The strains can now be expressed by means of curvatures and twist of the middle surface. In the case of pure bending of prismatic bar a rigorous solution was obtained by assuming that cross-sections of the bars remain plane after bending and rotate so as to remain perpendicular to the deflected neutral axis. Combination of such bending in two perpendicular directions brings us to pure bending of plates.

Let Fig. (2a) represent a thin rectangular plate loaded by uniformly distributed bending
$M_x$, $M_y$ per unit length at its edges. These moments are considered positive when they are directed as shown in the figure, i.e. when they produce compression in the upper surface of the plate and tension in the lower. Let also (Fig. 2b) be a rectangular element cut out of the plate with sides $dx$, $dy$, $t$. The thickness $t$ is considered very small compared with the other dimensions. Obviously the stress conditions at the edges of all such elements will be identical to that of Fig. 2a. Assume now that the lateral sides of the element remain plane during bending and rotate about the axes so as to remain normal to the deflected middle surface. Due to symmetry the middle surface does not undergo any extension and it is therefore the neutral surface.

From the geometry of the above described form of deformation the displacements in the $x$, $y$, $z$ directions can be found as follows:

A point $B$ on the middle surface has been displaced to $B'$ by $w$ in the $z$-direction. An element of surface $dxdy$ has rotated by an angle equal to the slope of the deflected middle surface in the direction so as to remain normal to the middle surface. See Fig. 3.

This angle for small displacements is obviously equal to $\frac{\partial w}{\partial x}$. Thus the horizontal displacement $u_x$ in the $x$-direction of a point at distance $z$ from the middle surface is:

$$u_x = -z \frac{\partial w}{\partial x} \quad \text{(The sign indicates negative displacement for positive $z$).}$$

In a similar manner we find the displacement in the $y$-direction. The complete displacement system is:

$$u_x = -z \frac{\partial w}{\partial x}, \quad u_y = -z \frac{\partial w}{\partial y}, \quad u_z = w(x,y)$$

The corresponding strains are:

$$\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = \frac{z}{E}$$

$$\sigma_{xy} = 2z \frac{\partial^2 w}{\partial x \partial y} = \frac{2z}{E}$$

Since we treat the problem of plates as a plane stress problem, we find by means of Hook’s law

$$\sigma_x = \frac{6z}{(1-\nu)} \left( \frac{1}{R_x} + \frac{\nu}{R_y} \right) \frac{\partial^2 w}{\partial x^2}$$

$$\sigma_y = \frac{6z}{(1-\nu)} \left( \frac{1}{R_y} + \frac{\nu}{R_x} \right) \frac{\partial^2 w}{\partial y^2}$$

$$\sigma_z = \frac{6z}{(1-\nu)} \left( -\frac{1}{R_x} - \frac{\nu}{R_y} \right) \frac{\partial^2 w}{\partial z^2}$$

These normal stresses are linearly distributed over the plate thickness. Their resultants must be equal to $M_x$ and $M_y$ respectively:

$$\int_{-h/2}^{h/2} \sigma_x z \ dydz = M_x dy$$

$$\int_{-h/2}^{h/2} \sigma_y z \ dxdz = M_y dx$$

Substituting from (4) we find:

$$M_x = -D \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}$$

$$M_y = -D \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}$$

where $D = \frac{Eh^3}{12(1-\nu)}$ is the flexural rigidity of the plate. If besides the flexural moments $M_x$, $M_y$, there are uniformly distributed twisting moments $M_{xy}$ and $M_{yx}$ along the sides of the plate of Fig. 2a, these must be equal to the resultant of distributed shear forces $\sigma_{xy}$, $\sigma_{yz}$ along the sides of the element of Fig. 2b.

From eq. (3) we obtain:

$$\sigma_{xy} = \sigma_{yx} = 2Gz \frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} dx = \int_{-h/2}^{h/2} \sigma_{xy} z \ dxdz , \quad M_{yx} dy = \int_{-h/2}^{h/2} \sigma_{yx} z \ dydz$$

$$D(1-\nu) \frac{\partial^2 w}{\partial x^2}$$

Equations (5) give the moments per unit length for pure bending and twisting of a plate.
A18.11 The Differential Equation of the Deflection Surface.

To develop the theory of small deflections of thin plates we make one more assumption. At the boundary of the plate we assume that its edges are free to move in the plane of the plate.

Thus the reactive forces at the edges due to the supports are normal to the plate. With these assumptions we can neglect any strain in the middle plane during bending.

Let us consider, Fig. 4, an element $dx\,dy$ of the middle plane. Along its edge the moments $M_x$, $M_y$, $M_{xy}$ are distributed. These are the resultants of the bending and twisting stresses distributed linearly along the thickness of the plate (see eqs. 4 and 5).

If the plate is loaded by external forces normal to the middle plane in addition to the above moments there are vertical shearing forces $Q_x$, $Q_y$, acting on the sides of the element of Fig. 4.

$$Q_x = \frac{h}{2} \sigma_{xz} dx, \quad Q_y = \frac{h}{2} \sigma_{yz} dx - (6)$$

Let $q$ be the transverse load per unit area acting normally to the upper face of the plate. Considering the force equilibrium in the $z$-direction of the element of Fig. 4 we find:

$$\frac{\partial Q_x}{\partial x} \, dx\,dy + \frac{\partial Q_y}{\partial y} \, dy\,dx + qdx\,dy = 0 \quad \text{or} \quad$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad \text{or} \quad (6a)$$

Taking the equilibrium of the moments acting in the $x$-direction we obtain:

$$\frac{\partial M_{xy}}{\partial x} \, dx\,dy - \frac{\partial M_y}{\partial y} \, dx\,dy + Q_x \, dx\,dy = 0 \quad \text{or} \quad$$

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_x = 0 \quad \text{or} \quad (6b)$$

From the moment equilibrium in the $y$-direction we find:

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} = Q_y = 0 \quad \text{or} \quad (6c)$$

By eliminating $Q_x$, $Q_y$ from $a$, $b$, $c$, we find the equilibrium relation between the moments:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = q \quad \text{or} \quad (7)$$

To represent this equation in terms of the deflections $w$ of the plate, we make the assumption that the expression (5) derived for pure bending holds approximately also in the case of laterally loaded plates. This assumption is equivalent to neglecting the effect on bending of the shearing forces and the compressive stress $q_y$. This is an extension of the engineering theory of bending of beams. As in the case of beams it gives good approximation for bending of plates under transverse loads.

Introducing equation (4) into (7) we find:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial x \partial y} = q \quad \text{or} \quad (8)$$

The problem of bending of plates is thus reduced to integrating eq. (8) for $w$. The corresponding shearing forces in terms of the displacements are found from eqs. 4 and 6b and c:

$$Q_x = \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_x}{\partial x} = - D \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \quad \text{or} \quad (8a)$$

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = - D \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} \quad \text{or} \quad (8b)$$

The above analysis is sufficient to seek solutions of specific problems. The general procedure is to find approximate solution of the fourth order differential equation (8) which satisfies the given boundary displacement and force conditions.

A18.12 Strain Energy in Pure Bending of Plates.

In evaluating the strain energy of a thin plate we shall ignore the contribution of the shear strains which are generally small for small deflections.
The strain energy stored in a plate element is obtained by calculating the work done by the moments $M_{xy}$, and $M_{x}$ on the element during bending. Since the sides of the element remain plane during bending the work done by $M_{xy}$ is obtained by taking half the product of $M_{xy}$ and the relative angle of rotation of the two sides of the element. Since the curvature in the x-direction is $-\frac{\partial^2 w}{\partial x^2}$, the relative angle of rotation of the sides 1 and 2 of distance $dx$ will be $-\frac{\partial^2 w}{\partial x^2} dx$.

Thus the work due to $M_{xy}$ is:

$$dV_x = \frac{1}{2}M_{x} \frac{\partial^2 w}{\partial x^2} dx dy \hspace{3cm} (a)$$

Similarly the work due to $M_{y} dx$ is:

$$dV_y = \frac{1}{2}M_{y} \frac{\partial^2 w}{\partial y^2} dx dy \hspace{3cm} (b)$$

The twisting moment $M_{xy}dy$ also does work against rotation of the element about the x-axis. The relative angle of rotation of the two sections 1, 2 is obviously $\frac{\partial^2 w}{\partial x \partial y} dx$. Thus the work done by $M_{xy}dy$ is:

$$dV_{xy} = \frac{1}{2}M_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy \hspace{3cm} (c)$$

and the work due to $M_{xy}dx$ = $M_{xy}dy$ is:

$$dV_{xy} = \frac{1}{2}M_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy \hspace{3cm} (d)$$

(It is noted that the twist does not affect the work produced by the bending moments, neither the bending affect the work produced by the torsional moments).

Thus the total strain energy per unit volume of the plate is:

$$V_0 = \frac{dV_x + dV_y + dV_{xy} + dV_{yx}}{dx dy} =$$

$$\frac{1}{2} \left[ -M_{x} \frac{\partial^2 w}{\partial x^2} - M_{y} \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \hspace{3cm} (10)$$

Substituting $M_{x}$, $M_{y}$, $M_{xy}$ in terms of the displacement from eqs. (5) we find:

$$V_0 = \frac{1}{2} D \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] -$$

$$2(1-u) \left[ \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] \hspace{3cm} (11)$$

This expression will be modified later when we will consider the superposition of compressive loads in the plane of the plate which are related to the problem of plate buckling.

A18.13 Bending of Rectangular Plates.

The general differential equation for bending plates was given in section C2.3 (eq. 9). Two very useful methods of solution have been widely used, namely, the Fourier Series Method and the Energy Method. Both methods will be developed in the following for rectangular plates and various edge-supporting conditions. The edge support conditions are classified as follows:

a) Built-in edge or Fixed: The deflection along the built-in side is zero and the tangent plane to the deflected middle surface is horizontal. Thus if for instance the x-axis coincides with the built-in edge these conditions are:

$$(w)_{y=0} = 0 \hspace{2cm} \left( \frac{\partial w}{\partial y} \right)_{y=0} = 0 = 0 \hspace{3cm} (12a)$$

b) Simply supported edge: The deflection along the simply-supported side is zero and the bending moment parallel to this side is also zero. Thus if the plate is simply supported along the x-axis we have:

$$(w)_{y=0} = 0 \hspace{2cm} (M_{y})_{y=0} = 0 \hspace{2cm} \left( \frac{\partial^2 w}{\partial y^2} \right)_{y=0} = 0 \hspace{3cm} (12b)$$

c) Free edge: The bending moment, twisting moment and shear force along the free side is zero. Thus if the free side coincides with the straight line $x = a$ we have:

$$M_{x} \mid x=a = 0 \hspace{2cm} M_{xy} \mid x=a = 0 \hspace{2cm} Q_{x} \mid x=a = 0 \hspace{2cm} (12c)$$

However, as was proved by Kirchhoff, two boundary conditions are only necessary to find a unique solution of the bending problem. He has shown that the two last equations of the above conditions can be replaced by one condition.
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\[
\left( \frac{\partial^3 w}{\partial x^2} + (2-\nu) \frac{\partial^2 w}{\partial x  \partial y} \right)_{x=a} = 0
\]

Expressing the condition \( M_x \) \( x=a \) in terms of \( w \) we find the final form of the boundary conditions along the free edge:

\[
\left( \frac{\partial^3 w}{\partial x^2} + (2-\nu) \frac{\partial^2 w}{\partial x  \partial y} \right)_{x=a} = 0 \quad \text{and} \quad \left[ \frac{\partial^3 w}{\partial x^2} + (2-\nu) \frac{\partial^2 w}{\partial x  \partial y} \right]_{x=a} = 0
\]

In the following solutions for various edge conditions will be developed.

1. Simply supported rectangular plates

Let a plate with sides \( a \) and \( b \) and axes \( x, y \), as shown in Fig. 6, be simply supported around the whole periphery and loaded by a distributed load \( q = f(x,y) \).

Two methods of solutions will be developed:

a) Navier solution by means of double Fourier Series:

We can always express \( f(x,y) \) in the form of a double trigonometric (Fourier) series:

\[
q = f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

where:

\[
a_{mn} = \frac{2}{ab} \int_{0}^{a} \int_{0}^{b} f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy
\]

These boundary conditions are satisfied if we take:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

By substituting in eq. (13b) with \( q \) given by eq. (13) we find:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0
\]

\[
\frac{1}{a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ( \frac{m^2}{a^2} + \frac{n^2}{b^2} ) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0
\]

This relation is identity if:

\[
c_{mn} = \frac{16\pi}{ab} \frac{m^2}{a^2} \frac{n^2}{b^2}
\]

and thus:

\[
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16\pi}{ab} \frac{m^2}{a^2} \frac{n^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

In the case of a load \( q \), uniformly distributed over the whole surface we have:

\[
f(x,y) = q, \quad q = \text{const.}
\]

\[
a_{mn} = \frac{4qa}{ab} \int_{0}^{a} \int_{0}^{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy = \frac{16q}{ab} \frac{m^2}{a^2} \frac{n^2}{b^2}
\]

where \( m, n \) are both odd integers. If either or both are even \( a_{mn} = 0 \) and substituting in (14c):

\[
w = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16q}{ab} \frac{m^2}{a^2} \frac{n^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

with maximum deflection at the center,

\[
w_{\text{max}} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16q}{ab} \frac{m^2}{a^2} \frac{n^2}{b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

This is a rapidly converging series and a satisfactory approximation is obtained by taking only the first term. For a square
plate this approximation becomes:

\[ w_{\text{max}} \approx \frac{4qa^4}{\pi^2D} = 0.0454 \frac{qa^4}{Eh^3} \quad \text{(for } \nu = 0.3) \]

which is by 2-1/2% error with the exact solution.

The expressions for bending and twisting moments are not so quickly convergent. To improve the solution another series solution can be developed as follows:

b) Levy alternate single series solution:

The method will be developed for uniform load \( q_0 = \text{const.} \) Levy suggested a solution in the form:

\[ w = \sum_{m=1}^{\infty} Y_m(y) \sin \frac{m\pi x}{a} \quad \text{(16)} \]

where \( Y_m \) is a function of \( y \) only. Each term of the series satisfies the boundary conditions \( w = 0, \frac{\partial^2 w}{\partial x^2} = 0 \) at \( x = a \). It remains to determine \( Y_m \) so as to satisfy the remaining two boundary conditions \( w = 0, \frac{\partial^2 w}{\partial y^2} = 0 \) at \( y = b \).

A further simplification can be made if we take the solution in the form

\[ w = w_1 + w_2 \quad \text{(17a)} \]

where

\[ w = \frac{q_0}{24D} \left( x^2 - 2ax^3 + a^2x \right) \quad \text{(17b)} \]

is the deflection of a very long strip with the long side in the \( x \)-direction loaded by a uniform load \( q_0 \) supported at the short sides \( x = 0, x = a \), and free at the two long sides. Since (17b) satisfies the differential equation and the boundary conditions at \( x = 0, x = a \), the problem is solved if we find the solution of:

\[ \frac{\partial^4 w_1}{\partial x^4} + 2 \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_2}{\partial y^4} = 0 \quad \text{(18)} \]

with \( w_x \) in the form of (16) and satisfying together with \( w_1 + w_2 \) of eq. (17b) the boundary conditions \( w = 0, \frac{\partial^2 w}{\partial y^2} = 0 \) at \( y = \frac{b}{2} \) (see Fig. 7).

Substituting (16) into (18) we obtain:

\[ \sum_{m=1}^{\infty} \left( \frac{Y_m}{a^2} \right) \frac{\partial^4 Y_m}{\partial y^4} \sin \frac{m\pi x}{a} = 0 \quad \text{(19)} \]

where for symmetry \( m = 1, 3, 5 \ldots \)

This equation can be satisfied for all values of \( x \) if:

\[ Y_m \frac{\partial^2 Y_m}{\partial x^2} + \frac{m^4\pi^4}{a^4} Y_m = 0 \quad \text{--- --- --- (19b)} \]

The general solution of (19b) is:

\[ Y_m(y) = \frac{G a^4}{D} \left[ \frac{A_m \cos h \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sin h \frac{m\pi y}{a}}{a \sin h \frac{m\pi a}{a}} + C_m \sin h \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cos h \frac{m\pi y}{a} \right] \quad \text{--- --- --- (20)} \]

Since the deflection is symmetric with respect to the \( x \)-axis it follows that \( C_m = D_m = 0 \). Thus:

\[ w = \frac{q_0}{24D} \left( x^4 - 2ax^3 + a^2x \right) + \frac{qa^4}{D} \sum_{m=1,3,5} \left( \frac{A_m \cos h \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sin h \frac{m\pi y}{a}}{a \sin h \frac{m\pi a}{a}} \right) \sin \frac{m\pi y}{a} \]

or

\[ w = \frac{qa^4}{D} \sum_{m} \left( \frac{4}{\pi^4 m^4} + A_m \cos h \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sin h \frac{m\pi y}{a} \right) \sin \frac{m\pi y}{a} \sin \frac{m\pi a}{a} \]

where \( m = 1, 3, 5 \ldots \). Substituting this expression into the boundary conditions:

\[ w = 0, \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for } y = \pm \frac{b}{2} \text{ we find:} \]

\[ A_m \cos h a_m + B_m a_m \sin h a_m + \frac{4}{\pi^4 m^4} = 0 \quad \text{--- (21a)} \]

\[ (A_n + 2B_m) \cos h a_m + B_m a_m \sin h a_m = 0 \]

where \( a_m = m\pi b/2a \)

From these equations we find:
\[ A_m = \frac{2(\alpha_m \tan \alpha_m + 2)}{n^s m^s \cos \alpha_m} \]

\[ B_m = \frac{2}{n^s m^s \cos \alpha_m} \]

Thus:

\[ w = \frac{4qa^*}{n^s D} \sum_{m=1,3,5} \frac{1}{m^s} \left[ 1 - \frac{\alpha_m \tan \alpha_m + 2}{2 \cos \alpha_m} \right] \cos h \frac{2amY}{b} + \frac{\alpha_m}{2 \cos \alpha_m} \frac{2y \sin h \frac{2amY}{b}}{b} \sin \frac{amx}{a} \]

The maximum deflection occurs at the middle \( x = \frac{a}{2}, y = 0 \):

\[ w_{\text{max}} = \frac{4qa^*}{n^s D} \sum_{m=1,3,5} \frac{1}{m^s} \left[ 1 - \frac{\alpha_m \tan \alpha_m + 2}{2 \cos \alpha_m} \right] \]

The summation of the first series of terms corresponds to the solution of the middle of a uniformly loaded strip, eq. (17b). Thus:

\[ w_{\text{max}} = \frac{5}{384} \frac{qa^*}{D} - \frac{4qa^*}{n^s D} \sum_{m=1,3,5} \frac{1}{m^s} \left( \frac{\alpha_m \tan \alpha_m + 2}{2 \cos \alpha_m} \right) \]

This series converges very rapidly. Taking a square plate, \( a/b = 1 \), we find from (21e):

\[ a_1 = \frac{\pi}{2}, \quad a_3 = \frac{3\pi}{2} \quad \text{e.t.c.} \]

\[ w_{\text{max}} = \frac{5}{384} \frac{qa^*}{D} - \frac{4qa^*}{n^s D} (0.08562 - 0.00025 + \ldots) \]

\[ \geq 0.00406 \frac{qa^*}{D} \]

We observe that only the first term of the series in (21e) need to be taken into consideration.

The bending moments are found by substituting (21c) into eqs. (5). The maximum bending moments at \( x = \frac{a}{2}, y = 0 \) are:

\[ (M_x)_{\text{max}} = \frac{qa^*}{3} - (1-\nu)qa^* n^s \sum_{m=1,3,5} \frac{m-1}{m} \]

\[ (M_y)_{\text{max}} = \frac{q}{3} \frac{qa^*}{n^s} \sum_{m=1,3,5} \frac{m-1}{m} \]

(c) Solution by means of the principle of virtual work

From the discussion of the method (a) we can represent \( w \) in double Fourier Series:

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{amx}{a} \sin \frac{bny}{b} \]

The coefficients \( C_{mn} \) may be considered as the co-ordinate defining the deflection surface. A virtual displacement will have the form:

\[ \delta w = \delta C_{mn} \sin \frac{amx}{a} \sin \frac{bny}{b} \]

The strain energy \( V_1 \) can be found by substituting (22a) into eq. (6). After a few algebraic manipulations we find:

\[ V_1 = \int_0^a \int_0^b V_0 dx dy = \frac{n^s a^4 b^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \]

\[ (\frac{m^s}{a^4} + \frac{n^s}{b^4}) \]

Let us now examine the deflection of the plate of Fig. 5 with a concentrated load \( P \) at the point with co-ordinates \( x = \xi, y = \eta \). The increment of strain energy due to the increment of the deflection by:

\[ \delta W = 5C_{mn} \sin \frac{amx}{a} \sin \frac{bny}{b} \]

is found from (23):

\[ \delta V_1 = \frac{n^s a^4 b^2}{4} C_{mn} \left( \frac{m^s}{a^4} + \frac{n^s}{b^4} \right) \delta C_{mn} \]

The increment of the work of the load \( P \) is:

\[ \delta W = P \delta C_{mn} \sin \frac{amx}{a} \sin \frac{bny}{b} \]

From \( \delta V_1 = \delta W = 0 \) we obtain:
Let us consider the case of a uniform load \( q \). We write the deflection in the form:

\[
W = W_1 + W_2 - - - - - - - - - - - - - - - -(26a)
\]

where \( W_1 \) is the deflection of a simply supported strip of length \( a \), which for the system of axes of Fig. 8 can be written (see Levy's method in previous section):

\[
W_1 = \frac{4qa^4}{\pi^4} \sum_{m=1,3,5} \frac{1}{m^4} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b},
\]

and \( W_2 \) is represented by the series:

\[
W_2 = \sum_{m=1,3,5} \frac{2}{\pi^4} \frac{\partial^4 W_2}{\partial x^4} + 2 \frac{\partial^4 W_2}{\partial x^2 \partial y^2} = 0
\]

is found as in the previous section:

\[
Y_m = \frac{qa^4}{\pi^4} \left( A_m \cos \frac{\pi m x}{a} + B_m \cos \frac{\pi m y}{b} \right)
\]

where \( A_m, B_m \) are determined so as to satisfy the last four boundary conditions. Using the conditions (25b) we obtain:

\[
A_m = -\frac{4}{\pi^4 m^4} , \quad C_m = -D_m
\]

By the conditions (25c) we find:

\[
B_m = \frac{1}{\pi^4}, \quad C_m = \frac{1}{\pi^4}, \quad D_m = \frac{1}{\pi^4}
\]

Substituting \( A_m, B_m, C_m \) and \( D_m \) in eq. (26d) we find the deflection. The maximum deflection occurs at the middle of the free edge.

A18.14 Combined Bending and Tension or Compression of Thin Plates.

In developing the differential equations of equilibrium in previous pages, it was assumed that the plate is bent by transverse loads normal to the plate and the deflections were so small that the stretching of the middle plane can be neglected. If we consider now the case where only edge loads are active coplanar
with the middle surface (Fig. 9) we have a plane stress problem. If we assume that the stresses are uniformly distributed over the thickness and denote by $N_x$, $N_y$, $N_{xy}$, $N_{yz}$ the resultant force of these stresses per unit length of linear element in the $x$ and $y$-directions (Fig. 9) it is obvious that:

$$N_x = \sigma x, \quad N_y = \sigma y$$

$$N_{xy} = N_{yx} = \sigma_{xy}$$

(27)

Fig. 9

The equations of equilibrium in the absence of body forces can be written now in terms of these generalized stresses $N_x$, $N_y$, $N_{xy}$ by substituting from eq. (27) to the equations of equilibrium

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

(28)

On the other hand if the plate is loaded by transverse loads the stresses give rise to pure bending and twisting moments only. The equations of equilibrium for the latter have been given in before (see eqs. 5, 7, 8). If both transverse loads and coplanar edge loads are acting simultaneously, then for small vertical deflections the state of stress is the superposition of the stresses due to $N_x$, $N_y$, $N_{xy}$ and $M_x$, $M_y$, $M_{xy}$. For large vertical deflection of the plate, however, there is interaction of the coplanar stresses and the deflections. These stresses give rise to additional bending moments due to the non-zero lever arm of the edge loads from the deflected middle surface, as in the case of beams. When the edge loads are compressive this additional moments might cause instability and failure of the plate due to excessive vertical deflections.

In this chapter the problem of instability of plates will be examined.

When the edge loads are compressive and give rise to additional bending moments eq. (8) must be modified.

Consider an element of the middle surface $dx, dy$ (Fig. 10a). The conditions of force equilibrium in the $x, y$-directions are given by eq. (28). Consider now the projection of the stresses $N_x$, $N_y$, $N_{xy}$ in the $z$-direction:

(a) Projection of $N_x$: From Fig. 10a it follows that the resultant projection is:

$$- N_x \frac{\partial w}{\partial x} + (N_x + \frac{\partial w}{\partial x} dx) \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} dx$$

and neglecting terms of higher order:

$$\left(N_x \frac{\partial w}{\partial x} + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} \right)$$

(29)

(b) Projection of $N_y$: By similar argument we find that this projection is equal to:

$$\left(N_y \frac{\partial w}{\partial y} + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} \right)$$

(30)

(c) Projection of $N_{xy}$ and $N_{yx}$: From Fig. 10b we find:

$$- N_{xy} \frac{\partial w}{\partial y} + (N_{xy} + \frac{\partial N_{xy}}{\partial x} dx) \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} dx$$

and neglecting terms of higher order:

$$\left(N_{xy} \frac{\partial w}{\partial x} + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial x} dx \right)$$

Similarly we find the projection of $N_x = N_{xy}$.
\( \left( \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) \ dx \ dy \) - - - - - - - - - (d)

Thus in eq. (6a) the terms given by (a), (b), (c) and (d) should be added (divided of course by \( dx \ dy \)):

\[
\frac{\partial^2 \sigma}{\partial x \partial y} + \frac{\partial^2 \sigma}{\partial y^2} + q = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \\
\left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \ \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \ \frac{\partial w}{\partial y} = 0 - - - - - (e)
\]

But due to the equation of equilibrium (28) the two terms inside the parentheses in (e) are zero. Thus:

\[
\frac{\partial^2 \sigma}{\partial x \partial y} + \frac{\partial^2 \sigma}{\partial y^2} + q = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 - - - - - (29)
\]

Eq. (29) replaces eq. (6a) when edge loads are present. Eqs. (6b) and (6c) are, however, still valid since they express moment equilibrium of the element \( dx \ dy \) in which the contribution of \( N_x, N_y, N_{xy} \) is zero. Thus eliminating \( q_x, q_y \) between (6b) and (29) we find:

\[
\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial y} = \\
\frac{1}{\mu} \left( q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) - - - - - (30)
\]

Eq. (30) replaces eq. (28) when edge loads are present and the deflections are large so that instability might occur.

The distribution of the coplanar stresses \( N_x, N_y, N_{xy} \) can be found from eqs. (28) by solving the plane stress problem. In the following the above theory will be applied to rectangular plates.

### A18.15 Strain Energy of Plates Due to Edge Compression and Bending

The energy expression for pure bending, eq. (11), must be complemented to include the contribution of the edge coplanar loads. Assume that first the edge loads are applied. Obviously the strains due to the stresses \( N_x, N_y, N_{xy} \) are:

\[
\varepsilon_x = \frac{1}{\mu} (N_x - V N_y), \quad \varepsilon_y = \frac{1}{\mu} (N_y - V N_x), \\
\gamma_{xy} = \frac{N_{xy}}{\mu} - - - - - - - (31)
\]

The strain energy is:

\[
V_{1N} = \frac{1}{2} \int \left( N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \gamma_{xy}^0 \right) \ dx \ dy = \\
\frac{1}{2} \left( \frac{N_x^2}{\mu} + \frac{N_y^2}{\mu} - 2N_x N_y (1+V) N_{xy}^2 \right) - - - - (a)
\]

During bending due to transverse loads or/and due to buckling we assume that the edge loads and consequently \( N_x, N_y, N_{xy} \), remain constant. Its variation is thus zero and we do not consider it in the following. Let us apply now the transverse load that produces bending. (We can also consider bending due to other transverse disturbance, which is the case of buckling). If \( u, v \), are the displacements of the middle surface due to the coplanar loads (which are assumed constant across the thickness) and \( w \) the bending deflection of the plate it can be shown that the strains are:

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2}, \\
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2}, \\
\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} - - - - (b)
\]

Let us apply now bending with constant coplanar stresses. Due to stretching of the middle surface the energy is:

\[
\int \left( N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy} \right) \ dx \ dy - - - - (c)
\]

Introducing (b) into (c) and adding the strain energy due to bending, eq. (11), we find the total change of strain energy due to bending which is:

\[
V_1 = \int \left[ N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \ dx \ dy + \\
\frac{1}{2} \int \left[ N_x \left( \frac{\partial^2 w}{\partial x^2} \right) + N_y \left( \frac{\partial^2 w}{\partial y^2} \right) + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \ dx \ dy + \\
\frac{1}{2} \int \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \ dx \ dy - - - - (d)
\]

Here \( u, v \) are the additional coplanar displacements after bending has started. It can be shown by integrating by parts that the first integral is the work done during bending by the edge loads. For instance taking a rectangular plate this integral becomes:

\[
\int \left[ N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xy} \frac{\partial u}{\partial y} + N_{xy} \frac{\partial v}{\partial x} \right] \ dx \ dy = \\
\int \left( |N_x| \frac{\partial u}{\partial x} + |N_y| \frac{\partial v}{\partial y} + |N_{xy}| \frac{\partial u}{\partial y} + |N_{xy}| \frac{\partial v}{\partial x} \right) \ dx \ dy - - - - (e)
\]
Obviously the first two integrals represent the work done by the edge loads while the second integral is zero due to the equilibrium equations (28). Thus the work of the edge loads is:

\[ W_N = \frac{1}{2} \int \left[ N_x \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} + N_{xy} \left( \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x} \right) \right] \, dx \, dy \quad - \quad (r) \]

We assume now that for small deflections the stretching of the middle surface of the plate is negligible. (This is the so-called inextensional theory of plates). In this case by zeroing the strains in eq. (b) and substituting in (r) we find:

\[ W_N = \frac{1}{2} \int \left[ N_x (\frac{\partial w}{\partial x})^2 + N_y (\frac{\partial w}{\partial y})^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \, dx \, dy \quad - \quad (32a) \]

In the strain energy expression, eq. (a) the first two terms cancel each other and the strain energy is only due to bending:

\[ V_I = \frac{1}{2} \int \left\{ \frac{3}{2} (\frac{\partial w}{\partial x})^2 + \frac{3}{2} (\frac{\partial w}{\partial y})^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x \partial y} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \right\} \, dx \, dy \quad - \quad (32b) \]

In the absence of transverse loads the work of external forces is simply due to the edge loads:

\[ W = W_N \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (32c) \]

Expressions (32b) and (c) will be used in solving the buckling problem by means of the principle of virtual work.

**A18.16 Buckling of Rectangular Plates with Various Edge Loads and Support Conditions.**

**General discussion.**

In calculating critical values of edge loads for which the flat form of equilibrium becomes unstable and the plate begins to buckle, the same methods and corresponding reasonings as for compressed bars will be employed.

The critical values can be obtained by assuming that the plate has a slight initial curvature or a small transverse load. These values of the edge loads for which the lateral deflection \( w \) becomes infinite are the critical values (see Part I for similar treatment in columns).

Another way of investigating such instability is to assume that the plate buckles due to a certain external disturbance and then to calculate the edge loads for which such a buckled configuration (deflections different from zero) is possible. It was found in the case of column that this latter solution (Euler's solution) approaches asymptotically the first at the limit where the deflections become extremely large but for even small deflections the edge load acquires a value very near the Euler's critical value. The latter technique is mathematically more convenient and it gives for plates also a very good estimate of their compressive strength. In the following we shall use this latter approach by assuming a plate with edge loads and no transverse load. Eq. (30) becomes in this case:

\[ \frac{3}{2} \int \left\{ \frac{3}{2} (\frac{\partial w}{\partial x})^2 + \frac{3}{2} (\frac{\partial w}{\partial y})^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x \partial y} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \right\} \, dx \, dy \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (33) \]

By solving eq. (33) we will find that the assumed buckling mode is possible (\( w \neq 0 \)) for certain definite values of the edge loads, the smallest of which determines the critical load. The energy method can also be used in investigating buckling problems. In this method we assume that the plate is initially under the plane stress conditions due to the edge loads and the stress distribution is assumed as known. We then consider the buckled state as a possible configuration of equilibrium. The change of the work is given by eq. (32a). We interpret here \( w \) as a virtual displacement though we do not use the variation symbol \( \delta \). Thus the increment of work \( \delta W \) is given by (32a) and the increment of strain energy \( \delta V_I \) is given by eq. (32b). If \( \delta W = \delta V_I \) for every possible shape of buckling the flat equilibrium is stable. If \( \delta W \neq \delta V_I \) for a certain shape of buckling then the flat configuration is unstable and the plate will buckle under any load above the critical load. If \( \delta W = \delta V_I \), the equilibrium is neutral and from this equation we find the critical load. The critical load therefore is found from the equation:

\[ - \frac{1}{2} \left[ N_x (\frac{\partial w}{\partial x})^2 + N_y (\frac{\partial w}{\partial y})^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \, dx \, dy = \]

\[ D \left\{ \frac{3}{2} (\frac{\partial w}{\partial x})^2 + \frac{3}{2} (\frac{\partial w}{\partial y})^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x \partial y} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \right\} \, dx \, dy \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (34) \]

Here \( w \) is a certain assumed deflection which satisfies the boundary conditions (virtual deflection).

**A18.17 Buckling of Simply Supported Rectangular Plates Uniformly Compressed in One Direction.**

Let a plate of sides \( a \) and \( b \) (Fig. 11) simply supported around its periphery be compressed by load \( N_x \) uniformly distributed along the sides \( x = 0 \) and \( x = a \). From the
obvious solution of the corresponding plane stress problem we find that the state of stress is everywhere a simple compression equal to \( N_x \) (the load at the periphery).

The deflection surface of a simply supported plate when bending takes place have been found previously (see eq. 14a). Its general expression can be written in a double series form:

\[
W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

The increment of strain energy found by substituting (35a) in the right-hand side of (34) is:

\[
\delta V = \frac{n^4 a^4 b^4}{3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2
\]

The increment of work done by the external forces is found by substituting (35a) into the left-hand side of eq. (34) and \( N_x = \text{const}, N_y = N_{xy} = 0 \). Thus:

\[
\delta W = \frac{1}{2} N_x \int_{a/2}^{a/2} \int_{b/2}^{b/2} \delta u \delta y \, dx \, dy = \frac{n^4 a^4 b^4}{3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2
\]

From the equality \( \delta W = \delta V \), solving for \( N_x \), we obtain:

\[
N_x = \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 C_{mn}^2}
\]

This fraction has some intermediate value between the maximum and the minimum of the fractions (a). It follows that if we wish to make the fraction (b), which is similar to the fraction of eq. (35d), a minimum, we must take only one term in the numerator and the corresponding term in the denominator. Thus to make the fraction of eq. (35d) minimum, we must put all the parameters \( C_{11}, C_{12}, C_{13}, \ldots \) except one, zero. This is equivalent to assuming that the buckling configuration is of simple sinusoidal form in both directions, i.e.

\[
W_{mn} = C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.
\]

The minimum expression for \( C_{mn} \) obtained by dropping all the terms except \( C_{mn} \) becomes:

\[
N_x = \frac{n^4 a^4 b^4}{3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2
\]

It is obvious that the smallest value of \( N_x \) is obtained by taking \( n = 1 \). This means that the plate buckles always in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. Thus for \( n = 1 \), eq. (35e) becomes:

\[
(N_x)_{cr} = \frac{n^4 a^4 b^4}{3} \left( m + \frac{1}{m} \frac{a^2}{b^2} \right)
\]

The value of \( m \) (in other words the number of half-waves) which makes this critical value the smallest possible depends on the ratio \( a/b \) and can be found as follows:

Let us express (36a) in the form:

\[
(N_x)_{cr} = \frac{n^4 a^4 b^4}{3} \left( m + \frac{1}{m} \frac{a^2}{b^2} \right)
\]

where \( k \) is a numerical factor depending on \( \frac{a}{b} \). From (36a) and (36b) we have:

\[
k = \frac{b^2}{a^2} \left( m + \frac{1}{m} \frac{a^2}{b^2} \right)
\]

If we plot \( k \) against \( \frac{a}{b} \) for various values of the integer \( m = 1, 2, 3, \ldots \), we obtain the curves of Fig. (12). From these curves the critical load factor \( k \) and the corresponding number of half-waves can readily be determined. It is only necessary to take the corresponding point \( \frac{a}{b} \) as the axis of abscissas and to choose that curve which gives the smallest \( k \). In Fig. 12 the portion of the various curves which give the critical values of \( k \) are shown by full lines. The transition from \( m \) to \( (m + 1) \) half-waves occurs at the intersection of the two corresponding lines. From eq. (36c) we find:
A18.22 THEORY OF THE INSTABILITY OF COLUMNS AND THIN SHEETS

\[ m_b + \frac{a}{mb} = \frac{(m+1)b}{a} + \frac{a}{(m+1)b} \quad \text{or:} \]
\[ \frac{a}{b} = \sqrt{m (m+1)} \quad - - - - - - - - (36a) \]

Thus the transition from one to two half-waves occur for:
\[ \frac{a}{b} = \sqrt{1 (1+1)} = \sqrt{2} \]
from two to three for
\[ \frac{a}{b} = \sqrt{2 (2+1)} = \sqrt{6} \]

\[ - - - - - - - - \quad \text{and so on.} \]

The number of half-waves increases with the ratio \( \frac{a}{b} \) and for very long plates \( m \) is very large.
\[ \frac{a}{b} = m \]

This means that a very long plate buckles in half-waves the lengths of which approach the width of the plate. The buckled plate is subdivided into squares.

The critical value of the compression stress is:
\[ \sigma_{xy} = \frac{(N_x)}{A} = \frac{kn_3E}{h} \cdot \frac{t^2}{b^2} \quad - - - - (36a) \]
\[ (t = \text{thickness}) \]

A18.18 Buckling of Simply Supported Rectangular Plate Compressed in Two Perpendicular Directions.

Let (Fig. 13) \( N_x, N_y \) the uniformly distributed edge compressions. Using the same as before expression for the deflections (eq. 35a) and applying the energy equation (33) with \( N_x, N_y \) constants (which is the solution of the corresponding plane stress problem) we find:

\[ N_x \frac{a^2}{A} + N_y \frac{b^2}{B} = D \left( \frac{a^4}{A} + \frac{b^4}{B} \right) \quad - - - - (37a) \]

and by introducing the parameter:
\[ \sigma_o = \frac{N_x}{A} \quad - - - - - - - - (37b) \]

we obtain:
\[ \sigma_{xm} + \sigma_{yn} \frac{b^2}{A} = \sigma_o \left( m^2 + n^2 \frac{b^2}{A} \right) \quad - - - - (37c) \]

Taking any integer \( m \) and \( n \) the corresponding deflection surface of the buckled plate is given by:
\[ \omega_{mn} = \omega_{mn} \sin \frac{mnx}{A} \cdot \sin \frac{mny}{B} \quad - - - - (38) \]

and the corresponding \( \sigma_x, \sigma_y \) are given by (37c) which is a straight line in the diagram \( \sigma_y, \sigma_x \) (Fig. 14). By resulting and plotting such lines for various pairs of \( m \) and \( n \) we find the region of stability and the critical combination of \( \sigma_x, \sigma_y \) which is on the periphery of the polygon formed by the full lines of Fig. 14.

A18.19 Buckling of Simply Supported Rectangular Plate Under Combined Bending and Compression.

Let us consider a simply supported rectangular plate (Fig. 15). Along the sides \( x = 0, x = a \) there are linearly distributed edge loads given by the equation:
\[ N_x = N_o \left( 1 - \frac{a-x}{B} \right) \quad - - - - - - - - (39a) \]
which is a combination of pure bending and pure compression. Let us take the deflection again in the form:

\[ w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \]  

(39b)

\[ \sigma_{cr} = \frac{(N_0)_{cr}}{h} \]  

(39g)

We examine for each value of \( m \) the solutions of the system (39f). Starting from \( m = 1 \) and denoting:

\[ \sigma_{cr} = \frac{(N_0)_{cr}}{h} \]  

(39g)

we obtain from (39f):

\[ C_{m1} \left[ \left( 1 + \frac{a^2}{b^2} \right) \sigma_{cr} \frac{a}{b} \left( 1 - \frac{1}{n^2} \right) \right] \]

\[ \Delta \sigma_{cr} \frac{a}{b} \sum_{m=1}^{\infty} \frac{C_{m1} n_1}{(n^2 - 1)^{1/2}} \]  

(39h)

These are homogeneous equations in \( a_11, a_{12}, ... \) etc. The system possesses a non-zero solution (which indicates the possibility of buckling of the plate) if the determinant of eq. (39m) is zero. From this condition an equation is obtained for \( \sigma_{cr} \). Obviously the system (39h) is of infinite number of equations \( n = 1, 2, 3, ... \). A sufficient approximation is obtained by taking a large but finite number of terms and find the solution of the finite determinant (using for example digital computers). Thus a curve of \( \sigma_{cr} \) versus \( a/b \) is obtained for \( m = 1 \) like that of Fig. 12. Repeating the same calculation for \( m = 2, 3, ... \) etc., we find similar curves of two, three, etc. half-wave lengths. The regions of the curves with minimum ordinates define the region of stability as in Fig. (12).

A18.20 Inelastic Buckling of Thin Sheets

The problem of the inelastic buckling of thin sheets has been extensively studied by various authors. The main difficulty in such studies is in reference to the stress-strain relations of plasticity under complex states of stress. Many controversial discussions have appeared in literature without resolving the theoretical difficulties. For this reason we will not develop the theory of inelastic buckling in this chapter. Some of the better references on this subject are listed below.

Chapter C4 presents the plasticity correction factors to use in calculating the inelastic buckling strength of thin sheets.

A18.21 References.

This multiple exposure photograph of a Boeing supersonic transport model shows the variable-sweep wing in three configurations: forward for takeoff and landing, swept part way back for transonic flight, and swept completely back as an arrow wing for 1800-mile-an-hour supersonic cruise.

**SPECIFICATIONS (Basic Design)**

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<td>Takeoff Speed</td>
<td>175 miles an hour</td>
</tr>
<tr>
<td>Approach Speed</td>
<td>136 miles an hour</td>
</tr>
<tr>
<td>Wing Span (forward position)</td>
<td>173 feet, 4 inches</td>
</tr>
<tr>
<td>(aft position)</td>
<td>86 feet, 4 inches</td>
</tr>
<tr>
<td>Length</td>
<td>203 feet, 10 inches</td>
</tr>
<tr>
<td>Height</td>
<td>48 feet, 4 inches</td>
</tr>
</tbody>
</table>
A19.1 Typical Wing Structural Arrangement

For aerodynamic reasons, the wing cross-section must have a streamlined shape commonly referred to as an airfoil section. The aerodynamic forces in flight change in magnitude, direction and location. Likewise in the various landing operations the loads change in magnitude, direction and location, thus the required structure must be one that can efficiently resist loads causing combined tension, compression, bending and torsion. To provide torsional resistance, a portion of the airfoil surface can be covered with a metal skin and then adding one or more internal metal webs to produce a single closed cell or a multiple cell wing cross-section. The external skin surface which is relatively thin for subsonic aircraft is efficient for resisting torsional shear stresses and tension, but quite inefficient in resisting compressive stresses due to bending of wing. To provide strength efficiency, spanwise stiffening units commonly referred to as flange stringers are attached to the inside of the surface skin. To hold the skin surface to airfoil shape and to provide a medium for transferring surface air pressures to the cellular beam structure, chordwise formers and ribs are added. To transfer large concentrated loads into the cellular beam structure, heavy ribs, commonly referred to as bulkheads, are used.

Figs. A19.1 and A19.2 illustrate typical structural arrangements of wing cross-sections for subsonic aircraft. The surface skin is relatively thin. In general the wing structural flange arrangement can be classified into two types; (1) the concentrated flange type where flange material is connected directly to internal webs and (2) the distributed flange type where stringers are attached to skin between internal webs.

Fig. A19.3 shows several structural arrangements for wing cross-sections for supersonic aircraft. Supersonic airfoil shapes are relatively thin compared to subsonic aircraft.

Distributed Flange Type of Wing Beam.

![Distributed Flange Type of Wing Beam](image)

Concentrated Flange Type of Wing Beam.

Dashed line represents secondary structure. In many cases this portion is fabric covered.
To withstand the high surface pressures and to obtain sufficient strength much thicker wing skins are usually necessary. Modern milling machines permit tapering of skin thicknesses. To obtain more flange material integral flange units are machined on the thin skin as illustrated in Fig. k.

Fig. j

Fig. k

Fig. l

Light Weight Core

Fig. A19.3 Wing Sections - Supersonic Aircraft

In a cantilever wing, the wing bending moments decrease rapidly spanwise from the maximum values at the fuselage support points. Thus thick skin construction must be rapidly tapered to thin skin for weight efficiency, but thinner skin decreases allowable compressive stresses. To promote better efficiency sandwich construction can be used in outer portion of wing (Fig. 1). A light weight sandwich core is glued to thin skin and thus the thin skin is capable of resisting high compressive stresses since the core prevents sheet from buckling.

A19.2 Some Factors Which Influence Wing Structural Arrangements

1. Light Weight:

The structural designer always strives for the minimum weight which is practical from a production and cost standpoint. The thinner the ultimate allowable stresses, the lighter the structures. The concentrated flange type of wing structures as illustrated Fig. (a, b and c) of Fig. A19.1 permits high allowable compressive flange stresses since the flange members are stabilized by both web and covering sheet, thus eliminating column action, which permits design stresses approaching the crippling stress of the flange members. Since the flange members are few in number, the size or thickness required is relatively large, thus giving a high crippling stress. On the other hand, this type of design does not develop the effectiveness of the metal covering on the compressive side, which must be balanced against the saving in the weight of the flange members.

In the distributed type of flange arrange-
of the structural box is seldom obtained in actual airplane design due to cut-outs in the wing surface for such items as retractable landing gears, mail compartments and bomb and gun bays. If the distributed flange type of box beam is used, they are interrupted at each cut-out, which requires that means must be provided for drifting the flange loads around the opening, an arrangement which adds weight because conservative overlapping assumptions are usually made in the stress analysis. The additional structure and riveting to provide for the transfer of flange load around large openings adds considerably to the production cost.

For landing gears as well as many other installations, the wing cut-outs are confined to the lower surface, thus a structural arrangement as illustrated in Fig. A19.4 is quite common. The upper surface is of the distributed flange type whereas the lower flange material is concentrated at the two lower corners of the box. In the normal flying conditions, the lower surface is in tension and thus cell sheet covering between the cut-outs is equally effective in bending if shear lag influence is discounted. For negative accelerated flying conditions, the lower surface is in compression thus sheet covering between corner flanges would be ineffective in bending. However, since the load factors in these flight conditions are approximately one-half the normal flight load factors, this ineffectiveness of the lower sheet in bending is usually not critical.

Cut-outs likewise destroy the continuity of intermediate interior shear webs of such sections as illustrated in Figs. (c and 1), and the shear load in these interrupted webs must be transferred around the opening by special bulkheads on each side of the cut-out, which means extra weight.

In many cases, cut-outs in the leading edge are necessary due to power plant installations, landing gear wells, etc. Furthermore, in many airplanes, it is desirable to make the leading edge portion removable for inspection of the many installations which occupy this space in the portion of the wing near the fuselage. If such is the case, then an interior web should be located near the front of the wing section.

Inspection doors for the central portion of the box beam structure are usually located on the lower side of the wing. They are usually fastened to two spanwise stringers with screws and the removable panel is effective in resisting bending and shear load. (See Fig. A19.5)

Cutouts in the wing structural box destroys the continuity of the torsional resistance of the cell and thus special consideration must be given to carrying torsional forces around the cut-out. This special problem is discussed later.

(4) Folding-Wings:

For certain airplanes, particularly Carrier based Naval airplanes, it is necessary that provision be made to fold the outer wing panels upward. This dictates definite hinge points between the outer and center wing panels. If a distributed flange type of structure is used, the flange loads must be gathered and transferred to the fitting points, thus a compromise solution consisting of a small number of spanwise members is common practice.

(5) Wing Flutter Prevention:

With the high speeds now obtained by modern airplanes, careful attention to wing flutter prevention must be given in the structural layout and design of the wing. In general, the critical flutter speed depends to a great extent on the torsional rigidity of the wing. When the mass center of gravity moves aft of the 25 per cent of chord point, the critical flutter speed decreases, thus it is important to keep weight of the wing forward. At high speeds where "compressibility" effects become important, the torsional forces on the wing are increased, which necessitates extra skin thickness or a larger cell. Designing for flutter prevention is a highly specialized problem.

(6) Ease and Cost of Production:

The airplane industry is a mass production industry and therefore the structural layout of the wing must take into account production methods. The general trend toward this time is to design the wing and body structure, so that sub-assemblies of the various parts can be made, which are finally brought together to form the final assembly of the wing panel. To make this process efficient requires careful consideration in the details and layout of the wing structure. Photograph A19.6 illustrates the sub-assembly
Photograph A19.6

Designing To Facilitate Production.

Photographs by courtesy of North American Aviation Co.
break-down of the structural parts of an airplane of a leading airplane company. Fabrication and assembly of these units permits the installation of much equipment before assembly of the units to the final assembly.

A19.3 Wing Strength Requirements

Two major strength requirements must be satisfied in the structural design of a wing. They are: (1) Under the applied or limit loads, no part of the structure must be stressed beyond the yield stress of the material. In general, the yield stress is that stress which causes a permanent strain of 0.002 inches per inch. The terms applied or limit refer to the same loads, which are the maximum loads that the airplane should encounter during its lifetime of operations. (2) The structure shall also carry design loads without rupture or collapse or in other words failure. The magnitude of the Design Loads equals the Applied Loads times a factor of safety (F.S.). In general, the factor of safety for aircraft is 1.6, thus the structure must withstand 1.6 times the applied loads without failure. In missiles, since no human passengers are involved, the factor of safety is less and appears at this time to range between 1.15 to 1.25.

Aircraft factors of safety are rather low compared to other fields of structural design, chiefly because weight saving is so important in obtaining a useful transportation vehicle relative to useful load and performance. Since safety is the paramount design requirement, the correctness of the theoretical design must be checked by extensive static and dynamic tests to verify whether the structure will carry the design loads without failure.

A19.4 Wing Stress Analysis Methods

In many of the previous chapters of this book, internal forces were calculated for both statically determinate and statically indeterminate structures. The internal loads in a statically determinate structure can be found by the use of the static equilibrium equations alone. The over-all structural arrangement of members is necessary, but the size or shape of no individual member is required. In other words, design consists of finding internal loads and then supplying a member to carry this load safely and efficiently. In a statically indeterminate structure, additional equations beyond the static equilibrium equations are necessary to find all the internal stresses. The additional equations are supplied from a consideration of structural distortions, which means that the size and shape and kind of material for members of the structure must be known before internal stresses can be determined. This fact means a trial and error method is necessary. Another important distinction is that a statically determinate wing structure has just enough members to produce stability and if one member is removed or fails, the entire structure will usually fail, whereas a statically indeterminate structure has one or more additional members than are necessary for static stability and thus some members could fail without causing the entire structure to collapse. In other words, the structure has a fail safe characteristic in that a redistribution of internal stress can take place if some members are over loaded. In general, statically indeterminate structures can be designed lighter and with smaller overall deflections.

METHODS OF STRESS ANALYSIS FOR STATICALLY INDETERMINATE WING STRUCTURES

Two general methods are commonly used, namely,

(1) Flexural beam theory with simplifying assumptions.

(2) Solving for redundant forces and stresses by applying the principles of the elastic theory by various methods such as virtual work, strain energy, etc.

The second method is no doubt more accurate since less assumptions are necessary. A wing structure composed of several cells and many spanwise stringers is a many degree redundant structure. Before the development of high speed computing machinery, so-called rigorous methods were not usable because the computing requirements were impossible or entirely impractical. However, present day computer facilities have changed the situation and rigorous methods are now being more and more used in aircraft structural analysis. Art. A7.5 and A8.10 in Chapters A7 and A8 present matrix methods for finding deflections and stresses to be used with computer facilities.

A19.5 Example Problem 1. 3-Flange ~ Single Cell Wing.

Fig. A19.7 shows a portion of a cantilever wing. To provide torsional strength a single closed cell (1) is formed by the interior web AB and the metal skin cover forward of this web. Thin sheet is relatively weak in resisting compressive stresses thus 3 flange stringers A, B and C are added to develop efficient bending resistance. The structure to the rear of spar AB is referred to as secondary structure and consists of thin metal or fabric covering attached to chordwise wing ribs. The air load on this portion is therefore carried forward by the ribs to the single cell beam.

A wing is subjected to many flight condi-
tions. The engineers who calculate the applied loads on the wing usually refer the resulting shears and moments to a set of convenient x, y and z axes. Fig. A19.7 shows the location of these reference axes. The job of the stress engineer is to provide structure to resist these loads safely and efficiently. The general procedure is to find the stresses or loads in all parts of the cell cross-section at several stations along the spanwise direction and form these loads or stresses proportion the required areas, thicknesses and shapes.

In this example, the internal loads will be calculated for only one section, namely, that at Station 240. It will be assumed that the design critical loads from the critical flight condition are as follows.

- \( M_X = 1,100,000 \text{ in} \cdot \text{lb} \)
- \( V_Z = 11,500 \text{ lb} \)
- \( M_Z = 80,000 \text{ in} \cdot \text{lb} \)
- \( V_X = 700 \text{ lb} \)
- \( M_Y = 460,500 \text{ in} \cdot \text{lb} \)

Fig. A19.8 shows these shears and moments referred to the reference axes with origin at point (O). Moments are represented by vectors with double arrow head. The sense of the moment follows the right-hand rule.

**SOLUTION.**

**ASSUMPTIONS: -** It will be assumed that the 3 flange stringers A, B and C develop the entire resistance to the bending moments about the Z and X axes. For skin under compression this assumption is nearly correct since the skin will buckle under relatively low stress. Since skin can take tensile stresses, this assumption is conservative. However, since the thin sheet cover must resist the shear stresses we will make this conservative assumption. The main advantage of this approximate assumption is that it makes the structure statically determinate.

Fig. A19.8 shows the wing cut at Station 240. The unknown forces are the three axial loads in the stringers A, B and C and the three shear flows \( q_{ab}, q_{bc} \) and \( q_{ac} \) on the three sheet panels, making a total of 6 unknowns and since there are 6 static equilibrium equations available for a space structure, the structure is statically determinate.

Since the size and shape of flange members A, B and C are unknown, we guess their centroid locations as indicated by the dots in Fig.A19.8. The axial load in each of the 3 stringers has been replaced by its x, y and z components as shown on the figure. The external applied loads are given at the reference origin (O) as shown in Fig. A19.8.

We now apply the equations of equilibrium to find the 6 unknowns.

To find \( C_Y \) take moments about Z axis through points A and B,

\[
EM_{Z(ab)} = -22 C_Y + 60000 = 0, \text{ whence } C_Y = 2727.27 \text{ lb}.
\]

The result comes out with a plus sign thus indicating that the assumed sense of tension was correct.

To find \( B_Y \) take moments about an X axis thru (A).

\[
EM_{X(A)} = -11 B_Y + 1,100,000 + 3636 x 0.125 = 0
\]
whence, $B_y = 100,043$ lb. tension as assumed.

To find $A_y$ take $\Sigma F_y = 0$

$\Sigma F_y = -100,043 - 3636 + A_y = 0,$

whence, $A_y = 103679$ lb. and compression as assumed.

Since the direction of the 3 stringers is known, we can find the $X$ and $Z$ components of the stringer loads by simple geometry.

The $y$, $x$, and $z$ length components of the three stringers from the dimensions given in Fig. A19.7 are found to be,

<table>
<thead>
<tr>
<th>Member</th>
<th>$y$</th>
<th>$x$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>240</td>
<td>6.2</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>240</td>
<td>6.2</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>240</td>
<td>20.93</td>
<td>3</td>
</tr>
</tbody>
</table>

The force components are therefore:

$A_z = 103679 \times 3/240 = 1296$ lb.

$A_x = 103679 \times 6.2/240 = 2679$ lb.

$B_z = 100043 \times 3/240 = 1250$ lb.

$B_x = 100043 \times 6.2/240 = 2584$ lb.

$C_z = 3636 \times 3/240 = 45$ lb.

$C_x = 3636 \times 20.93/240 = 316$ lb.

Fig. A19.9 shows these forces applied in the plane of the cross-section at Station 240, together with the unknown shear flows and the external forces act in the plane of the cross-section.

whence $q_{bc} = \frac{-3065}{665} = -13.66$ lb./in. and having the sense as assumed.

In the above moment equation the moment of the shear flow $q_{bc}$ about point (a') equals $q_{bc}$ times twice the area of the cell or 665.

To find $q_{ac}$ take $\Sigma F_x = 0$

$\Sigma F_x = -2679 + 2684 + 316 - 22 q_{ac} - 22 \times 13.66 - 700 = 0$

whence $q_{ac} = 35.45$ lb./in. with sense as assumed.

To find $q_{ab}$ take $\Sigma F_Z = 0$

$\Sigma F_z = -1296 - 1250 + 45 + 35.45 \times 0.5 - 11.5 \times 13.66 + 11500 - 11 q_{ab} = 0$

whence, $q_{ab} = 806$ lb./in.

The loads on the stringers and sheet panels have now been determined. The axial load in the stringers is practically equal to their $y$ component since axial load equals the $y$ force component divided by the cosine of a small angle. The stress engineer would find similar stresses at a number of stations along the span. These 6 stresses are generally referred to as primary stresses. Usually in most structures there are secondary stress effects which must be considered before final member sizes can be determined. For example, internal webs of a box type beam are designed usually as tension field beams. Tension field beam theory shows that the flange members are subjected to secondary stresses and the strength design of members and their connections to carry given stress loads is taken up in detail in Volume II.

A19.6 Example Problem 2. Metal Covered Wing With Single External Brace Strut.

In Chapter A2, the stress analysis of an externally braced fabric covered monoplane wing was considered. To provide sufficient torsional strength and rigidity, two external brace struts were necessary. However, if a wing is metal covered, a single external brace strut can be used, since the closed cell or cells formed by the metal sheet covering and the internal webs provide the torsional resistance and the wing can be designed as a simply supported beam with cantilever overhang. An excellent example of this type of wing structure is the Cessna aircraft Model 180 as shown in the photograph. An excellent airplane relative to performance, ease of manufacture and maintenance.

To introduce the student to the general approach of stress analyzing such a wing struc-
Fig. A19.10 and A19.11 shows the wing dimensions and general structural layout of a monoplane wing with one external brace strut. The wing panel is attached to fuselage by single pin fittings at points A and B with pin axes parallel to x axis. The mating lugs of the fittings at point A are made snug fit but those at B with some gap, thus drag reaction of wing loads on fuselage is resisted entirely at fitting A. Since the fittings at A and B cannot resist moments about x axis, it is necessary to add an external brace strut DC to make structure stable. The panel structure consists of a main spar ACE and a rear spar BF. The entire panel is covered with metal skin forward of the rear spar.

A simplified air load has been assumed as shown in Fig. A19.12, namely, a uniform load \( w = 30.27 \text{ lb./in. of span} \) acting at the 30 percent of chord point. When this resultant load is resolved into \( z \) and \( x \) components the results are \( w_z = 30 \text{ lb./in.} \) and \( w_x = 4 \text{ lb./in.} \) as shown in Fig. A19.12.

The general physical action of the wing structure in carrying these air loads can be considered as 3 rather distinct actions, namely, (1) The front spar ACE resists the bending moments and shears due to load \( w_z \), (2) The skin and webs of the two-cell tube resists all moments about \( y \) axis or broadly speaking torsional moments, (3) the entire panel cross-section resists the bending moment and flexural shear due to drag load \( w_x \), with the top and bottom skin acting as webs and the two spars as the flanges of this box beam.

General Calculations:

The unknown external reactions (see Fig. A19.11) are \( A_y, A_z, A_x, B_y, B_z \) and DC., or a total of 6. Since 6 static equations of equilibrium are available, the reactions are statically determinate. Reaction DC is also the load in brace strut DC.

To find reaction DC take moments about \( x \) axis through points A, B

\[ \Sigma M_x(AB) = (- 170 x 30 x 170/2) + DC(80/99.4)60 = 0 \]

whence, DC = 8979 lb. (The sign comes out plus so the sense assumed in Fig. A19.11 was correct.) The load in the strut is therefore 8979 lb. tension.

To find \( B_y \), take moments about a \( z \) axis through point \( A \).

\[ \Sigma M_z(A) = -(4 x 170 x 170/2) + 27 B_y = 0 \]

whence, \( B_y = 2141 \text{ lb. acting with sense assumed.} \)
To find $A_y$ take $E_F = 0$

$E_F = 2141 - 8979 (80/99.4) + A_y = 0$

whence $A_y = 5085$ lb.

To find $B_z$ take moments about y axis through (A).

$M_y(A) = -(30 \times 170 \times 3) - 27 B_z = 0$

whence $B_z = 567$ lb. acting as assumed.

To find $A_z$ take $E_F = 0$

$E_F = 170 \times 30 - 567 = (8979 \times 59/99.4) + A_z = 0$

whence $A_z = 796$ lb. acting as assumed.

The fittings at A and B should be designed to take the reactions at these points as found above. The external strut DC and its end fittings must carry the tension load of 8979 lb.

The next step is to find the stresses and loads on the structural parts of the wing panel.

Since the spar ACE can resist the bending moments about x axis the airloads in Fig. A19.12 are moved to the spar centerline as shown in Fig. A19.13.

The torsional moment of 90 in. lb. per inch of spar is resisted by the cellular tube made up of two cells (1) and (2). In many designs the leading edge cell is neglected in resisting the torsional moments due to many cutouts, etc., thus cell one could be assumed to provide the entire torsional shear resistance and the shear flow for this case would be $q = M_y/2A$ where $M_y$ equals the torsional moment at a given panel section and A the enclosed area of cell (1). If both cells were considered effective then the sheet thickness is necessary before solution for shear flow can be computed. Refer to Chapter A6 for computing torsional shear flow in multiple cell tubes.

The maximum torsional moment would be at the fuselage end of the wing panel and its magnitude would be $170 \times 90 = 15300$ in. lb. Since the top and bottom skin is not attached to fuselage, this torsional moment must be thrown off on a rib at the end of the panel and this rib in turn transfers this moment in terms of a couple reaction on the spars at points A and B. This couple force equals the moment divided by distance between spars or

$15300/27 = 568.7$ lb.

Front spar (ACF) loads due to $w_z = 30$ lb/in:

$w = 30$ lb/in.

Fig. A19.14 shows free body of front spar ACE. To find strut load DC take moments about (A).

$M_A = (-30 \times 170 \times 170/2) + 60 (DC \times 80/99.4) = 0$

whence DC = 8979 lb. Tension which checks value previously found.

The $y$ and $z$ components of the strut reaction at C will then be,

$C_y = 8979 (80/99.4) = 7226$ lb.

$C_z = 8979 (59/99.4) = 5329$ lb.

These values are indicated on Fig. A19.14.

To find $A_y$ take $E_F = 0$

$E_F = -7226 + A_y = 0$, hence $A_y = 7226$ lb.

In finding the reaction $A_y$ previously, the value was 5085 lb. The difference is due to the drag bending moment which tends to put a tension load on front spar and compression on the rear spar.

Fig. A19.15 shows the air drag load of 4 lb/in. The bending moment on panel at a distance $y$ from the wing tip equals

$w_y (y/2) = 4y^2/2 = 2y^2$.

The axial load $P_y$ in either spar at any distance $y$ from tip thus equals bending moment divided by arm of 27" or $P_y = 2y^2/27 = 0.074y^2$. The axial load at points A and B thus equals $0.074 \times 170^2 = 2141$ lb. (tension in front spar and compression in rear spar). Thus each spar is subjected to an axial load
increasing from zero at tip to 2141 lb. at fuselage attachment points and varying as \( y^2 \). At point C on from spar the axial tension load would be 0.74% (90°) = 600 lb.

The design of the front spar between points C and E would be nothing more than a cantilever beam subjected to a bending forces plus an axial tensile load plus a torsional shear flow. The design of the spar between points C and A is far more complicated since we have appreciable secondary bending moments to determine, which must be added to the primary bending moments. Fig. A19.15a shows a free body of spar portion AC.

\[
\begin{align*}
\text{Fig. A19.15a} \\
\text{w = 30} \\
\text{5085} \\
\text{M} = 30 \times 90 \times 45
\end{align*}
\]

The lateral load of 30 lb./in. bends the beam upward, thus the axial loads at A and C will have a moment arm due to beam deflection which moments are referred to as secondary moments. To find deflections the beam moment of inertia must be known, thus the design of this beam portion would fall in the trial and error procedure. Articles A5.28 to 29 of Chapter A5 explain and illustrate solution of problems involving beam-column action and such a procedure would have to be used in actually designing this beam portion.

The rear spar BF receives two load systems, namely a varying axial load of zero at F to 2141 lb. at B and the web of this spar receives a sheared load from the torsional moment. The rear spar is not subjected to bending moments.

In Fig. A9.10 the secondary structure consisting of chordwise ribs and spanwise light stringers riveted to skin are not shown. This secondary structure is necessary to hold wing contour shape and transfer air pressures to the box structure. This secondary structure is discussed in Chapter A21. The broad subject of designing a member or structure to withstand stresses safely and efficiently is considered in detail in later chapters.

A19.7 Single Spar - Cantilever Wing - Metal Covered

A single spar cantilever wing with metal covering is often used particularly in light commercial or private pilot aircraft.

Suppose in the single spar externally braced wing of Fig. A19.11, that the external brace strut DC was removed. Obviously the wing would be unstable as it would rotate about hinge fittings at points A and B. To make the structure stable the single pin fitting at (A) would have to be replaced by two fittings, one on the upper flange and the other on the lower flange in order to be able to resist a \( M_x \) moment. Fig. A19.16 shows this modification. The fitting at B could remain as before, a single pin fitting.

The stress analysis of this wing would consist of the spar AE resisting all the \( M_x \) moments and the \( V_z \) shears and acting as a cantilever beam. The torsional moment about a y axis coinciding with spar AE would be resisted by shear stresses in the cellular tubes formed by the skin and the spar webs. The drag bending and shear forces would be resisted by the beam whose flanges are the front and rear spars and the web being the top and bottom skin.

A19.8 Stress Analysis of Thin Skin - Multiple Stringer Cantilever Wing. Introduction and Assumptions.

The most common type of wing construction is the multiple stringer type as illustrated by the six illustrative cross-sections in Fig. A19.2. A structure with many stringers and sheet panels is statically indeterminate to many degrees with respect to internal stresses. Fortunately, structural tests of complete wing structures show that the simple beam theory gives stresses which check fairly well with measured stresses if the wing span is several times the wing chord, that sweep back is minor and wing is free of major cutouts and discontinuities. Thus it is common procedure to analyze and design a wing overall by the beam theory and then investigate those portions of the wing where the beam theory may be in error by using more rigorous analysis methods such as those explained and illustrated in Art. A3.10 of Chapter A8.

ASSUMPTIONS - BEAM THEORY

In this chapter the wing bending and shear stresses will be calculated using the unsymmetrical beam theory. The two main assumptions in this theory are:

(1) Transverse sections of the beam originally plane before bending remain plane after bending of beam. This assumption means that longitudinal strain varies directly as the distance from
the neutral axis or strain variation is linear.

(2) The longitudinal stress distribution is directly proportional to strain and therefore from assumption (1) is also linear. This assumption actually means that each longitudinal element acts as if it were separate from every other element and that Hooke's law holds, namely, that the stress-strain curve is linear.

Assumption (1) neglects strain due to shear stresses in skin, which influence is commonly referred to as shear lag effect. Shear lag effects are usually not important except near major cut-outs or other major discontinuities and also locations where large concentrated external forces are applied.

Assumption (2) is usually not correct if elastic and inelastic buckling of skin and stringers occur before failure of wing. In applying the beam theory to practical wings, the error of this assumption is corrected by use of a so-called effective section which is discussed later.

A19.9 Physical Action of Wing Section in Resisting External Bending Forces from Zero to Failing Load.

Fig. A19.17 shows a common type of wing cross-section structural arrangement generally referred to as the distributed flange type.

![Fig. A19.17](image)

The corner members (a) and (b) are considerably larger in area than the stringers (c). The skin is relatively thin. Now assume the wing is subjected to gradually increasing bending forces which place the upper portion of this wing section in compression and the bottom portion in tension. Under small loading the compressive stresses in the top surface will be small and the stress will be directly proportional to strain and the beam formula \( \sigma_c = Mx/L^2 \) will apply and the moment of inertia \( I_x \) will include all of the cross-section material. As the external load is increased the compressive stresses on the thin sheets starts to buckle the sheet panels and further resistance decreases rapidly as further strain continues, or in other words, stress is not directly proportional to strain when sheet buckles. Buckling of the skin panels however does not cause the beam to fail as the stringers and corner members are low stressed compared to their failing stress. The stringers (c) are only supported transversely at wing rib points and thus the stringers act as columns and fail by elastic or inelastic bending.

corner flange members (a) and (b) are stabilized in two directions by the skin and webs and usually fail by local crippling.

Now continuing the loading of the wing after the skin has buckled, the stringers and corner members will continue to take additional compressive stress. Since the ultimate strength of the stringers is less than that of the corner members, the stringers (c) will start to bend elastically or inelastically and will take practically no further stress as additional strain takes place. The corner members still have considerable additional strength and thus additional external loading can be applied until finally the ultimate strength of the corner members is reached and then complete failure of the top portion of the beam section takes place. Therefore, the true stress - strain relationship does not follow Hooke's Law when such a structure is loaded to failure.

In the above discussion the stringers (c) were considered to hold their ultimate buckling load during considerable additional axial strain. This can be verified experimentally by testing practical columns. Practical columns are not perfect relative to straightness, uniformity of material, etc. Fig. A19.18 shows the load versus lateral deflection of column midpoint as a column is loaded to failure and fails by elastic bending. Fig. A19.19 shows similar results when the failure is inelastic bending.

![Fig. A19.18](image)

The test results show that a compression member which fails in bending, normally continues to carry approximately the maximum load under considerable additional axial deformation. Thus in the beam section of Fig. A19.17 when the stringers (c) reach their ultimate load, failure of the beam does not follow since corner members (a) and (b) still have remaining strength.

A19.10 Ultimate Strength Design Requirement

The strength design requirements are:

1. Under the applied or limit loads, no structural members shall be stressed above the material yield point, or in other words, there must be no permanent deformation or deflection of any part of the structure.
(2) Under the design loads which equal the applied loads times a factor of safety, no failure of the structure shall occur. The usual factor of safety for conventional aircraft is 1.5, or the structure must carry loads 50 percent greater than the applied loads without failure. The higher the stress at failure of a member the less material required and therefore the less structural weight. The stress engineer thus tries to design members which fall in the inelastic zone.

The bending stress beam formula $\sigma_b = Mc/I$ does not take care of this non-linear stress-strain action and thus some modification of the moment of inertia of the beam cross-section is necessary if the ultimate strength of a wing section is to be computed fairly accurately. The stress engineer usually solves this problem by using a modified cross-section, usually referred to as the effective cross-section.

A19.11 Effective Section at Failing Load.

In order to use the beam formula which assumes linear stress-strain relation, corrections to take care of skin buckling and stringer buckling must be introduced. The effectiveness of the skin panels will be considered first.

When a compressive load is applied to a sheet-stringer combination, the thin sheet buckles at rather low stress. For further loading the compressive stress varies over the panel width as illustrated in Fig. A19.20. The stress in the sheet at the stringer attachment line is the same as in the stringer since it cannot buckle and therefore suffers the same strain as the stringer. Between the stringers the sheet stress decreases as shown by the dashed line in Fig. A19.20. This variable stress condition is difficult to handle so the stress engineer makes a convenient substitution by replacing the actual sheet with its variable stress by a width of sheet carrying a uniform stress equal to the stringer stress. In Fig. A19.21 $2w$ is the effective sheet width to go with each stringer. The total stress on the effective widths carrying a uniform stress equal to the stringer stress equals the total load on the sheet panels carrying the actual varying stress distribution. The equation for effective width is usually written in the form

$$2w = kt \left( \frac{E}{\sigma_{st}} \right)^{\frac{1}{2}}$$

a widely used value for $k = 1.3$, whence

$$2w = 1.30 \times \left( \frac{E}{\sigma_{st}} \right)^{\frac{1}{2}}$$

$\sigma_{st} =$ stress in stringer

Therefore if we know the stress in the stringer we can find the width of sheet to use with the stringer to obtain an effective section to take care of the sheet buckling influence.

Effective Factor for Buckled Stringers.

Consider the beam section in Fig. A19.17. If we take a stringer (c) and attach a piece of sheet to it equal to $2w$, the effective width and test it in compression and brace it in a plane parallel to the sheet, the resulting test stress versus strain shortening curve (c) of Fig. A19.22 will result. The length of the test specimen would equal the rib spacing in the wing. The corner members (a) or (b) in Fig. A19.17 being stabilized in two directions will fail by local crippling, thus if a short piece of this member is tested to failure in compression the test curve (A) in Fig. A19.22 is obtained. Curve (C) shows that the stringer holds approximately its maximum load for a considerable strain period. Curve (A) shows that for the same unit strain member (a) can take considerable higher stress. If we take a unit strain of .006, the point at which the maximum stress of 40000 is obtained in member (a) (see point (3) on Curve A) the stress at the same strain for member (c) will be 35000 (see point (2) on Curve C).

* The buckling of sheets is taken up in detail in Volume II.
For stress analysis procedure using the beam formula, we assume a linear stress variation from zero to 47000 psi. Since stringers (c) can only take 38000 psi at the same strain, or in other words, stringer (c) is less effective than members (a) and (b). The effective-ness factor for stringer (c) equals its ultimate strength divided by the ultimate strength of member (a) or 38000/47000 = .808.

**A19.12 Example Problem**

The wing section in Fig. A19.23 is subjected to a design bending moment about the x axis of 500,000 in.lbs., acting in a direction to put the upper portion in compression. The problem is to determine the margin of safety for this design bending moment. The material is 2424 aluminum alloy.

![Diagram](image)

**SOLUTION**

The beam formula for bending stress at any point is \( \sigma = Mz/I_X \). To solve this equation we must have the effective moment of inertia of the beam cross-section. The bottom surface being in tension under the given design bending moment is entirely effective, however the top surface has a variable effectiveness since the skin, stringers and corner flange members have different ultimate failing stresses.

From equation (1) the effective width of sheet to use with each rivet line depends on the stress in the stringer to which it is attached. The failing stress for the stringer will be used, which means that the failing stress of the stringers and corner flange must be known before the effective width can be found. For this example the zee stringers have been selected of such a size as to give an ultimate column failing stress of 35000 psi, and the corner flange members have been made of such size and shape as to give a falling stress of 47000 psi. These failing stresses can be computed by theory and methods as given in Volume II. The sizes have purposely been selected to given strengths represented by the test curves of Fig. A19.22.

The effective width with each rivet line from equation (1) would be,

\[
2w = 1.90 x .04 (10.5 x 10^5/38000)^{1/2} = 1.28 inches
\]

Thus the area of effective skin = 1.25 x .04 = .05 in.². The area of the zee stringer is 0.135 sq. in, which added to the effective skin area gives 0.185 sq. in, which value is entered in Column (2) of Table I opposite zee stringers numbered 2, 3 and 4 in Fig. A19.22. The same procedure is done for the corner members 1 and 5 with the resulting effective areas as given in Table I. On the bottom side which is in tension all material is effective. The skin width equal to one-half the distance to adjacent stringers is assumed to act with each stringer. Taking the area of the angle section as 0.11 and adding skin area equal to 6 x .035 = .21 and a total of 0.32 sq. in. which value is shown in Column 2 of Table I opposite stringers 7, 8 and 9. The areas of the lower corner members plus bottom skin and web skin would come out as recorded in the Table.

The next step is to correct for stringer effectiveness in compression. The failing stress for the zee stringer was given as 38000 psi, and for the corner members 47000 psi. The effectiveness factor for the zee stringer thus equals 37000/47000 = .808. This factor is recorded in Column 3 of Table I. For the corner members 1 and 5 and all the tension members the factor is of course unity. The balance of Table I gives the calculation of the effective moment of inertia \( I_{ax} \) about the x neutral axis.

The compressive stress intensity at the centroid of the zee stringers thus equals,

\[
\sigma_0 = \frac{Mz}{I_X} = 500,000 x 5.57/59.50 = 46600 psi.
\]

The true stress equals the effectiveness factor times this stress = .808 x 46600 = 37400 psi. The failing stress equals 38000 psi hence margin of safety = (38000/37400) - 1 = .01 or one percent.
The stress analysis of this wing would show the resulting bending and shear stresses for a number of spanwise stations for the critical design load conditions. In this example solution the bending longitudinal stresses will be determined on cross-sections at two stations, namely, stations 20 and 47.5, and the shear stresses will be determined for station 20. In this example problem, the leading edge cell will be considered ineffective as well as any structure to rear of rear beam, hence structure is a one cell beam with multiple stringers. A second solution including the leading edge cell to form a two cell beam will also be presented.

**ANALYSIS FOR BENDING LONGITUDINAL STRESSES**

Longitudinal stresses (tension or compression) are produced by external forces normal to the cross-section and by bending moments about x and z axes. The stress equations are:

\[ \sigma_x = \frac{P/A}{-9.9} \]
\[ \sigma_z = \frac{P/A}{-9.9} \]

where \( \sigma_x \) = longitudinal stress

\[ P = \text{external load acting through centroid of effective wing cross-section} \]
\[ A = \text{effective area of cross-section} \]

For any given flange member with area (a) the load \( P_a \) on the member would be,

\[ P_a = \sigma_x a \]

The stresses due to bending moments are from Chapter A13, Art. A13.5:

\[ \sigma_x = -\left( K_x M_x - K_z M_z \right) a \]

where \( \sigma_x = \text{bending stress with plus being tension} \)

Kx = Ixz/(Ix Iy - Ixz)
Kz = Iz/(Ix Iy - Ixz)
Ky = Iy/(Ix Iy - Ixz)

The normal component of the axial load in a flange member equals \( \sigma_y \) a where (a) is the area of the flange member. Since the angle between the normal to the beam section and the centroidal axis of a stringer is generally quite small, the difference between the cosine of a small angle and unity is negligible and thus the normal component can be considered as the axial load in the stringer.

Before equations 2, 3 and 4 can be solved, the effective cross-section area must be known as well as the moments of inertia and product of inertia about x and z centroidal axes.
Fig. A19.24 Structural Layout of Tapered Cantilever Wing
ANALYSIS OF WING STRUCTURES

Fig. A19.25  WING SECTION AT STATION 20. EFFECTIVE AREAS.

Fig. A19.26  WING SECTION STATION 47.5
Fig. A19.25 shows the cross section at Station 20 divided into 14 longitudinal units numbered 1 to 14. Since the external load condition to be used places the top surface in compression, the skin will buckle and thus we use the effective width procedure to obtain the skin portion to act with each stringer. Fig. A19.25 shows the effective skin which is used with each flange member to give the total area of numbers (3) to (7). The skin on the bottom surface being in tension is all effective and Fig. A19.25 shows the skin area used with each bottom flange member.

The next factor to decide is the stringer effectiveness as discussed and explained in the previous example problem. For the structure of Fig. A19.25 we will assume that the compressive falling stress of the stringers is the same as that for the corner members, thus we will have no correction factor to take care of the situation of flange members having different ultimate strengths.

Table A19.2, columns 1 to 11, and the calculations below the table give the calculations for determining the section properties at Station 20, namely A, Ix, Iy and Ixz. Table A19.3 gives the same for wing section at Station 47.5. The areas in column (2) are less since sizes have changed between Stations 20 and 47.5.

Calculation of longitudinal stress due to Mx and My bending moments:

The design bending moments will be assumed and are as follows:

| Station 20 | Mx = 1,300,000 in.lbf. | My = -285,000 in.lbf. |
| Station 47.5 | Mx = 1,000,000 in.lbf. | My = -215,000 in.lbf. |

The moments about the y axes are not needed in the bending stress analysis but are needed in the shear analysis which will be made later.

To solve equation (4) the constants Ka, Kb, and Kc must be known.

For Station 20 from Table A19.2, Ix = 230.3, Iy = 1030 and Ixz = -50, whence:

Kx = Ixz/(Ix Iy - Ixz²) = -50/(230.3 x 10.30 - 50²) = -50/235500 = -0.0002125
Ky = Iy/235500 = 1030/235500 = 0.004375
Kz = Ix/235500 = 230.3/235500 = 0.00098

Substituting in equation (4)

σb = -[0.0098 x 255000 - (-0.0002125 x 1300000)] x [-0.004375 x 1,300,000 - (-0.0002125 x 255000)] z

whence, σb = 3.3 x 5639 z = - - - - - - - - - - - - - - - (5)

Column 12 of Table A19.2 gives the results of this equation for values of x and z in columns 10 and 11. Multiplying these bending stresses by the stringer areas, the stringer loads are given in column 13. The sum of the loads in this column should be zero since the total tension must equal total compression on a section in bending.

Stresses at station 47.5:

Ix = 157.4, Iy = 700, Ixz = -35.4 (Table A19.3)
Kx = 35.4/(157.4 x 700 - 35.4²) = -35.4/106950 = -0.000324
Ky = 700/106950 = 0.00643
Kz = 157.4/106750 = 0.001447
σb = -[0.01447 x -215000 - (-0.00324 x 1,000,000)] x [-0.00643 x 1,000,000 - (-0.00324 x -215000)] z

whence, σb = -14.5 x 6360 z

Column 12 of Table A19.3 gives the results of this equation and column 13, the total stringer loads at station 47.5.

The stresses in column 12 of each table would be compared to the falling stress of the flange members to obtain the margin of safety.

ANALYSIS FOR SHEAR STRESSES IN MEMS AND SKIN

The shear flow distribution will be calculated by using the change in axial load in the stringers between stations 20 and 47.5, a method commonly referred to as the 4P method. For explanation of this method, refer to Art. A15.16 of Chapter A15.

The shear flow in the y direction at a point n of the cell wall equals,

qyn = qo + Σ AP o d

where qo is a known value of shear flow at some point o and the second term is the change in shear flow between points o and n.
## TABLE A19.2
### SECTION PROPERTIES ABOUT CENTROIDAL X AND Z AXES

<table>
<thead>
<tr>
<th>Flange Area No. A</th>
<th>Z'</th>
<th>AZ'</th>
<th>AZ'^2</th>
<th>X'</th>
<th>AX'</th>
<th>AX'^2</th>
<th>AX·Z'</th>
<th>Z = Z' - Z</th>
<th>X = X' - X</th>
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<th>P = cdA</th>
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General Notes:
- See Fig. A19.25 for Section at Station 20.
- Reference axes X'X' and Z'Z' are assumed as shown.
- Properties are calculated with respect to these axes and transferred to the centroidal X and Z axes.

### TABLE A19.3
### SECTION PROPERTIES ABOUT CENTROIDAL X AND Z AXES

<table>
<thead>
<tr>
<th>Flange Area No. A</th>
<th>Z'</th>
<th>AZ'</th>
<th>AZ'^2</th>
<th>X'</th>
<th>AX'</th>
<th>AX'^2</th>
<th>AX·Z'</th>
<th>Z = Z' - Z</th>
<th>X = X' - X</th>
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General Notes:
- See Fig. A19.26 for section at Station 47.5.
- Reference axes X'X' and Z'Z' are assumed as shown.
- cd = -14.5 X = 6360 Z

General Notes:
- See Fig. A19.26 for section at Station 47.5.
- Reference axes X'X' and Z'Z' are assumed as shown.
- cd = -14.5 X = 6360 Z
dp equals the change in stringer axial load over a distance d in the y direction.

Since the cell in our problem is closed the value of \( q_y \) at any point is unknown, we assume it zero on web 1-14 by imagining that the web is cut as shown in Fig. A19.27. Equation (6) thus reduces to,

\[
q_y = - \frac{\Delta P}{27.5} \tag{7}
\]

Columns 1 to 5 of Table A19.4 show the solution of equation (7). The shear flow values of \( q_y \) in column 5 are plotted on the cell wall in Fig. A19.27, remembering that \( q_y = q_x = q_2 \). For rules giving direction of \( q_x \) and \( q_z \) refer to Chapter A14, Art. A14.6.

![Fig. A19.27](image)

**TABLE A19.4**

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For equilibrium of all the forces in the plane of the cross-section \( \Sigma M_y \) must equal zero. For convenience we will select a moment y axis through the c.g. of the cross-section. The moment of the shear flow \( q_y \) on any sheet element equal \( q \) times double the area of the triangle formed by joining the c.g. with lines going to each end of the sheet element. These double areas are referred as \( m \) values (See Fig. A19.27). Column 6 of Table A19.4 records these double areas which were obtained by use of a planimeter. Column (7) gives the moment of each shear flow about the c.g. and the total of this column gives the moment about the c.g. of the complete shear flow system of Fig. A19.27 or a value of 266060 in. lb.

The double areas \( m \) can be found approximately as follows:

The moment of the shear flow \( q \) on the web (2-3) about point 0 equals \( q \times x \times z \) times twice the area \( A_1 + A_2 \). In most cases, the area \( A_2 \) can be neglected.

By simple geometry, the area

\[
A_1 = \frac{1}{2} (x_2 z_2 - x_1 z_1).
\]

The moment of the shear flow \( q \) on the web (2-3) thus equals \( q \times x_2 z_2 - x_1 z_1 \). Since values of \( x \) and \( z \) for all flange points with reference to section c.g. are given in the Table A19.2, it is unnecessary to use the planimeter except for regions of sharp curvature.

**MOMENT OF EXTERNAL LOADS ABOUT C.G. OF STATION 20**

As stated before the engineers in the applied loads calculation group supply the shears and moments at various spanwise stations. We will assume that these loads are: \( V_z = 12000 \) lb., \( V_x = +2700 \) lb., \( M_y = -390,000 \) in. lb. The location of the reference y axis used by the loads group will be assumed as located at point 0 in Fig. A19.28 relative to cross-section at Station 20.

![Fig. A19.28](image)

Therefore moment of external loads about c.g. is,

\[
E M_{c.g.} = 12000 \times 33.3 - 2700 \times 11.8 - 390000 = 41800 \text{ in.} \text{lb.}
\]

Moments Produced by Inclination of Flange Loads With Beam Section.

Since the flange members in general are not normal to the beam sections, the flange loads
have components in the Z and X directions. Columns (4) and (7) of the Table A19.5 give the values of these in plane components. The slopes dx/dy between stations 20 and 47.5 are found by scaling from Fig. A19.24. Fig. A19.29 shows these induced in plane forces as found in Table A19.5.

### Table A19.5

<table>
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<tr>
<th>Flange No.</th>
<th>P (kips)</th>
<th>dx/dy</th>
<th>P&lt;sub&gt;x&lt;/sub&gt; (kips)</th>
<th>M&lt;sub&gt;c.g. + P&lt;sub&gt;x&lt;/sub&gt;&lt;/sub&gt;</th>
<th>dx/dy</th>
<th>P&lt;sub&gt;z&lt;/sub&gt; (kips)</th>
<th>M&lt;sub&gt;c.g. + P&lt;sub&gt;z&lt;/sub&gt;&lt;/sub&gt;</th>
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<td>0.18</td>
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</table>

### Notes:
- Column (2): P from Table A19.2
- Column (5) and (8): Values of Z and X are found in Columns 10, 11 of Table A19.2

Fig. A19.29 In plane forces produced by flange axial loads.

The moments of these in plane components about the section c.g. are given in Columns (5) and (8) of Table A19.5. In general, these moments are not large.

**Total Moments of All Forces About Section c.g. at Station 20:**

- Due to flanges = 7160 - 2224 = 4936 in-lb.  (Ref. Table A19.5)
- Due to assumed static shear flow = 255060 in-lb.  (Ref. Table A19.4)

Due to external loads = 41800 in-lb.

Then the total unbalanced moment = 4936 + 255060 + 41800 = 302796 in-lb.

For equilibrium, this must be balanced by a constant shear flow q<sub>r</sub>

\[
q_r = \frac{M}{2A} = \frac{-302796}{3 \times 461.5} = -328 \text{ lb./in.}
\]

(Note: 461.5 = total area of cell)

The shear stresses q<sub>r</sub> are listed in Column (6) of Table A19.4.

The final or resultant shear flow q<sub>r</sub> at any point therefore equals

\[
q_r = q + q_r
\]

The resulting values are given in Column 9 of Table A19.4. Fig. A19.30 illustrates the results graphically.

**Example Problem**

To avoid repetition of similar type calcu-
lations as was used in the previous single cell problem, the bending and shear stresses will be determined for the same structure as in the previous example except that the leading edge cell is considered effective, thus making a 2-cell structure. Since there are no spanwise stringers in the leading edge, very little skin on the compressive side will be effective. On the tension side, the leading edge skin would be effective in resisting bending axial loads and thus the moment of inertia would be slightly different from that found in example problem 1. Since this problem is only for the purpose of illustrating the use of the equations, the leading edge skin will be neglected in computing the bending flexural stresses. With this assumption, the bending stresses and flange loads at stations 20 and 47.5 are the same as for the previous problem. (See values in column 12 and 13 of Tables A19.2 and A19.3.)

Shear Flow Calculations:

To compute the static shear flow, each cell is assumed cut at one point as shown in Fig. A19.31, and thus the shear flow is zero at points (a) and (b).

Fig. A19.31

1 2 3 4 5 6 7

Cell (1) Area = 93.5

Cell (2) Area = 461.5 sq in.

14 13 12 11 10 9 8

Table A19.6 (Column 5) gives the value of the static shear flow under these assumptions.

The first 7 columns of this table are the same as in Table A19.4, since no stringers have been added to cell (1), and the shear q is assumed zero in cell (1).

To make the twist of each cell the same and also to make the summation of all torsional forces zero will require two unknown constant shear flows, q1 in cell (1) and q2 in cell (2). Thus two equations will be written, namely:

\[ \theta_1 = \frac{1}{2G} \left( \int q_1 L/t + q_2 L/t \right) \]

The twist \( \theta \) per unit length of a cell equals

\[ \theta = \frac{L}{2G} \int q \text{ L/t} \]

The modulus of rigidity \( G \) will be assumed constant and thus will be omitted.

Consider cell (1): (Refer to Table A19.6, Columns 10 and 11)

Area of cell (1) = 93.5 sq in.

\[ \theta_1 = \frac{1}{2G} \left[ \int q_1 L/t + \int q_2 L/t \right] \]

\[ \theta_1 = 0 + 1276 q_1 - 230 q_2 \]

whence

\[ \theta_1 = 7.65 q_1 - 1.379 q_2 \]

For cell (2):

Area of cell (2) = 461.5 sq in.

### TABLE A19.6

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<th>Flange No.</th>
<th>P at Sta,</th>
<th>Q at Sta,</th>
<th>G at Sta,</th>
<th>q at Sta,</th>
<th>L (in.)</th>
<th>t</th>
<th>L/t</th>
<th>g(L/t)</th>
<th>q1</th>
<th>q2</th>
<th>qr = 2q1 - q2</th>
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<td>0.32</td>
<td>166</td>
<td>33400</td>
<td>0 -317 -115.9</td>
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**Notes:**

L = length of web sheet between flange members.

t = web thickness.
\[ q_x = \frac{1}{2 \times 461.5} \left( 733030 - 230 \times q_x - 2469 \times q_x \right) \]
\[ q_x = 794 - 2485 q_1 + 2.678 q_x \quad \text{(b)} \]

For continuity \( q_1 \) must equal \( q_x \), hence equating (a) and (b):

\[ 7.889 q_x - 4.086 q_x - 794 = 0 \quad \text{(c)} \]

For equilibrium the summation of all moments in the plane of the cross-section about the section (c.g.) must be equal to zero, or

\[ M_{c.g.} = 0 \]

The moment of the external loads about the section c.g. is the same as in previous problem.

\[ M = 41800 \text{ in.lb.} \]

The induced moment due to the in-plane components of the flange axial loads is likewise the same as in previous problem (see Table A19.5).

\[ M_{\text{due to flange loads}} = 4936 \text{ in.lb.} \]

The torsional moment due to the static shear flow from Column (7) of Table A19.5 equals 255050 in.lb. The torque due to the unknown constant shear flows of \( q_1 \) and \( q_x \) is equal to twice the enclosed area of each cell times the shear flow in that cell. Hence

\[ M_{\text{(due to } q_1 \text{ and } q_x)} = 2 \times 233.5 \times q_1 + 2 \times 461.5 \times q_x = 167 \times q_1 + 923 \times q_x \]

Therefore \[ M_{c.g.} = 167 q_1 + 923 q_x = 302796 = 0 \quad \text{(d)} \]

Solving equations (c) and (d), we obtain,

\[ q_1 = -62.8 \text{ lb./in.}, q_x = -317 \text{ lb./in.} \]

These values are listed in columns (12) and (13) of Table A19.5. The final or resultant shear flow \( q_x \) on any sheet panel equals the sum of \( q + d_1 + q_x \). The results are shown in column (14) of Table A19.6. Fig. A19.32 shows the dotted shear flow pattern. Comparing this figure with Fig. A19.30 shows the effect of adding the leading edge cell to the single cell of the previous problem.

**A19.15** Bending Strength of Thick Skin - Wing Section

Figs. 1 and k of Fig. A19.3 illustrate approximate shapes for airfoils of supersonic aircraft. Such airfoils have relatively low thickness ratios and since supersonic military aircraft have comparatively high wing loadings, it is necessary to go to thick skin in order to resist the wing bending moments efficiently. The ultimate compressive stress of such structures can be made rather uniform and occurring at stresses considerably above the yield point of the material. Since structures must carry the design loads without failure, it is necessary to be able to calculate the ultimate bending resistance of such a wing section if the margin of safety is to be given for various load conditions.

The question of the ultimate bending resistance of beam sections that fail at stresses beyond the elastic stress range is treated in Article A13.10 and example problem 7 of Chapter A13 and should be studied again before proceeding with the following example problem.

**A19.16** Example Problem

To illustrate the procedure of Art. A13.10, a portion of a thick skin wing section as illustrated in Fig. A19.34 will be considered.

![Fig. A19.34](image)

For simplicity the section has been symmetrical about the x-x axis. The material is aluminum alloy. In this problem the material stress-strain curves will be assumed the same in both tension and compression. The problem is to determine the margin of safety for this beam section when subjected to a design bending moment \( M_x = 1,850,000 \text{ in.lb.} \)

**SOLUTION:**

Since it is desirable to use the beam formula \( \sigma_b = \frac{M_x}{I_x} \), it is necessary to obtain a modified beam section to correct for the non-linear stress-strain relationship since the give structure will fail under stresses in the inelastic zone. The maximum compressive stress at surface of beam will be assumed at 50000 psi. This value could be calculated from a consider-
The values in column (3) of Table A19.7 represents the true compressive stress at the midpoint of a strip area when the beam is resisting its maximum or failing bending moment. The values in column (4) represent the compressive stress at the midpoint of the strip areas if the bending stress is linear and varying from zero at neutral axis to 30000 psi at edge of beam section (curve B of Fig. A19.35a).

To illustrate, consider strip area number (2) in Fig. b of A19.35. Project a horizontal dashed line from midpoint of this strip until it intersects curves A and B at points (a) and (b) respectively. From these intersection points project downward to read values of 48000 and 40600 psi respectively.

In using the linear beam formula, the stress intensity on strip (2) would be 40600 but actually it is 46000. The ratio between the two is given the symbol K. Thus to modify the linear stress to make it equal to the nonlinear stress we increase the true strip areas by the factor K, giving the results of column (6).

The modified moment of inertia (column 8) equals $I_X = 115.52$.

The design bending moment was 1,850,000 in.lb.

Consider point at midpoint of strip (1) $Z = 2.8125$ inches

$\sigma_o = M_Z I_X = (1,850,000 \times 2.8125)/115.52 = 45100$ psi.

This stress is based on the modified strip areas. The true stress $\sigma_t$ on strip 1 thus equals $K \sigma_o = 1.049 \times 45100 = 47400$. The allowable stress at failure equals 49000 from Table A19.7. Hence margin of safety $= (49000/47400) - 1 = .04$ or 4 percent.

The margin of safety for points other on the beam sections will likewise be 4 percent. For example at midpoint of strip (3), $Z = 2.0625$. K=1.34 whence $\sigma_{ot} = [(1,850,000 \times 2.0625)/115.52] 1.34 = 44400$.

whence margin of safety = (46000/44400) - 1 = .04

The moment of inertia without modifying the strip area would come out to be $I_X = 104.42$ hence the stress at midpoint of strip (1) would calculate to be $\sigma_o = (1,850,000 \times 2.8125)/104.42 = 49900$. The allowable stress for linear stress variation would be 48600 from column (4) of Table. Hence margin of safety would be
(46900/49900) - 1 = -0.06. The elastic theory thus gives a margin of safety 10 percent less than the strength given when true stress-strain or non-linear relationship is used.

If the same comparison was made for bending about Z axis of this same beam section the difference would be considerably more than 10 percent as more beam area is acting in the region of greater discrepancy between curves A and B.

A19.17 Application to Practical Wing Section

A practical wing section involves these facts: - (1) The section is unsymmetrical; (2) external load planes change their direction under different flight conditions; (3) the material stress-strain curves are different in tension and compression in the inelastic range.

Since the stress analyst must determine critical margins of safety for many conditions, it would be convenient to have an interaction curve involving $M_x$ and $M_y$ bending moments which would cause failure of the wing section. This interaction curve could be obtained as follows:

(1) Choose a neutral axis direction and its location.

(2) Assuming that plane sections remain plane, and taking the maximum strain as that causing failure of the compressive flange, use the stress-strain curve to determine the longitudinal stress and then the internal load on each element of the cross-section. A check on the location of the assumed neutral axis is that the total compression on cross-section must equal total tension. Since the location was assumed or guessed, the neutral axis must be moved parallel to itself to another location and repeated until the above check is obtained.

(3) Find the internal resisting moment about the neutral axis and an axis normal to the neutral axes. Resolve these moments into moments about x and z axes or $M_x$ and $M_z$. These resulting values of $M_x$ and $M_z$ are bending moments which acting together will cause failure of the wing in bending.

(4) Repeat steps 1, 2 and 3 for several other directions for a neutral axis which results will give additional combinations of $M_x$ and $M_y$ moments to cause wing failure. Thus an interaction curve involving values of $M_x$ and $M_y$ which cause failure of wing in bending is obtained and thus the margin of safety for any design condition is readily obtainable.

A19.18 Shear Lag Influences

In the beam theory, the assumption is made that plane sections remain plane after bending. In a beam involving sheet and stringer panels, this assumption means that the shear panels have infinite shearing rigidity, which of course is not true as shearing stresses produce shearing strains. The effect of shear panel shear strains is to cause some stringers to resist less axial load than those calculated by beam theory. This decreased effectiveness of stringers is referred to as "shear lag" effect, since some stringers tend to lag back from the position they would take if plane sections remain plane after bending.

In general, the shear lag effect in sheet-stringer structures is not appreciable except for the following situations:

(1) Cutouts which cause one or more stringers to be discontinued.

(2) Large abrupt changes in external load applications.

(3) Abrupt changes in stringer areas.

In Chapters A7 and A8, strains due to shearing stresses were considered in solving for distortions and stresses in structures involving sheet-stringer construction. Even in these so-called rigorous methods, simplifying assumptions must be made as for example, shear stress is constant over a particular sheet panel and estimates of the modulus of rigidity for sheet panels under a varying state of buckling must be made. The number of stringers and sheet panels in a normal wing is large, thus the structure is statically indeterminate to many degrees and solutions necessitate the use of high speed computers. Before such analyses can be made, the size and thickness of each structural part must be known, thus rapid approximate methods of stress analysis are desirable in obtaining accurate preliminary sizes to use in the more rigorous elastic analysis.

To illustrate the shear lag problem in its simplest state, consider the three stringer-sheet panel unit of Fig. A19.36. The three stringers are supported rigidly at B and equal loads P are applied to the two edge stringers labeled (1) at point (A). The center stringer (2) has zero axial load at (A), but as end B is approached, the sheet panels transfer some of the load P to the center stringer by shear stresses in the sheet. At the support points B the transfer of load from side stringers to center stringer is such as to make the load in all three strings approximately equal or equal to $2P/3$. 
A19.19 Application of Shear Lag Approximation to Wing with Cut-Out.

Fig. A19.38 shows the top of a multiple stringer wing which includes a cut-out in the surface. The stringers (5), (6) and (7) must be discontinued through the cut-out region.

It is assumed that the effectiveness of these 3 interrupted stringers is given by the triangles in the figure. At beam section 1-1 these stringers have zero load. The stringer load is then assumed to increase linearly to full effectiveness when it intersects the sides of this triangle whose height equals 3b. At beam section 2-2 stringers (5) and (7) have become effective since they intersect triangle at points (a) on section 2-2. At section 3-3 point c, stringer (6) becomes fully effective.

To handle shear lag effect in a practical wing problem another column would be inserted in Table A19.1 between columns (3) and (4) to take care of the shear lag effect. The shear lag effectiveness factor which we will call R would equal the effectiveness obtained from a triangle such as illustrated in Fig. A19.38.

For example, the shear lag factor R at beam section 1-1 in Fig. A19.38 would be zero for stringers (5), (6) and (7) and one for all other stringers. At beam section 2-2 stringers (5) and (7) have a factor R = 1.0 since they are fully effective at points (a). Stringer (6) is only 50 percent effective since section 2-2 is halfway from section 1-1 to point (c), thus R = 0.5 for stringer (6). At beam section 3-3, stringer (6) becomes fully effective and thus R = 1.0 for all stringers. The final modified stringer area (A) in column (4) of Table A19.1 would then equal the true stringer area plus its effective skin times the factors Xr. The procedure from this point would be the same as discussed before. Thus shear lag approximations can be handled quite easily by modifying the stringer areas. Using these modified stringer areas, the true total loads in the stringers are obtained. The true stresses equal these loads divided by the true stringer area, not the modified area.

A19.20 Approximate Shear Lag Effect in Beam Regions where Large Concentrated Loads are Applied.

Wing and fuselage structures are often required to resist large concentrated forces as for example power plant reactions, landing gear reactions, etc. To illustrate, Fig. A19.39 represents a landing condition, with vertical load. The wing is a box beam with 7 stringers.
and flange members. Fig. (a) shows the bending moment diagram due to the landing gear reaction alone. The internal resistance to this bending moment cannot be uniform on a beam section adjacent to section A-A because of the shear strain in the sheet panels or what is called shear lag effect. To approximate this stringer effectiveness, a shear lag triangle of length 3b is assumed, and the same procedure as discussed in the previous article on cut-outs is used in finding the longitudinal stresses. It should be understood that the bending moments due to the distributed forces on the wing such as air loads and dead weight inertia loads are not included in the shear lag considerations, only the forces that are applied at concentrated points on the structure and must be distributed into the beam. A load on the gear or power plant would produce a localized couple plus an axial force besides a shear force as in Fig. A19.39. The resistance to this couple and axial force would likewise be based on the effectiveness triangle in Fig. A19.39.

A19.21 Approximation of Shear Lag Effect for Sudden Change in Stringer Area

Stringers of one size are often spliced to stringers of smaller size thus creating a discontinuity because of the sudden change in stringer area.

Fig. A19.40 shows the stringer arrangement in a typical sheet-stringer wing. Stringer A is spliced at point indicated. The stringer area \( A_s \) is decreased suddenly by splicing into a stringer with less area \( A_t \).

![Fig. A19.40](image)

To approximate the shear lag effect, assume the area of stringer B at splice point to be the average area of the two sides or \( (A_t + A_s)/2 \). This average area is then assumed to taper to \( A_s \) and \( A_t \) at a distance 3b from the splice point. The shear lag effectiveness factor \( R \) will therefore be greater than 1.0 on the side toward the smaller stringer \( A_t \), and less than one on the side toward the stringer with the greater area \( A_s \), since the average area was used for the splice point.

A19.22 Problems

1. Fig. A19.41 shows a cantilever, 3 stringer, single cell wing. It is subjected to a distributed airload of 2 lb./in.\(^2\) average intensity acting upward in the z direction and 0.25 lb./in.\(^2\) average intensity acting rearward in the x direction. The center of pressure for z forces is on the 25 percent of chord line measured from the leading edge and at mid-height of spar AB for the x air forces. Assume the 3 stringers A, B, C develop the entire resistance to external bending moments. Find axial loads in stringers A, B, C and the shear flow in the 3 sheet panels of cell (1) at wing stations located 50°, 100° and 150° from wing tip. Consider structure to rear of cell (1) as only carrying airloads forward to cell (1) and not resisting wing torsion or bending.

![Fig. A19.41](image)

2. Fig. A19.42 shows a monoplane wing with one external brace strut. The wing is fastened to fuselage by single pins at points (a) and (b). The fitting at (b) is designed to take off drag reaction. The airloads are \( w_0 = 40 \text{ lb./in.} \) of wing span, with center of pressure at 50 percent of chord from leading edge and acting upward.
and \( w_x = 5 \text{ lb./in.} \), acting to rear and located at mid-depth of wing. Find reactions at points (a), (b) and (d). Find axial loads on front and rear spars. Find primary bending moments on front spar. Find shear flow on webs and walls. Neglect structure forward of front spar and rearward of rear spar.

(3) Fig. A19.43 shows a portion of a single cell - multiple stringer cantilever wing. The external air loads are:

\[ w_a = 100 \text{ lb./in.} \] acting upward and whose center of pressure is along a y axis coinciding with stringer (3).

\[ w_x = 6 \text{ lb./in.} \] acting to rear and located at mid-depth of wing.

Table A gives the stringer areas at stations 0 and 150. Assume stringers have linear variation in area between these two stations. Use 30t as effective skin with compression stringers.

Find axial loads in stringers at stations 150 and 130 and determine shear flow system at station 150.

(4) Same as problem (3) but add an internal web of .04 thickness connecting stringers (3) and (8).

(5) Same as problem (4) but add a leading edge split with radius equal to one-half the front spar depth. Take skin thickness as .04 inches.

![Diagram of cantilever wing](image)

### Table A

<table>
<thead>
<tr>
<th>Flange No.</th>
<th>Area (in.²) @ Sta. 150</th>
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![Diagram of wing tip](image)

Fig. A19.43


Douglas DC-8 Wing. View Showing Lower Surface of Outer Wing Panel Between Center and Rear Spar.
CHAPTER A20
INTRODUCTION TO FUSELAGE STRESS ANALYSIS

A20.1 General. In general the purpose of an airplane is to transport a commercial payload or a military useful load. The commercial payload of a modern airliner may be 100 or more passengers and their baggage. These passengers must be transported safely and comfortably. For example, an airliner flies at high altitudes where temperatures may be far below zero and where the air density is such as not to sustain human life. These facts mean that the body which carries the passengers must be heated, ventilated and pressurized to provide the necessary safety. Air travel must be acceptable to the passengers, thus the airplane body must shield the passengers from excessive noise and vibration, and furthermore efficient, restful and attractive furnishings must be provided to make travel enroute comfortable and enjoyable. The portion of the airplane which houses the passengers on payload is referred to as the fuselage. Fuselages vary greatly in size and configuration. For example, the fuselage of a supersonic military airplane may house only one passenger, the pilot, the remainder of the fuselage interior space being used to house the power plant, to provide retracting space for landing gear, and to house the many mechanical and electronic installations which are necessary to fly the airplane and carry out the various operations for which the airplane was designed to accomplish. Many groups of engineers with various backgrounds of training and experience are therefore concerned with the design of the fuselage. The structures engineer plays a very important part because he is responsible for the strength, rigidity and light weight of the fuselage structure.

A20.2 Loads. Basic Structure.

The wing, being the lifting body is subjected to large distributed surface air forces, whereas the fuselage is subjected to relatively small surface air forces. The fuselage is subjected to large concentrated forces such as the wing reactions, landing gear reactions, empennage reactions, etc. In addition the fuselage houses many items of various sizes and weights which therefore subject the fuselage to large inertia forces. In addition, because of high altitude flight, the fuselage must withstand internal pressures, and to handle these internal pressures efficiently requires a circular cross-section or a combination of circular elements.

The student should refer to Chapter A4 for further discussion of loads on aircraft and also to Chapter A5, where example calculations of fuselage shears and moments are presented.

The basic fuselage structure is essentially a single cell thin walled tube with many transverse frames or rings and longitudinal stringers to provide a combined structure which can absorb and transmit the many concentrated and distributed applied forces safely and efficiently. The fuselage is essentially a beam structure subjected to bending, torsional and axial forces. The ideal fuselage structure would be one free of cut-outs or discontinuities, however a practical fuselage must have many cut-outs. Fig. (a) shows the basic interior fuselage structure of a small airplane with skin removed. It consists of transverse frames and longitudinal stringers. Photographs 1, 2 and 3 illustrate fuselage construction of late model large aircraft.
PHOTO, NO. 2
View Looking Inside of Rear Portion of Fuselage
of Beechcraft Twin-Bonanza Airplane.

PHOTO, NO. 3
Fuselage Construction of Boeing
707 Jet Airliner.

(FOR GENERAL DETAILS OF DOUGLAS DC-8 FUSELAGE CONSTRUCTION SEE PAGE A15.32)
A20.3 Stress Analysis Methods. Effective Cross-Section.

It is common practice to use the simplified beam theory in calculating the stresses in the skin and stringers of a fuselage structure. If the fuselage is pressurized, the stresses in the skin due to this internal pressure must be added to the stresses which resist the flight loads. In wings the skin in the middle region of the airfoil is relatively flat and thus the skin is usually considered as made up of flat sheet panels. In fuselages, however, the skin is curved and curved sheet panels have a higher critical compressive buckling stress than flat panels of the same size and thickness. In small airplanes, the radius of curvature of the fuselage skin is relatively small and thus the additional buckling strength due to this curvature may be appreciable. A simple procedure of approximately including the effect of sheet curvature will now be explained.

Fig. A20.1 illustrates a distributed stringer type of fuselage section. Assume that external loads are applied which produce bending of the beam about the Y axis with compression on the upper portion of the cell.

$$t_e = t \left( \frac{\sigma_{cr}}{\sigma_b} \right)$$  \hspace{1cm} (1)

or an effective area can be written

$$A_e = b't \left( \frac{\sigma_{cr}}{\sigma_b} \right)$$  \hspace{1cm} (2)

where $b'$ is the width of curved sheet between the effective sheet widths $w_4, w_5,$ etc. (See Fig. A20.1).

To illustrate this approach in obtaining the effective cross-section of a fuselage section, an example problem will be presented. The example problem will be broadened to some extent for the purpose of introducing the student to design procedure.

A20.4 Example Problem.

Let it be required to determine the stringer arrangement for the approximate elliptical shaped fuselage section shown in Fig. A20.2.

The following data will be assumed:

- Design bending moment about y axis = 160,000 in. lb. (producing compression on upper portion).
- Zee stringers, one inch deep and with an area equal to 0.12 sq. in. shall be used.
- The ultimate compressive strength of the zee stringer plus its effective skin and a length equal to fuselage frame spacing is assumed to be 32000 psi. The skin thickness is .032 and all material is 2024-T4 aluminum alloy with $E = 10,500,000$ psi. The fuselage stringers are to be symmetrical about section center lines.
FUSELAGE STRESS ANALYSIS

\[ w = 1.7 \times 0.23 \sqrt{10,300,000/32000} = 0.975 \text{ in.} \]

which equals a width of \(0.975/0.23 = 30.5\) sheet thicknesses. Since the bending stress decreases to zero as the neutral axis is approached, and since the curved sheet between the Z strings can carry loads up to its buckling strength, a preliminary value of effective width \(w = 40\) will be assumed acting with each stringer. Thus total area of stringer plus effective skin equals \(0.12 \times 40 \times 0.023^2 = 0.16\) sq. in. The number of stringers required is therefore \(2.00/0.16 = 12\).

Fig. A20.2 shows how the stringers were placed to give 12 stringers on the top half. Since the skin on the lower half is in tension and therefore fully effective, the neutral axis will fall below the center line and thus the two stringers on the center line will be considered as part of the required 12 stringers. A fuselage cross-section has now been obtained. The desired final result is that the maximum compressive stress will be near but not over 3000 psi. The procedure from this point is still a trial and error process since the effective sheet on the compressive side depends on the magnitude of the compressive bending stress which in turn is influenced by the amount of effective sheet and the buckling load carried by the curved sheet.

Using the preliminary stringer arrangement of Fig. A20.2, Tables A20.1 and A20.2 give the calculation of the effective moment of inertia of the section about the horizontal neutral axis, Table A20.1 deals with the stringers and the effective sheet elements and Table A20.2 deals with the curved buckled sheet elements.

In the trial No. 1, the following assumptions are made:

1. A width of 30 thicknesses of skin act with each stringer on the upper or compressive side.

2. The area of the curved sheet between the effective sheet widths as found in (1) is modified to give an effective area by multiplying by a \(K\) factor of \(\sigma_w/\sigma_b\), where \(\sigma_b\) is the buckling compressive stress and \(\sigma_w\) is the bending stress at the center of the curved sheet element assuming 32000 at the extreme upper fiber of the beam section and zero at the horizontal center line, with linear variation in between these points.

NOTE: Since the entire skin on the lower half is effective, a more logical assumption would be to guess at the location of the neutral axis and use a variation of \(\sigma_b\) between the neutral axis and the extreme fiber. This approach will not be used in this example.

Solution:

The first thing to do is to determine approximately how many Z stringers will be required so that a section can be obtained to work with. Since the internal resisting moment must equal the external bending moment, one can guess at the internal resisting couple in terms of total compressive flange stress and an effective internal couple arm.

For elliptical and circular sections with distributed flange material, the approximate effective resisting arm of the internal couple can be taken as 0.75 times the height \(h\), and the average tension or compressive stress as \(2/3\) the maximum stress. Thus equating the external bending moment to the internal resisting moment an approximate total area \(A_c\), for the compressive side of the fuselage section can be obtained.

\[ M_y = A_c \times (0.67 \times \sigma_b)(0.75 \times h), \text{ whence,} \]

\[ A_c = \frac{M_y}{(0.75 \times 50 \times 0.67 \times 32000)} = 2.00 \text{ sq. in.} \]

Part of this total area is provided by the effective skin area. The effective width to use with each rivet line equals \(w = Ct \sqrt{\sigma_b/\sigma_{st}}\). We will take \(C = 1.7\) which is a commonly used value.
### TABLE A20.1

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**Explanatory Notes for Table A20.1**

1. **Col. 1** for numbering of stiffeners and sheet elements, see Fig. A20.2.
2. **Col. 2** Stiffener area = 0.12 + 0.30 x 0.032 x 0.032 / 0.15 sq. in. For stiffeners 2, 4, 6, 8, 10, below the centerline each stiffener is considered acting separately. The entire skin between stiffeners is considered as a unit.
3. **Col. 3** All arms z are measured to horizontal centerline axis.
4. **Col. 4** z = distance to neutral axis as found from results of Trial No. 1.
5. **Col. 5** Cb = 1800000 x z/1470

### TABLE A20.2

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**Explanatory Notes for Table A20.2**

1. **Col. 1** Trial No. 1, 2, 3, 4 (see Fig. A20.1 for meaning of terms)
2. **Col. 2** = 10,000,000 for aluminum alloys
3. **Col. 3** Cb varies as a straight line from 32000 at top of cell to zero at centerline
4. **Col. 4** z = distance from centroid of element to centerline axis of cell

**Results of Trial No. 1**

Considering results of both Tables and multiplying by z since only one half of cell was considered:

- z' = total effective area = (2.49 + .260)12
- 5.51 sq. in.
- \( \varepsilon = (z' x 2) / 2 = -18.62 \)
- \( \varepsilon = \varepsilon' / z' = -18.62 / 5.51 = -3.38^\circ \)
- \( \varepsilon = 2(14.3 + 52) - 5.51 x 3.38^2 = 1470 \) in.
Due to the symmetry of the section, Tables A20.1 and A20.2 give calculations for only one half of the material, thus the results are multiplied by two. General explanatory notes are given below each table.

The results of trial No. 1 give a neutral axis 3.38" below the center line and a moment of inertia of 1470 in.\(^4\). In trial No. 2, the effective sheet widths are based on the moment of inertia of 1470. The results of trial No. 2 give a moment of inertia of 1489 in.\(^4\) with a neutral axis .135" above the first location. If a third trial were used, making use of the 1489 moment of inertia, the change would be quite small since the effect of a small change in stress on the effective sheet width is negligible.

The compressive stress on stringer No. 2 using the resulting moment of inertia and neutral axis location, therefore becomes

\[
\sigma_b = \frac{M_y I_y}{I_y} = 150000 \times 27.45/1499 = 22600 \text{ psi}
\]

The allowable stress was 32000, hence the margin of safety is (32000/22600) = 1.40 or eight percent. If a smaller margin of safety was desired some material would be eliminated and the calculations of Tables A20.1 and A20.2 would be repeated.

Calculation of Shear Stress in Skin at Neutral Axis

The equation for the shear flow at some point on the skin is,

\[
q = q_0 + \frac{V_x}{I_y} \cdot \frac{E}{a_z} \cdot \frac{x^2}{x^2}
\]

Due to symmetry of cross-section about the Z axis the shear flow \(q_0\) is zero at a point on the center line Z axis. The summation of the term \(a_z\) between a point on the Z axis and the neutral axis is given in Table A20.3. The values of areas (a) and arms (z) are taken from Tables A20.1 and A20.2.

| TABLE A20.3 |
|------------------|------------------|
| Element No. | AREA |
| (1) | .0013 (28.28 - .13) | 0.59 |
| (2) | .0023 (27.58 - .13) | 1.17 |
| (3) | .0038 (27.58 - .13) | 1.05 |
| (4) | .0038 (25.38 - .13) | 2.86 |
| (5) | .0038 (23.48 - .13) | 2.52 |
| (6) | .0048 (21.58 - .13) | 3.34 |
| (7) | .0048 (19.68 - .13) | 3.28 |
| (8) | .0048 (18.68 - .13) | 2.46 |
| (9) | .0048 (16.78 - .13) | 1.75 |
| (10) | .0058 (14.88 - .13) | 0.41 |
| (11) | .0068 (12.98 - .13) | 0.35 |
| (12) | .0078 (11.08 - .13) | 0.09 |
| Total AREA | 13.97 |

Substituting in equation (3)

\[
q_{\text{M.A.}} = q_0 + V_x \frac{E}{a_z} \frac{x^2}{x^2}
\]

\[
0 + \left( \frac{V_x}{1489} \right) 19.57 = 0.0132 \frac{V_x}{\text{lb./in.}}
\]

The shear stress \(\tau = q/t\) = .0132 \(\frac{V_x}{V_y}\) = .413 \(\frac{V_x}{V_y}\)

The average shear stress on the section would be \(\tau_{\text{avg.}} = \frac{V_x}{2ht} = \frac{V_x}{2} \times 50 \times 0.062 = .312 \frac{V_x}{V_y}\).

Thus for this shape of cross-section and stringer arrangement the maximum shear stress is 0.413/0.312 times the average shear stress or approximately 4/3 times as large.

The procedure as given above is quite conservative relative to the true or actual margin of safety, because a linear variation of stress with strain has been assumed and failure of the section is assumed to occur when the most remote stringer reaches its ultimate compressive stress. Actually in a static test of a fuselage to destruction, the fuselage section as a whole will not collapse when one stringer buckles, but will continue to take increasing load until other stringers have reached their ultimate strength. Furthermore, in a typical fuselage structure, stringers of various sizes, shapes and therefore different compressive strengths are used, and thus to obtain a better measure of the ultimate strength of a fuselage section, modifications in stress procedures are made to measure stringer effectiveness. This subject was discussed in some detail in Arts. 11 and 12 of Chapter A19. To illustrate stringer effectiveness in fuselage bending stress analysis, a simple example problem will be presented.

A20.5 Ultimate Bending Strength of Fuselage Section.

Example Calculation.

Fig. A20.3 shows the cross-section of a circular fuselage. The 2 stringers are arranged symmetrically with respect to the center line Z and X axes.

Three sizes of Z stringers are used as illustrated in Fig. A20.4 and are labeled S\(_1\), S\(_2\) and S\(_3\). These symbols are used on Fig. A20.3 to indicate where each type of stringer is used. The stringers on each side of the section are numbered 1 to 13 as shown on Fig. A20.3. Fig. A20.5 shows a plot of the stress-strain curve for the three stringer types loaded in compression and with a column length equal to the fuselage frame spacing. Fig. A20.5 also shows a tension stress-strain diagram for the material which is aluminum alloy (2024). The ultimate bending strength will be calculated for bending which places the upper portion in compression.

\[
q_{\text{M.A.}} = q_0 + V_x \frac{E}{a_z} \frac{x^2}{x^2}
\]

\[
0 + \left( \frac{V_x}{1489} \right) 19.57 = 0.0132 \frac{V_x}{\text{lb./in.}}
\]
Since the location of the neutral axis is unknown, a location will be assumed, namely, 7 inches below the center line axis as shown in Fig. A20.3. The entire calculations for determining the effective moment of inertia can best be done in Table form, as shown in Table A20.4. Due to symmetry about the Z axis only one-half of the structure need be considered since the results can be multiplied by two.

Column (1) lists the stringer numbers relative to location and Column (2) according to types $S_3$, $S_2$, and $S_1$. Column (3) gives the stringer area. On the tension or lower side of the section, the skin is all effective and that area of skin halfway to each adjacent stringer is assumed to act with stringers numbered 9 to 13 and this skin area is recorded in Column (5). On the compressive side the skin is only partially effective. The effective width $w$ for each stringer rivet line depends on the stringer stress. We will take the effective width $w = 1.9t \sqrt{\frac{G}{S}}$. The effective area $A_e$ will then equal $w t$. These effective skin areas are recorded in Column (6). In solving this equation the stringer stress $S_e$ has been taken as $-36500$ psi on stringer number (1), and then varying linearly to zero at the neutral axis as indicated in Column (4). This assumption is not true but accurate enough to obtain effective skin areas. To illustrate, consider stringer number (1). The effective area $A_e$ equals $w t = 1.9 \times 0.022 \times (10,500,000/36500) = 0.092$ sq. in. Column (5) gives the sum of the stringer and effective skin areas or $A_s + A_e$.

In this example problem, the effectiveness of the curved sheet panels between the sheet effective widths will be neglected since its influence is small. It could be included as illustrated in the previous example problem. Column (7) lists the distances from the assumed neutral axis to the centroid of each stringer-skin unit. We now assume that plane sections remain plane or a linear strain variation. Referring to Fig. A20.5, it is noticed that when a unit strain of .006 is obtained in stringer $S_3$, type the compressive stress is 36500, which represents its ultimate stress. Stringer (1) is of $S_3$ type and is located farthest from the neutral axis. Sub. Fig. (a) of Fig. A20.3 shows the strain diagram with .006 at stringer (1) and varying as a straight line to zero at the neutral axis. Column (8) of Table A20.4 records the unit strain at each stringer centroid. The true stress at each stringer point due to these strain values is read from the curves on Fig. A20.6 and recorded in column (9) of the Table. It should be noted that larger (3) although closer to the neutral axis than stringer (1) carries a higher stress than stringer (1). This is possible because when stringer (1) reaches its maximum stress, it bends but continues to hold the same stress with
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<td>Effective Skin Area $A_e$</td>
<td>Total Stringer Area $A$</td>
<td>Arm $Z$ (in)</td>
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<td>Stress $\sigma$</td>
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Increasing strain, but stringer (3) which has not reached its maximum strength of 39000 continues to take increasing load.

Since we wish to use the beam formula $\sigma_b = M_Z/IX$ in computing stresses, we must modify the stringer areas to give a linear stress variation since the formula is based on a linear stress variation. The stringer modification factor $K$ equals the ratio of the true stress in column (9) of Table to linear stress value in column (4) or $K = \sigma/\sigma_c$. The results are recorded in column (10). The modified stringer areas are then equal to $KA$ and are recorded in column (11). Column (12) gives the first moment of the modified areas about the assumed neutral axis, giving a total value of $-3.16$.

The distance $Z$ from the assumed neutral axis to the true neutral axis is thus,

$$Z = Z \text{ KAZ}^3/\text{KA}$$

$$= -3.16 = -0.76^\prime$$

The true N.A. would fall about .70 inches below assumed position. The effect on total sum of Column (13) would be negligible, thus Table A20.4 will not be revised.

Column (13) gives the calculation of the effective moment of inertia with $Z'$ being equal to $Z$. The effective moment of inertia is therefore twice the sum of Column (13) or 3252.

**Calculation of Ultimate Resisting Moment.**

The maximum stress at the most remote stringer which is number (1) is 36500. From the beam formula,

$$M_X = \sigma_1X/Z$$

$$= (36500 \times 3252)/35.7 + 0.7$$

$$= 3,250,000 \text{ in.lb.}$$

This bending strength when compared to any design bending moment about the $X$ axis would give the margin of safety relative to bending strength.

If the moment of inertia had been computed without regard to non-linear stress variation, or in other words, using $K$ equal 1 for all stringers the neutral axis would have come out 4.9 inches below the centerline axis and the moment of inertia would have calculated to be 2382 in.\(^4\). The resisting moment developed would then be $(36500 \times 3252)/33.6 = 2,600,000$ in.lbs. Thus the true strength is 25 percent greater than the strength for linear stress variation. This result explains why such structures test overstrength if designed on linear stress variation basis.

After stringer stresses are obtained
using the modified areas of Table A20.4, the true stringer areas must be used to find the true stringer loads, which must be used in the shear flow analysis.

A20.6 Shear Flow Analysis for Fuselage Structures

The shear flow analysis can be made once the effective cross-sections of the fuselage are obtained. The procedure is the same as was illustrated for wing structures in Chapter A19. To illustrate, two example problems will be presented.

Example Problem 1. Symmetrical Tapered Section.

Fig. A20.6 shows a portion of a tapered circular shaped fuselage structure that might be representative of the rear portion of a fuselage for a small airplane. Since this example is only for the purpose of illustrating shear flow analysis, it will be assumed that the 16 stringers are the only effective material. In an actual stress analysis, the effective cross-section would have to be used as illustrated in previous articles A20.3 to A20.5.

The problem will be to determine the stringer stresses and the skin shear flow stress system at Station (0) under a given load system at Station (150) as shown in Fig. A20.6.

Solution No. 1 - Solution by Considering Beam Properties at Only One Section.

If the change in longitudinal stringer or flange material is fairly uniform this method can be used with little error in the resulting shear flow stresses.

Moment of Inertia of section at station (0) about centroidal Y axis:

\[ I_y = (15^\circ x 1.1 x 2) + (13.86^\circ + 10.61^\circ + 5.74^\circ) x 4 = 180 \text{ in.}^4 \]

Table A20.5 gives the necessary calculations for determining the flange bending stresses and the net total shear load to be taken by the cell skin. Since the cell is tapered, the stringers have a \( z \) component, thus the stringer axial loads help resist the external shear load. The summation of column (8) of Table A20.5 gives \( -333.4 \text{ lb.} \) for a summation for half the fuselage section.

Hence, net web shear at station (0) equals:

\[ V_{web} = V_{ext.} + V_{flange} = 2000 + (2 x -333) = 1333.2 \text{ lb.} \]

The results in this particular problem show that at station (0) the flange stringer system resists one third of the external shear load. At station (150) the web system will resist the entire external shear load of 2000 lb. since the load in the stringers is zero.

In actual design the net web shear should be used since in many cases it will decrease the sheet thickness required one or more gauges.

Calculation of Flexural Shear Flow.

\[ q = q_0 - \frac{V_{z(web)} \Sigma a z}{I_y} = q_0 - \frac{1333.2 \Sigma a z}{150} \]

Due to symmetry of the section about the \( z \) axis, the flexural shear flow in the web at the center line is zero. Therefore, \( q_0 \) will be taken as zero and the summation in equation (A) will
### TABLE A20.5

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<td>Area</td>
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<td>$P_y = C_{h}A_{h}$ (3)(4) lbs.</td>
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<td>$P_x = -P_y$ dx dy &amp; $P_y = -P_x$ dx dy</td>
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</table>

Shear taken by stringers = -333.4 0

NOTES:

Col. 4 $C_{h} = -M_{z}/I_{y} = -2000 \times 150 \times z / 180 = -1667z$

Col. 5 Total x component of load in stringer member. For practical purposes, it equals axial load in stringers since cosine of a small angle is practically one.

Col. 6 The slope of the stringers in the z and y directions can be calculated from the dimensions of the two end sections and the length of the cell (see Fig. A20.6).

Col. 8 The in-plane components of the stringer axial loads at station 0.

start with stringer (1)

$q_{x} = 0 - 7.40 \times .05 \times 15 = -5.55 \text{ lb./in.}$

$q_{y} = -5.55 - 7.40 \times .1 \times 13.86 = -15.80$

$q_{z} = -15.80 - 7.40 \times .1 \times 10.61 = -22.66$

$q_{y} = -22.66 - 7.40 \times .1 \times 5.74 = -27.91$

The torsional moment $T$ about the centroid of the section at station (0) equals $S \times 2000 = 10000 \text{ in. lb. (clockwise when looking toward station 150).}$ Due to the symmetry of the section at station 0, the in-plane components of the stringer loads produce zero moment about the section centroid.

For equilibrium a constant internal shear flow $q_{y}$ is necessary to make $\Sigma M_{x} = 0$

$$q_{y} = \frac{T}{2A} = \frac{-10000}{2 \times \pi \times 18} = -7.06 \text{ lb./in.}$$

Adding the torsional shear flow $q_{y}$ to the flexural shear flow $q$, the following results are obtained:

$q_{x} = -7.06 + 5.55 = -1.51 \text{ lb./in.}$

$q_{z} = -7.06 + 15.80 = 8.74$

$q_{x} = -7.06 + 23.66 = 16.60$

$q_{z} = -7.06 + 27.91 = 20.85 \text{ etc.}$

Fig. A20.7 shows the results in graphical form, on the left side of the section the shears are of the same sign and therefore add together.

Solution No. 2. Shear Flow by Change in Stringer Loads Between Adjacent Stations. AP Method.

The shear flow will be calculated by considering the change in the axial load in the longitudinal stringers between fuselage sections at stations (0) and (30).

Fig. A20.8 shows the beam section at station 30, the stringer areas being the same as at station 0, but the section as a whole is smaller due to the taper of the cell.

area of each stringer = .10

Fig. A20.8

Table A20.6 gives the calculations for the flexural shear system. The procedure is the same as illustrated for wing structures in Chapter A19.

Comparing the results of column 13 with the flexural shear flow as found by solution No. 1, we find the second solution gives a maximum shear flow of 28.96 lb./in. against a value of 27.91 for the first solution. The first method deals with the properties at only one section and this cannot include the effects of change in moment of inertia on the shear flow. The
### TABLE A20.6

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<th>Sta. 30 Area sq. in.</th>
<th>Sta. 0 Arm z</th>
<th>Sta. 30 Arm z</th>
<th>Sta. 0 ( \sigma_b ) = 16672 (psl.)</th>
<th>Sta. 30 ( \sigma_b ) = 15222 (psl.)</th>
<th>( \Delta P )</th>
<th>Panel Taper Corr. Factor ( K )</th>
<th>Flexural Shear Flow ( q = \frac{\Delta P K}{30} ) lb./in.</th>
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<td>17700</td>
<td>15050</td>
<td>1770</td>
<td>1505</td>
<td>-8.83</td>
</tr>
<tr>
<td>9</td>
<td>.05</td>
<td>.05</td>
<td>-15.00</td>
<td>-14.00</td>
<td>25000</td>
<td>21300</td>
<td>1250</td>
<td>1085</td>
<td>-8.17</td>
</tr>
</tbody>
</table>

**NOTES:**

Col. 6 \( \sigma_b = -3000 \times 150 z/180 = -1067 \) z

Col. 7 \( \sigma_b = -3000 \times 120 z/157.2 = -1522 \) z

Col. 10 Change in axial load in each stringer between stations 0 and 30 divided by distance between Stations. This result represents the average shear flow induced by the loading up of each stringer between stations 0 and 30.

Col. 11 The width of a skin panel at Station 0 is 5.88 inches and 5.5 inches at Station 30. The shear flow on the edge of the panels at Station 0 equals (5.5/5.88) \( \Delta P/30 \). (See Art. A15.18 of Chapter A15 for explanation). This refinement is usually neglected and the average values as given in Col. 10 are used which are conservative.

Col. 13 Due to symmetry of structure, the shear flow is zero on z axis. Thus shear flow at any station equals the progressive summation of the shear flow values in Col. 12.

The second method is recommended for practical analysis procedure.

Since the section is symmetrical, there are no moments induced by the in-plane components of the stringer forces at station 0.

The torsional shear flow forces are the same as in solution method No. 1 and these are added to the values of column 13 of Table A20.6 and give a pattern similar to Fig. A20.7.

### A20.7 Example Problem. Tapered Circular Fuselage with Unsymmetrical Stringer Areas.

Fuselage cross-sections are seldom all symmetrical relative to stringer and skin areas because the practical fuselage has cut-outs such as doors, etc. To illustrate the unsymmetrical case a simplified case will be presented.

Fig. A20.9 shows a portion of a tapered fuselage. The stringer areas are such as to make the cross-sections unsymmetrical relative to bending material. Again for simplicity, we will assume the stringers are the only effective material. In actual design practice the effectiveness of the skin and each stringer would have to be considered as explained in Articles A20.4 and 5.

The problem will be to determine the stringer stresses and the skin shear flow values at station (0) due to the given external loads of \( P_z = 4000 \) lb., \( P_y = 1000 \) lb. and \( P_x = 1500 \) acting at station (150) as shown in Fig. A20.9.

**SOLUTION:**

Since we choose to use the \( \Delta P \) method in finding the shear flow system at station (0), we will find the stringer loads at two stations, namely, station (0) and station (30). The first step is to find the moment of inertia of each fuselage section about centroidal z and y axes and the product of inertia about these axes. Table A20.7 (Columns 1 to 11) gives the calculations of the section properties for station
The skin stringers are located symmetrically with respect to the centerline axes, however the stringer areas as given in ( ) on the figure are not symmetrical with these axes. It is assumed in this problem that the stringers taper uniformly between the values as given for station 0 and 150. The cell would of course have interior transverse frames which are not shown on the figure.

TABLE A20.7

<table>
<thead>
<tr>
<th>Stringer No.</th>
<th>Area (in²)</th>
<th>Arm (in)</th>
<th>Arm (in)</th>
<th>az (in³)</th>
<th>az²</th>
<th>ay (in³)</th>
<th>ay²</th>
<th>az ay</th>
<th>z - z</th>
<th>y - y</th>
<th>σ₀</th>
<th>P₀ = F/2a</th>
<th>σ₃ = aσ₀ - σ₀</th>
<th>P₃ = (2)(12 + 13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>.60</td>
<td>10.5</td>
<td>-12.00</td>
<td>6.30</td>
<td>66.20</td>
<td>-7.20</td>
<td>86.50</td>
<td>-75.50</td>
<td>11.36</td>
<td>-14.46</td>
<td>-1500</td>
<td>-369.2</td>
<td>1150.80</td>
<td>441</td>
</tr>
<tr>
<td>b</td>
<td>.10</td>
<td>18.98</td>
<td>-8.48</td>
<td>1.90</td>
<td>36.00</td>
<td>-0.85</td>
<td>7.20</td>
<td>16.10</td>
<td>19.84</td>
<td>-10.94</td>
<td>-21960</td>
<td>-441</td>
<td>307.0 - 936.1</td>
<td>-2239</td>
</tr>
<tr>
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<td>.10</td>
<td>22.50</td>
<td>0</td>
<td>2.25</td>
<td>59.80</td>
<td>0</td>
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<td>16742</td>
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<td>-22304</td>
<td>-441</td>
<td>307.0 - 936.1</td>
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<tr>
<td>d</td>
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<td>18.98</td>
<td>8.48</td>
<td>1.90</td>
<td>36.00</td>
<td>0.85</td>
<td>7.20</td>
<td>16.10</td>
<td>19.84</td>
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<td>16742</td>
<td>-441</td>
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<td>88.10</td>
<td>9.60</td>
<td>115.10</td>
<td>100.80</td>
<td>11.36</td>
<td>9.54</td>
<td>7892</td>
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<td>307.0 - 936.1</td>
<td>-1719</td>
</tr>
<tr>
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<td>9.60</td>
<td>115.10</td>
<td>100.80</td>
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<td>-1719</td>
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<tr>
<td>g</td>
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<td>8.48</td>
<td>-8.40</td>
<td>88.10</td>
<td>9.60</td>
<td>115.10</td>
<td>100.80</td>
<td>11.36</td>
<td>9.54</td>
<td>7892</td>
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<td>307.0 - 936.1</td>
<td>-1719</td>
</tr>
<tr>
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<td>-1719</td>
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<td>14.40</td>
<td>37.80</td>
<td>-9.85</td>
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<td>4592</td>
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<td>-1719</td>
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<td></td>
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<td></td>
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<td>-403.2</td>
<td>-37.70</td>
<td>-1500</td>
<td></td>
</tr>
</tbody>
</table>

**Reference Axes Z' and Y' are taken as the centerline axes.**

(See Fig. A20.9)

**Location of centroid and transfer of properties to centroidal axes.**

\[ z = -2.90/3.40 = -0.85" \]

\[ \bar{y} = 8.40/3.40 = 2.46" \]

\[ L_y = 643.9 - 3.40 \times 855^2 = 641.4 \]

\[ L_z = 403.2 - 3.40 \times 2.46^2 = 382.6 \]

\[ L_{yz} = -37.7 - 3.40 \times 2.46 \times -0.85 = -30.55 \]

**General Notes:**

Col. 12 \( \sigma_b = 307.0y - 936.1x \)

Col. 14. Since the total tensile stresses equal to total compressive stresses in bending, the sum of Col. 14 should equal the external applied normal load.
TABLE A20.8

<table>
<thead>
<tr>
<th>Stringer No.</th>
<th>Area (a)</th>
<th>Arm (z')</th>
<th>Arm (y')</th>
<th>(a')</th>
<th>(a'^2)</th>
<th>(a'y')</th>
<th>(a'y'^2)</th>
<th>(z = z' - \bar{z})</th>
<th>(y = y' - \bar{y})</th>
<th>(\sigma_y = \frac{F}{Iz} a)</th>
<th>(\sigma_z = \frac{F}{Iy} a)</th>
<th>(P_s = \sigma_y a (\sigma_z - \sigma_y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.50</td>
<td>9.80</td>
<td>-11.2</td>
<td>4.90</td>
<td>48.1</td>
<td>-5.60</td>
<td>62.8</td>
<td>54.9</td>
<td>10.90</td>
<td>-13.31</td>
<td>-14800</td>
<td>-503</td>
</tr>
<tr>
<td>b</td>
<td>0.10</td>
<td>17.72</td>
<td>-7.92</td>
<td>1.77</td>
<td>31.4</td>
<td>-0.79</td>
<td>6.3</td>
<td>-14.0</td>
<td>18.82</td>
<td>-10.03</td>
<td>-21090</td>
<td>-503</td>
</tr>
<tr>
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<td>21.00</td>
<td>0</td>
<td>2.10</td>
<td>44.1</td>
<td>0</td>
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<td>d</td>
<td>0.10</td>
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<td>7.92</td>
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<td>0.79</td>
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<td>-503</td>
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<td>83.0</td>
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<td>9.09</td>
<td>-7088</td>
<td>-503</td>
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<tr>
<td>f</td>
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<td>11.20</td>
<td>-6.47</td>
<td>53.2</td>
<td>7.40</td>
<td>83.0</td>
<td>-72.3</td>
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<td>9.09</td>
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<td>-503</td>
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<tr>
<td>g</td>
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<td>62.3</td>
<td>1.58</td>
<td>12.6</td>
<td>-28.0</td>
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<td>5.81</td>
<td>-17563</td>
<td>-503</td>
</tr>
<tr>
<td>h</td>
<td>0.20</td>
<td>-21.00</td>
<td>0</td>
<td>-4.20</td>
<td>58.2</td>
<td>0</td>
<td>0</td>
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<td>2.11</td>
<td>17943</td>
<td>-503</td>
<td>3489</td>
</tr>
<tr>
<td>i</td>
<td>0.20</td>
<td>17.72</td>
<td>-7.92</td>
<td>3.54</td>
<td>62.8</td>
<td>-1.56</td>
<td>12.6</td>
<td>28.0</td>
<td>16.60</td>
<td>12135</td>
<td>-503</td>
<td>2327</td>
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<td>-9.8</td>
<td>-11.20</td>
<td>7.55</td>
<td>25.0</td>
<td>2.92</td>
<td>32.6</td>
<td>28.6</td>
<td>8.7</td>
<td>3570</td>
<td>-503</td>
<td>797</td>
</tr>
<tr>
<td>Sum</td>
<td>2.98</td>
<td>-3.29</td>
<td>520.2</td>
<td>6.28</td>
<td>299.2</td>
<td>-26.3</td>
<td>-1500</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

NOTES:

Reference \(z'\) and \(y'\) axes are taken as the centerline axes.

\[
\bar{z} = -3.29/2.98 = -1.09
\]
\[
\bar{y} = 6.28/2.98 = 2.11
\]

\[
I_y = 520.2 - 2.98 \times 1.10^2 = 516.6
\]
\[
I_z = 299.2 - 2.98 \times 2.11^2 = 286.0
\]
\[
I_{yz} = -28.3 - 2.98 \times 2.11 \times -1.10 = -19.4
\]

(0) and the similar columns of Table A20.8 gives the calculations for station (30).

Before the bending and shear stresses can be calculated, the external bending moments, shears and normal forces at stations (0) and (30) must be known.

At station (0):

- The bending moment about \(y\) neutral axis at station (0) equals,

\[
M_y = P_y (150) + P_x (7.35) = 4000 \times 150 + 1500 \times 7.35 = 611500 \text{ in.lb.}
\]

\[
M_z = P_y (150) - P_x (2.46) = -1000 \times 150 + 1500 \times 2.46 = -146310 \text{ in.lb.}
\]

The shears at station (0) are \(V_x = P_z = 4000 \text{ lb.}\) and \(V_y = P_y = -1000 \text{ lb.}\)

The normal load \(P_n\) at station (0) referred to centroid of section equals \(2P_x = -1500 \text{ lb.}\)

In a similar manner, the values at station (30) are,

\[
M_y = 4000 \times 120 + 1500 \times 8.10 = 492150 \text{ in.lb.}
\]

\[
M_z = -1000 \times 120 + 1500 \times 2.11 = 116820 \text{ in.lb.}
\]

\[
P_n = -1500 \text{ lb.}, \ V_z = 4000 \text{ lb.}, \ V_y = -1000 \text{ lb.}
\]

Calculation of Bending Stresses.

Station (0):

\[
\sigma_y = -(K_1 M_z - K_2 M_y) y - (K_3 M_y - K_4 M_z) z
\]

where

\[
K_1 = I_{yz} / (I_y z^2 - I_y z) * 1
\]

\[
K_2 = I_z / (I_{yz} - I_y z) *
\]

\[
K_3 = I_y / (I_{yz} - I_y z) *
\]

Substituting values from Tables A20.7 and A20.8:

\[
K_1 = -30.55 / (641.4 \times 382.5 - 30.55) = -30.55 / 244670 = -0.0001248
\]

\[
K_2 = 382.5 / 244670 = 0.00156
\]

\[
K_3 = 641.4 / 244670 = 0.00262
\]
Substituting $K$ values in equation for $\sigma_b$:

$$\sigma_b = \sqrt{\left(0.00282 x - 146310 - (-0.001248 x + 611800) y - 0.00156 x 611500 + (-0.0001248 x - 146310) z \right)}$$

whence

$$\sigma_b = 307.0 y - 936.1 z \text{ (plus } \sigma_b \text{ is tension)}$$

Station (30):

$$K_x = -19.4 / (516.6 x 286 - 19.4 x) = -19.4 / 147620 = -0.001315$$

$$K_y = 296 / 147620 = 0.001936$$

$$K_z = 516.6 / 147620 = 0.0035$$

$$\sigma_b = \sqrt{\left(0.0035 c - 116800 - (-0.001315 x + 492150) y - 0.001936 x 492150 - (-0.0001315 x + 116800) z \right)}$$

whence

$$\sigma_b = 344.3 y - 937.7 z$$

Column (12) in Tables A20.7 and A20.8 gives the results of solving the equations for $\sigma_b$.

Since an external load of 1500 lb. is acting normal to the sections and through the section centroids, an axial compressive stress $\sigma_c$ is produced on the sections. (See Columns 13). The total load $P_y$ in each stringer equals the area of the stringer times the combined bending and axial stresses. (See column 14 of each table).

Calculation of Flexural Shear Flow $q$.

Table A20.9 gives the necessary calculations to determine the shear flow at station (0) based on the change in stringer loads between stations (0) and (30). The correction of the average shear due to the taper in the skin panels as was done in example problem (1), Table A20.6, column (11), is omitted in this solution since it tends toward the conservative side. Since the effective cross section is unsymmetrical, the value of the flexural shear flow $q$ at any point is unknown thus a value for $q$ at some point is assumed. In Table A20.9 the shear flow $q$ in the web $aj$ is assumed zero. Column (5) gives the results at other points under this assumption.

Moment of Shear Flow about Intersection of Centerline Axes

For equilibrium in the plane of the cross section at station (0), the summation of the moments in the plane, of all internal and external forces must be zero. Column (7) of Table A20.9 gives the moment of the flexural shear about this point. (See notes and Fig. below Table for explanation.)

<table>
<thead>
<tr>
<th>Stringer No.</th>
<th>$P_y$ at Sta. 0 (lb)</th>
<th>$\Delta P_y$ at Sta. 30 (lb)</th>
<th>$q$ at Sta. 30 (lb/sq.in.)</th>
<th>$m$ (lb-sq.in.)</th>
<th>$mq$</th>
<th>$q_y$ = $q + q_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-2312</td>
<td>7651</td>
<td>55.37</td>
<td>35.37</td>
<td>800</td>
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<td>2159</td>
<td>5.47</td>
<td>5.47</td>
<td>600</td>
<td>52</td>
</tr>
<tr>
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<td>-2004</td>
<td>2159</td>
<td>5.47</td>
<td>5.47</td>
<td>600</td>
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<td>1615</td>
<td>3.47</td>
<td>3.47</td>
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<td>52</td>
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<td>120.52</td>
<td>600</td>
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<td>8.57</td>
<td>600</td>
<td>52</td>
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<td>10.67</td>
<td>600</td>
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<td>600</td>
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<td>0</td>
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<td>52</td>
</tr>
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</table>

| Sum         | 92670               |

**NOTES:**
- Col. (2) and (3) from tables A20.7 and A20.8
- Col. (4) $\Delta P_y = \left( P_y \text{at Sta. 0} - P_y \text{at Sta. 30} \right)$
- Col. (6) $m = \text{double areas (see Fig. a)}$
- Col. (7) $mq = \text{moment of shear flow} q \text{ on each web element about} O'$

**Moments Due to In Plane Components of Stringer Loads.**

Since the stringers are not normal to the section at station (0), the stringers have in-plane components which may produce a moment about the intersection of the symmetrical axes which has been selected as a moment center. Table A20.10 gives the calculations for the in-plane components and their moments about point $O'$.

**Moment of External Load System About Point $O'$.**

The 1000 lb. load at station (160) acting in the Y direction has a moment arm of $7''$ about the point $O'$ of station (0).

Hence external moment $= 1000 \times 7 = 7000$ in.lb.

Therefore the total moment about the assumed moment center $O' = \ldots$
### TABLE A20.10

<table>
<thead>
<tr>
<th>Stringer No.</th>
<th>Px (lbs.)</th>
<th>dy/dx</th>
<th>Py = dy/dx</th>
<th>M0 =</th>
<th>dx/dx</th>
<th>Pz = dy/dx</th>
<th>M0 =</th>
<th>Pz y'</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2600</td>
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<td>-</td>
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<td>173</td>
<td>- .1820</td>
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<td>152</td>
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<td>245</td>
<td>- .2970</td>
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<td>214</td>
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<td>.0421</td>
<td>111</td>
<td>941</td>
<td>-</td>
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<tr>
<td>j</td>
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<td>.0266</td>
<td>33</td>
<td>346</td>
<td>.0233</td>
<td>29</td>
<td>348</td>
<td>-</td>
</tr>
</tbody>
</table>

| Sum          | 1634      | 1612   | -22*       | 0    | 150    | 957        | 957  | 150    |

**NOTES:**
- Col. (2) from Table A20.7
- Col. (3) equals the slope of stringers in y and z directions. (see Fig. A20.9)
- Col. (5) and (8) from Table A20.7
- Col. (8) shows the Py and Pz components from Cols. (4) and (7).

Total moment about y' = -1634 = 1612 = -22*

Looking Toward Sta. 150

**92670 due to shear flow q**

-22 due to in plane components of stringers.

7000 due to the external loads.

Total = 99648 in. lb.

Therefore for equilibrium a moment of -99648 is required which can be provided by a constant shear flow q around the cell, hence

\[
q = \frac{T}{2A} = \frac{-99648}{2 \times 957} = -52 \text{ lb./in. (957 = enclosed area of cell)}
\]

This value of q is entered in column 8 of Table A20.9. The resulting shear flow in any web portion qs equals the algebraic sum of q and q1. (See Col. 9, Table A20.9). Fig. A20.11 shows the results in graphical form.

**A20.8 Discontinuities - Shear Lag - Pressurization Stresses - Combined Stresses.**

A practical fuselage has many cut-outs. The approximate effect of these discontinuities as well as the shear lag effect at sections where large concentrated loads are applied can be determined by the procedure given in Articles 18 to 20 of Chapter A19. A more rigorous analysis can be made by the application of the basic theory as given in Chapter A8.

The problem of shear stresses due to internal pressures is presented in Chapter A16. The strength design of the fuselage skin involves a question of combined stresses. The broad problem of the strength design of structural elements and their connections under all types of stress conditions is covered in Volume II.

**A20.9 Problems.**

(1)
All material is aluminum alloy. $E = 10,500,000$ psi. The ultimate compressive strength of stringer plus its effective skin is 35000 psi. For effective sheet width use $w = 1.9t$ ($3.057$). For buckling strength of curved panels use $Q_{cr} = 0.3 Et/r$. Determine the ultimate bending moment that the fuselage section will develop for bending about horizontal neutral axis. Use linear stress distribution. Follow procedure as given in example problem in Art. A20.4.

(2) Fig. A20.13 shows the cross-section of a rectangular fuselage. The dots represent stringer locations. Three types of stringers are used, namely, $S_1$, $S_a$, and $S_s$. Fig. A20.14 shows the ultimate compressive stress-strain curve for each of the three stringer types and also the tension stress-strain curve of the material.

Determine the ultimate bending resistance of the fuselage section about the horizontal neutral axis if the maximum unit compressive strain is limited to .008. Refer to Art. A20.5 for method of solution.

ADDITIONAL DATA. Area stringer $S_1 = .12$ sq. in.; $S_a = .25$ sq. in.; $S_s = .08$ sq. in. $E = 10,500,000$ psi.

(3) Fig. A20.15 shows a tapered circular fuselage with 8 stringers. The area of each stringer is 0.1 sq. in. Assume stringers develop entire bending resistance. Find the axial load in stringers at station (110) due to $P_z$ and $P_x$ loads at station (0). Also find shear flow system at station 110 using $AP$ method. Use properties at station (90) in obtaining average shear flows.

(4) Same as Problem (3) but change area of stringer no. (2) to 0.3 sq. in., thus making an unsymmetrical section.

A20.10 Secondary Stresses in Fuselage Stringers and Rings.

The stresses that are found in the stringers or longerons of a typical fuselage by use of the modified beam theory or by the more rigorous theory of Chapter A9, are referred to as primary stresses. Because of the necessity of weight saving, most fuselage structures are designed to permit skin buckling, which means that shear loads in the skin are carried by diagonal semi-tension field action. This diagonal tension in the skin panels produces additional stresses in the stringers and also in the fuselage rings. These resulting stresses are referred to as secondary stresses and must be properly added to the primary stresses in the strength design of the individual stringer or ring. Chapter A11 covers the subject of these secondary stresses due to diagonal semi-tension field action in skin panels. It is suggested to the student that after studying Chapters A19 and A20, that Chapters A10 and C11 be referred to in order to obtain a complete stress picture for skin covered structures.
CHAPTER A21
LOADS AND STRESSES ON RIBS AND FRAMES

A21.1 Introduction. For aerodynamic reasons the wing contour in the chord direction must be maintained without appreciable distortion. Unless the wing skin is quite thick, spanwise stringers must be attached to the skin in order to increase the bending efficiency of the wing. Therefore to hold the skin-stringer wing surface to contour shape and also to limit the length of stringers to an efficient column compressive strength, internal support or brace units are required. These structural units are referred to as wing ribs. The ribs also have another major purpose, namely, to act as a transfer or distribution unit. All the loads applied to the wing are reacted at the wing supporting points, thus these applied loads must be transferred into the wing cellular structure composed of skin, stringers, spars, etc., and then reacted at the wing support points. The applied loads may be only the distributed surface airloads which require relatively light internal ribs to provide this carry through or transfer requirement, to rather rugged or heavy ribs which must absorb and transmit large concentrated applied loads such as those from landing gear reactions, wing reactions, tail reactions, power plant reactions, etc. The dead weight of all the payload and fixed equipment inside the fuselage must be carried to frames by other structure such as the fuselage floor system and then transmitted to the fuselage shell structure. Since the dead weight must be multiplied by the design acceleration factors, these internal loads become quite large in magnitude.

Another important purpose or action of ribs and frames is to redistribute the shear at discontinuities and practical wings and fuselages contain many cut-outs and openings and thus discontinuities in the basic structural layout.

A21.2 Types of Wing Rib Construction.

Figs. A21.1 to 6 illustrate the common types of wing construction. Fig. 1 illustrates a sheet metal channel for a leading edge 3

Since the airplane control surfaces (vertical and horizontal stabilizer, etc.) are nothing more than small size wings, internal ribs are likewise needed in these structures.

The skin-stringer construction which forms the shell of the fuselage likewise needs internal forming units to hold the fuselage cross-section to contour shape, to limit the column length of the stringers and to act as transfer agents of internal and externally applied loads. Since a fuselage must usually have clear internal space to house the payload such as passengers in a commercial transport, these internal fuselage units which are usually referred to as frames are of the open or ring type. Fuselage frames vary in size and strength from very light former type to rugged heavy types which must transfer large concentrated loads into the fuselage shell such as those from landing gear reactions, wing reactions, tail reactions, power plant reactions, etc.
stringer, single spar, single cell wing structure. The rib is riveted, or spot-welded, or glued to the skin along its boundary. Fig. 2 shows the same leading edge cell but with spanwise corrugations on the top skin and stringers on the bottom. On the top the rib flange rests below the corrugations, whereas the stringers on the bottom pass through cutouts in the rib. Fig. 3 illustrates the general type of sheet metal rib that can be quickly made by use of large presses and rubber dies. Figs. 4 and 5 illustrate rib types for middle portion of wing section. The rib flanges may rest below stringers or be notched for allowing stringers to pass through. Ribs that are subjected to considerable torsional forces in the plane of the rib should have some shear ties to the skin. For ribs that rest below stringers this shear tie can be made by a few sheet metal angle clips as illustrated in Fig. 5. Fig. A21.7 shows an artist’s drawing of the wing structure of the Beechcraft Bonanza commercial airplane. It should be noticed that various types and shapes of ribs and formers are required in airplane design. Photographs A21.1 to 3 illustrate typical rib construction in various type aircraft, both large and small. Since ribs compose an appreciable part of the wing structural weight, it is important that they be made as light as safety permits and also be efficient relative to cost of fabrication and assembly. Rib development and design involves considerable static testing to verify and assist the theoretical analysis and design.

A21.3 Distribution of Concentrated Loads to Thin Sheet Panels.

In Art. A21.1 it was brought out that ribs were used to transmit external loads into the wing cellular beam structure. Concentrated external loads must be distributed to the rib before the rib can transfer the load to the wing beam structure. In other words, a concentrated load applied directly to the edge of a thin sheet would cause sheet to buckle or cripple under the localized stress. Thus a structural element usually called a web stiffener or a web flange is fastened to the web and the concentrated load goes into the stiffener which in turn transfers the load to the web. To get the load into the stiffener usually requires an end fitting. In general the distributed air loads on the wing surface are usually of such magnitude that the loads can be distributed to rib web by direct bearing of flange normal to edge of rib web without causing local buckling, thus stiffeners are usually not needed to transfer air pressures to wing ribs.

EXAMPLE: PROBLEM ILLUSTRATING TRANSFER OF CONCENTRATED LOAD TO SHEET PANEL.

Fig. A21.8 shows a cantilever beam composed of 2 flanges and a web. A concentrated load of 1000 lb. is applied at point (A) in the direction shown. Another concentrated load of 1000 lb. is applied at point (E) as shown.

To distribute the load of 1000 lb. at (A), a horizontal stiffener (AB) and a vertical stiffener (CAD) are added as shown. A fitting would be required at (A) which would be attached to both stiffeners. The horizontal component of the 1000 lb. load which equals 800 lb. is taken by the stiffener (AB) and the vertical component which equals 600 lb. is taken by the
Fig. A21.7 General Structural Details of Wing for Beechcraft "Bonanza" Commercial Airplane.

vertical stiffener CD. The vertical load at E would be transferred to stiffener EF through fitting at E. The problem is to find the shear flows in the web panels, the stiffener loads and the beam flange loads.

SOLUTION: It will be assumed that the beam flanges develop the entire resistance to beam bending moments, thus shear flow is constant on a web panel.

The shear flows on web panels (1) and (2) will be computed treating each component of the 1000 lb. load as acting separately and the results added to give the final shear flow.

Figs. A21.9 and A21.10 show free bodies of that portion including web panels (1) and (2) and stiffeners CA and A5 and the external load at (A). In Fig. A21.9 the shear flows \( q_1 \) and \( q_2 \) on the top and bottom edges respectively have been assumed with the sense as shown. Taking moments about point E,

\[
ZM_c = 800 \times 3 - 12 \times 10q_1 = 0, \text{ whence } q_1 = 20 \text{ lb./in.}
\]

\[
ZF_x = 600 - 20 \times 10 - 10q_2 = 0, \text{ whence } q_2 = 50 \text{ lb./in.}
\]
PHOTO. A21.1 Type of Wing Ribs Used in Cessna 180 Model Airplane, a 4 Place Commercial Airplane.

PHOTO. A21.2 Rib Type Used in Outer Panel-Fuel Tank Section- of Douglas DC-8 Commercial Jet Airliner.

Referring to Fig. A21.10,

\[ ZF_G = 600 \times 10 - 10 \times 10q_x = 0, \text{ whence } q_x = \frac{50 \text{ lb. in.}}{10} \]

\[ ZF_X = -50 \times 10 + 10q_x = 0, \text{ whence } q_x = \frac{50 \text{ lb./in.}}{10} \]

Combining the two shear flows for the two loads,

\[ q_x = 20 + 50 = 70 \text{ lb./in.} \]

\[ q_x = 50 - 50 = 10 \text{ lb./in.} \]

Fig. A21.11 shows the results. Fig. A21.12 shows stiffener AB as a free body, and Fig. A21.13 the axial load diagram on stiffener AB, which comes directly from Fig. A21.11 by starting at one end and adding the shear flows.

Fig. A21.14 shows a free body of the vertical stiffener CAD, and Fig. A21.15 the axial load diagram for the stiffener.

Solving equations (1) and (2) gives,

\[ q_x = 70 \text{ lb./in.}, \quad q_x = 10 \text{ lb./in.} \]

which checks first solution.

The shear flow \( q_x \) in web panel (3) is obtained by considering stiffener EBF as a free body, see Fig. A21.16.

\[ ZF_y = 12q_x + 10 \times 3 - 9 \times 70 - 1000 = 0 \]

whence, \( q_x = 133.33 \text{ lb./in.} \)

The shear flow \( q_x \) could also be found by treating entire beam to right of section through panel (3). For this free body,

\[ ZF_y = -600 - 1000 + 12q_x = 0 \]

whence, \( q_x = 133.33 \)

Fig. A21.17 shows diagram of axial load in stiffener EF as determined from Fig. A21.16 by starting at one end and adding up the forces to any section.

After the web shear flows have been determined the axial loads in the beam flanges follow as the algebraic sum of the shear flows. Fig. A21.18 shows the shear flows along each beam flange as previously found. The upper and lower beam flange loads are indicated by the diagrams adjacent to each flange.

In this example problem the applied external load at point (A) was acting in the plane of the beam web, thus two stiffeners were sufficient to take care of its two components. Often
loads are applied which have three rectangular components. In this case, the structure should be arranged so that line of action of applied force acts at intersection of two webs as illustrated in Fig. A21.19 where a load P is applied at point (O) and its components \( P_x, P_y \) and \( P_z \) are distributed to the web panels by using three stiffeners \( S_x, S_y, \) and \( S_z \) intersecting at (O).

In cases where a load must be applied normal to the web panel, the stiffener must be designed strong enough or transfer the load in bending to adjacent webs.

In this chapter, the webs are assumed to resist pure shear along their boundaries. In most practical thin web structures, the webs will buckle under the compressive stresses due to shear stresses and thus produce tensile field stresses in addition to the shear stresses. The subject of tension field beams is discussed in detail in Volume II. In general the additional stresses due to tension field action can be superimposed on those found for the non-buckling case as explained in this chapter.

**STRESSES IN WING RIBS**

**A21.4 Rib for Single Cell 2 Flange Beam.**

Fig. A21.20 illustrates a rib in a 2-flange single cell leading edge type of beam. Assume that the air-load on the trailing edge portion (not shown in the figure) produces a couple reaction \( P \) and a shear reaction \( R \) as shown. These loads are distributed to the cell walls by the rib which is fastened continuously to the cell walls. Let \( q = \) shear flow per inch on rib perimeter which is necessary to hold rib in equilibrium under the given loads \( P \) and \( R \).

Taking moments about some point such as (1) of all forces in the plane of the rib:

\[ \sum M_1 = -Ph + 2Aq = 0 \]

hence

\[ q = \frac{Ph}{2A}. \quad (A = \text{enclosed area of cell}) \]

**Fig. A21.20**

With \( q \) known the shear and bending moment at various sections along the rib can be determined. For example, consider the section at B-B in Fig. A21.20. Fig. A21.21 shows a free body of the portion forward of this section.

The bending moment at section B-B equals:

\[ M_B = 2qA_1, \quad \text{where} \quad A_1 = \text{area of the shaded portion}. \]

Let \( F_X \) equal the horizontal component of the flange load at this section.

**Fig. A21.21**

\[ F_X = \frac{M_B}{a} = 2qA_1/a \]

The true upper flange load \( F_U = F_X \cos \theta_1 \) and the lower flange load equals \( F_L = F_X \cos \theta_2 \).

The vertical shear on the rib web at B-B equals the vertical component of the shear flow \( q \) minus the vertical components of the flange loads. Hence

\[ V_{\text{web}} = q \cdot a - F_X \tan \theta_1 - F_X \tan \theta_2 \]

\[ = q \cdot a - \frac{2qA_1}{a} (\tan \theta_1 + \tan \theta_2) \]

**Illustrative Problem**

The rib in the leading edge portion of the wing as illustrated in Fig. A21.22 will be analyzed.

A distributed external load as shown will be assumed.
Solution:

The total air load at of beam = 8 x 10/2 = 160 lb. The arm to its c.g. location from the beam equals 40/3 = \(13.33^\circ\). Hence the reactions at the beam flange points due to the loads on the trailing edge portion equals:

\[
P = 160 \times 13.33/10 = 213.2 \text{ lb.} \quad \text{(See Fig. A21.23)}
\]

Shear reaction \(V_r = 160 \text{ lb.}\)

Let \(q\) be the constant flow reaction of the cell skin on the rib perimeter which is necessary to hold the rib in equilibrium under the applied air loads.

Take moments about some point such as the lower flange (1).

\[
EM = -213.2 \times 10 + 9 \times 15 \times 7.5 + 2 \times 139.3 \times q = 0
\]

whence, \(q = \frac{1232}{878.6} = 4.42 \text{ lb./in.}\)

A21.5 Stresses in Rib for 3 Stringer Single Cell Beam.

Fig. A21.25 shows a rib that fits into a single cell beam with 3 stringers labeled (a), (b) and (c). An external load is applied at point (a) whose components are 3000 and 3000 lb. as shown. Additional reactions from a trailing edge rib are shown at points (b) and (c). A vertical stiffener ad is necessary to distribute the load of 3000 lb. at (a). The following values will be determined:

1. Rib web shear loads on each side of stiffener ad.
2. Rib flange load at section ad.
3. Rib flange and web load at section just to left of line bc.

SOLUTION. It will be assumed that the 3 stringers develop the entire wing beam bending resistance; thus the wing shear flow is constant between the stringers. The wing rib is riveted to the wing skin and thus the edge forces on the rib boundary will be assumed to be the same as the shear flow distribution. In other words, the three shear flows \(q_{ab}, q_{ba}\) and \(q_{bc}\) hold the external loads in equilibrium. The sense of these 3 unknown shear flows will be assumed as shown in Fig. A21.25.

To find \(q_{ab}\), take moments about point (b)
To find web shear $q_{ad}$ take $EF_Z = 0$

$\phi = 0$

$q_{ad} = 463 \text{ lb./in.}$

whence, $C' = 5054 \text{ lb.}$

To find flange load $C'$ take $EF_X = 0$

$EF_X = -C' + 3000 + 2158(16/17) = 0$

whence, $C' = 5054 \text{ lb.}$

To find web shears $q_{ad}$ take $EF_Z = 0$

$EF_Z = 2158(6/17) - 157.3\times 9.3 + 9.3q_{ad} = 0$

whence, $q_{ad} = 463 \text{ lb./in.}$

The above values are the same as previously obtained.

The rib flange loads and web shear will be calculated for a section just to left of line.
cb. Fig. A21.29 shows the free body for the rib to left of this section.

\[
\begin{align*}
E_b &= -157.3 \times 2 \times (160 + 60) + 5000 \times 15 \\
   &\quad -11.5 \times C = 0 \\
\therefore C &= 500 \text{ lb.}
\end{align*}
\]

To find flange load T take \( \Sigma F_x = 0 \)

\[
\begin{align*}
\Sigma F_x &= 3000 - 157.3 \times 15 - 15 \times 42.7 - 500 + T = 0 \\
\therefore T &= 500 \text{ lb.}
\end{align*}
\]

To find \( q_{c_b} \) take \( \Sigma F_z = 0 \)

\[
\begin{align*}
\Sigma F_z &= 5000 - 157.3 \times 11.5 - 11.5 \times q_{c_b} = 0 \\
\therefore q_{c_b} &= 275 \text{ lb./in.}
\end{align*}
\]

The above results could have been obtained with less numerical work by considering the forces to right of section cb in Fig. A21.29.

A21.6 Stress Analysis of Rib for Single Cell Multiple Stringer Wing.

When there are more than three spanwise stringers in a wing, there are four or more panels in the cell walls, thus the reactions of the cell walls upon the rib boundary cannot be found by statics as was possible in the 3 stringer case of the previous example problem.

Fig. A21.30 illustrates a wing section consisting of four spanwise flange members. The concentrated loads acting at the four corners of the box might be representative of reactions from the engine mount or nacelle structure and the reactions from a rib which supports the wing flap. These loads must be distributed into the walls of the wing box beam which necessitates a rib. Before the rib can be designed, the bending and shear forces on the rib must be determined. The calculations which follow illustrate a method of procedure.

![Fig. A21.30](image)

**SOLUTION:**

The total shear load on the wing in the Z direction equals \( V_Z = -5000 - 5000 + 2000 = -8000 \text{ lb.} \) and \( V_X = -8500 + 7500 - 4000 + 4500 = -500 \text{ lb.} \)

The boundary forces on the rib will be equal to the shear force system on the cell walls due to the given external force system.

From Chapter A14, page A14.8, equation (14), the expression for shear flow is,

\[
q_Y = - (K_a Y_a - K_b Y_b) \times Z_a - (K_a Y_a - K_b Y_b) \times Z_b
\]

(1)

The constants \( K \) depend on the section properties of the wing cross-section. Table A21.1 gives the calculation of the moment of inertia and product of inertia about centroidal Z and X axes. In this example the 4 stringers a, b, c and d have been considered as the entire effective material in resisting wing bending stresses.

**TABLE A21.1**

<table>
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<th>Flange No.</th>
<th>Area</th>
<th>( Z )</th>
<th>( X )</th>
<th>( Z^2 )</th>
<th>( X^2 )</th>
<th>( Z^2 - X^2 )</th>
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<th>( Z \times Z )</th>
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<td>3.55</td>
<td>30.94</td>
<td>15.72</td>
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<td>12160</td>
<td>-156.9</td>
<td>182.8</td>
<td>-34.7</td>
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</tbody>
</table>

\[
Z = Z_{AZ}/Z_a = -5.27/6.10 = -.865
\]

\[
X = X_{AZ}/Z_a = 72.0/6.10 = 11.8
\]

Centroidal x and z moments of inertia:

\[
I_Z = 230.3 - 6.10 \times .865 = 225.8
\]

\[
I_Z = 2160 - 6.10 \times 11.8 = 1310
\]

\[
I_{XX} = -156.9 - 610 \times .865 \times 11.8 = -34.7
\]
With the wing section properties known, the constants K can be calculated.

\[ K_1 = \frac{I_{xz}}{(I_x I_z - I_{xz}^2)} = \frac{-94.7(225.8 \times 1310 - 94.7^2)}{-94.7(236700)} = \frac{-94.7}{236700} = -0.00033 \]

\[ K_2 = \frac{I_z}{236700} = \frac{1310}{236700} = 0.00546 \]

\[ K_3 = \frac{I_x}{236700} = \frac{225.8}{236700} = 0.000786 \]

Substituting in equation (1),

\[ Q_y = \left[ \frac{0.000786 (-500)}{0.00033} \right] - \left[ \frac{0.0046 (-9000)}{-0.00033} \right] \]

\[ \Sigma x - \left[ 0.000786(-500) - \frac{-0.00033(-9000)}{0.00033} \right] \]

\[ \Sigma x = 3.363 \Sigma z + 41.205 \Sigma zA - - - - - - - (2) \]

Since the shear flow at any point on the cell walls is unknown, it will be assumed zero on web ad, or imagine the web is cut as shown in Fig. A21.31. The static shear flows can now be found.

\[ Q_{ab} = 3.363 (-11.8)(2.0) + 41.205 x 6.36 \]

\[ x = 2 = 444 \text{ lb./in.} \]

\[ Q_{bc} = 444 + 3.363 x 18.2 x 1.25 + 41.205 \]

\[ x = 4.1 x 1.25 = 748 \text{ lb./in.} \]

\[ Q_{cd} = 748 + 3.363 x 18.2 x 1.15 + 41.205 \]

\[ (-7.54)(1.15) = 461 \text{ lb./in.} \]

These shear flows are plotted on Fig. A21.31. Refer to Chapter A14 regarding sense of shear flows.

The moments of the forces in the plane of the rib will now be calculated:

Taking moments about the c.g. of the beam cross section (See Fig. A21.31):

\[ Z_{M, g} = -11000 x 11.8 - 3500 x 6.36 - 7500 x 5.64 - 4500 x 7.54 - 2000 x 13.2 - 444 x 2 x 90 - 748 x 2 x 108.7 - 461 x 2 x 39 = -648400 \text{ in.lb.} \]

For equilibrium, \( Z_{M, g} \) must equal zero, therefore a constant flow shear \( q_s \) acting around the rib perimeter is necessary which will produce a moment of 648400 in.lb.

\[ q_s = \frac{M}{2A} = \frac{648400}{2 \times 368.5} = 880 \text{ lb./in.} \]

(Note: 368.5 = total area of cell)

Adding this shear flow to that of Fig. A21.31., the resulting force system of Fig. A21.32 is obtained. The reactions of the beam cell walls on the rib have been determined and the bending moments and shears on the rib can now be calculated.

To illustrate, consider the rib section B-B which passes through the c.g. of the beam section. Fig. A21.33 shows a free body of the bulkhead portion to the left of section B-B.

Moments at section B-B will be referred to the point (O):

\[ Z_{M, g} = -11000 x 11.8 - 3500 x 6.36 - 7500 x 5.64 + 436 \]

\[ x = 2 x 36 + 880 x 2 x 70.8 + 419 x 2 x 38.3 = 3800 \text{ in.lb.} \]
The resultant external shear force along the section B-B equals the summation of the z components of all the forces.

\[ V = ZF_y = -11000 + 12 \times 880 - 436 \times 0.36 + 419 \times 0.96 = -195 \text{ lb.} \]

The resultant load normal to the section B-B equals the summation of the force components in the x direction.

\[ H = ZF_x = -8500 + 7500 + (436 - 419) \times 11.8 = -800 \text{ lb.} \]

Fig. A21.34 shows these resultant forces referred to point (O) of the cross-section. If we assume that the rib flanges develop the entire resistance to normal stresses, we can find flange loads by simple statics.

To find upper flange load \( F_u \) take moments about lower flange point.

\[ ZM = 12.6 \cdot F_u - 38200 - 800 \times 5.6 = 0 \]

whence \( F_u = 3443 \text{ lb. tension} \)

To find \( F_L \) use \( ZF_x = 0 \)

\[ ZF_x = 3443 - 800 - F_L = 0, \text{ whence } F_L = 2643 \text{ lb. compression.} \]

The shear flow on web equals \( V/12.6 = 195/12.6 = 15.5 \text{ lb./in.} \). This result neglects effect of flanges not being normal to section B-B, which inclination is negligible in this case.

If the entire cross-section of rib is effective in bending, then the web thickness and flange sizes of the rib would be needed to obtain the section moment of inertia which is necessary in the beam equation for bending stresses. The forces at (O) would then be referred to neutral axis of section before bending and shear stresses on the rib section could be calculated.

To obtain a complete picture of the web and flange forces, several sections along the rib span should be analyzed as illustrated for section B-B.

A21.7 Rib Loads Due to Discontinuities in Wing Skin Covering.

As referred to before, ribs in addition to transmitting external loads to wing cell structure are also a means of re-distributing the shear forces at a discontinuity, the most common discontinuity being a cut-out in one or more of the webs or walls of the wing beam cross section. The usual procedure in finding the boundary forces on a rib located adjacent to a cut-out is to find the applied shear flows in the wing on two sections, one on each side of the rib. Then the algebraic sum of these two shear flows will give the rib boundary forces. With the boundary forces known the rib web and flange stresses can be found as previously illustrated. The procedure can best be illustrated by example problems.

A21.8 Example Problem. Wing with Cut-Out Subjected to Torsion.

Fig. A21.35 shows a rectangular single cell wing beam with four stringers or flanges located at the four corners. The upper surface skin is discontinuous in the center bay (2). The wing is subjected to a torsional moment of

![Figure A21.35](image)

80000 in.lb. at Station (70) and a couple force at Station (60) as shown in Fig. A21.35. The problem will be to determine the applied forces on rib (A).

**SOLUTION:**

The applied shear flow on the cell walls will be found for two cross-sections of the wing, one on each side of rib (A).

In bay (1) the torsional moment \( M = 80000 \text{ in.lb.} \). The applied shear flow on a cross-section of the wing in bay (1) thus equals,

\[ q = \frac{M}{2A} = \frac{80000}{2 \times 10 \times 40} = 100 \text{ lb./in.} \]

This shear flow system is shown on Fig. A21.36 which is a free body of rib (A). In bay (2), since the top skin is removed, the torsional moment must be taken by the front and rear vertical webs, since any shear flow in the bottom skin could not be balanced.

The torsional moment in bay (2) is,
LOADS AND STRESSES ON RIBS AND FRAMES

\[ M = 80000 + 4000 \times 10 = 120000 \text{ in.lb.} \]

The total shear load on each vertical web thus equals \( 120000/40 = 3000 \text{ lb.} \), which gives a shear flow \( q' = 3000/10 = 300 \text{ lb./in.} \) on each web. This applied shear is shown on the free body of rib (A) in Fig. A21.36. On the left end of the rib a shear flow of 100 is acting up and on the other side a shear flow of 300 is acting down, thus the rib web must take the difference or 200 acting down. On the right end of the rib the load on the rib web is 200 lb./in. up. The loads on the top and bottom flanges of the rib is obviously 100 lb./in. Fig. A21.37 shows the loads applied to the rib boundary when the torsion in bay (1) and the external couple force is transferred to the cross-section of bay (2).

ADDITIONAL EFFECTS DUE TO DIFFERENTIAL BENDING OF BEAMS IN BAY (2).

The torsion in bay (1) and the external couple force are thrown off as couple force on the front and rear beams of middle bay (2), with the total shear load on each beam being 3000 lb. as previously calculated. These beam shear loads must be transmitted to bay (3) and thus cause bending of the beams in bay (2). Since each beam is attached to relative rigid box structures at each end, namely bays (1) and (3), the beams tend to bend with no rotation of their ends. If we neglect the deflections of these end box structures, we can assume that the beams bend with no rotation of their ends or each beam is fixed ended. Fig. A21.38a illustrates the deflection of the front beam in bay (2) under the assumption of no end rotation. The beam elastic curve has a point of inflection at the span midpoint. Figs. 38b, c show the beams bending moment and shear diagrams.

The end moments are \( M = VL/2 = 3000 \times 30/2 = 45000 \text{ in.lb.} \). Assuming the beam flanges develop the entire bending resistance the beam flange loads at the beam ends are \( P = 45000/10 = 4500 \text{ lb.} \) (See Fig. a).

The deflection of the rear beam would be the reverse of Fig. a, and thus all forces would also be reversed.

Fig. A21.35 shows bay (1) of the wing as a free body acted upon by the flange loads due to bending of the beams in bay (2). These internal flange forces from bay (2) must be held in equilibrium by the internal stresses in the adjacent wing structure of bay (1).

According to the well known principle of mechanics formulated by Saint Venant, the stresses resulting from such an internal force system will be negligible at a distance from the forces. This distance in case of a cut-out is usually assumed as approximately equal to the width of the cut-out, or in general to the width of the adjacent wing bay. Thus in Fig. A21.39 the flange loads of 4500 lbs. each are assumed to be dissipated at a uniform rate for a distance of 20 inches. Thus the shear flow created by each stringer load which equals the change in axial load per inch in the stringer in bay (1) equals 4500/20 = 225 lb.

Fig. A21.40 shows a segment 1 inch wide cut from wing bay (1) with the dP load in each
To find the shear flow on the cross-section, the front web is first assumed cut, and thus the static shear flow \( q_s = 225 \) from cut face where \( q_s \) is zero. Fig. A21.40 shows this static shear flow.

For equilibrium of the cross-section, the moment of the forces in the plane of the cross-section must equal zero. Taking moments about lower left hand corner of the \( q_s \) force system,

\[ M = 225 \times 40 \times 10 = 90,000 \text{ in.} \text{lb.} \]

For equilibrium a moment of -90000 is necessary. Therefore a constant shear flow system \( q \) must be added to develop a moment of -90000. Thus

\[ q = \frac{M}{2A} = \frac{-90000}{2 \times 40 \times 10} = -112.5 \text{ lb.} / \text{in.} \]

Adding this shear flow to that for \( q_s \) in Fig. A21.40 gives the final values in Fig. A21.41. This shear flow system represents the stress system caused on cross-section of bay (1) due to the differential bending of the beams in bay (2). This shear flow system must therefore be resisted by rib (A) as it must terminate at end of bay (1). Therefore the shear flows in Fig. A21.41 are applied boundary loads to rib (A) and these must be added to the rib loads in Fig. A21.37 to give the final rib loads of Fig. A21.42. With the final rib loads known, the rib flange and web stresses can be found as previously explained.

A21.9 Example Problem. Wing with Cut-Out Subjected to Bending and Torsional Loads.

The middle bay (2) has no skin on the bottom surface, or in other words, the middle bay has a channel cross-section, which fact often happens in practical wing design as for example a space or well for a retractable landing gear. The problem will be to find the shear flow in bays (1) and (2) and the boundary loads on rib (A) between bays (1) and (2).

Solution No. 1

This method of solution will make use of the shear center location for bay (2) in order to obtain the true torsional moment on bay (2). With this torsional moment known, the procedure is similar to the previous example involving wing torsion only.

We will first calculate the shear flow in wing bay (1). Fig. A21.44 shows the cross-section.
The section moments of inertia are needed in calculating shear flows.

\[ I_x = (1 \times 6^2 \times 2) + (0.5 \times 4^2 \times 2) = 38 \text{ in}^3 \]
\[ X = \frac{2AX}{ZA} = \frac{(1 \times 30)}{3} = 10 \text{ in.} \]
\[ I_z = (2 \times 10^2) + (1 \times 20^2) = 600 \text{ in}^3 \]
\[ V_x = 8800 \text{ lb}, \quad V_y = 2400 \text{ lb} \]
\[ q_y = -\frac{V_x}{I_x} zA - \frac{V_y}{I_z} EXA, \text{ substituting} \]
\[ q_y = -100 \text{ EZA} - 4 \text{ EXA} \]

Since the shear flow is unknown at any point on cell, we will assume front web (ad) as cut or carrying zero shear.

\[ q_{ac} = 100 (-6)(1) - 4 (-10)(1) = 640 \text{ lb/in} \]
\[ q_{bc} = 640 - 100 (-4)(0.5) - 4 (20) 0.5 = 600 \]
\[ q_{ba} = 800 - 100 (4)(0.5) - 4 (20) 0.5 = 560 \]

Fig. A21.45 shows these static shear flows.

\[ \Sigma F_x = 2400 - 30 q_{ab} = 0, \text{ whence } q_{ab} = 80 \]
\[ \Sigma F_d = 2400 x 3 - 8800 x 15 - 30 x 30 x 12 q_{bc} \]
\[ (8 x 30) = 0, \text{ whence } q_{bc} = 640 \]
\[ \Sigma F_z = 8800 - 3 x 640 - 2 x 50 - 12 q_{ad} = 0 \]
\[ \text{whence } q_{ad} = 350 \]

Fig. A21.47 shows the result. This shear flow system is the final or true shear on bay (2).

Since we have a channel or open wing cross-section in bay (2), any torsional moment on this bay must be transmitted by differential bending of the front and rear beams. To obtain the torsional moment on bay (2), the shear center location must be known.

Horizontal location of shear center: Assume the section bends about centroidal X axis without twist under a V_y load of 8800 lb.

\[ q = -\frac{V_x}{I_x} zA, \text{ or } q = -100 \text{ EZA} \]
\[ q_{cb} = -100 (-4)(0.5) = 200 \]
\[ q_{ba} = 200 - 100 (4)(0.5) = 0 \]
\[ q_{ad} = 0 - 100 (5)(1) = 400 \]

Fig. A21.49 shows the shear flow results for bending about x-x without twist. The line of action of the resultant of this shear flow force system locates the horizontal position of the shear center.

\[ \bar{x} = (200 x 8 x 30)/8800 = 5.45 \text{ in.} \]
Vertical position of shear center:

Assume section bends about centroidal $z$ axis without twist under a load of $V_x = 2400$ lb.

$$q = -\frac{V_x}{l_z} = -4 \text{ ExA}$$

$$Q_{cb} = -4 \times 20 \times 0.5 = -40 \text{ lb.}/\text{in.}$$

$$Q_{ba} = -40 - 4 \times 20 \times 0.5 = -80$$

$$Q_{ad} = -80 - 4 \times (-10) = -40$$

Fig. A21.50 shows the shear flow results.

The vertical distance $z$ from point (a) to the line of action of the resultant which locates the vertical position of shear center is,

$$z = Z_{sa}/2400 = (40 \times 8 \times 30)/2400 = 4 \text{ in.}$$

Fig. A21.51 shows the shear center location and the external loads. The moment about the shear center which equals the torsion on the wing bay (2) equals,

$$M_{c.c.} = -8800 \times 9.55 - 2400 \times 13 = -115240 \text{ in.}$$

This torsional moment must be resisted by front and rear beams. Hence shear load on each beam = $115240/30 = 3841$ lb.

As in the previous example problem involving torsion, the beams in bay (2) will be assumed to bend without rotation of their ends, or in other words the bending moment at mid-point of bay is zero. The flange loads at points a, b, c and d on bay (1) from the differential bending of beams in bay (2), thus equal the beam shear times half the span of bay (2) divided by the beam depth.

For front beam $P = 3841 \times 12.5/12 = 4000$ lb.

For rear beam $P = 3841 \times 12.5/8 = 6000$ lb.

Fig. A21.52 shows these flange loads applied to bay (1). These loads are dissipated uniformly in bay (1) over a distance of 30 inches, or the shear flow per inch produced by these flange loads equals $\Delta P = P/30$, whence

$$\Delta P_a = \Delta P_d = 4000/30 = 133.3 \text{ and } \Delta P_b = \Delta P_c = 6000/30 = 200 \text{ lb.}$$

Fig. A21.53 shows an element of bay (1) one inch wide with these $\Delta P$ loads. The shear flow $q$ assuming the front web cut equals $2\Delta P$. The resulting static shear flows which equals $2\Delta P$ is shown in Fig. A21.53.

The moment of this shear flow system about point (d) = $133.3 \times 30 \times 12 - 66.7 \times 8 \times 30 = 31980$. For $ZM = 0$, we need a constant shear flow $q = -31980/2 \times 300 = -53.3$ lb./in. Adding this constant shear flow to that of Fig. A21.53 gives the shear flow system of Fig. A21.54. These results represent the effect on bay (1).
of removing the bottom skin in bay (2). Adding the shear flows of Fig. A21.54 to those of Fig. A21.46, we obtain the final shear flows in bay (1) as shown in Fig. A21.55.

**Boundary Loads on Fig. A13 (A)**

The boundary loads on rib (A) will equal the difference between the shear flows in bays (1) and (2). Fig. A21.56 shows a free body of rib (A) with the shear flows obtained from Figs. A21.55 and A21.49.

![Fig. A21.56](image)

The resulting applied boundary forces to the rib equal the algebraic sum of the shear flows on each side of the rib which gives the values in Fig. A21.57.

![Fig. A21.57](image)

With the rib boundary loads known, the stresses in the rib can be found as previously illustrated in this chapter.

**Solution No. 2**

This method of solution first finds the shear flow in all bays assuming bottom skin is not removed in center bay (2). This gives a shear flow in the bottom skin. However, the skin in bay (2) is actually removed so a corrective set of shear flows on bay (2) along the boundary lines of the bottom skin must be applied to eliminate the shear flows found in the bottom skin. The problem then consists of finding the influence of these corrective shear flows upon the shear flows as found for bays (1) and (2) when bottom skin in bay (2) was not removed.

The first step is to find the shear flows in all bays assuming bottom skin in bay (2) is not removed. The calculations would be exactly like those in solution (1) and the shear flow in all bays would be those in Fig. A21.46. The bottom skin in Fig. A21.46 has a shear flow of 192 with sense as shown. Since this skin is missing, we reverse this shear flow and find the resisting shear flows on the other three sides of the bay cross-section. Fig. A21.58 shows the section, with the 3 unknown shear flows \( q_{ab} \), \( q_{bc} \) and \( q_{ad} \).

![Fig. A21.58](image)

To find \( q_{ab} \) use \( ZF_x = 0 \), \( 192 \times 30 - 30q_{ab} = 0 \)

whence, \( q_{ab} = 192 \)

\[ En_1 = -30 \times 192 \times 12 + 8q_{bc} \times 30 = 0 \]

whence, \( q_{bc} = 288 \)

\[ SF_2 = 4 \times 192 \times 8 \times 288 + 12q_{ad} = 0 \]

whence, \( q_{ad} = 128 \)

Adding the shear flows of Fig. A21.58 to those of Fig. A21.46 gives the final shear flows in bay (2) as shown in Fig. A21.59. These results check the results in Fig. A21.48 obtained in solution method (1).

![Fig. A21.59](image)

Fig. A21.60 shows the corrective shear flows of Fig. A21.58 applied to bay (2). On the bottom skin the corrective shear flow is shown on the boundary of the cut-out. These shear
flows cause differential bending of the front and rear beams in bay (2). If we make the assumption that the beam and suffer no rotation, the bending moment is zero at midpoint of the bay and thus the flange loads at points a, b, c and d of bay (1) equal the algebraic sum of the shear flows on each side of a flange times half the span of bay (2) or 12.5 inches. Thus from Fig. A21.60,

\[ P_a = (129 + 126)12.5 = 4000 \text{ lb. compression} \]
\[ P_b = (293 + 126)12.5 = 6000 \text{ lb. tension} \]
\[ P_c = (293 + 126)12.5 = 6000 \text{ compression} \]
\[ P_d = (129 + 126)12.5 = 4000 \text{ tension} \]

Referring to Fig. A21.52, we find that the F values above are the same as the P values obtained by solution (1). Thus the remainder of solution (2) would be identical to that in solution (1), and therefore the calculations will not be repeated here.

### A21.10 Fuselage Frames

Frames in a fuselage serve the same purpose as ribs in wing structures. Ribs are usually of beam or truss construction and can be stress analyzed fairly accurately by statics. Fuselage frames however, are of the closed ring type of structure and are therefore statically indeterminate relative to internal stresses. Once the applied loads on a frame are known the internal stresses can be found by the application of the elastic theory as covered in Chapters A8, A9, A10 and A11. The loads on fuselage frames due to discontinuities in the fuselage structure, such as those due to windows and doors, can be approximately determined by the procedures previously presented for wing ribs.

The photographs on page 32 of Chapter A15 show some of the frame construction of the Douglas DC-8 airliner. Other pictures of fuselage construction are given in Chapter A20. Photographs A21.4 and 5 illustrate typical frame construction and arrangement.

### A21.11 Supporting Boundary Forces on Fuselage Frames

When external concentrated loads are applied to a fuselage frame through a suitable fitting or connection, the frame is held in equilibrium by reacting fuselage skin forces which are usually transferred to the frame boundary by rivets which fasten fuselage skin to frame. Since the fuselage shell is usually stress analyzed by the beam theory, it is therefore consistent to determine the distribution of the supporting skin forces by the same theory.

### Example Problem 1

Fig. A21.61 illustrates a cross-section of a circular fuselage. Two concentrated loads of 2000 lb. each are applied to the fuselage frame at the points indicated. The problem is to determine the reacting shear flow forces in the fuselage skin which will balance the two externally applied loads. This fuselage section might be considered as the aft portion of a medium size fuselage and the loads are due to air loads on the horizontal tail surfaces. To make the numerical calculations short the fuselage stringer arrangement has been assumed symmetrical.

#### Solution:

In this solution the fuselage skin resisting forces will be assumed to vary according to the general beam theory. The general flexural shear flow equation for bending about the Y axis is,
\[ q = \frac{V_2}{I_y} \zeta A , \text{ where } V_2 = 4000 \text{ lb.} \]

The moment of inertia \( I_y \) of the fuselage cross-section is required. In this simplified illustration, the area of each stringer plus its effective skin will be taken as .15 sq.in. The student should of course realize after studying Chapters A19 and A20 that the true effective area should be used on the compressive side and that the skin on the tension side of the fuselage is entirely effective. These facts would tend to make the effective cross-section unsymmetrical about the Y axis. Since the only purpose of this illustrative solution is to show how the frame loads are balanced, the section being assumed as symmetrical which will greatly decrease the amount of calculations required.

Fig. A21.61

Moment of inertia of fuselage section about Y axis which is the neutral axis under our simplified assumptions.

\[ I_y = .15 (17.6^2 + 16.2^2 + 13.5^2 + 10^2 + 5^2) \times 4 = 637 \text{ in.}^2 \]

Due to symmetry of effective section and external loading, the shear flow in the fuselage skin on the z axis or between stringers 1 and 11 and 11 or 11 will be zero. Thus starting with stringer (1) the shear flow in the skin resisting the external loads of 4000 lb. can be written around the circumference of the section.

\[ q = \frac{V_2}{I_y} \zeta A = \frac{4000 \zeta A}{637} = 6.275 \zeta A \]

\[ q_{-11} = 6.275 \times .15 \times 17.6 = -16.57 \frac{\text{lb.}}{\text{in.}} \]

\[ q_{11} = -6.275 \times .15 \times 16.2 = -31.82 \]

Due to symmetry of effective cross-section, the shear flow is symmetrical about the Y axis.

As a check on the above work, the summation of the z components of the shear flow on each skin panel between the stringers should equal the external load of 4000 lb.

\[ EF_z = [1.4 \times 16.57 + 2.7 \times 31.82 + 3.5 \times 57.22] + 5 \times 66.62 + 5 \times 71.32 = 4000 \text{ lb.} \]

Fig. A21.62 shows the frame with its balanced load system. The internal stresses can now be found by the methods of Chapters A8 to A11.

Fig. A21.62

Example Problem 2. Unsymmetrical Vertical Loading

In certain conditions in flying and landing, unsymmetrical concentrated loads are applied to the fuselage or tail structure. For example, Fig. A21.63 shows the same section and frame as was used in Problem 1. Due to an unsymmetrical load on the horizontal tail, the reactions from the tail on the fuselage are as illustrated in the figure. The total load in the z direction is still 4000 lb. but the loads are not symmetrical about the z axis. For analysis purposes, consider the loads as transferred to the
The moment of the two loads about the c.g. =
1500 x 11.5 - 2500 x 11.5 = -11500 in. lb. The
shear load Vz = 4000 produces the same shear
flow pattern as Fig. A21.62. To balance the
moment of -11500, a constant shear flow q, around the frame is necessary.

\[ q = \frac{M}{2A} = \frac{11500}{2 \times 1 \times 18^2} = 5.65 \text{ lb./in.} \]

(A = area of fuselage cross-section)

Adding this constant force system to that of Fig. A21.62, gives the final boundary
supporting forces on the frame as illustrated in Fig. A21.65. The elastic stress analysis
of the frame can now proceed.

Add the constant force system to that of Fig. A21.62, giving the final boundary
supporting forces on the frame as illustrated in Fig. A21.65. The elastic stress analysis
of the frame can now proceed.

Fig. A21.67 shows a wing rib inserted in a
3 flange single cell wing beam, which is subjected to the external loads as shown.

1. Find rib flange loads at (c) and (d).
2. Find rib web shear flow on each side of stiffener cd.
3. Find rib flange and web loads at section 5" to left of line ab.

Fig. A21.68 shows a 3 stringer single cell wing beam. A rib is inserted to distribute the
concentrated loads as shown.

1. Find shear flows in rib web panel (1) (2) and (3).
2. Find rib flange loads at sections dc and ab.
(4) Fig. A21.69 shows a 2 stringer, 2 cell wing beam. A rib is inserted to transfer 1000 lb. load to beam structure.

Find shear flow in rib web in each cell adjacent to line ab. Also rib flange loads adjacent to points (a) and (b).

(5) Fig. A21.70 shows 3 bays of a cantilever single cell, 4 stringer wing beam. The bottom skin in bay (2) is removed. Find the shear flows in all bays and boundary loads on ribs (A) and (B) when the external wing loads are as follows: \( T = 56000 \text{ in. lb.}, P_1 = 0, P_2 = 0, P_3 = 2000 \text{ lb.}, P_4 = 2000 \text{ lb.}, P_5 = 0. \)

(6) Same as problem (5) but upper skin in bay (2) is removed instead of the lower skin.

(7) Same as problem (5) but with the following external loads:

\[ T = 56000 \text{ in. lb.}, P_1 = 5000 \text{ lb.}, P_2 = 2000 \text{ lb.}, P_3 = P_4 = 0 \text{ and } P_5 = 1000 \text{ lb.} \]

(8) Same as problem (7) but with top skin removed instead of lower skin.

(9) Same as (5) but with read spar web removed instead of bottom skin.

(10) Same as problem (7) but with rear spar web removed instead of bottom skin.

(11) In Fig. A21.71 the external bulkhead loads \( P_1 \) and \( P_2 \) equal 4000 lb. each and \( P_3 \) equals zero. The fuselage stringer material consists of four omega sections with an area of .25 sq. in. each. Determine the skin resisting forces on the bulkhead in balancing the above loads. Neglect any effective skin in this problem.

(12) Same as problem (11) but make \( P_1 = 4000 \) and \( P_2 = 6000. \)

(13) Same as problem (12) but add \( P_3 = 3000 \) lb.

(14) In a water landing condition the hull frame of Fig. A21.72 is subjected to a normal bottom pressure of 200 lb. per in. The area of the bulb angle stringers is .11 sq. in. each and they are 7/8 in. deep. The area of the Z stringers is .18 sq. in. each and the depth 1.5 in. The area of the stringers a, b, c, d and e is .20 sq. in. each. Neglecting any effective skin determine the skin resisting forces on the frame in balancing the bottom water pressures.

(15) Same as problem (14) but consider that the water pressure is only acting on one side of the bottom of the frame.
CHAPTER A22
ANALYSIS OF SPECIAL WING PROBLEMS
ALFRED F. SCHMITT

A22.1 Introduction.

In a previous chapter (A19) analyses of wing beams were carried out using the engineering theory of bending and rational modifications thereof. As discussed there, wing configurations which depart radically from the usual conception of a "beam" present the engineer with the choice of making approximate and/or empirical corrections to beam theory, or of following a complete analytic treatment of the structure.

This chapter illustrates the latter approach to several special problems associated with aircraft wing structures, viz., *

Art. A22.2 - stresses around a panel cutout
Art. A22.3 - shear lag problem
Art. A22.4 - cutout in a box beam
Art. A22.5 - swept wing box beam

Aside from presenting one analytic treatment of these problems, a discussion is given of the physical nature of each phenomenon. An understanding of the nature of the problem is of prime importance, since no one analytic technique can be all-powerful in the solution of stress problems. The analyst must exercise judgment and ingenuity in approaching each new situation.

In this chapter all analyses are made using the matrix formulation of the Method of Dummy Unit Loads (Chapters A7, A8), a familiarity with which is assumed.

Such problems as those listed above are too unwieldy to be studied here in great detail; hence no attempt at exhaustive analyses has been made. To bring into relief the main features of each problem, the structure selected for analysis is one which is simple in construction and so loaded as to exhibit clearly the phenomenon under study. Many practical details, such as the effects of sheet wrinkling, rivet and fitting "give", stress concentrations, etc., have been side-stepped so as not to cloud the objective. Further, the problems of idealization of the original structure, into the one finally analyzed, are treated only lightly; additional references are cited where appropriate.

The analyses shown are strictly applicable to (reasonably) thin-skinned wings only, where-in the "constant shear flow" assumptions are valid, viz.

1 - the sheet carries shear stresses only
2 - normal (direct) stresses are carried in the flanges (spar caps and stringers) with effective areas of skin lumped in. In all cases handled here the skin was assumed fully effective (stresses below skin buckling stress — see Art. A19.11, Chapter A19).

To enhance the usefulness of these problems, all the structures chosen for analysis were taken from referenced NACA (National Advisory Committee for Aeronautics) publications wherein the reader may find detailed discussions of the problems, other methods of analysis and data obtained from tests upon the specimens. Where available, these data have been used herein for comparison.

A22.2 Stresses Around a Panel Cutout

"Cutouts in wings and fuselages constitute one of the most troublesome problems confronting the aircraft designer. Because the stress concentrations caused by cutouts are localized, a number of valuable partial solutions of the problem can be obtained by analyzing the behavior, under load, of simple skin-stringer panels" (1)**

Thus, in the case of a wing beam with a panel cutout of the upper surface (Fig. A22.1), it would be feasible to analyze the section immediately around the cutout as a flat sheet-stringer panel under the action of axial stringer loads and edge shears (coming from the spar webs). The axial stringer forces could be computed with sufficient accuracy by the engineering theory of bending (E.T.B.) since these are removed sufficiently far from the cutout proper. The edge shear flows are readily computed by those elementary considerations which give the spar-web shear flows.

* One other important special problem - the so-called "bending stresses due to torsion" - is not treated here specifically. As indicated in Chapter A8, the general box beam analysis presented there encompasses this problem (Example Prob. 18, p.p. A8. 24 through A8. 27).

** Numbers in parentheses refer to the bibliography at the end of the chapter.
The sheet-stringer panel may, in general, contain a large number of longitudinal elements (stringers). The labor involved in treating this multi-element structure in detail is prohibitive, and thus an appropriate idealization must be made. First, it is likely that the panel may be considered to be symmetric about a longitudinal axis, so that only the half-panel need be handled. Second, the complex, multi-stringer structure is replaced by one having but three stringers. As indicated in Fig. A22.2, these stringers are:

1. a substitute stringer having for its area all the effective area of the fully continuous members to one side of the "comb stringer" (the stringer bordering the cutout) and placed at the centroid of the area of material for which it substitutes. The stress which this stringer develops is then the average stress for the material it replaces.

2. the combing stringer, being simply the main continuous stringer bordering the cutout.

3. another substitute stringer, this one replacing all of the effective material made discontinuous by the cutout. It is located at the centroid of the material it replaces, and its stress is the average stress for this same material.*

The sheet thicknesses used are the same as those of the actual structure.**

* An alternate idealization, in which stringers #1 and #3 are located along the lines AB and CD, respectively (Fig. A22.2), was used in Reference (3) for a box beam loaded in torsion.

** When the longitudinal members themselves contribute to the shear stiffness of the cover (as is the case for "hat" section stringers riveted to the skin so as to form small closed cross sections), an effective thickness must be used. This point is discussed in Reference (3). In this source, however, the increase in shear stiffness is accounted for, not by increasing skin thickness, but by decreasing the panel width—an equivalent procedure.

Fig. A22.2 Idealization of the half panel by use of substitute stringers

Fig. A22.3 gives the geometry of the idealized panel.

\[
A_1 = 0.703 \text{ in}^2 \\
A_2 = 0.212 \text{ in}^2 \\
A_3 = 1.045 \text{ in}^2 \\
AR = 0.25 \text{ in}^2 \\
t_1 = 0.031 \text{ in} \\
t_2 = 0.031 \text{ in} \\
b_1 = 5.96 \text{ in} \\
b_2 = 7.56 \text{ in}
\]

SOLUTION:

Fig. A22.4 is an exploded view of the half-panel showing the placement and numbering of the internal generalized forces (Art. A7.3, Chapter A7) and the external loading. Note that the applied axial stresses were assumed to be constant chordwise, giving stringer loads proportional to the stringer areas; their sum is \( P_x \), one of external loads.

The applied edge shear flows, coming from the spar web, were assumed constant spanwise, as from a constant shear load. Other load distributions may be handled by allowing these applied shears to vary from panel to panel. For very extreme load variations additional transverse members could be inserted to create more spanwise panels allowing a better fit to the spar shear variation. The applied shear flows were considered as the other external load and designated \( P_y \).

Panels on the centerline have zero shear due to symmetry (Fig. A22.3).
Experience has shown that for symmetric panels symmetrically loaded it is satisfactory to consider transverse members to be rigid in their own planes (4). Thus, in this problem, member flexibilities for forces $q_{x}$, $q_{y}$, and $q_{z}$ may be taken to be zero. In the actual NACA test specimen with which results are to be compared, those transverse members bordering the cutout appeared to have been heavily reinforced (to an extent unknown to the writer). Hence it is logical to take their stiffnesses as great.

Member flexibility coefficients were collected in matrix form as below using the formulas of Chapter A7. Note that the coefficients for subscripts 5, 6, 7 and 8 were set equal to zero (rigid transverse members).

\[
\begin{align*}
& \begin{bmatrix}
T_{i} & r_{i}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
4.755 & 0.316 & -1.66 & -0.840
9.510 & 0.316 & -0.840 & -1.66
-19.76 & -1.316 & 1.66 & 0.840
-39.51 & -1.316 & 0.840 & 1.66
0 & 0.268 & 1.00 & 0
0 & 0 & 1.00 & 0
-12.09 & -0.268 & 0 & 1.00
0 & 0 & 0 & 1.00
-1.317 & -0.0450 & 0.112 & -0.056
-1.317 & 0 & -0.056 & 0.056
0.711 & -0.0450 & -0.056 & -0.112
0 & 0.0355 & 0 & 0
-1.569 & -0.0355 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
16.16 & 0.359 & 0 & 0
4.86 & 0.359 & 0 & 0
23.96 & 0.359 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
& \begin{bmatrix}
T_{i} & r_{i}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
4.755 & 0.316 & -1.66 & -0.840
9.510 & 0.316 & -0.840 & -1.66
-19.76 & -1.316 & 1.66 & 0.840
-39.51 & -1.316 & 0.840 & 1.66
0 & 0.268 & 1.00 & 0
0 & 0 & 1.00 & 0
-12.09 & -0.268 & 0 & 1.00
0 & 0 & 0 & 1.00
-1.317 & -0.0450 & 0.112 & -0.056
-1.317 & 0 & -0.056 & 0.056
0.711 & -0.0450 & -0.056 & -0.112
0 & 0.0355 & 0 & 0
-1.569 & -0.0355 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
16.16 & 0.359 & 0 & 0
4.86 & 0.359 & 0 & 0
23.96 & 0.359 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
& \begin{bmatrix}
T_{i} & r_{i}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
4.755 & 0.316 & -1.66 & -0.840
9.510 & 0.316 & -0.840 & -1.66
-19.76 & -1.316 & 1.66 & 0.840
-39.51 & -1.316 & 0.840 & 1.66
0 & 0.268 & 1.00 & 0
0 & 0 & 1.00 & 0
-12.09 & -0.268 & 0 & 1.00
0 & 0 & 0 & 1.00
-1.317 & -0.0450 & 0.112 & -0.056
-1.317 & 0 & -0.056 & 0.056
0.711 & -0.0450 & -0.056 & -0.112
0 & 0.0355 & 0 & 0
-1.569 & -0.0355 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
16.16 & 0.359 & 0 & 0
4.86 & 0.359 & 0 & 0
23.96 & 0.359 & 0 & 0
\end{bmatrix}
\end{align*}
\]

The "off diagonal" values have negative sign because the sense of those internal generalized forces having subscripts (14), (15), (17) and (18) was taken opposite to that used in the derivation in Art. A7.10. A change in sign requires a change in sign in off-diagonal coefficients only.)
The following matrices were formed:

For eq. (17) of Chapter A8:

\[
\begin{bmatrix}
-5206 & -262.8 \\
-6633 & -262.8
\end{bmatrix}
\]

For eq. (18) of Chapter A8:

\[
\begin{bmatrix}
392.1 & 293.5 \\
293.5 & 392.1
\end{bmatrix}
\]

The inverse of this last was found,

\[
\begin{bmatrix}
-4.343 & 5.302 \\
-5.302 & 4.343
\end{bmatrix}
\]

Finally, per eq. (23) of Chapter A8, the unit load stress distribution was,

\[
\begin{bmatrix}
m & 1 & 2 \\
1 & -10.93 & -65 \\
2 & -18.34 & -65 \\
3 & -4.08 & -351 \\
4 & -11.66 & -351 \\
5 & 1.40 & 115 \\
6 & 1.40 & 383 \\
7 & 3.79 & 115 \\
8 & 15.98 & 383 \\
9 & -0.270 & 0.019 \\
10 & -0.510 & 0 \\
11 & -1.15 & -0.219 \\
12 & 0 & 0.0355 \\
13 & -1.60 & -0.0355 \\
14 & 0 & 0.1084 \\
15 & 0 & 0.0383 \\
16 & 16.16 & 0.359 \\
17 & 4.96 & 0.108 \\
18 & 23.66 & 0.359
\end{bmatrix}
\]

Fig. A22.5 Comparison between calculated and measured stresses (psi) on the half-panel.

\[
\begin{bmatrix}
5 & 18.03 & 5.04 \\
6 & 5.04 & 10.08 \\
7 & 18.03 & 5.04 \\
8 & 5.04 & 10.08
\end{bmatrix}
\]

The problem was solved retracing the same steps as before, but using the modified matrix, to yield the stresses:

\[
\begin{bmatrix}
m & 1 & 2 \\
1 & -11.11 & -0.219 \\
2 & -16.63 & -0.219 \\
3 & -3.90 & -0.379 \\
4 & -13.17 & -0.379 \\
5 & 2.14 & 0.104 \\
6 & 2.14 & 0.372 \\
7 & 2.52 & 0.104 \\
8 & 14.61 & 0.372 \\
9 & -0.26 & -0.017 \\
10 & -0.62 & 0 \\
11 & -1.08 & -0.017 \\
12 & 0 & 0.036 \\
13 & -1.60 & -0.036 \\
14 & 0 & 0.359 \\
15 & 0 & 0.108 \\
16 & 0 & 0.036 \\
17 & 16.11 & 0.359 \\
18 & 4.96 & 0.108 \\
19 & 23.66 & 0.359
\end{bmatrix}
\]

The above analytical results are compared with NACA test data in Fig. A22.5 for the loading \( P_1 = 1 \), \( P_2 = 0 \). Agreement is seen to be good.

EFFECT OF RIB FLEXIBILITY:

To investigate the influence of rib flexibility, the problem was reworked assuming aluminum rib caps, of constant area \( A = 0.25 \text{ in}^2 \), as the transverse members bordering the cutout. The appropriate member flexibility coefficients were inserted in the \( A_{ij} \) matrix (in place of the zeroes used above) were
Comparison of this result with the previous one for the rigid ribs reveals that the most important effect of rib flexibility was to increase the concentration of stresses in the comong stringer bordering the cutout. It should be noted, however, that for this symmetric panel, the use of a very flexible rib as compared with a rigid rib led to stress increases of the order of only 10% in the combing stringer. Thus, the "rule of thumb" that transverse flexibilities may be neglected in symmetric panels is re-affirmed.

A22.3 Shear Lag Analysis of Box Beams

"The bending stresses in box beams do not always conform very closely to the predictions of the engineering theory of bending. The deviations from the theory are caused chiefly by the shear deformations in the cover of the box that constitutes the flange of the beam. The problem of analyzing these deviations from the engineering theory of bending has become known as the shear lag problem, a term that is convenient though not very descriptive." (3)

Fig. A22.6 illustrates the basic problem. The beam cover sheet is loaded along the edges by shear flows from the spar webs. These shear flows are resisted by axial forces developed in the longitudinal members (spar caps and stringers). According to elementary considerations, the stringer stresses should be uniform chordwise at any given beam station ("elementary theory" in the figure). Actually, the central stringers tend to "lag behind" the others in picking up the load because the intermediate sheet, which transfers the loads in from the edges, is not perfectly rigid in shear. The action may be comprehended readily by visualizing an extreme case: a large degree of "lag" would occur if the load transferring skin were made of a highly flexible material such as a plastic sheet or even rubber. In such a case the inside stringers would be out of action almost entirely! With the inside stringers lagging, the outside stringers and spar caps must carry an over-stress to maintain equilibrium ("actual" in the figure).

Fig. A22.7 shows the beam analyzed herein.

![Diagram of beam analysis]

The beam is an idealization of one tested by the NACA and reported in reference (3). Note that the beam has no lower cover sheet and that it is symmetric about a vertical axis. Transverse bulkheads are located at stations 12", 24", 36" and 48" from the root.

The actual beam specimen had three more stringers than shown in the idealized structure, these being located one each midway between the pairs of longitudinals shown on the beam cover of Fig. A22.7. In the idealizing process, these extra stringer areas were divided equally between adjacent longitudinals. The stringer areas shown are the effective areas, with those in the top cover tapering linearly from root to tip. All skin was considered effective in carrying direct stresses.

Some detailed discussions of the techniques of idealization of practical beams are given in references (5) and (5a).

SOLUTION:

To permit the handling of the calculation in a limited space, it was elected to analyze...
the structure for a single transverse (vertical) tip load symmetrically placed. In that case, because of symmetry, it was necessary to treat only one-half of the structure. In addition, no shear flows could appear in the middle panels. Further, it is known that the influence of rib flexibility on shear lag is slight for symmetric systems, so that the ribs were considered rigid in their own planes; hence no generalized forces were needed on the ribs to describe their strain energies.

Fig. A22.8 shows the placement and numbering of the generalized forces on the half-beam.

![Diagram of forces](image)

**Fig. A22.8 Choice of generalized forces for shear lag problem.**

Member flexibility coefficients were computed with the formulas of Chapter A7 and arranged in a matrix.

The shear flows \( q_x, q_y, q_{iz} \) and \( q_{iz} \) were selected as redundants. Setting these equal to zero, the stress distribution due to a one-half pound load at the tip (a unit load divided equally between beam halves) was readily computed.

\[
\begin{pmatrix}
.07692 \\
0 \\
-.9230 \\
.9230 \\
0 \\
.07692 \\
0 \\
-1.3461 \\
1.3461 \\
0 \\
.07692 \\
0 \\
-2.769 \\
2.769 \\
0 \\
.07692 \\
0 \\
-3.692 \\
3.692 \\
0
\end{pmatrix}
\]

Next, the unit redundant stress distribution was computed. Fig. A22.9 illustrates a typical calculation, showing the stresses in the tip bay for \( q_x = 1, q_y = q_{iz} = q_{iz} = 0 \).
The complete redundant stresses were:

\[
\begin{array}{c|cccc}
\mathbf{f} & 2 & 7 & 12 & 17 \\
\hline
1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 \\
4 & -12 & 0 & 0 & 0 \\
5 & 12 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 \\
7 & 0 & 1 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 \\
9 & -12 & -12 & 0 & 0 \\
10 & 12 & 12 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 1 & 0 \\
13 & 0 & 0 & 0 & 0 \\
14 & -12 & -12 & -12 & 0 \\
15 & 12 & 12 & 12 & 0 \\
16 & 0 & 0 & 0 & 0 \\
17 & 0 & 0 & 0 & 1 \\
18 & 0 & 0 & 0 & 0 \\
19 & -12 & -12 & -12 & -12 \\
20 & 12 & 12 & 12 & 12 \\
\end{array}
\]

\[\mathbf{t}_{ir} = \left( \begin{array}{cccc}
-2175 & -1848 & -1260 & -460.7 \\
40,210 & 16,810 & 9,536 & 3,041 \\
16,810 & 32,160 & 9,536 & 3,041 \\
9,536 & 9,536 & 25,030 & 3,041 \\
3,041 & 3,041 & 3,041 & 18,650 \\
\end{array} \right)\]

\[\mathbf{t}_{rs} = 10^{-6} \left( \begin{array}{cccc}
3.266 & -1.505 & -.656 & -.183 \\
-1.505 & 4.219 & -1.000 & -.279 \\
-.656 & -1.000 & 4.687 & -.496 \\
-.183 & -.279 & -.496 & 5.512 \\
\end{array} \right)\]

and, finally, per eq. (23) of Chapter A8,

\[\mathbf{G}_{im} = \left( \begin{array}{cccc}
.07692 & .035453 & - .9230 & .50856 \\
.4144 & .07692 & .03136 & -1.846 \\
1.055 & .72907 & .07692 & .02401 \\
-2.7691 & 1.690 & 1.079 & .07692 \\
-3.692 & 2.493 & 1.199 & \end{array} \right)\]

The following matrix products were formed (per eqs. (17), (18) of Chapter A8):

* This is an obvious place in which to use combinations of redundants to decrease the structural coupling (reference Chapter A8, pp. A8.29, 30). (Recommended as an exercise for the student.)

Fig. A22.10 shows the above computed stresses and those reported by the NACA as obtained by test. Agreement is seen to be quite good. 4.589

\[
\begin{array}{c}
\text{CALCULATED} \\
3.146 \quad 2.364 \quad 1.278 \\
\text{MEASURED} \\
5.0 \quad 2.0 \quad 1.1 \\
\end{array}
\]

Fig. A22.10 Comparison between calculated and measured stresses (psi) in a box beam.
A22.4 Stress Analysis of a Box Beam With a Cutout.

In Article A22.2, one technique was employed for computing the stresses around a cutout. In that analysis the effect of the cutout was assumed to have been localized about the cutout region; consequently, the problem was treated by isolating the affected panel. Quite often, when the cutout is placed well inboard on the wing, its influence on the root stresses is appreciable. Therefore, it is desirable to be able to consider the overall problem of the box beam with a cutout for such cases.

"The most convenient and the most rapid method of analyzing structures with cutouts is the indirect, or inverse, method. The analysis by the indirect method is made in two steps. First, the structure is analyzed for the basic condition that exists before the cutout is made. The results of this basic analysis are used to calculate the internal forces that exist along the boundary of the proposed cutout. External forces equal and opposite to these internal forces are then introduced; these external forces reduce the stresses to zero along the boundary of the proposed cutout, and consequently the cutout can now be made without disturbing the stresses." (3)

It is desired to modify the calculation of the previous article (A22.3, "Shear Lag Analysis") to allow for the presence of two cutouts symmetrically placed. Panel "q1s" was removed while the single tip load remained. Since the unit remained symmetric, the data from the previous analysis, in which the transverse rib stiffnesses were taken to be infinitely great, should still yield satisfactory results.

SOLUTION:

The calculation was accomplished in three steps:

1) The stress distribution was found in the "basic structure" (no cutout). This work was carried out in Article A22.3 where it was found that $q_{1s} \approx .02401$ lbs/inch.

2) A stress distribution was found for an "applied load" of $q_{1s} = 1$. Such a loading has zero external resultant.

3) .02401 times this last stress distribution was subtracted from the first.

The calculation for step (2) above, will now be carried out by considering a unit shear flow, $q_{1s} = 1$, applied while all other loads are zero. Under this loading condition the relative displacements at redundant cuts 2, 7 and 17 are equal to $\approx \frac{s_{1,12}}{s_{1,12}} \approx \frac{s_{1,18}}{s_{1,18}}$ and $\approx \frac{s_{1,17}}{s_{1,17}}$ respectively, where these coefficients have been computed previously in Article A22.3.

To restore continuity (reduce the relative displacements to zero) three redundant forces are applied, one each, at the cuts 2, 7 and 17. The appropriate equations specifying continuity are

$$
\begin{bmatrix}
\approx s_{2,2} & \approx s_{2,7} & \approx s_{2,17} \\
\approx s_{7,2} & \approx s_{7,7} & \approx s_{7,17} \\
\approx s_{17,2} & \approx s_{17,7} & \approx s_{17,17}
\end{bmatrix}
\begin{bmatrix}
q_2 \\
q_7 \\
q_{17}
\end{bmatrix}
= 
\begin{bmatrix}
\approx s_{1,12} \\
\approx s_{1,18} \\
\approx s_{1,17}
\end{bmatrix}
$$

Note that these equations say simply that the deflections at the redundant cuts due to the (unknown) redundant forces must be equal and opposite to the deflections due to $q_{1s} = 1$.

All coefficients in the above equation were computed in Art. A22.3. Specifically,

$$
\begin{bmatrix}
40,210 & 16,810 & 3,041 \\
16,810 & 32,150 & 3,041 \\
3,041 & 3,041 & 18,850
\end{bmatrix}
\begin{bmatrix}
q_2 \\
q_7 \\
q_{17}
\end{bmatrix}
= 
\begin{bmatrix}
9,536 \\
9,536 \\
3,041
\end{bmatrix}
$$

The above matrix is the $[\approx r_{s}]$ of Art. A22.3, with the "$q_{1s}$ row" and "$q_{1s}$ column" removed.

The equations were solved to give the values of the redundants for a unit applied load, $q_{1s} = 1$, as

$$
\begin{bmatrix}
q_2 \\
q_7 \\
q_{1s}
\end{bmatrix}
= 
\begin{bmatrix}
- .1399 \\
- .2134 \\
1.00
\end{bmatrix}
$$

The $q_{1s}$ force is included in the above for later convenience. Then the complete stress distribution, due to applying a unit shear flow $q_{1s} = 1$, was

* The procedure described here is quite generally useful for studies of the effect of removing one or more members; such might be required for an analysis of the effects of structural damage.

** See Appendix for a method of "extracting" the inverse of this matrix from that previously found for the complete $[\approx r_{s}]$ matrix.
\[
\begin{bmatrix}
G_{1,1} \\
0 \\
-1.399 \\
1.879 \\
-1.879 \\
0 \\
-2.134 \\
0 \\
4.240 \\
-4.240 \\
0 \\
1.00 \\
0 \\
-7.760 \\
7.760 \\
0 \\
-1.055 \\
0 \\
-6.494 \\
6.494
\end{bmatrix}
\]

\( \left[ \mathbf{G}_{1} \right] \) taken from Art. A22.3

To obtain the stresses in the loaded beam with the cutout, 0.02401 in. was the above stresses were subtracted from the "basic" stress distribution of Art. A22.3, as previously explained.

\[
\begin{bmatrix}
G_{1}^{m} \text{ CUTOUT} \\
-0.02401(G_{1,1})
\end{bmatrix} = \begin{bmatrix}
G_{1}^{m} \\
\text{BASIC}
\end{bmatrix}
\]

Note that \( q_{1,1} \) is now zero and the cutout panel may be "lifted out".

In the case of a structure under a variety of external loadings \( (m = 1, 2, 3 \ldots) \), the more general equation, corresponding to the above, is

\[
\begin{bmatrix}
G_{1}^{m} \text{ CUTOUT} \\
\text{BASIC}
\end{bmatrix} = \begin{bmatrix}
G_{1}^{m} \\
\text{BASIC}
\end{bmatrix} - G_{1,1} - \begin{bmatrix}
G_{1,1,1} \\
\text{m}
\end{bmatrix}
\]

where the row matrix \( G_{1,1,1} \) is simply the "12 row" of \( \left[ G_{1,1,1} \right] \) BASIC

**COMPARISON WITH TEST DATA:**

Reference 3 reports test data for the case analyzed, the stringer stresses being plotted below in Fig. A22.11.

In the actual test specimen a stringer passing through the cutout was severed, it having zero stress at stations 12 and 24, therefore. However, during the idealization process discussed in Art. A22.3 (for the beam without cutout) the area for this stringer was placed partly with the comber stringer and partly with the spar cap. In the same way, some sheet from the midportion of the panel, now made discontinuous by the cutout, was added also to the spar cap and combing stringer.

It follows then that the full idealized areas of the comber stringer and spar cap should not be used in figuring the stresses at stations 12 and 24 (produced by forces \( q_{1}, q_{1}, q_{1,1}, q_{1,2} \)). With these areas reduced by the appropriate subtractions, the stresses were computed and are plotted in Fig. A22.11. Agreement with test data is seen to be quite satisfactory.

![Fig. A22.11 Comparison between calculated and measured stresses (psi) in a box beam with cutouts.](image)

**A22.5 Analysis of a Swept Box Beam.**

"Experimental investigations of swept box beams have shown that the stresses and distortions in a swept wing can be appreciably different in character from those that would exist if the root were normal to the wing axis. The principle effect of swesback on the stresses occurs under bending loads and consists in a concentration of bending stress and vertical..."
shear in the rear spar near the fuselage. With regard to distortions, the effect of sweep is to produce some twist under loads that would produce only bending of an unswept wing and some bending under loads that would produce only twist of an unswept wing.\(^6\)

In the following example a swept box beam is analyzed by the matrix methods of Chapter A8 and, in particular, by the specific techniques of reference \((7)\). The method accounts for the interaction between the swept cover panels and the longitudinal members. It is this action that is responsible for the distinctive structural characteristics of the swept box beam.

Again, we emphasize that the method used here is strictly applicable to thin-skin fin wings of beam-like proportions only. Considering the wide variety of structural layouts which may be employed in swept wing configurations, a comprehensive treatment cannot be given here. An excellent review of methods better adapted to thick-skin construction and to "plate-like" (very thin, wide) wings, may be found in reference \((6)\). One method of analyzing such wings is given in Chapter A28.

THE STRUCTURE:

The structure shown is Fig. A22.12 is an idealization of the NACA test beam of references \((6)\) and \((9)\), in which a single substitute stringer has been employed along the cover sheet to allow for the anticipated shear lag effect. The figure shows only one-half of the complete unit, which was built symmetrically about the axis corresponding to the longitudinal axis of the airplane.

Only tip loads were to be applied (at points A and B). The outer section of the beam was assumed to carry stresses which could be calculated reasonably well by the engineering theory of bending (E.T.B.). For this purpose it was judged satisfactory to consider the outer 66" of the beam as a single bay (A-B-D). If loads were to have been applied inboard of the tip, it would have been necessary to consider additional bay divisions between A-B and C-D (that is, insert additional ribs at stations

![Fig. A22.12](image-url)

![Fig. A22.13](image-url)
of load application). Rib C-D was located at one of the actual rib locations in the NACA test specimen and was assumed rigid in its own plane.

The choice of bay C-D-F-E as a single bay was somewhat arbitrary. For improved accuracy, additional ribs inboard of C-D could have been used. Note that any ribs placed inboard of point F will produce triangular skin panels in the cover sheets. Examples of treatments for such panels may be found in references (9), (10) and (11).

Rib E-F was considered flexible in its own plane, it being known that the flexibility of a rib is important at a location where a structure changes direction. Note that this rib was made of steel in the test specimen.

Effective areas of longitudinals as shown in Fig. A22.12 were computed by considering all of the skin to be effective. The spar cap areas are equal to the sum of the areas of the angle member at the cap location, plus one-half of the effective area of material between the cap and the substitute stringer (this area includes several stringers as well as skin) plus one-sixth of the attached spar web area. The substitute stringer area was collected in like manner from the half-panels to either side.

The method used in calculating the effective areas of the rib caps (E-F) is given in detail in reference (9), from which the value used here was taken. The "carry-through bay" cover sheet thickness is equal to that used on the specimen (.050") plus a weighted increase to allow for the presence of splice plates along the plane of symmetry (see reference 9).

**INTERNAL GENERALIZED FORCES:**

Fig. A22.13 shows the choice and numbering of the generalized forces.

The beam was rigidly supported at points E, F and at the two corresponding points on the other beam half. These might correspond to the fuselage ring attach points in an airplane. The vertical end caps on rib E-F were considered rigid axially, so that no flexibility coefficients were associated with the reactions \( q_{ax} \) and \( q_{ax} \). Flexibility of these members affects total deflections only and can be omitted in a stress analyses where deflections are not sought.

Since only symmetric loadings were considered in this analysis no shear was transmitted by the carry-through bay and hence no shear flows were shown in that portion.

Sets of additional axial forces \( q_{ax} \) through \( q_{ax} \) were applied to the ends of the flanges and stringers adjacent to the obliquely cut ends of the cover sheet panels in bay C-D-F-E. These forces are necessary to account for the interaction between the swept covers and the longitudinals. As shown in Fig. A22.14, the pure shear flow on the oblique edge is obtained by superposing onto the panel a zero-resultant system consisting of a uniform tensile stress of intensity \( 2q \) plus a pair of concentrated balancing loads. The balancing loads must be contributed by the bordering longitudinals and hence react on these as tensile loads (Fig. A22.14c). The balancing loads applied to the stringers are shown dashed since they are internal forces within the bay and are not to be entered into the equilibrium equations for the structure.

![Fig. A22.14 Showing how the uniform shear stress on an oblique panel end (b) is created by superposition of a uniform tensile stress plus two balancing forces (a). The balancing forces react as tensile loads on the bordering longitudinals (c).](image)

From an energy viewpoint, these dashed forces account for the additional strain energy stored by the axial components of shear flows in the non-rectangular panels. This energy is stored in the cover panels themselves (and is accounted for in this manner since the longitudinals contain the effective area from the cover sheet***) and in the longitudinals which react against these components.

"Dashed loads" are applied to the longitudinals adjacent to any obliquely cut panel end. Similar dashed loads would be applied to

***This much of the energy could be accounted for in another fashion by modifying the member flexibility coefficients for the sheet panel. See Reference 12, where this was done. However, that reference incorrectly neglects the additional energy stored in the longitudinals, as was demonstrated in Reference 13.
the outboard ends of the panels in bay C-D-F-E if they too were cut obliquely. Such panel
configurations arise often in swept wing con-
struction having ribs parallel to the air-
stream. Formulae for more general quadrilateral
panels are given in Reference 7.

THE STRESS DISTRIBUTIONS

For the symmetric loadings considered
here the structure was indeterminate only two
times since the outer bay was assumed to be
determinate by the E.T.B.

Stresses in outer bay by E.T.B.:

Flange stresses at rib C-D (for both
P_f = 1 and P_e = 1)

\[ \begin{align*}
M &= 66'' \\
I &= 98.57 \text{ in.}^4; C = 3.5'' \\
f_b &= \frac{MC}{I} = 2.608 \text{ psi}
\end{align*} \]

Therefore,

\[ \begin{align*}
q_a &= q_e = 2.608 \times 1.121 = 2.924 \text{ lbs.} \\
q_a &= 2.608 \times 1.373 = 3.581 \text{ lbs.}
\end{align*} \]

For a unit transverse load at the shear
center (midpoint, because of symmetry)

\[ \begin{align*}
q_a &= \frac{3.581}{2 \times 66} = 0.02713 \text{ lbs/in} \\
q_b &= \frac{2.924}{66} = 0.07143 \text{ lbs/in}
\end{align*} \]

The unit load was shifted 15" to either side
by application of a torque, T = 15 in-lbs.
The uniform shear superposed was

\[ q = \frac{T}{2A} = \frac{15}{2 \times 210} = 0.03571 \text{ lbs/in} \]

Finally, superposition gave the stresses
for the outer bay as

\[ \begin{bmatrix}
q_{a1} \\
q_{a2} \\
q_{b1} \\
q_{b2}
\end{bmatrix} =
\begin{bmatrix}
1.071 \\
0.06284 \\
0.05858 \\
0.05720 \\
2.924 \\
3.581 \\
2.924
\end{bmatrix}
\]

Stresses in inner bays:

According to the discussion of Art. A8.12,
Chapter A8, the determinate stress distribution,
\[ [\sigma_{1m}] \], may be any stress distribution in equili-

brum with the applied loads, and preferably
one close to the final true stress distribution.
The magnitude of the redundant forces is re-
duced by use of a satisfactory estimate of the
true stresses.

The stresses in the two inner bays were
determined for both \( \sigma_{1m} \) and \( \sigma_{2m} \) simultaneously.
Since this inner portion of the structure is
two times indeterminate we can estimate two
loads. For this purpose the two flange loads
\( q_{a1} \) and \( q_{a2} \) were written as

\[ q_{1m} = q'_{1m} + q_2, \]

\[ q_{2m} = q'_{2m} + q_1, \]

where the (single) primed values are approxi-

mate values determined by the E.T.B. and the
double primed values are the unknown corrections
(the redundants). Using \( My/I \) at stations 66" and
118" from the tip gave

\[ q'_{1m} = 3.228 P_f + 5.223 P_e \]

\[ q'_{2m} = 3.899 P_f + 3.399 P_e \]

The equilibrium equations for the elements
of the structure were written next by summing
forces and moments.

Joint F

\[ q_{af} + q_{ae} = 0 \]

\[ q_{af} + q_{ae} = 1.414 q_{as} = 5.513 P_f + 5.513 P_e + 1.414 q_{as} \]

Joint E

\[ q_{ae} + q_{ae} = 0 \]

\[ q_{ae} + q_{ae} = 1.414 q_{as} = 7.392 P_f + 7.392 P_e + 1.414 q_{as} \]

EM about F-E

\[ q_{af} + q_{ae} + q_{ae} = 11.918 P_f + 8.808 P_e \]

* This is a rather crude way to estimate these loads and is
used here only for simplicity. The analyst is generally better
advised to exercise a little more ingenuity in making these
estimates, even to the extent of being guided by other swept
wing solutions.
Shear Flows around the non-rectangular panels. (Check by summing moments about E, G and F.)

\[
\begin{align*}
\text{Cap EC} & \quad q_s - 0.7115 q_e = 0.04431 P_1 + 0.04431 P_a + 0.01923 q''_s \\
\text{Rib Vertical at E} & \quad q_s - q_{s1} = -0.1403 P - 0.211 P_a \\
\text{Cap BC} & \quad -21.2 q_s - 21.2 q_{s2} - q_{s0} + q_{s1} = 0 \\
\text{Joint G} & \quad 0.707 q_{s2} - q_{s0} = 0 \\
& \quad 0.707 q_{s3} - q_{s0} + q_{s1} = 0 \\
\text{Cap DF} & \quad q_{s0} + q_{s1} + 0.04717 (q_{s2} - q_{s3}) = 0 \\
\text{Rib Vertical at F} & \quad q_{s2} + q_{s3} = 0.2831 P_1 + 0.3540 P_a \\
\text{Cap DF} & \quad q_{s1} + 1.682 q_{s0} = 0.04432 (P_1 + P_a) + 0.04545 q''_s \\

\text{The first five of the above equations were readily solved by substitution, yielding:} \\
q_{s2} = q_{s0} = 2.756 P + 2.756 P_a + 0.707 q''_s \\
-q_{s0} = q_{s2} = 3.696 P + 3.696 P_a + 0.707 q''_s \\
q_{s1} = 5.466 P + 2.436 P_a - 0.707 q''_s \\
-q_{s3} = -0.707 q''_s \\
\text{Also, from equilibrium of joint G:} \\
q_{s3} = 7.729 P_a + 3.444 P_a - q''_s - q''_s \\
\text{The remaining seven equations were arranged for solution thus:}
\end{align*}
\]

\[
\begin{align*}
1 & \quad 0 & 0 & 0 & -1 & 0 & 0 & \quad \begin{pmatrix} q_s \\ q_s \\ \vdots \\ q_{s2} \\ q_{s3} \\ \vdots \\ q_{s8} \end{pmatrix} \\
1 & \quad -0.7115 & 0 & 0 & 0 & 0 & 0 & \quad \begin{pmatrix} q_s \\ q_s \\ \vdots \\ q_{s2} \\ q_{s3} \\ \vdots \\ q_{s8} \end{pmatrix} \\
0 & \quad 0 & 21.2 & 0 & 21.2 & 0 & 1 & \quad \begin{pmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{s3} \\ q_{s4} \\ \vdots \\ q_{s8} \end{pmatrix} \\
0 & \quad 0 & 1.682 & 1 & 0 & 0 & 0 & \quad \begin{pmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{s3} \\ q_{s4} \\ \vdots \\ q_{s8} \end{pmatrix} \\
0 & \quad 0 & 0 & 1 & 0 & 0 & 0 & \quad \begin{pmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{s3} \\ q_{s4} \\ \vdots \\ q_{s8} \end{pmatrix} \\
0 & \quad -21.2 & 0 & 0 & -21.2 & 1 & 0 & \quad \begin{pmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{s3} \\ q_{s4} \\ \vdots \\ q_{s8} \end{pmatrix} \\
0 & \quad 0 & 0 & 0 & 0 & -1 & 1 & \quad \begin{pmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{s3} \\ q_{s4} \\ \vdots \\ q_{s8} \end{pmatrix} \\

= \quad \begin{pmatrix} -0.1403 & -0.2111 & 0 & 0 \\ 0.04431 & 0.04431 & 0.01923 & 0 \\ 2.756 & 2.756 & 0 & 0.707 \\ 0.04432 & 0.04432 & 0 & 0.04545 \\ 0.2831 & 0.3540 & 0 & 0 \\ -3.696 & -3.696 & -0.707 & 0 \\ -5.466 & -2.436 & 0.707 & 0.707 \end{pmatrix} \quad \begin{pmatrix} P_1 \\ P_a \\ q_{s1} \\ q_{s3} \end{pmatrix}
\end{align*}
\]

After inverting the matrix of coefficients on the left hand side of this last equation, and multiplying through thereby, the stresses were obtained as

\[
\begin{align*}
\begin{pmatrix} q_s \\ q_{s0} \\ q_{s1} \\ q_{s2} \\ q_{s3} \\
q_{s1} \\ q_{s2} \\ q_{s3} \\ q_{s4} \\ q_{s5} \\
q_{s3} \\ q_{s4} \\ q_{s5} \\ q_{s6} \\ q_{s7} \end{pmatrix} & \quad \begin{pmatrix} 0.1006 & 0.0295 & 0.0076 & -0.0076 \\ 0.0791 & -0.0208 & -0.0175 & -0.00950 \\ 0.0013 & -0.0411 & 0.00401 & 0.0230 \\ 0.0422 & 0.1135 & -0.00675 & 0.00675 \\ 0.2409 & 0.2409 & 0.00675 & -0.00675 \\ 3.087 & 0.964 & -0.9355 & -0.3444 \\ -2.379 & -1.473 & -2.284 & 0.3625 \end{pmatrix} \quad \begin{pmatrix} P_1 \\ P_a \end{pmatrix}
\end{align*}
\]

The complete determinate and unir-
redundant stress distributions, using the re-
sults up to this point, were therefore:
### DASHED LOAD CALCULATIONS:

In the above matrix, loads \( q_{1z} \), \( q_{1y} \), \( q_{1r} \), and \( q_{ir} \) were obtained from \( q_s \) and \( q_{ar} \) following formulas given in reference 7 for general quadrilateral panels. The equations applicable to a parallel-sided panel are:

\[
\begin{align*}
P_C &= c - \frac{d}{2} q_L x 2 \\
P_D &= c - \frac{d}{2} q_L x 2
\end{align*}
\]

Balancing Load

(= Dashed Load)

For the panel CBBH; \( c = 52.3" \), \( d = 73.5" \), \( c + d = 125.8" \), \( L = 15" \). Hence

\[
\begin{bmatrix}
q_{1z}
q_{1y}
q_{1r}
q_{ir}
\end{bmatrix}
= 
\begin{bmatrix}
P_C
P_D
\end{bmatrix}
= 
\begin{bmatrix}
0.1071 & 0.03572 & 0 & 0 \\
0.06284 & -0.00858 & 0 & 0 \\
0.00858 & -0.06284 & 0 & 0 \\
0.03572 & 0.1071 & 0 & 0 \\
0.222 & 0.222 & 0 & 0 \\
3.581 & 3.581 & 0 & 0 \\
2.324 & 2.324 & 0 & 0 \\
1.006 & 0.0295 & 0.00676 & -0.00676 \\
0.0751 & -0.0208 & -0.01753 & -0.00950 \\
0.0013 & -0.0411 & 0.00401 & 0.0230 \\
0.0422 & 0.1135 & 0.00675 & 0.00675 \\
5.228 & 5.228 & 1.0 & 0 \\
7.729 & 3.444 & -1.0 & -1.0 \\
8.399 & 3.899 & 0 & 1.0 \\
9.289 & 2.850 & 0 & 0 \\
-1.384 & 0.360 & 0.3668 & 1.662 \\
0.146 & 0.460 & -0.0449 & -0.2576 \\
-0.044 & 0.773 & -0.0754 & -0.4524 \\
2409 & 2409 & 0.00676 & 0.00676 \\
-3.696 & -3.696 & -0.707 & 0 \\
3.087 & 0.964 & 0.3355 & -0.3355 \\
-2.379 & -1.473 & -0.2284 & 0.3285 \\
2.756 & 2.756 & 0 & 0.707 \\
3.696 & 3.696 & 0 & 0.707 \\
5.466 & 2.436 & -0.707 & -0.707 \\
2.756 & 2.756 & 0 & 0.707
\end{bmatrix}
\]

Finally, the true stress distribution was found as,

\[
\begin{bmatrix}
\sigma_{rs} \\
\sigma_{rs}^{-1}
\end{bmatrix} =
\begin{bmatrix}
17.70 & 250.2 \\
-173.7 & 50.43
\end{bmatrix}
\begin{bmatrix}
114.5 & 42.96 \\
42.96 & 88.54
\end{bmatrix}
\begin{bmatrix}
0.01062 & -0.005563 \\
-0.005563 & 0.01493
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{rs} \\
\sigma_{rs}^{-1}
\end{bmatrix} =
\begin{bmatrix}
0.01062 & -0.005563 \\
-0.005563 & 0.01493
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{rs} \\
\sigma_{rs}^{-1}
\end{bmatrix} =
\begin{bmatrix}
0.01062 & -0.005563 \\
-0.005563 & 0.01493
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{rs} \\
\sigma_{rs}^{-1}
\end{bmatrix} =
\begin{bmatrix}
0.01062 & -0.005563 \\
-0.005563 & 0.01493
\end{bmatrix}
\]

Finally, the true stress distribution was found as,
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

MATRIX OF MEMBER FLEXIBILITY COEFFICIENT \( \phi_{ij} \) (\( E = 1 \)) (Formulas from Chapter 27 and Reference 7)

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  | 25  | 26  |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 | 15.400 | 102.990 | 19.000 |

COMPARISON WITH TEST DATA:

Fig. A22.15. Comparison Between Calculated and Measured Stresses (psi) in a Swept Box Beam.

The stresses shown are for a unit tip load, centrally placed (\( P_t = P_a = 1/2 \) lb.).

The stresses in the leading edge spar between stations 65 and 118 cannot duplicate the experimental variation since only a single bay was employed in this region in the idealization. The fact that the calculated root stresses run consistently above the test values is difficult to explain. Inasmuch as the calculated stresses satisfy equilibrium, the test values, all being lower, would seem to defy this fundamental requirement. More details concerning the testing techniques and method of data presentation would probably resolve this conflict. Both test and calculated values clearly exhibit the characteristic build-up of stresses in the rear spar of a swept wing.

RIB E-G-F RIGID:

As a matter of interest, it was decided to investigate the effect upon stresses when rib E-G-F is taken to be rigid. Such a calculation is readily achieved by putting the member flexibility coefficients for the rib equal to zero.

Thus, in the matrix those coefficients with subscripts 19, 20, 21, 22 and 23 were set equal to zero and the complete calculation was repeated.

The results, for spar cap loads at the wing root (a most sensitive point), were:

\[
\begin{bmatrix}
G_{1m}^{11} & 12 & 3.824 & 1.792 \\
14 & 6.983 & 6.810
\end{bmatrix}
\]

A comparison with the flexible rib calculations follows:

<table>
<thead>
<tr>
<th>LOADING</th>
<th>SPAR CAP FORCE</th>
<th>FLEX. RIB</th>
<th>RIGID RIB</th>
<th>% DIFF.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple</td>
<td>( q_{1x} )</td>
<td>6.311</td>
<td>5.616</td>
<td>11</td>
</tr>
<tr>
<td>Bending; ( P_t = P_a = 1 )</td>
<td>( q_{1x} )</td>
<td>12.61</td>
<td>13.60</td>
<td>8</td>
</tr>
<tr>
<td>Torsion; ( P_t = P_a = 1 )</td>
<td>( q_{1x} )</td>
<td>1.829</td>
<td>2.032</td>
<td>11</td>
</tr>
</tbody>
</table>

1.237
Considering that rib 26F was relatively rigid to begin with - being made of heavy gage steel - it may be seen that neglect of the flexibility of a corresponding all-aluminum rib could lead to serious errors.

REFERENCES


(2) Rosecrans, R., A Method for Calculating Stresses in Torsion-Box Covers with Cut-Outs, NACA TN 2280, Feb. 1951

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(6) Zender, G., and Libove, C., Stress and Distortion Measurements in a 45° Swept Box Beam Subjected to Bending and Torsion, NACA TN 1525


(9) Heldenfels, R., Zender, G., and Libove, C., Stress and Distortion Analysis of a Swept Box Beam Having Bulkheads Perpendicular to the Spars, NACA TN 2262.


CHAPTER A23
STRUCTURAL ANALYSIS OF A DELTA WING
ANALYSIS BY THE "METHOD OF DISPLACEMENTS"
ALFRED F. SCHMITT

A23.1 Introduction

The purpose of this chapter is two-fold: first, to present a discussion and an example of techniques for handling the structural analysis of highly redundant low-aspect ratio wings (typified by the delta wing) and second, to illustrate the "Displacement Method" of structural analysis (1), a method fundamentally different from those of Least Work or Dummy-Unit Loads.

The very low aspect ratio, thin wing is a structural configuration of relatively recent origin. It is a built-up structure of ribs, spars and a cover sheet but yet is so thin and so highly redundant that its structural characteristics are actually closer to those of a tapered flat plate than to a beam. In a beam the primary stresses are longitudinal; indeed, one of the basic assumptions of beam theory is that transverse stresses are negligible and that cross sections remain undeformed. For the very low aspect ratio wing however, chordwise stresses and deformations are of great importance. The degree of redundancy of these low aspect ratio wings is very high because of the multiple paths by which a load may be carried over the gridwork of spars and ribs.

In the wing beam problem examples of Chapters A8 and A22 redundant internal loads were determined by use of the Method of Dummy-Unit Loads - a variant of the Least Work theorem. While the method is perfectly general, it becomes increasingly difficult to obtain satisfactory accuracy as the degree of redundancy of the system increases. To some extent, accuracy may be retained by skillful choice of redundants and through the use of carefully chosen determinate stress distributions (see Art. A8-12, Chapter A8). A high degree of engineering judgment must be exercised, however. Even so, it has been found difficult to apply successfully these Least Work methods to the low aspect ratio multi-spar wing.

Several authors (2), (3), (4) have presented methods of analysis for the highly redundant wing based upon use of the "Displacement Method" of analysis. These references differ primarily in the techniques advocated for the breakdown and idealization of the structure into primary elements. In the following example, a method following that proposed by Levy (2), is applied to an idealized delta wing structure. The choice of this method for presentation is primarily for pedagogical reasons - it being the least detailed and consequently the easiest to grasp conceptually. The reader who is interested in actual application is recommended to Reference (4) for techniques which are probably better able to handle practical problems because of their greater generality, flexibility and growth potential.

A23.2 Basis of the "Method of Displacements"

The Method of Displacements draws its name from the fact that displacements rather than forces are dealt with as the independent variables. In earlier chapters the relation between structural deflections and applied loads was written through the use of the flexibility influence coefficients as (see Eqs. 24, 26, Chapter A7)

\[ \{ \delta_m \} = [A^{mn}] \{ F_n \} \]

(1)

where \( \delta_m \) is the deflection of point m and \( F_n \) is the external load at point n. If one forms the inverse of \([A^{mn}]\), this equation may be rewritten as

\[ \{ F_n \} = [A_{mn}]^{-1} \{ \delta_m \} = \left[ K_{mn} \right] \{ \delta_m \} \]

(2)

where \([K_{mn}]\) is called the stiffness matrix.

While the flexibility influence coefficients give the deflections per unit load, the stiffness matrix gives the loads per unit deflection. Thus, any one column (say the \( m^{th} \)) of \([K_{mn}]\) gives the values of the forces (reaction) developed at all numbered points of the structure when the corresponding point (point n) is deflected a unit amount, all other points held fixed.

Consider, for illustration, the grid-like structure of Fig. A23.1 into which a delta wing structure has been idealized. Note that the torsional stiffness of the shear-carrying cover sheet has been accounted for by the presence of
A23.2 STRUCTURAL ANALYSIS OF A DELTA WING

A quadrilateral "torque boxes" connecting points of intersection of the spars and ribs. This

Fig. A23.1 Idealized Delta Wing Structure

idealization is discussed in more detail in the example. Assume that the points of intersection have been numbered as in Fig. A23.2.

Fig. A23.2 Reactions Developed at a Net of Points when a Single Point is Displaced.

If one point, such as point S, is displaced a unit amount, the other points remaining fixed, reactions are developed at these various points as shown. These reactions may include moments as well as forces, should rotations have been taken as pertinent displacements in addition to the translations (see points S and 7 of Fig. A23.2). The values of reactions so developed would form the 6th column of \([K_{rr}]\) and the 6th row, since \([K_{rr}]\) is always symmetric.

Examination of Fig. A23.2 shows that the resistance to the displacement of a single point is the cumulative resistance of all members meeting at that point.* Thus, the stiffnesses are additive. If the stiffness matrices

* The resistance of a torque box having a corner at point 6 of Fig. A23.2 (say box 6-7-8-9) is a result of the tendency for the box to keep all four corners in the same plane. Obviously three-cornered "boxes" have no such resistance; hence none were shown in Fig. A23.1.

are written for the individual elements of the structure, they are simply added to give the stiffness matrix of the composite. Finally, the resulting stiffness matrix may be inverted to yield the matrix of flexibility influence coefficients (Eq. 2 "in reverse").

We note here, as an aside, that the Moment Distribution Method of Chapter A11, the Slope Deflection Method of Chapter A12 and the Method of Successive Corrections of Chapters A6 and A15 are all examples of the "Displacement Method" of structural analysis. In each of these methods the displacements are taken as the independent variables and these are adjusted to achieve equilibrium of the loaded structure. The "adjustment" may appear as a systematic relaxation of artificial constraints (Moment Distribution and Successive Corrections) or it may be done mathematically in one stroke by the solution of a set of simultaneous equations (Slope Deflection Method). The latter approach - solution of a set of simultaneous equations - is essentially that followed herein, the "solution" being effected by matrix inversion.

It is to be said of the Method of Displacements that it is complementary to the Least Work method in that it is better suited to the handling of the highly redundant problem, while Least Work is better suited to problems of few redundancies.

A23.3 A Delta Wing Example Problem

The idealized structure of Fig. A23.1 will be analyzed to determine

(a) the influence coefficients

(b) the internal stresses as a function of the applied loads.

The grid points are numbered as in Fig. A23.2. Note that the numbers increase to the rear and outboard.

IDEALIZATION:

In the structure under consideration member 1-2-3-4 lies on the airplane centerline. The bending stiffness of this member includes that of the half-fuselage plus one-half of the "carry-through structure." More detailed representations of the structure in this region are generally desirable, the oversimplified model used here being employed to limit the amount of data to be handled. Some techniques which may be applied in idealizing the structure in this region are given in Reference (6).

Idealization, particularly with regard to effective skin areas, has been discussed in some detail by Levy (2). The complexities of
this phase of the problem are too great to permit an expensive treatment here. Briefly, Levy's recommendations are:

(a) all skin may be considered effective between spanwise spars when computing the cap areas of such spars. This assumption is subject to modification, of course, if spanwise stresses are anticipated which will buckle the skin.

(b) for streamwise ribs an effective width of 0.362L, where L is the rib length, may be taken as acting with each rib cap (Fig. A23.3a).

(c) for the leading edge spar an effective width of 0.18L of the spar between spanwise spars is taken as acting with the cap of each such spar segment (Fig. A23.3b)

---

Beam Element Properties

(Moments of inertia, in [inches']^2, are assumed to vary linearly between numbered points)

<table>
<thead>
<tr>
<th>1/2 Fuselage Beam (1-2-3-4)</th>
<th>Spar 2-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 @ 48&quot;</td>
<td>48&quot;</td>
</tr>
<tr>
<td>I = 2000</td>
<td>2700</td>
</tr>
<tr>
<td>3400</td>
<td>2200</td>
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<tr>
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<td>18.25</td>
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<table>
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<tr>
<th>Rib 3-6-7</th>
<th>Spar 3-6-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>48&quot;</td>
<td>48&quot;</td>
</tr>
<tr>
<td>48&quot;</td>
<td>48&quot;</td>
</tr>
<tr>
<td>I = 9.24</td>
<td>20.79</td>
</tr>
<tr>
<td>3.59</td>
<td>47.23</td>
</tr>
<tr>
<td>40.82</td>
<td>12.33</td>
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</tbody>
</table>

<table>
<thead>
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<th>Rib 3-9</th>
<th>Spar 4-7-9-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>48&quot;</td>
<td>3 @ 48&quot;</td>
</tr>
<tr>
<td>I = 6.23</td>
<td>2.87</td>
</tr>
<tr>
<td>5.09</td>
<td>4.78</td>
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<tr>
<td>3.82</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Leading Edge Spar (1-6-8-10)

<table>
<thead>
<tr>
<th>3 @ 67.7&quot;</th>
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</thead>
<tbody>
<tr>
<td>I = 16.02</td>
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<tr>
<td>8.30</td>
</tr>
</tbody>
</table>

---

SIGN CONVENTION AND NOTATION:

The sign convention and notation adopted in conjunction with the grid numbering scheme of Fig. A23.2 is as follows:

Forces:

\[ \begin{align*}
P_m & \quad \text{transverse force at joint } m, \ + \text{ UP} \\
M_m & \quad \text{moment at joint } m \text{ acting about a pitching axis, } + \text{ NOSE UP} \\
N_m & \quad \text{moment at joint } m \text{ acting about a rolling axis, } + \text{ RIGHT WING UP} \\
Q_m & \quad \text{moment at joint } m \text{ acting in the plane of an oblique beam member (a member neither parallel nor normal to the streamwise direction), } + \text{ CLOCKWISE WHEN THE MEMBER IS VIEWED WITH ITS JOINT NUMBERS INCREASING LEFT TO RIGHT.}
\end{align*} \]
Displacements:

\[ A_m \] - transverse displacement of joint \( m \), + UP
\[ \theta_m \] - rotation of joint \( m \) about a pitching axis, + NOSE UP
\[ \phi_m \] - rotation of joint \( m \) about a rolling axis, + RIGHT WING UP
\[ T_m \] - rotation of joint \( m \) in the plane of an oblique member, + IN DIRECTION OF \( +Q_m \) (see page A23.3)

For a special purpose it will be convenient later to introduce another set of displacements to be called "relative displacements." Where any confusion can exist, the displacements defined above, which may be visualized as being measured with respect to a set of reference axes fixed in space, will be referred to as "absolute displacements."

With the above sign convention, any beam element (spar, rib or oblique member) which is viewed so that its joint numbers increase from left to right, will have positive forces and displacements taken in the same sense as every other member. This point is illustrated in Fig. A23.4.

![Diagram](image)

**Fig. A23.4 Illustrating the Homogeneity of the Sign Convention**

**CHOICE OF PERTINENT FORCES AND DISPLACEMENTS:**

Before beginning the analysis proper, the analyst must decide upon which forces and displacements are to be considered pertinent. One will, of necessity, have to consider transverse forces (and translatory displacements) at all net points (ten points in the example). In addition, rolling moments, \( N \), (and rotations \( \phi \)) must be considered along the airplane centerline on all spars carrying across the centerline. Wherever three or more beam-like members intersect (e.g. points 5 and 8 of Fig. A23.2), their bending stiffnesses will react against each other and hence couples in two planes \( (M, N) \) must be considered at these points. *

Wherever additional knowledge about the effects of various loadings are required, corresponding forces or couples should be sized. Thus, the influence of additional forces at points intermediate to the grid intersections, may be accounted for by the addition of appropriate extra forces along beam spans. Couples may be applied at any points where deflection slopes are required, e.g., the streamline slope at the trailing edge might be needed in an aero-elastic analysis, in which case couples \( M_x, M_y, M_z \) and \( M_{Q_x} \) (cf. Fig. A23.2) would be employed. The effect of couples from auxiliary aerodynamic surfaces and/or actuators may be desired, in which case appropriate additions may be made.

In the present example, because of space limitations, only the minimum number of forces and corresponding displacements are considered, viz:

- \( P_4 \) through \( P_{14} \) - forces on net points
- \( N_4 \) through \( N_6 \) - rolling couples on through-spars at the airplane centerline
- \( M_4, N_4, M_5, N_5, M_6, N_6 \) - pitching and rolling couples at points of intersection of three beam elements **

**A23.4 Calculation of Element Stiffness Matrices**

The task of computing the stiffness matrix for any one element of a configuration is a relatively straightforward structural problem. This problem may be either a statically determinate or a redundant one depending upon the geometry of the element and upon the number of pertinent forces and displacements associated with it. If it is a redundant problem, it is a small one by comparison with the overall structure of which it is an element. One might say that a feature of the Displacement Method of Analysis is that it "makes (many) little ones out of big ones!"

For instance, looking at Spar 2-5 of the present problem (Fig. A23.5a), we see that there are four forces (reactions) acting on it...

* The implication here is that beam torsional stiffnesses are not considered in such interactions. This assumption is probably quite satisfactory in general, bending stiffnesses being much greater than torsional stiffnesses for most beam elements. On occasion, a beam will have considerable torsional stiffness and it will then be necessary to account for it. The leading edge spar might be such a beam if it replaces the "D" nose section of the wing in the idealization.

** Couples \( Q_4, Q_5 \) and \( Q_6 \), in the plane of the oblique Leading Edge Spar, are included hereby, since any such couples may be resolved into \( M_x, N_x, M_y, N_y \) and \( M_z, N_z \) components. See "Transformation of Matrices for Oblique Beams", below.
element stiffness matrices involves first finding their flexibility influence coefficients and then inverting these matrices. By way of illustration, the calculation of the influence coefficients is carried out in detail below for spar 4-7-9-10. (While the calculation is effected here by the Least Work matrix method, any of numerous other methods (Moment Area, Elastic Weights, etc.) may be employed at this stage, should the analyst see fit.)

Fig. A23.4b shows the forces applied to spar 4-7-9-10, corresponding to those considered pertinent for the wing (see above).

For the immediate calculation of influence coefficients, forces \( P_x \) and \( P_z \) will be considered as fixed end reactions: deflections thus computed will be measured relative to the beam element ends. To distinguish the deflections thus obtained they will be referred to as relative deflections and will be denoted by the lower case Greek letters \( \delta \) and \( \phi \).

Fig. A23.7 shows the internal generalized forces for this spar.

\[
\begin{pmatrix}
\frac{q_1}{4} \\
\frac{q_2}{7} \\
\frac{q_3}{9} \\
q_4
\end{pmatrix}
\]

**Fig. A23.7 Internal Generalized Forces Used in Influence Coefficient Calculation for Spar 4-7-9-10**

Member flexibility coefficients for the tapered beam segments are computed next using the formulas of Art. A7.10 of Chapter A7. Collected in matrix form these become (\( S = 1 \))

\[
\begin{pmatrix}
3.213 & 1.623 & 0 & 0 \\
1.623 & 6.841 & 1.864 & 0 \\
0 & 1.864 & 8.55 & 4.923 \\
0 & 0 & 4.923 & 21.54
\end{pmatrix}
\]

Next, unit values of the loads \( N_x \), \( P_x \), and \( P_z \) are applied successively to obtain the unit stress distribution:

\[
\begin{pmatrix}
m \\
1 \\
2 \\
3 \\
4
\end{pmatrix}
\begin{pmatrix}
N_x \\
P_x \\
P_z
\end{pmatrix}
\]

Finally, the following matrix triple product is formed to give the influence coefficients (cf. Eq. 24, Chapter A7).
\[
\begin{bmatrix}
\Delta_{mn}
\end{bmatrix} = \begin{bmatrix}
\Theta_1
\end{bmatrix} \begin{bmatrix}
\alpha_{1}\beta
\end{bmatrix} \begin{bmatrix}
\phi_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
10.20 & -141.6 & -119.9 \\
-141.6 & 2776 & 2566 \\
-119.9 & 2566 & 3104
\end{bmatrix}
\]

or, written out as an equation;

For Spar 4-7-9-10

\[
\begin{bmatrix}
\phi \\
\Theta \\
\Theta_s
\end{bmatrix} = \begin{bmatrix}
10.20 & -141.6 & -119.9 \\
-141.6 & 2776 & 2566 \\
-119.9 & 2566 & 3104
\end{bmatrix} \begin{bmatrix}
N_s \\
P_r \\
P_s
\end{bmatrix}
\]

Note again that for this beam element the flexibility influence coefficients just computed are for relative deflections (note symbols used). The end transverse deflections \((\Delta^e, \Delta^s)\) and the end transverse reactions \((P^e, P^s)\) do not appear in these results.

**Beam Element Stiffness Matrices (Relative Deflections)**

The next step involves finding the beam element stiffness matrices for relative deflections by inverting the above influence coefficient matrix. The results after inverting will be of the form

\[
\begin{bmatrix}
N \\
\Theta \\
\Theta_s
\end{bmatrix} = \begin{bmatrix}
\kappa
\end{bmatrix} \begin{bmatrix}
\phi \\
\Theta \\
\Theta_s
\end{bmatrix}
\]

where \(\begin{bmatrix}
\kappa
\end{bmatrix}\) may be called the "relative stiffness matrix". Specifically, for Spar 4-7-9-10

\[
\begin{bmatrix}
N_s \\
P_r \\
P_s
\end{bmatrix} = \begin{bmatrix}
.3554 & .02303 & -.005306 \\
.02303 & .005313 & -.001600 \\
-.005306 & -.001600 & .001438
\end{bmatrix} \begin{bmatrix}
\phi \\
\Theta \\
\Theta_s
\end{bmatrix}
\]

* We note that the influence coefficient matrix to be inverted does not appear to be "well conditioned" in the sense of Art. A8.12 (see p. A8.29). The situation is more apparent than real, however, and arises because of a difference in units among the coefficients of Eq. (3). Thus, the appearance is readily altered by "scaling" (factorizing out appropriate constants). For instance, we may write

\[
\begin{bmatrix}
\Theta_s / \alpha \\
\Theta / \beta \\
\Theta_s / \beta_s
\end{bmatrix} = \begin{bmatrix}
10.20 & -141.6 & -119.9 \\
-141.6 & 2776 & 2566 \\
-119.9 & 2566 & 3104
\end{bmatrix} \begin{bmatrix}
10N_s \\
10P_r \\
10P_s
\end{bmatrix}
\]

Scaling in this fashion does not actually enhance the condition of the matrix (which is basically related to the size of the determinant of the matrix) but it does permit one to assay the problem better.

**Complete Beam Element Stiffness Matrices**

The stiffness matrix given above relates the deflections relative to the beam element ends to the loads applied to the beam element, end reactions excluded. It is desired next to obtain the "complete" beam element stiffness matrix in which absolute beam deflections are related to all the loads applied to the beam element including the end reactions.

Consider deflections first. It is desired to transform from the relative deflections of Fig. A23.8a to the absolute deflections of Fig. A23.8b.

\[
\delta_m = \delta_m^r + \delta_m^s - \delta_m
\]

\[
\delta_{m+r} = - \frac{L}{L} \Delta_m^r + \Delta_m^r - \frac{L}{L} \Delta_m^s
\]

\[
\delta_{m+s} = \delta_m^r + \delta_m^s - \delta_m
\]

For comparison with the matrix form to follow, these equations are rewritten as

\[
\phi_m = \frac{L}{L} \Delta_m + \frac{1}{L} \Delta_m^s
\]

\[
\delta_{m+r} = \frac{L}{L} \Delta_m^r + \Delta_m^r - \frac{L}{L} \Delta_m^s
\]

\[
\delta_{m+s} = - \frac{1}{L} \Delta_m + \frac{1}{L} \Delta_m^s
\]

In matrix form these equations are written so as to provide a transformation from absolute to relative displacements. In matrix form the index (subscript) is understood to increase monotonically down the column, and inasmuch as the grid numbering scheme gives this same property to the joint numbers on an element one has:
\[ \begin{bmatrix} \rho_m \\
\phi_m \\
P_m \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \rho_m \\
\phi_m \\
P_m \end{bmatrix} \]  

\[ \text{where} \]

\[ \begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} \\
- \frac{1}{L} \begin{bmatrix} 1 \\
L_m - L \\
L \end{bmatrix} \end{bmatrix} \\
\begin{bmatrix} 0 \\
\frac{L_m}{L} \\
0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} \\
\frac{1}{L} \begin{bmatrix} 1 \\
0 \\
L_m - L \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\
\frac{1}{L} \\
0 \end{bmatrix} \end{bmatrix} \]

Here \[ \begin{bmatrix} I \end{bmatrix} \] denotes a unit matrix. The matrices are shown "partitioned", those portions which transform the end rotations being shown separated from those which transform the transverse deflections.

The transformation matrix \[ \begin{bmatrix} T \end{bmatrix} \] in Eq. (5) is not square, having two more columns than rows to allow for the added two deflections, one at each beam end.

Consider next the loads. The applied loads may be transformed, using the equations of statics to yield the end reactions.

\[ \begin{align*}
M_m &= L_m + r \\
M_m &= L_m + r \\
P_m &= \frac{N_m}{L} - \frac{1}{L} \begin{bmatrix} 1 \\
L_m - L_r \\
L \end{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}
\end{align*} \]

Fig. A23.9

From Fig. A23.9 one has

\[ \begin{align*}
P_m &= \frac{N_m}{L} - \frac{1}{L} \begin{bmatrix} 1 \\
L_m - L_r \\
L \end{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} \\
P_m + s &= \frac{N_m}{L} - \frac{1}{L} \begin{bmatrix} 1 \\
L_m - L_r \\
L \end{bmatrix} \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix} + \frac{N_m + s}{L}
\end{align*} \]

In matrix form, the general expression which introduces the end reactions in terms of the other applied loads is

\[ \begin{bmatrix} N_m \\
P_m \\
N_m \end{bmatrix} = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} N_m \\
P_m \\
N_m \end{bmatrix} \]  

\[ \text{with end reactions} \]

\[ \begin{bmatrix} S \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\
\frac{L}{L_m - L} \\
\frac{1}{L} \\
0 \\
r \end{bmatrix} \\
\frac{1}{L} \\
1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\
\frac{L}{L_m - L} \\
\frac{1}{L} \\
0 \\
r \end{bmatrix} \\
\frac{1}{L} \\
1 \end{bmatrix} \begin{bmatrix} 0 \\
\frac{1}{L} \\
0 \end{bmatrix} \end{bmatrix} \]

The above transformation matrix \[ \begin{bmatrix} S \end{bmatrix} \] is not square, having two more rows than columns, these extra rows yielding the end reactions.

Provided the sign convention is observed as originally adopted, and provided the grid numbering scheme is such that joint numbers increase to the rear and outboard, then the above transformation matrices will apply equally well to ribs, spars and oblique beams:

replace \( N \) by \( M \) or \( Q \), \( \phi \) by \( \theta \) or \( T \) and \( \phi \) by \( \theta \) or \( T \).

For the case of beam elements having couples applied along the beam (e.g., the L.S. spar, Fig. A23.4c), simple modifications of these transformation matrices are necessary.

The relative stiffness equation, Eq. (4),

\[ \begin{bmatrix} N \end{bmatrix} = \begin{bmatrix} \phi \end{bmatrix} \]

is the relation between loads and relative deflections of a beam element. Substituting from eqs. (5) and (6), into (4), one obtains

\[ \begin{bmatrix} N \end{bmatrix} = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} \phi \end{bmatrix} \]

\[ \text{with end reactions} \]

\[ \begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix} \]

**T** is introduced here as a symbol for the relative rotation of a joint in the plane of an oblique member.
Eq. (7) is the relation between absolute deflections and the complete loading on the element. We let

\[ \mathbf{K} = \begin{bmatrix} \mathbf{S} \cdot \mathbf{K} \cdot \mathbf{T} \end{bmatrix} \]  

be the "complete" element stiffness matrix.

Note that \( \mathbf{T} \) is the transpose of \( \mathbf{K} \), as it must be if \( \mathbf{T} \) is to be symmetric.

To continue the illustration, the \( \mathbf{T} \) and \( \mathbf{S} \) matrices for Spar 4-7-9-10 are now written. For this member the rotation at station 10 is not considered and the couple at that end is zero, hence the last row and column of \( \mathbf{T} \) and \( \mathbf{S} \) are omitted in writing them from the general expressions.

Equation (5) written for Spar 4-7-9-10:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & -0.06945 & 0 & 0 & 0.06945 \\ 0 & -0.667 & 1.0 & 0 & -0.333 \\ 0 & -0.333 & 0 & 1.0 & -0.667 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

Equation (6), written for Spar 4-7-9-10:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.06945 & -0.667 & -0.333 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

The matrix \( \mathbf{K} \) is now formed per eq. (8).

For Spar 4-7-9-10:

\[ \begin{bmatrix} \mathbf{N} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0.3584 & -0.01606 & 0.02353 & -0.02306 & -0.01656 \\ -0.01606 & 0.35852 & -0.002397 & 0.02332 & -0.24513 \\ 0.02303 & -0.001537 & 0.002004 & -0.003600 & -0.25363 \\ -0.02332 & 0.02303 & -0.021650 & 0.004138 & -0.43322 \\ -0.01656 & 0.01606 & 0.22328 & -0.42362 & 0.23238 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

In the manner of the above illustrative case one goes through the calculations for the remaining beam elements to obtain:

\[ \begin{bmatrix} \mathbf{N} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 4.321 & -1.218 & 1.218 & 1.555 \\ -1.218 & -0.04208 & -0.04208 & -0.08054 \\ 1.218 & -0.04208 & -0.04208 & 0.02854 \\ 1.555 & -0.08054 & 0.08054 & 2.317 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

For Spar 3-6-3:

\[ \begin{bmatrix} \mathbf{N} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 3.256 & -0.0542 & 0.1055 & -0.2523 & -0.5520 \\ -0.0542 & 0.02784 & -0.04112 & -0.00258 & -0.1172 \\ 0.1055 & -0.04112 & 0.22717 & -0.02056 & -0.05654 \\ -0.2523 & -0.00258 & -0.02056 & 0.03988 & 1.75 \\ -0.5520 & -0.1172 & -0.05654 & 0.33808 & 1.53 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

For 1/2 Fuselage Beam:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 158.5 & -4.312 & 5.560 & -1.533 & 2.652 \\ -4.312 & 1.557 & -2.335 & 0.09933 & -0.01718 \\ 5.560 & -2.335 & 4.452 & -2.701 & 0.07429 \\ -1.533 & 0.09933 & -2.701 & -1.027 & 0.05281 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

For Rib 9-6-7:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} -0.259 & 0.259 & -0.575 & -0.05056 \\ -0.259 & -0.259 & 0.575 & 0.05056 \\ -0.575 & 0.575 & 1.150 & -0.100 \\ 0.05056 & -0.05056 & 0.100 & 0.200 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

For Rib 8-9:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0.322 & -0.06910 & 0.06910 \\ -0.06910 & 0.1437 & -0.1437 \\ 0.06910 & 0.1437 & 0.1437 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

For Leading Edge Spar:

\[ \begin{bmatrix} \mathbf{M} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} 0.3584 & -0.01606 & 0.02353 & -0.02306 & -0.01656 \\ -0.01606 & 0.35852 & -0.002397 & 0.02332 & -0.24513 \\ 0.02303 & -0.001537 & 0.002004 & -0.003600 & -0.25363 \\ -0.02332 & 0.02303 & -0.021650 & 0.004138 & -0.43322 \\ -0.01656 & 0.01606 & 0.22328 & -0.42362 & 0.23238 \end{bmatrix} \begin{bmatrix} \mathbf{\phi} \\ \mathbf{\psi} \end{bmatrix} \]

** Because of its size, this matrix relation (and some to follow) is written in a condensed tabular form. The correct interpretation should be obvious.

* The notation .0y means there are N zeroes following the decimal point and preceding the first significant figure, e.g., .0276 = .0000276.
Transformation of Matrices for Oblique Beams:

The complete stiffness matrix of any beam element not either parallel to or normal to the streamline direction, such as the leading edge spar in the present example, requires an additional modification to yield expressions relating couples and rotations about the pitching and rolling axes, rather than in the plane of the member. Such a modification is readily made.

In the present example, a pitching rotation of joint 5 \( \theta_z \) results in a rotation of joint 5 in the plane of the leading edge spar given by

\[ T_z = \sin \alpha \theta_z = 0.707 \theta_z \]

where \( \alpha \) is the sweep angle of the spar. Likewise, a rolling rotation of joint 5 \( \phi_z \) has a rotational component in the plane of the leading edge spar given by

\[ T_{\phi_z} = \cos \alpha \theta_z = 0.707 \phi_z \]

Then, when joint 5 experiences both pitching and rolling rotations, the total rotation in the plane of the leading edge spar is

\[ T_z = 0.707 \theta_z + 0.707 \phi_z \]

This last equation, and similar ones for joints 1 and 3, when put in matrix form, yield the matrix transformation for the displacements.

The above two matrix transformations - one for displacements and one for loads - (and note that they are the transpose of each other) are now applied to the complete stiffness matrix equation for the leading edge spar as given in tabular form in the last section. The operation involves premultiplying by one transformation matrix and postmultiplying by the other. The result is:

Next, consider resolution of the in-plane beam element couples. Couples \( Q_x \), \( Q_y \), and \( Q_z \) have components in the pitching and rolling directions. For example, at joint 5

\[ M_x = \sin \alpha \theta_z = 0.707 Q_z \]

and

\[ N_z = \cos \alpha \theta_z = 0.707 Q_z \]

Similar relations for the other couples along this member lead to the matrix transformation for this element’s loads:

\[
\begin{bmatrix}
M_x \\
N_x \\
P_x \\
M_x \\
N_x \\
P_x \\
T_{\phi_z}
\end{bmatrix}
= \begin{bmatrix}
0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
Q_x \\
P_x \\
Q_x \\
P_x \\
Q_x \\
P_x \\
T_{\phi_z}
\end{bmatrix}
\]
Torque Box Stiffness Matrices

In the case of the torque boxes, the stiffness matrices may be computed directly (they are determinate). The following approximate analysis is suggested as being satisfactory for most torque boxes. A more detailed analysis may be in order for boxes with extreme geometries. An alternate method is presented in Reference (2) and an appropriate discussion may be found in Reference (6). A completely different treatment of the cover skin is given in Reference (4).

Consider the quadrilateral box of Fig. A23.10

![Fig. A23.10](image)

For purposes of this immediate analysis, the box is considered cantilevered from one end, such as r-s. An elastic axis (e.a.) is assumed to exist midway between the long sides and an "effective root" is employed. The torque on the box is

\[ T = P_m b_m - P_n b_n \]

An (assumed) average rate of twist, \( \Phi \), is computed approximately by using the GJ at a representative section half way along the box.

\[ \Phi = \frac{T}{GJ} = \frac{P_m b_m - P_n b_n}{GJ} \]

Then the deflection \( \delta_n \) and \( \delta_m \) are given by

\[ \delta_n = \frac{\Phi L_n b_n}{GJ} = \frac{P_m b_m - P_n b_n}{GJ} L_n b_n \]

\[ \delta_m = \frac{\Phi L_m b_m}{GJ} = \frac{P_m b_m - P_n b_n}{GJ} L_m b_m \]

By summing moments about the effective root, \( P_n \) is found in terms of \( P_m \) and then eliminated from the above equations to give

\[ \delta_n = \frac{P_m L_n b_n}{GJ} L_n b_m + L_m b_n \]

\[ \delta_m = \frac{P_m L_m b_m}{GJ} L_n b_n + L_m b_m \]

\[ \delta_{TOTAL} = \delta_m + \frac{L_n}{L_m} \delta_n \]

Since it is only the deflection of one corner (say, point m) which is to be related to the corner reactions, the box is now rotated about the effective root axis to reduce \( \delta_n \) to zero. The resulting total deflection of point m is

\[ \delta_{TOTAL} = \delta_m + \frac{L_m}{L_n} \delta_n \]

\[ \delta_m = \frac{P_m L_m b_m}{GJ} \left( \frac{L_n b_m + L_m b_n}{L_n} \right) (b_m + b_n) \]

Re-solving this, we write

\[ P_m = \frac{GJ}{L_m (b_n + b_m)} \left( \frac{L_n b_m + L_m b_n}{L_n} \right) \delta_{TOTAL} \]

This last expression relates the load at point m to the deflection at that point, with the other three corners undeflected. The reactions developed at these other corners are now found in terms of \( P_m \) using the equations of statics (summation of forces and summation of moments about two axes). The result, expressed in matrix form is

\[ \begin{bmatrix} P_m \\ P_n \\ P_r \\ P_s \end{bmatrix} = \begin{bmatrix} 1 & -L_n/L_m \\ -L_m/L_n & A_1 & A_2 \\ A_1 & A_2 & 1 \end{bmatrix} \]

where

\[ A_1 = \frac{L_n b_m + L_m b_n + b_n (L_m - L_n)}{L_n (b_r + b_s)} \]

\[ A_2 = -\frac{(L_n b_m + L_m b_n) + b_r (L_m - L_n)}{L_n (b_r + b_s)} \]

If now the corner points have absolute displacements \( \Delta_m, \Delta_n, \Delta_r, \) and \( \Delta_s \), one can find from the geometry of the unit that the total relative displacement of point m may be written

\[ \delta_{TOTAL} = \begin{bmatrix} \Delta_m \\ \Delta_n \\ \Delta_r \\ \Delta_s \end{bmatrix} \]

When these last three equations are combined to relate the four corner reactions to the absolute corner displacements, the torque box stiffness matrix is seen to be the square, symmetric matrix,
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[
\mathbf{K} = \frac{GJ_L}{(L_n+b_n)(L_m+b_m)} \begin{bmatrix}
1 & \frac{L_n}{L_m} & A_1 & A_2 \\
(L_n+b_n)(L_m+b_m) & A_1 & A_2 \\
L_m & L_m & A_1 & A_2
\end{bmatrix} \quad \text{(9)}
\]

In the above derivation the torsional stiffness \(GJ\) at a representative section is used. The stiffness is obtained from Bredt’s equation for the twist of a single cell thin-walled tube (Eq. 18, Chapter A6):

\[
\Phi = \frac{T}{4A*G} \int \frac{ds}{t}
\]

from which, by definition

\[
GJ = \frac{4A*G}{\int \frac{ds}{t}}
\]

Here \(A\) is the area enclosed within the tube cross section by the median line of the tube wall and the integration is carried out around the tube perimeter (index \(s\) gives distance along the perimeter). For the torque boxes encountered in delta wings it is probably satisfactory to neglect the \(ds/t\) contribution from the vertical webs, it being small compared with the corresponding contribution from the cover sheets.

In the example wing three boxes (2-3-5-6, 3-4-6-7 and 6-7-8-9) are to be used. These boxes are each 48 inches square in plan and have average depths (assumed here to be the uniform depths of the representative sections) of 7.26", 5.32" and 4.27", respectively. Fig. A23.11 shows the assumed representative cross section of box 2-3-5-6 and its \(GJ\) calculation.

| A = 7.26 x 48 = 348 in² |
| \(h_{AV} = 7.26"\) |

Note:

\[
\int \frac{ds}{t} = 2 \times 48 = 96 \times 10^3 \quad G = \frac{E}{2.6}
\]

\(GJ_{e1} = \frac{4(348) \times 1.88 \times 10^3}{2.6} = 99.3 \text{ lb-in.}^2\)

\((ds/t) \text{ contribution of vertical webs neglected}\)

Fig. A23.11 Calculation of \(GJ\) for the Representative Section of Box 2-3-5-6

In similar fashion one finds:

For box 3-4-5-6

\(GJ_{e1} = 53.3\)

For box 6-7-8-9

\(GJ_{e1} = 34.3\)

Finally, the stiffness matrices for the three boxes of the example become: (cf eq. 9)

For box 2-3-5-6

\[
[\mathbf{K}] = \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

For box 3-4-6-7

\[
[\mathbf{K}] = \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

For box 6-7-8-9

\[
[\mathbf{K}] = \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

A23.5 Complete Wing Stiffness Matrix

The stiffness matrix for the composite wing now may be obtained by forming the sum of the complete stiffness matrices for the beam elements and the torque boxes. For this purpose a large matrix table is laid out and entries from the individual stiffness matrices are transferred into the appropriate locations. Wherever multiple entries occur in a box these are summed.

Before the large wing matrix is laid out it is necessary to observe first that the matrix which would be obtained as just indicated would be singular, i.e., its determinant would be zero, and hence it could not be inverted to yield the flexibility influence coefficients (see Appendix A). This condition arises due to the fact that the equations represented (19 in number in the example problem) are not independent; three of these equations can be obtained as linear combinations of the others.

That there are three such interdependencies may be seen from the existence of the three equations of statics which may be applied to the wing (summation of normal forces and summation of moments in two vertical planes): hence three of the reactions expressed by the structural equations in the matrix may be found
from the others by the equations of statics. To remove the "singularity" from the stiffness matrix it is only necessary to drop out three equations - achieved by removing three rows and corresponding columns (so as to retain a symmetric stiffness matrix).

The act of removing the three equations selected is also equivalent to assuming the corresponding deflections to be zero. In this way a reference base for the deflections is also established. The choice of reference base is somewhat arbitrary, but, following a suggestion of Williams (7), a triangular base will be employed as shown in Fig. A23.12.

![Fig. A23.12 Deflection Reference Base](image)

Here the deflections at points 1 and 5 are zero, fixing the reference triangle, since the point corresponding to 5 in the other half of the wing (say, 5') will have zero deflection also due to symmetry. The third condition is applied to point 2: point 2 will be assumed to have zero rolling rotation ($\delta_4 = 0$) for symmetric wing loadings and to have zero transverse deflection ($\Delta_4 = 0$) for antisymmetric loadings.

Hence the following equations (rows and columns) are to be omitted from the wing stiffness matrix:

\[
\begin{align*}
& P_{15}, \Delta_1 \\
& P_{55}, \Delta_5 & \text{for symmetric loadings} \\
& N_{55}, \delta_5 \\
& P_{15}, \Delta_1 \\
& P_{55}, \Delta_5 & \text{for antisymmetric loadings} \\
& P_{55}, \delta_5
\end{align*}
\]

The 16 x 16 wing stiffness matrices thus obtained can now be inverted as they are non-singular. However, if this step is carried out one finds that the resultant influence coefficients are those for only the half-wing acting alone and supported by constraints as assumed above. To account for the presence of the other half of the wing, it is necessary to specify additional geometric conditions along the airplane centerline. This step is accomplished by assuming the following deflections zero (eliminating their corresponding rows and columns from the matrix):

\[
\begin{align*}
& \delta_1 & \text{for symmetric loadings (zero lateral slope or rolling rotation along the airplane centerline)} \\
& \delta_4 & \text{for antisymmetric loadings (zero transverse deflections and pitching rotations along the airplane centerline)}
\end{align*}
\]

It will be seen that in both cases an additional 3 equations are eliminated from the 16 x 16 matrices reducing them to 13 x 13's. (In general, there is no reason why the matrices for the symmetric and antisymmetric cases have to be the same size. A rotation $\delta_1$, for instance, would be zero in one case only.)

Written below is the 13 x 13 wing stiffness matrix for the antisymmetric case ($\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \theta_1 = 0$). As explained earlier, each entry therein is the sum of all corresponding stiffnesses for all elements meeting at the point. A typical multiple entry occurs for joint 5 - (row $P_{55}$, column $\delta_5$). These comprise:

\[
\begin{align*}
0.007178 & \text{ from spar 3-6-8} \\
0.001332 & \text{ from rib 3-5-6} \\
0.000310 & \text{ from box 6-7-8-9} \\
0.000988 & \text{ from box 2-3-5-6} \\
0.000462 & \text{ from box 3-4-6-7} \\
0.01070 & \text{ TOTAL}
\end{align*}
\]

* The antisymmetric loading pattern is one wherein the wing is loaded equally, but in opposite sense, on corresponding points of the two wing halves. Any general loading may be resolved into the sum of one symmetric and one antisymmetric loading.
### ANÁLISIS Y DISEÑO DE VEHÍCULOS DE VUELO

**Matrix de Rigidez, Caso Antisimétrico** *(Nota: Los espacios en blanco indican ceros)*

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \Delta_1 )</th>
<th>( \Delta_2 )</th>
<th>( \Delta_3 )</th>
<th>( \Delta_4 )</th>
<th>( \Delta_5 )</th>
<th>( \Delta_6 )</th>
<th>( \Delta_7 )</th>
<th>( \Delta_8 )</th>
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<tbody>
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<td>0.01076</td>
<td>-0.001460</td>
<td>-0.39378</td>
<td>0.310</td>
<td>0.02760</td>
<td>0.1068</td>
<td>0.00634</td>
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<td>0.02323</td>
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**Matriz de Influencia de Ala:**

La matriz de influencia de la estructura del ala es ahora calculada a través de la inversa de la matriz de rigidez superior (ver Ecuación 2). Un sistema de cómputo digital automático es el método esencial para esta etapa. Los detalles del procedimiento y técnicas utilizadas en el cálculo de matrices inversas no son totalmente conocidos, pero los resultados se presentan. Se supone que este tipo de trabajo puede ser realizado por personal de cómputo.

### Matriz de Influencia de Ala, Caso Antisimétrico

<table>
<thead>
<tr>
<th>( A_{mm} )</th>
<th>( A_{PP} )</th>
<th>( A_{PP} )</th>
<th>( A_{PP} )</th>
<th>( A_{PP} )</th>
<th>( A_{PP} )</th>
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<td>1236</td>
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<td>-30.83</td>
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<td>4001</td>
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<td>-0.2828</td>
<td>-0.3472</td>
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*In spite of the exercise of great care and much ingenuity, problems of such great size and such a nature will arise occasionally as to defy satisfactory inversion. For these problems, such clever physical concepts as "block" solutions of portions of the structure are available. The interested reader is referred to a discussion of Ref. 4 and to techniques of block and group relaxations in the literature on Relaxation Methods (e.g. Ref. 8).*
A23.14 STRUCTURAL ANALYSIS OF A DELTA WING

A23.6 Wing Internal Stresses

Following the calculation of the wing influence coefficients, a relationship is available between the applied wing loads (assumed given) and the wing deformations. In a symbolic fashion,

\[
\begin{pmatrix}
\Delta_m \\
\Delta_n \\
\Delta_m
\end{pmatrix}
= [A_{mn}] \text{WING}
\begin{pmatrix}
P_m \\
N_m \\
M_m
\end{pmatrix}
\text{WING}
\]

Earlier in the analysis, deformations of a structural element (beam or box) were related to the forces acting on that element by the (complete) element stiffness matrix:

\[
\begin{pmatrix}
P \\
M \\
N
\end{pmatrix}
= [K]_{\text{ELEMENT}}
\begin{pmatrix}
\Delta \\
\Theta \\
\Phi
\end{pmatrix}
\text{ELEMENT}
\]

Now since the deformations of an element must conform to those of the wing,

\[
\begin{pmatrix}
\Delta_m \\
\Delta_n \\
\Delta_m
\end{pmatrix}
= \begin{pmatrix}
\Delta_m \\
\Delta_n \\
\Delta_m
\end{pmatrix}
\text{ELEMENT}
\begin{pmatrix}
P_m \\
N_m \\
M_m
\end{pmatrix}
\text{WING}
\]

Hence, the previous equations may be combined immediately to yield

\[
\begin{pmatrix}
P \\
M \\
N
\end{pmatrix}
= [K]_{\text{ELEMENT}} [A]_{\text{WING}}
\begin{pmatrix}
P \\
M \\
N
\end{pmatrix}
= (10)
\]

Equation (10) provides the desired relation between external loads applied to the wing and the forces acting on a specific element. These forces on the element might be looked upon as those necessary to hold the element in the deflected shape conforming to that of the wing. Of course, with these element forces known the finer details of the stress distribution within the element are readily found by standard techniques.

Note that the matrix \([K]_{\text{ELEMENT}}\) in Eq. (10) will have to be "blown up" to an appropriate size before it may be premultiplied onto \([A]_{\text{WING}}\). This enlargement is accomplished by the insertion of columns of zeroes at each column location corresponding to a wing deformation which does not affect the specific element under consideration.

For illustration, the indicated steps are now carried through for spar 4-7-8-10. Referring to previous work we find for the element matrix:

\[
[K]_{\text{ELEMENT}} = \begin{pmatrix}
N_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Expanding (and rearranging to agree in numerical order with the wing matrix),
Then multiplying out with $\left[ \frac{\text{Amn}}{\Delta m} \right]_{\text{WING}}$ per Eq. (10), we get the five forces on spar 4-7-9-10 (see Fig. A23.4-b) in terms of the 13 wing applied loads as:

\[
\begin{bmatrix}
N_x & -0.357 & -0.0177 & 0.794 & -0.234 & 0.02694 & -0.1736 & -0.3834 & 0.2706 & 0 & -0.871 & 1.00 & 0 & -0.555 \\
F_x & -0.304 & -0.1945 & 1.812 & 0.2190 & 0.4985 & -0.320 & -0.001237 & -0.180 & -0.249 & 0.01465 & -0.2698 & -0.567 & -0.0434 \\
F_y & 0.02294 & 0.3735 & -0.2038 & -0.5191 & -0.2815 & 0.9408 & 0.00160 & -0.230 & -0.625 & -0.00299 & 0.00148 & 0.1706 & -0.007071 \\
M_x & -0.3275 & -0.1550 & 0.3466 & -0.3787 & -0.4174 & -0.1407 & 0.975 & 0.1255 & -0.486 & 0.01461 & -0.0281 & -0.135 & -0.001107 \\
M_y & -0.00305 & -0.1364 & -0.3347 & -0.2706 & -0.1392 & -0.111 & -0.682 & -0.349 & -0.1202 & -0.6815 & -0.01686 & -0.2258 & -0.001618
\end{bmatrix}_{\text{WING}}
\]

Each column of forces in the above matrix must satisfy the equations of statics on the element. A variety of checks on the accuracy of such a result are thus available.

The student will find it instructive to study carefully this last result to observe in what manner a load applied to the wing appears on this spar element. For instance, one sees that of the load $F_x = 1$ applied to the wing, .3735 goes onto spar 4-7-9-10. Examination of Fig. A23.1 reveals that the remainder, 1.00 - .3735 = .6265 must be taken up by rib 5-6-7 and the torque boxes 3-4-7-5 and 6-7-3-9.

**REFERENCES**


3) Schuerch, H., Delta Wing Design Analysis, Society of Automotive Engineers National Aeronautical Meeting Preprint No. 141, September 1953.


COMMERCIAL TRANSPORT FOR THE 1970s? -- This is an artist's conception of a passenger transport of the future -- cruising at three to five times the speed of sound at 60,000 feet or higher. This is one of hundreds of configurations considered by Convair Division of General Dynamics Corporation at San Diego, Calif., in its studies of supersonic airliners.
PART B
FLIGHT VEHICLE MATERIALS AND THEIR PROPERTIES
CHAPTER B1
BASIC PRINCIPLES AND DEFINITIONS

B1.1 Introduction.

The flight vehicle structures engineer faces a major design requirement of a high degree of structural integrity against failure, but with as light structural weight as possible. Structural failure in flight vehicles can often prove serious relative to the loss of life and the vehicle. However, experience has shown that if a flight vehicle, whether military or commercial in type, is to be satisfactory from a payload and performance standpoint, major effort must be made to save structural weight, that is, to eliminate all structural weight not required to insure against failure.

Since a flight vehicle is subjected to various types of loading such as static, dynamic and repeated, which may act under a wide range of temperature conditions, it is necessary that the structures engineer have a broad knowledge of the behavior of materials under loading if safe and efficient structures are to be obtained. This Part B provides information for the structures design engineer relative to the behavior of the most common flight vehicle materials under load and the various other conditions encountered in flight environment, such as elevated temperatures.

B1.2 Failure of Structures.

A flight vehicle like any other machine, is designed to do a certain job satisfactorily. If any structural unit of the vehicle suffers effects which in turn affect in some manner the satisfactory performance of the vehicle, the unit can be considered as having failed. Failure of a structural unit is therefore a rather broad term. For example, failure of a structural unit may be due to too high a stress or load causing a complete fracture of the unit while the vehicle is in flight. If this unit should happen to be one such as a wing beam or a major wing fitting, the failure is a serious one as it usually involves loss of the airplane and loss of life. Likewise the collapse of a strut in landing gear structure during a landing operation of the airplane can be a very serious failure. Failure of a structural unit may be due to fatigue and since fatigue failure is of the fracture type without warning indications of impending failure, it can also prove to be a very serious type of failure.

Failure in a different manner can result from a structural unit being too flexible, and this flexibility might influence aerodynamic forces sufficiently as to produce unsatisfactory vehicle flying characteristics. In some cases this flexibility may not be serious relative to the loss of the vehicle, but it is still a degree of failure because changes must be made in the structure to provide a satisfactory operating vehicle. In some cases, excessive distortion such as the torsional twist of the wing can be very serious as this excessive deflection can lead to a build-up of aerodynamic-dynamic forces to cause flutter or violent vibration which can cause failure involving the loss of the airplane.

To illustrate another degree of failure of a structural unit, consider a wing built in fuel tank. The stiffened sheet units which make up the tank are also a part of the wing structure. In general these sheet units are designed not to wrinkle or buckle under airplane operating conditions in order to insure against leakage of fuel around riveted connections. Therefore if portions of the tank walls do wrinkle in operation resulting in fuel leakage, which in turn require repair or modification of the structure, we can say failure has occurred since the tank failed to do its job satisfactory, and involves the item of extra expense to make satisfactory. To illustrate further, the flight vehicle is equipped with many installations, such as the control systems for the control surfaces, the power plant control system etc., which involves many structural units. In many cases excessive elastic or inelastic deformation of a unit can cause unsatisfactory operation of the system, hence the unit can be considered as having failed although it may not be a serious failure relative to causing the loss of the vehicle. Thus the aero-astro structures engineer is concerned with and responsible for preventing many degrees of failure of structural units which make up the flight vehicle and its installations and obviously the greater his knowledge regarding the behavior of materials, the greater his chances of avoiding troubles from the many degrees of structural failure.

B1.3 General Types of Loading.

Failure of a structural member is influenced by the manner in which the load is
applied. Relative to the length of time in applying the load to a member, two broad classifications appear logical, namely, (1) Static loading and (2) Dynamic loading. For purposes of explanation and general discussion, these two broad classifications will be further broken down as follows:

Static Loading

- Continuous Loading.
- Gradually or slowly applied loading.
- Repeated gradually applied loading.

Dynamic Loading

- Impact or rapidly applied loading.
- Repeated impact loading.

Static Continuous Loading. A continuous load is a load that remains on the member for a long period of time. The most common example is the dead weight of the member or the structure itself. When an airplane becomes airborne, the weight of the wing and its contents is a continuous load on the wing. A tank subjected to an internal pressure for a considerable period of time is a continuous load. Since a continuous load is applied for a long time, it is a type of loading that provides favorable conditions for creep, a term to be explained later. For airplanes, continuous loadings are usually associated with other loads acting simultaneously.

Static Gradually or Slowly Applied Loads. A static gradually applied load is one that slowly builds up or increases to its maximum value without causing appreciable shock or vibration. The time of loading may be a matter of seconds or even hours. The stresses in the member increase as the load is increased and remain constant when the load becomes constant. As an example, an airplane which is climbing with a pressurized fuselage, the internal pressure loading on the fuselage structure is gradually increasing as the difference in air pressure between the inside and outside of the fuselage gradually increases as the airplane climbs to higher altitudes.

Static Repeated Gradually Applied Loads. If a gradually applied load is applied a large number of times to a member it is referred to as a repeated load. The load may be of such nature as to repeat a cycle causing the stress in the member to go to a maximum value and then back to zero stress, or from a maximum tensile stress to a maximum compressive stress, etc. The situation envolving repeated loading is important because it can cause failure under a stress in a member which would be perfectly safe, if the load was applied only once or a small number of times. Repeated loads usually cause failure by fracturing without warning, thus repeated loads are important in design of structures.

Dynamic or Impact Loading. A dynamic or impact loading when applied to a member produces appreciable shock or vibration. To produce such action, the load must be applied far more rapidly than in a static loading. This rapid application of the load causes the stresses in the member to be momentarily greater than if the same magnitude of load was applied statically, that is slowly applied. For example, if a weight of magnitude W is gradually placed on the end of a cantilever beam, the beam will bend and gradually reach a maximum end deflection. However if this same weight of magnitude W is dropped on the end of the beam from even such a small height as one foot, the maximum end deflection will be several times that under the same static load W. The beam will vibrate and finally come to rest with the same end deflection as under the static load W. In bringing the dynamic load to rest, the beam must absorb energy equal to the change in potential energy of the falling load W, and thus dynamic loads are often referred to as energy loads.

From the basic laws of Physics, force equals mass times acceleration (F = Ma) and acceleration equals time rate of change of velocity. Thus if the velocity of a body such as an airplane or missile is changed in magnitude, or the direction of the velocity of the vehicle is changed, the vehicle is accelerated which means forces are applied to the vehicle. In severe flight, airplane maneuvers like pulling out of a dive from high speeds or in striking a severe transverse air gust when flying at high speed, or in landing the airplane on ground or water, the forces acting externally on the airplane are applied rather rapidly and are classed as dynamic loads. Chapter 4 discusses the subject of airplane loads relative to whether they can be classed as static or dynamic and how they are treated relative to design of aircraft structures.

Bl.4 The Static Tension Stress-Strain Diagram.

The information for plotting a tension stress-strain diagram of a material is obtained by loading a test specimen in axial tension and measuring the load with corresponding elongation over a given length, as the specimen is loaded statically (gradually applied) from zero to the failing load. To standardize results standard size test specimens are specified by the (ASTM) American Society For Testing Materials. The speed of the testing machine cross-head should not exceed 1/15 inch per inch of gage length per minute up to the yield point of the material
and it should not exceed 1/2 inch per inch of
gage length per minute from the yield point to
the rupturing point of the material. The
instrument for measuring the elongation must be
calibrated to read 0.0002 inches or less.
The information given by the tension stress-
strain diagram is needed by the engineer since it
is needed in strength design, rigidity
design, energy absorption, quality control and
many other uses.

Fig. B1.1 shows typical tensile stress-
strain diagrams of materials that fall in three
broad classifications. In the study of such
diagrams various facts and relationships have
been noted relative to behavior of materials
and standard terms and symbols have been pro-
vided for this basic important information.
These terms will be explained briefly.

Modulus of Elasticity (E). The mechanical
property that defines resistance of a material
in the elastic range is called stiffness and
for ductile materials is measured by the value
term called modulus of Elasticity, and, designated
by the capital letter E. Referring to Fig.
B1.1, it is noticed that the first part of all
three diagrams is a straight line, which indi-
cates a constant ratio between stress and
strain over this range. The numerical value
of this ratio is referred to as the modulus of
Elasticity (E). E is therefore the slope of
the initial straight portion of the stress-
strain diagram and its numerical value is
obtained by dividing stress in pounds per
square inch by a strain which is non-dimensional
or E = f/E, and thus E has the same units as
stress, namely pounds per square inch.

The clad aluminum alloys have two E values
as indicated in the lower diagram of Fig. B1.1.
The initial modulus is the same as for other
aluminum alloys, but holds only up to the pro-
portional limit stress of the soft pure
aluminum coating material. Immediately above
this point there is a short transition stage
and the material then exhibits a secondary
modulus of Elasticity up to the proportional
limit stress of the stronger core material.
This second modulus is the slope of the second
straight line in the diagram. Both modulus
values are based on a stress using the gross
area which includes both core and covering
material.

Tensile Proportional Limit Stress, \( \sigma_{p} \). The
proportional limit stress is that stress which
exists when the stress strain curve departs
from the initial straight line portion by a
unit strain of 0.0001. In general the pro-
portional limit stress gives a practical
dividing line between the elastic and inelastic
range of the material. The modulus of
elasticity is considered constant up to the

Proportional Limit

Ultimate Tensile Stress

Yield Stress

Stress - PSI

Strain - Inches Per Inch

(a) Material Having a Definite
Yield Point (such as some Steels)

Ultimate Tensile Stress

Proportional Limit

Yield Stress

Stress - PSI

Strain - Inches Per Inch

(b) Materials not Having a
Definite Yield Point (such as
Aluminum Alloys, Magnesium,
and Some Steels)

Primary Modulus Line

Secondary Modulus Line

Ultimate Tensile Stress

Yield Stress

Stress - PSI

Strain - Inches Per Inch

(c) Clad Aluminum Alloys

Fig. B1.1

Tensile Yield Stress \( (F_{Y}) \). In referring
to the upper diagram in Fig. B1.1, we find
that some materials show a sharp break at a
stress considerably below the ultimate stress
and that the material elongates considerably
with little or no increase in load. The
stress at which this takes place is called
the yield point or yield stress. However many
materials and most flight vehicle materials do
not show this sharp break, but yield more
gradually as illustrated in the middle diagram
of Fig. B1.1, and thus there is no definite
yield point as described above. Since
permanent deformations of any appreciable
amount are undesirable in most structures or
machines, it is normal practice to adopt an
arbitrary amount of permanent strain that is
considered admissible for design purposes.
Test authorities have established this value
of permanent strain or set as 0.002 and the
stress which existed to cause this permanent
strain when released from the material is
called the yield stress. Fig. B1.1 shows how
B1.4  BEHAVIOR OF MATERIALS AND THEIR PROPERTIES

It is determined graphically by drawing a line from the 0.002 point parallel to the straight portion of the stress-strain curve, and where this line intersects the stress-strain curve represents the yield strength or yield stress.

Ultimate Tensile Stress ($F_{tu}$). The ultimate tensile stress is that stress under the maximum load carried by the test specimen. It should be realized that the stresses are based on the original cross-sectional area of the test specimen without regard to the lateral contraction of the specimen during the test, thus the actual or true stresses are greater than those plotted in the conventional stress-strain curve. Fig. B1.2 shows the general relationship between actual and the apparent stress as plotted in stress-strain curves. The difference is not appreciable until the higher regions of the plastic range are reached.

Figs. B1.3 and B1.4 compare the shapes of the tension stress-strain curves for some common aircraft materials.

B1.5 The Static Compression Stress-Strain Diagram.

Because safety and light structural weight are so important in flight vehicle structural design, the engineer must consider the entire stress-strain picture through both the tension and compressive stress range. This is due to the fact that buckling, both primary and local, is a common type of failure in flight vehicle structures and failure may occur under stresses in either the elastic or plastic range. In general the shape of the stress-strain curve as it departs away from the initial straight line portion, is different under compressive stresses than when under tensile stresses. Furthermore, the various flight vehicle materials have different shapes for the region of the stress-strain curve adjacent to the straight portion. Since light structural weight is so important, considerable effort is made in design to develop high allowable compressive stresses, and in many flight vehicle structural units, these allowable ultimate design compressive stresses fall in the inelastic or plastic zone.

Fig. B1.5 shows a comparison of the stress-strain curves in tension and compression for four widely used aluminum alloys. Below the proportional limit stress the modulus of elasticity is the same under both tension and compressive stress. The yield stress in compression is determined in the same manner as explained for tension.

Compressive Ultimate Stress ($F_{cu}$). Under a static tension stress, the ultimate tensile stress of a member made from a given material is not influenced appreciably by the shape of the cross-section or the length of the member, however under a compressive stress the ultimate compressive strength of a member is greatly influenced by both cross-sectional shape and length of the member. Any member, unless very short and compact or tends to buckle laterally as a whole or to buckle laterally or cripple locally when under compressive stress. If a member is quite short or restrained against lateral buckling, then failure for some materials such as stone, wood and a few metals will be by definite fracture, thus giving a definite value for the ultimate compressive stress. Most aircraft materials are so ductile that no fracture is encountered in compression, but the material yields and swells out so that the increasing cross-sectional area tends to carry increasing load. It is therefore practically impossible to select a value of the ultimate compressive stress of ductile materials without having
The slope of this tangent gives the local rate of change of stress with strain. The secant modulus $E_s$ is determined by drawing a secant (straight line) from the origin to the point in question. This modulus measures the ratio between stress and actual strain. Curves which show how the tangent modulus varies with stress are referred to as tangent modulus curves. Fig. Bl.5 illustrates such curves for four different aluminum alloys. It should be noted that the tangent modulus is the same as the modulus of elasticity in the elastic range and gets smaller in magnitude as the stress gets higher in the plastic range.

Bl.7 Elastic - Inelastic Action.

If a member is subjected to a certain stress, the member undergoes a certain strain. If this strain vanishes upon the removal of the stress, the action is called elastic. Generally speaking, for practical purposes, a material is considered elastic under stresses up to the proportional limit stress as previously defined. Fig. Bl.7 illustrates elastic action. However, when the stress is removed, a residual strain remains, the action is generally referred to as inelastic or plastic. Fig. Bl.8 illustrates inelastic action.

Bl.8 Ductility.

The term ductility from an engineering standpoint indicates a large capacity of a material for inelastic (plastic) deformation in tension or shear without rupture, as contrasted with the term brittleness which indicates little capacity for plastic deformation without failure. From a physical standpoint, ductility is a term which measures...
the ability of a material to be drawn into a wire or tube or to be forged or die cast. Ductility is usually measured by the percentage elongation of a tensile test specimen after failure, for a specified gage length, and is usually an accurate enough value to compare materials.

Percent elongation = \( \left( \frac{L_a - L_o}{L_o} \right) \times 100 \)  = measure of ductility.

where \( L_o \) = original gage length and \( L_a \) = gage length after fracture. In referring to ductility in terms of percent elongation, it is important that the gage length be stated, since the percent elongation will vary with gage length, because a large part of the total strain occurs in the necked down portion of the gage length just before fracture.

**Bl.9 Capacity to Absorb Energy. Resilience. Toughness.**

**Resilience.** The capacity of a material to absorb energy in the elastic range is referred to as its resilience. For measure of resilience we have the term modulus of resilience, which is defined as the maximum amount of energy per unit volume which can be stored in the material by stressing it and then completely recovered when the stress is removed. The maximum stress for elastic action for computing the modulus of resilience is usually taken as the proportional limit stress. Therefore for a unit volume of material (1 cu. in.) the work done in stressing a material up to its proportional limit stress would equal the average stress \( f_p/2 \) times the elongation (\( \varepsilon_p \)) in one inch. If we let \( U \) represent modulus of resilience, then

\[
U = \left( \frac{f_p}{2} \right) \varepsilon_p
\]

But \( \varepsilon_p = f_p/E \), hence

\[
U = \left( \frac{f_p}{2} \right) \left( \frac{f_p}{E} \right) = f_p^2/2E
\]

Under a condition of axial loading, the modulus of resilience can be found as the area under the stress-strain curve up to the proportional limit stress. Thus in Fig. Bl.9, the area OAB represents the energy absorbed in stressing the material from zero to the proportional limit stress.

High resilience is desired in members subjected to shock, such as springs. From equation (1), a high value of resilience is obtained when the proportional limit stress is high and the strain at this stress is high, or from equation (2), when the proportional limit stress is high and modulus of elasticity is low.

In Fig. Bl.9 if the stress is released from point D in the plastic range, the recovery diagram will be approximately a straight line DB parallel to AO, and the area ODE represents the energy released, and often referred to as hyper-elastic resilience.

**Toughness.** Toughness of a material can be defined as its ability to absorb energy when stressed in the plastic range. Since the term energy is involved, another definition would be the capacity of a material for resisting fracture under a dynamic load. Toughness is usually measured by the term modulus of Toughness which is the amount of strain energy absorbed per unit volume when stressed to the ultimate strength value.

In Fig. Bl.9, let \( f \) equal the average stress over the unit strain distance \( \Delta \varepsilon \) from \( F \) to \( G \). Then work done per unit volume in stressing \( F \) to \( G \) is \( f \Delta \varepsilon \) which is represented by the area FOHI. The total work done in stressing to the ultimate stress \( f_u \) would then equal \( f_u^2 f \), which is the area under the entire stress-strain curve up to the ultimate stress point, or the area O A J K O in Fig. Bl.9 and the units are in. lb. per cu. inch. Strictly speaking it should not include the elastic resilience or the energy absorbed in the elastic range, but since this area is small compared to the area under the curve in the plastic range it is usually included in toughness measurements.

It should be noted that the capacity of a member for resisting an axially applied dynamic load is increased by increasing the length of a member, because the volume is increased directly with length. However, the ultimate strength remains the same since it is a function of cross-sectional area and not of volume of the material.
Toughness is a desirable characteristic when designing to resist impact or dynamic loads as it gives a reserve strength or factor of safety against failure by fracture when over-loading in actual use should happen to cause the member to be stressed fairly high into the plastic zone.

The property of ductility helps to produce toughness, but does not alone control toughness as illustrated in Fig. Bl.10, which shows the stress-strain curves for three different materials. Material (A) is strong but brittle, whereas material (C) is weak and ductile, and material (B) represents average strength and ductility. However, all three materials have the same modulus of toughness since the area under all three curves is the same.

Bl.10 Poisson's Ratio.

When a material is stressed, it will deform in the direction of the stress and also at right angles to it. For axial loading and for stress below the proportional limit stress, the ratio of the unit strains at right angles to the stress, to the unit strain in the direction of the stress is called Poisson's ratio. It is determined by direct measurement in a tensile or compressive test of a specimen, and is approximately equal to 0.3 for steel and 0.33 for non-ferrous materials. In many structures there are members which are subjected to stresses in more than one direction, say along all three coordinate axes. Poisson's ratio is used to determine the resultant stress and deformation in the various directions.

Poisson's Ratio in Plastic Range. Information is somewhat limited as to Poisson's ratio in the plastic range and particularly during the transition range from elastic to plastic action. For the assumption of a plastically incompressible isotropic solid, Poisson's ratio assumes a value of 0.5. Gerard and Weidform found from their research, that the transition of Poisson's ratio from the elastic value of around 1/3 to 1/2 in the plastic range is gradual and is most pronounced in the yield point region of the stress-strain curve.

Bl.11 Construction of a Stress-Strain Curve Through a Given Yield Stress by Using a Known Test Stress-Strain Curve.

Materials in general are produced to satisfy certain guaranteed minimum strength properties such as yield stress, ultimate tensile stress, etc. Thus in the design of important members in a composite structure, the minimum guaranteed properties must be used to provide the required degree of safety. In general, most materials will give properties slightly above that guaranteed values, thus the test stress-strain curve of purchased material cannot be used for design purposes. The stress-strain curve passing through a given yield point stress can be readily obtained for a test stress-strain curve as follows:

In Fig. Bl.11, the heavy curve O B C represents a known typical stress-strain curve for the given material. Let the minimum guaranteed yield stress be the value as shown at point A, using the 0.2 percent method. Then proceed as follows:

1. Draw a straight line through point O and point A which will intersect the typical curve at point B. Point B may be above or below the typical curve.
2. Locate any other point on the typical curve such as point C, and draw line from O through C.
3. Locate point D on line O C by the following ratio:
   \[ \frac{OD}{OC} = \frac{OA}{OB} \]
4. Repeat step 3 to obtain a number of points as shown by dots on Fig. Bl.11, and draw smooth curve through these points to obtain desired stress-strain curve.

Bl.12 Non-Dimensional Stress-Strain Curves.

The structural designer is constantly confronted with the design of structural units which fail by inelastic instability. The
solution of such problems requires information given by the compressive stress-strain curve. Since flight vehicles make use of many different materials, and each material usually has many different states of manufacture which give different mechanical properties, the question of time required to obtain certain design information from stress-strain curves becomes important. For example, in the aluminum alloys alone there are about 100 different alloys, and when elevated temperatures at various time exposures are added, the number of stress-strain curves required is further greatly increased.

Fortunately, this time consuming work was greatly lessened when Ramsberg and Osgood (Ref. 1) proposed an equation to describe the stress-strain curve in the yield range. Their proposed equation specifies the stress-strain curve by the use of three parameters, the modulus of elasticity $E$, the secant yield stress $F_{o,y}$, which is taken as the line of slope 0.75 drawn from origin (see Fig. Bl.12), and a parameter $n$ which describes the shape of the stress-strain curve in the yield region. In order to evaluate the term $n$, another stress $F_{o,\infty}$ is needed, which is the intersection of the curve by a line of slope of 0.85E through the origin (see Fig. Bl.12).

![Stress-Strain Curve](image)

Fig. Bl.12

The Ramsberg and Osgood proposed three parameter representation of stress-strain relations in the inelastic range is:

$$\frac{F_{y}}{F_{o,\infty}} = \left(\frac{F_{y}}{F_{o,y}}\right)^{n}$$

(3)

The equation for $n$ is:

$$n = 1 + \log_{10} \left(\frac{17/7}{\log_{10} \left(F_{o,y}/F_{o,\infty}\right)}\right)$$

(4)

Fig. Bl.13 is a plot of equation (4). The quantities $E$, $n$, and $F_{o,y}$ are non-dimensional and may be used in determining the non-dimensional curves of Fig. Bl.14. $E$, $n$, and $F_{o,y}$ must be known to use these curves in obtaining values on the stress-strain curve. Table Bl.1 gives the values of $F_{o,y}$, $F_{o,\infty}$, $n$, etc., for many flight vehicle materials. Notice that the shape parameter varies widely for materials, being as low as 4 and as high as 90.

Bl.13 Influence of Temperature on Material Properties

Before the advent of the supersonic airplane or the long range missile, the aeronautical structures engineer could design the airframe of aircraft using the normal static mechanical properties of materials, since the temperature rise encountered by such aircraft had practically no effect on the material strength properties. The development of the turbine jet and rocket jet power plants provided the means of opening up the whole new field of supersonic and space flight. The flight environmental conditions were now greatly expanded, the major change being that aerodynamic heating caused by high speeds in the atmosphere caused surface temperature on the airframe which would greatly effect the normal static material strength properties and thus temperature and time became important in the structural design of certain types of flight vehicles.

Bl.14 Creep of Materials

It is obvious that temperature can weaken a material because if the temperature is high enough the material will melt or flow and thus have no load carrying capacity as a structural member. When a stressed member is subjected to temperature, it undergoes a change of shape in addition to that of the well known thermal expansion. The term creep is used to describe this general influence of temperature and time on a stressed material. Creep is defined in general as the progressive, relatively slow change in shape under stress when subjected to an elevated temperature. A simple illustration of creep is a person standing on a bituminous road surface on a very hot summer day. The longer he stands on the same spot the deeper the shoe soles settle into the road surface, whereas on a cold winter day the same time of standing on one spot would produce no noticeable penetration of the road surface.

High temperature, when used in reference to creep, has different temperature values for different materials for the same amount of creep. For example, mercury, which melts at \(-38^\circ F\), may creep a certain amount at \(-75^\circ F\), whereas tungsten, which melts at \(6170^\circ F\), may not creep as much at \(2000^\circ F\) as the mercury under \(-75^\circ F\). All materials creep under conditions of temperature, stress and time of stress application. The simplest manner in which to obtain the effects of creep is to study its effect on the static stress-strain diagram for the material.
### Table B1.1 Values of $F_p$, $F_{pT}$, $E_p$, $F_{pT}$, $F_{pT}^0$,  $n$, for Various Materials under Room & Elevated Temperatures (From Ref. 6)

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>Temp. Exp.</th>
<th>Temp. off</th>
<th>$e_%$</th>
<th>$F_p$</th>
<th>$F_{pT}$</th>
<th>$F_{pT}^0$</th>
<th>$E_p$</th>
<th>$F_{pT}$</th>
<th>$n$</th>
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</thead>
<tbody>
<tr>
<td><strong>STAINLESS STEEL</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AISI 301 1/4 Hard Sheet</td>
<td>1/2</td>
<td>RT</td>
<td>25</td>
<td>125</td>
<td>80</td>
<td>27.0</td>
<td>72</td>
<td>63</td>
<td>6.9</td>
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<tr>
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<td>25</td>
<td>125</td>
<td>43</td>
<td>26.0</td>
<td>28.2</td>
<td>33</td>
<td>5.2</td>
</tr>
<tr>
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<td>RT</td>
<td>15</td>
<td>150</td>
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<tr>
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<td>RT</td>
<td>15</td>
<td>118</td>
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<td>23.2</td>
<td>106.5</td>
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<td>8.2</td>
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<td>110</td>
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<td>94.5</td>
<td>93.5</td>
<td>8.0</td>
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<td>12</td>
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<td>16.2</td>
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<td>12</td>
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#### LOW CARBON & ALLOY STEELS

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<th>$F_{pT}$</th>
<th>$F_{pT}^0$</th>
<th>$E_p$</th>
<th>$F_{pT}$</th>
<th>$n$</th>
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#### HEAT RESISTANT ALLOYS

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<tr>
<td>AISI 301, Heat Treated</td>
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<td>RT</td>
<td>15</td>
<td>140</td>
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<tr>
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<td>140</td>
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</table>

Fig. Bl.15 (curves A and B) show the stress-strain curve for a material. Curve (A) is for a low elevated temperature condition and curve (B) that of a high elevated temperature condition. The results were obtained by a normal testing machine procedure requiring a short time test period, hence the results can be considered as independent of time.

![Stress-Strain Curve](image)

**Fig. Bl.15 (Ref. 2) - Effect of temperature and time on the strength characteristics of metals.**

The figure shows that the higher temperature (curve B) reduces the ultimate strength, yield strength and modulus of elasticity of the material as compared to curve (A) which is a test at a lower temperature.

**EFFECT OF TIME**

If temperature and stress are of such combination as to produce appreciable creep, then time becomes an important effect. For example, in Fig. Bl.15, if the material at low temperature (curve A) is stressed to a value P within the elastic limit of the material, and this stress is maintained for a considerable time period, very little creep, if any, will be detected, and when the stress is removed, the material will practically resume its original dimensions. However, if the material is stressed to the same value P but under a high temperature condition, creep will occur just as long as the stress P remains applied and the stress-strain curve will take a shape like curve (C) in Fig. Bl.15. This time dependent strain will follow a characteristic pattern. The material will never return to its original shape after creep has taken place regardless if the stress is removed. If both stress and high temperature continue, rupture produced by creep will finally occur.

**Bl.15. The General Creep Pattern.**

A typical manner of plotting creep-rupture test data is illustrated in Fig. Bl.16. For metals tested at high value of stress or temperature, three stages in the creep-time relation can be observed as shown in Fig. Bl.16. The initial stage, often called the stage of primary creep, includes the elastic deformation and that region where the rate of creep deformation decreases rather rapidly with time, which no doubt indicates an influence of strain hardening. The second stage, often referred to as the secondary creep stage, represents a stage where the rate of strain has decreased to a constant value (except for high stress) for a considerable time period, and this stage represents the period of minimum creep rate. The third stage, often called the tertiary creep stage, represents the period where the reduction in cross-sectional area leads to a higher stress, a greater creep rate and finally rupture.

**Transition Point.** The inflection point between the constant creep rate of the second stage and the increasing rate of the third stage is referred to as the transition point. Failure generally occurs in a relatively short time after the transition point. Transition points may not occur at very low stresses and may also not be definable at very high stresses. Minimum creep rate is that indicated in the second stage, where the creep rate is practically constant.

![Creep-Rupture Curve](image)

**Fig. Bl.16 Typical creep-rupture curve.**

Fig. Bl.17 shows a plot of creep-time curves for a material at constant temperature and several stress levels. It is noticed that increasing stress changes the creep-time relationship considerably.

**Bl.16 Stress-Time Design Charts.**

In many structural design problems involving elevated temperatures such as power
plants, the critical design factor is not strength but the permissible amount of creep that can be allowed to still permit the structure or machine to function or operate satisfactorily. Extensive tests are usually necessary to provide reliable creep design information and such test information is often recorded in the form as illustrated in Fig. B1.18.

Fig. B1.18 (Ref. 2) - Stress-time design chart at a single constant temperature for selecting limiting stress values.

B1.17 Effect of Time of Exposure.

In general materials can be roughly classified into those which time of exposure to elevated temperatures has great influence on the mechanical properties and those where such exposure produces relatively small effect.

This general fact is illustrated in Figs. B1.19 and B1.20. The yield strength of the aluminum alloy in Fig. B1.19 is far more influenced by time of exposure to elevated temperature than the steel alloy as shown in Fig. B1.20.

B1.18 Effect of Rapid Rate of Heating

Supersonic aircraft and missiles are subjected to rapid aerodynamic heating. The results of tests indicate that in general metals can withstand substantially higher

Fig. B1.19 (Ref. 3) - Effect of exposure at elevated temperatures on the room-temperature tensile yield strength (\(F_{\text{ty}}\)) of 7079-T6 aluminum alloy (hand forgings).

Fig. B1.20 (Ref. 3) - Effect of temperature on the tensile yield strength (\(F_{\text{ty}}\)) of 5 Cr-Mo-V aircraft steels.

stresses at a given temperature when heated from zero up to 2000°F per second under constant load conditions than when loaded after the material has been 1/2 hour or longer at constant temperature. Increasing the temperature rate from 200°F to 2000°F per second or more results in only a small increase in strength. Figs. B1.21 and B1.22 (Ref. 4) show the effect of temperature rates up to 1000°F per second upon the yield and ultimate

Fig. B1.21 - Tensile yield stress of 2024-T3 aluminum alloy for temperature rates from 0.2°F to 100°F per second and of stress-strain tests for 1/2-hour exposure.
strength respectively of aluminum alloy as compared to values when loaded after the material has been exposed 1/2 hour at constant temperature.


The development of the missile and the space vehicle brought another factor into the ever increasing number of environmental conditions that affect structural design, namely, extremely low temperatures. For example, in space the shady side of the flight vehicle is subjected to very low temperatures. Missiles carry fuels and oxidizers such as liquid hydrogen and oxygen which boil at -423 and -297°F respectively. In general, low temperatures increase the strength and stiffness of materials. This effect tends to decrease the ductility of the material or, in other words, produce brittleness, a property that is not desirable in structures because of the possibility of a catastrophic failure. In general, the hexagonal closely packed crystalline structures are best suited for giving the best service under low temperatures. The most important of such materials are aluminum, titanium, and nickel-base alloys. Fig. Bl.22 shows the effect of both elevated and low temperatures on the ultimate tensile strength of 2014-T6 aluminum alloy under various exposure times.

Bl.20 Fatigue of Materials.

Designing structures to provide safety against what is called fatigue failure is one of the most important and difficult problems facing the structural designer of flight vehicles. Fatigue failure is failure due to being stressed a number of times. For example, a beam may be designed to safely and efficiently carry a design static load and it will carry this static load indefinitely without failure. However, if this load is repeated a large enough number of times, it will fail under this static design load. The higher the beam stress under the static design load, the less the number of repeated loadings to cause failure.

To date no adequate theory has been developed to clearly explain the fatigue failure of materials. Fatigue failure appears to begin with a crack starting at a point of weakness in the material and progressing along crystal boundaries. A microscopic examination of metals indicates there are many small cracks scattered throughout a material. Under the action of repeated stress these small cracks open and close during the stress cycle. The cracks cause higher stress to exist at the base of the crack as compared to the stress if there were no crack. Under this repeated concentration of stress, the cracks will gradually extend across the section of the member and finally causing complete failure of the member.

Fatigue testing consists of 3 types: (1) the testing of material crystals, (2) the testing of small structural test specimens, and (3) the testing of complete composite structures. A tremendous amount of test information is available for the second type of testing. More and more attention is being given to the third type of testing. For example, a complete airplane wing or fuselage is often subjected to elaborate fatigue testing in order to insure the safe design life of the airplane.
The presence of cracks initially is not necessary to start a fatigue or progressive failure as irregularities such as slag inclusions, surface scratches, pitting, etc., can cause corrosive action to start, thus supplying the condition for the promotion of cracks and the resultant progressive failure.

The strength of ferrous metals under repeated stresses is often referred to as the endurance or fatigue limit. The endurance limit stress is the stress that can be repeated an infinite number of times without causing fracture of the material. Non-ferrous materials such as the aluminum alloys do not have an endurance limit as defined above but continue to weaken as the stress cycles are increased. Due to this fact and also since the required service life of structures and machines vary greatly, it is customary to refer to the strength under repeated stresses as endurance or fatigue strength instead of endurance limit. Thus the fatigue strength is the maximum stress that can be repeated for a specified number of cycles without producing failure of the structural unit.

The results of testing a specimen under repeated stresses such as tension, compression, bending, etc., is often plotted in a form which is referred to as the S-N (stress versus cycles) diagram, as illustrated in Fig. Bl.24.

![S-N Diagram](image)

Fig. Bl.24

The problem of fatigue design of aircraft airframes is covered in Chapter C13 of this book.


An impact load when applied to a structure produces appreciable shock or vibration. To produce such action, the load must be applied rapidly, that is, in a short interval of time. The effect of impact loads differs from that of static loads in that impact loads appreciably affect the magnitude of the stresses produced in a member and also the resistance properties or behavior of the material under load. The importance and effect of dynamic loads on the magnitude of stresses in aircraft structures is discussed briefly in Chapter A44. The limited discussion which follows will deal only with the effect of impact loads on the behavior of materials.

**IMPACT TESTING METHODS**

There are in general two types of tests to determine the behavior of materials under impact loads. The usual impact test which has been conducted for many years is referred to as the notched bar test and consists of subjecting notched specimens to axial, bending and torsional loads by the well known Charpy or Izod impact testing machines. In both of these machines an impact load is applied to the specimen by swinging a weight $W$ from a certain vertical height $(h)$ to strike and rupture the notched specimen and then stopping at a vertical height $(h')$. The energy expended in rupturing the specimen is then equal approximately to $(Wh - Wh')$. This type of test is primarily used for studying the influence of metallurgical variables.

The other type of impact testing is made on unnotched specimens and the general purpose is to obtain the stress-strain diagram of materials under impact load or the load-distortion diagram of a structural member or composite structure as the unit is completely fractured under an impact load.

B1.22 Examples of Some Results of Impact Testing of Materials.

Figs. Bl.25, 26 and 27 show the results of impact tests upon the stress-strain curve as compared to the static stress-strain diagram (Ref. 5).

![Stress-Strain Curve](image)

Fig. Bl.25 - Stress-Strain curves, 24ST aluminum alloy.
Fig. Bl.26 - Stress-strain curves, Dow metal X.

Fig. Bl.27 - Stress-strain curves, SAE 6140, drawn 1020°F.

Table Bl.2 shows additional impact testing results as compared to the static test results.

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<tr>
<td>Copper</td>
<td>1.483</td>
</tr>
</tbody>
</table>

-x Oil quenched from 1020°F.
-xx Cold-rolled.

REFERENCES

Ref. 4. NACA Technical Note 3462.
Ref. 5. NACA Technical Note 588.
Ref. 6. From Structures Manual, Convair Astronautics.
CHAPTER B2
MECHANICAL AND PHYSICAL PROPERTIES OF METALLIC MATERIALS FOR FLIGHT VEHICLE STRUCTURES

General Explanation. It would require several hundred pages to list the properties of the many materials used in flight vehicle structural design. The metallic materials presented in this chapter are those most widely used and should be sufficient for the use of the student in his structural analysis and design problems. All Tables and Charts in this chapter are taken from the government publication "Military Handbook, MIL-HDBK-5, August, 1962. Metallic Materials and Elements for Flight Vehicle Structures". This publication is for sale by the Supt. of Public Documents, Washington 25, D.C. The properties given in the various tables are for a static loading condition under room temperature. The effect of temperature upon the mechanical properties is given in the various graphs.

AISI ALLOY STEELS

Table B2.1 (AISI) Alloy Steels

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<td>Basis</td>
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<td></td>
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</table>

Mechanical properties:

| $F_{tu}$ ksi | 95 | 90 | 125 | 150 | 180 | 200 | 260 |
| $F_{cs}$ ksi | 75 | 70 | 103 | 132 | 163 | 175 | 217 |
| $F_{cr}$ ksi | 75 | 70 | 113 | 145 | 179 | 198 | 242 |
| $F_{cr}$ ksi | 55 | 55 | 82  | 95  | 109 | 119 | 149 |
| $F_{cr}$ ksi: | | | | | | | |
| ($e/D=1.5$) | 140 | 140 | 194 | 219 | 250 | 272 | 347 |
| ($e/D=2.0$) | 140 | 140 | 251 | 287 | 326 | 355 | 440 |
| $e$, percent. | | | | | | | |

See table 2.3.1.1(b) | See table 2.3.1.1(c) | $^*L$ 10 | $^*T$ 3 |

$E$, 10$^6$ psi | 29.0 |
$E_t$, 10$^6$ psi | 29.0 |
$G$, 10$^6$ psi | 11.0 |

Physical properties:

| $C$, Btu/(lb)(F) | 0.283 |
| $K$, Btu/(hr)(ft$^2$)(F).ft | 0.114 (at 32$^o$ F) |
| $a$, 10$^{-6}$ in./in./F | 22.0 (at 32$^o$ F) |
| $a$, 10$^{-6}$ in./in./F | 9.3 (0$^o$ to 200$^o$ F) |

B2.1
Fig. B2.1. Effect of temperature on $E$, $F_{ty}$, and $E$ of AISI alloy steel.

Fig. B2.2. Effect of temperature on the compressive yield strength ($F_{cy}$) of heat-treated AISI alloy steels.

Fig. B2.3. Effect of temperature on the ultimate shear strength ($F_{su}$) of heat-treated AISI alloy steels.

Fig. B2.4. Effect of temperature on the ultimate bearing strength ($F_{bry}$) of heat-treated AISI alloy steels.

Fig. B2.5. Effect of temperature on the bearing yield strength ($F_{byy}$) of heat-treated AISI alloy steels.
### Table B2.2 Design Mechanical and Physical Properties of 5Cr-Mo-V Aircraft Steel

<table>
<thead>
<tr>
<th>Basis</th>
<th>Mechanical properties:</th>
<th>(*)</th>
<th>(*)</th>
<th>(*)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_{tu}$ kpsi</td>
<td>240</td>
<td>260</td>
<td>280</td>
</tr>
<tr>
<td></td>
<td>$F_{ts}$ kpsi</td>
<td>220</td>
<td>240</td>
<td>260</td>
</tr>
<tr>
<td></td>
<td>$F_{yw}$ kpsi</td>
<td>145</td>
<td>155</td>
<td>170</td>
</tr>
<tr>
<td></td>
<td>$F_{ytr}$ kpsi</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(e/D=1.5)$</td>
<td>400</td>
<td>435</td>
<td>465</td>
</tr>
<tr>
<td></td>
<td>$(e/D=2.0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$F_{ytr}$ kpsi</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(e/D=1.5)$</td>
<td>315</td>
<td>340</td>
<td>365</td>
</tr>
<tr>
<td></td>
<td>$(e/D=2.0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$e$, percent:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bar, in 4D</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Sheet, in 2 in. (*)</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Sheet, in 1 in. (*)</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>$E$, 10^6 psi</td>
<td></td>
<td>30.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_s$, 10^6 psi</td>
<td></td>
<td>30.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$G$, 10^6 psi</td>
<td></td>
<td>11.0</td>
<td></td>
</tr>
</tbody>
</table>

**Physical properties:**

- $\omega$, lb/in.:
- $C$, Btu/(lb)/(F):
- $K$, Btu/(hr)/(ft^2)/(F)/(ft):
- $\alpha$, 10^-4 in./in./F:

### Footnotes:

- * Minimum properties expected when heat treated as recommended in section 2.1.2.
- * For sheet thickness greater than 0.080 inch.
- * Calculated values.

---

**Figures:**

- B2.3 Effect of temperature on the tensile and compressive modulus (E and Eq) of 5 Cr-Mo-V aircraft steels.
- B2.4 Effect of temperature on the compressive yield strength ($F_{cy}$) of 5 Cr-Mo-V aircraft steels.
- B2.5 Effect of temperature on the ultimate shear strength ($F_{su}$) of 5 Cr-Mo-V aircraft steels.
- B2.6 Effect of temperature on the ultimate tensile strength ($F_{tu}$) of Cr-Mo-V aircraft steels.
- B2.7 Effect of temperature on the tensile yield strength ($F_{ty}$) of 5 Cr-Mo-V aircraft steels.
- B2.8 Effect of temperature on the bearing yield strength ($F_{by}$) of 5 Cr-Mo-V aircraft steels.
# 17-7 PH STAINLESS STEEL

Table B2.3 Design Mechanical and Physical Properties of 17-7 PH Stainless Steel

<table>
<thead>
<tr>
<th>Alloy</th>
<th>17-7 PH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sheet, strip, and plate(*)</td>
</tr>
<tr>
<td>Condition</td>
<td>TH 1050</td>
</tr>
<tr>
<td>Thickness or diameter, in.</td>
<td>0.005 to 0.500</td>
</tr>
<tr>
<td>Basis</td>
<td>S</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{lim}$, ksi</td>
<td>180</td>
</tr>
<tr>
<td>$F_{s}$, ksi</td>
<td>150</td>
</tr>
<tr>
<td>$F_{y}$, ksi</td>
<td>138</td>
</tr>
<tr>
<td>$F_{t}$, ksi</td>
<td>117</td>
</tr>
<tr>
<td>$F_{s}$, ksi (ε/D=1.5)</td>
<td>297</td>
</tr>
<tr>
<td>$F_{s}$, ksi (ε/D=2.0)</td>
<td>380</td>
</tr>
<tr>
<td>$F_{t}$, ksi (ε/D=1.5)</td>
<td>225</td>
</tr>
<tr>
<td>$F_{t}$, ksi (ε/D=2.0)</td>
<td>247</td>
</tr>
<tr>
<td>$ε$, percent</td>
<td>See table 2.7.2.1(b)</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$E_{s}$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td></td>
</tr>
</tbody>
</table>

Physical properties:

| $ω$, lb/ln.ft.        | 0.276   |
| $C$, Btu/(ib)(F)      | 0.11  (ε) |
| $K$, Btu/(hr)(F)/(Ft) | 9.75 (at 300°F) |
| $σ$, 10^-4 in./in./F  | 6.3 (70°F to 600°F) for TH 1050. |
|                      | 6.3 (70°F to 600°F) for RH 950. |

* Test direction longitudinal for widths less than 9 in.; transverse for widths 9 in. and over.

* Test direction longitudinal; these properties not applicable to the short transverse (thickness) direction.

* Vendor guaranteed minimums for $F_{lim}$, $F_s$, and $ε$.

* Calculated value.

---

**Fig. B2.13.** Effect of temperature on the ultimate tensile strength ($F_{lim}$) of 17-7 PH (TH1050) stainless steel.

**Fig. B2.14.** Effect of temperature on the tensile yield strength ($F_{ty}$) of 17-7 PH (TH1050) stainless steel.
Fig. B2.15. Effect of temperature on the compressive yield strength ($F_{\text{cy}}$) of 17-7 PH (TH1050) stainless steel.

Fig. B2.18. Effect of temperature on the ultimate bearing strength ($F_{\text{bu}}$) of 17-7 PH (TH1050) stainless steel.

Fig. B2.16. Effect of temperature on the ultimate shear strength ($F_{\text{s}}$) of 17-7 PH (TH1050) stainless steel.

Fig. B2.19. Effect of temperature on the bearing yield strength ($F_{\text{by}}$) of 17-7 PH (TH1050) stainless steel.

Fig. B2.20. Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of 17-7 PH (TH1050) stainless steel.
Table B2.4 Design Mechanical and Physical Properties of 17-4 PH Stainless Steel

<table>
<thead>
<tr>
<th>Alloy</th>
<th>17-4 PH</th>
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</thead>
<tbody>
<tr>
<td>Form</td>
<td>Plate</td>
</tr>
<tr>
<td>Condition</td>
<td>H 900</td>
</tr>
<tr>
<td>Thickness or diameter, in...</td>
<td>8 and under</td>
</tr>
<tr>
<td>Basis</td>
<td>S(*)</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{ut}$, ksi</td>
<td>190</td>
</tr>
<tr>
<td>$F_{ty}$, ksi</td>
<td>170</td>
</tr>
<tr>
<td>$F_{ys}$, ksi</td>
<td>178</td>
</tr>
<tr>
<td>$F_{ts}$, ksi</td>
<td>123</td>
</tr>
<tr>
<td>$F_{sh}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$F_{sh}(e/D=1.5)$</td>
<td>313</td>
</tr>
<tr>
<td>$F_{sh}(e/D=2.0)$</td>
<td>380</td>
</tr>
<tr>
<td>$F_{sh}(e/D=2.5)$</td>
<td>255</td>
</tr>
<tr>
<td>$F_{sh}(e/D=3.0)$</td>
<td>280</td>
</tr>
<tr>
<td>$\varepsilon$, percent:</td>
<td></td>
</tr>
<tr>
<td>In 2 in.</td>
<td>10</td>
</tr>
<tr>
<td>In 4 D.</td>
<td>...</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$E_{ty}$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td></td>
</tr>
</tbody>
</table>

Physical properties:
- $a$, lb/in^2: 0.282.
- $C$, Btu/(lb)(F): 0.11 (32°F to 212°F).
- $K$, Btu/(hr)(ft^2)(F)/ft: 10.3 (at 900°F); 11.2 (at 500°F); 13.1 (at 900°F).
- $\sigma$, 10^6 in./in./F: 6.0 (70°F to 200°F); 6.1 (70°F to 400°F); 6.5 (70°F to 900°F).

* Vendors guaranteed minimums for $F_{ut}$, $F_{ty}$, and $\varepsilon$.
* Test direction longitudinal; these properties not applicable to the short transverse (thickness) direction.

- Fig. B2.21: Effect of temperature on the ultimate tensile strength ($F_{ut}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.22: Effect of temperature on the tensile yield strength ($F_{ty}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.23: Effect of temperature on the compressive yield strength ($F_{cy}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.24: Effect of temperature on the ultimate shear strength ($F_{sh}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.25: Effect of temperature on the ultimate bearing strength ($F_{ub}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.26: Effect of temperature on the bearing yield strength ($F_{by}$) of 17-4 PH (H900) stainless steel.
- Fig. B2.27: Effect of temperature on the tensile and compressive modulus (E and $E_{ty}$) of 17-4 PH (H900) stainless steel.
Table B2.5  Design Mechanical and Physical Properties of AM-350 Stainless Steel

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<tbody>
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<td>Form</td>
<td>Sheet and strip*</td>
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<tr>
<td>Condition</td>
<td>DA</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>0.187 and under</td>
</tr>
<tr>
<td>Basis</td>
<td>S</td>
</tr>
</tbody>
</table>

**Mechanical properties:**

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<tr>
<th>Property</th>
<th>DA</th>
<th>SCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{ty}$, ksi</td>
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<td>185</td>
</tr>
<tr>
<td>$F_{ty}$, ksi</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>$F_{ty}$, ksi</td>
<td>142</td>
<td>158</td>
</tr>
<tr>
<td>$F_{te}$, ksi</td>
<td>107</td>
<td>120</td>
</tr>
<tr>
<td>$F_{te}$, ksi  $(e/D=1.5)$</td>
<td>272</td>
<td>305</td>
</tr>
<tr>
<td>$F_{te}$, ksi  $(e/D=2.0)$</td>
<td>330</td>
<td>370</td>
</tr>
<tr>
<td>$F_{te}$, ksi  $(e/D=1.5)$</td>
<td>202</td>
<td>225</td>
</tr>
<tr>
<td>$F_{te}$, ksi  $(e/D=2.0)$</td>
<td>223</td>
<td>247</td>
</tr>
<tr>
<td>$e$, percent</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Property</th>
<th>DA</th>
<th>SCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$, $10^6$ psi</td>
<td>29.0</td>
<td></td>
</tr>
<tr>
<td>$E_v$, $10^6$ psi</td>
<td>30.0</td>
<td></td>
</tr>
<tr>
<td>$G$, $10^6$ psi</td>
<td>11.0</td>
<td></td>
</tr>
</tbody>
</table>

**Physical properties:**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$, lb/in.(^2)</td>
<td>0.282</td>
</tr>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.12 (32° to 212° F)</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(F)(Ft)</td>
<td>8.4 (at 100° F); 11.7 (at 800° F)</td>
</tr>
<tr>
<td>$\alpha$, $10^{-4}$ in./in./°F</td>
<td>8.3 (70° to 212° F); 7.2 (70° to 332° F)</td>
</tr>
</tbody>
</table>

* Test direction longitudinal for widths less than 9 in.; transverse for widths 9 in. and over.
Fig. B2.34 Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of AM-350 stainless steel (SCT).

Fig. B2.35 Effect of temperature on the bearing yield strength ($F_{by}$) of AM-350 stainless steel (SCT).

Fig. B2.36 Effect of temperature on the compressive yield strength ($F_{cy}$) of AM-350 stainless steel (SCT).

Fig. B2.37 Effect of temperature on the ultimate bearing strength ($F_{byu}$) of AM-350 stainless steel (SCT).

Fig. B2.38 Effect of temperature on the ultimate shear strength ($F_{su}$) of AM-350 stainless steel (SCT).
## AISI 301 Stainless Steel

Table B2.6 Design Mechanical and Physical Properties of AISI 301 Stainless Steel

<table>
<thead>
<tr>
<th>Alloy</th>
<th>AISI 301 *</th>
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<tbody>
<tr>
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<td>Plate ‡, sheet, and strip</td>
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<tr>
<td></td>
<td>Annealed</td>
</tr>
<tr>
<td>Form</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S</td>
</tr>
<tr>
<td>Condition</td>
<td></td>
</tr>
<tr>
<td>Basis</td>
<td></td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>* $F_{tu}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>75</td>
</tr>
<tr>
<td>T</td>
<td>75</td>
</tr>
<tr>
<td>$F_{ts}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>30</td>
</tr>
<tr>
<td>T</td>
<td>30</td>
</tr>
<tr>
<td>$F_{se}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>35</td>
</tr>
<tr>
<td>T</td>
<td>35</td>
</tr>
<tr>
<td>$F_{sw}$, ksi:</td>
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<tr>
<td>(e/D=1.5)</td>
<td>150</td>
</tr>
<tr>
<td>(e/D=2.0)</td>
<td>50</td>
</tr>
<tr>
<td>$e$, percent:</td>
<td></td>
</tr>
<tr>
<td>(e/D=1.5)</td>
<td>(%)</td>
</tr>
<tr>
<td>(e/D=2.0)</td>
<td>(%)</td>
</tr>
<tr>
<td>$E_s$, 10^4 psi:</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>29.0</td>
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<td>T</td>
<td>29.0</td>
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<tr>
<td>$E_m$, 10^4 psi:</td>
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</tr>
<tr>
<td>L</td>
<td>28.0</td>
</tr>
<tr>
<td>T</td>
<td>28.0</td>
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<tr>
<td>$G_s$, 10^3 psi:</td>
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</tr>
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<td>12.5</td>
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Physical properties:

<table>
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<tr>
<th></th>
<th>0.286.</th>
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</thead>
<tbody>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.108 (at 32°F).</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(°F)(Ft)</td>
<td>7.74 (at 32°F).</td>
</tr>
<tr>
<td>$\alpha$, 10^-4 in./in./F</td>
<td>9.2 (70°F to 200°F).</td>
</tr>
</tbody>
</table>

* Properties for annealed condition also applicable to annealed AISI 301, 303, 304, 321, and 347.

‡ Only annealed condition applicable to plate.

* See table B2.1.(b).

Note.—Yield strength, particularly in compression, and modulus of elasticity in the longitudinal direction may be raised appreciably by thermal stress-relieving treatment in the range 500°F to 800°F.
AISI 301 STAINLESS STEEL (Cont.)

Fig. B2.39 Effect of temperature on the ultimate tensile strength ($F_{tu}$) of AISI 301 (half-hard) stainless steel.

Fig. B2.40 Effect of temperature on the ultimate shear strength ($F_{ts}$) of AISI 301 (half-hard) stainless steel.

Fig. B2.41 Effect of temperature on the compressive yield strength ($F_{cy}$) of AISI 301 (half-hard) stainless steel.

Fig. B2.42 Effect of temperature on the ultimate tensile strength ($F_{tu}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.43 Effect of temperature on the ultimate bearing strength ($F_{bu}$) of AISI 301 (half-hard) stainless steel.

Fig. B2.44 Effect of temperature on the bearing yield strength ($F_{by}$) of AISI 301 (half-hard) stainless steel.

Fig. B2.45 Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of AISI 301 (half-hard) stainless steel.

Fig. B2.46 Effect of temperature on the ultimate tensile strength ($F_{tu}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.47 Effect of temperature on the tensile yield strength ($F_{ty}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.48 Effect of temperature on the compressive yield strength ($F_{cy}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.49 Effect of temperature on the ultimate shear strength ($F_{ts}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.50 Effect of temperature on the ultimate bearing strength ($F_{bu}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.51 Effect of temperature on the bearing yield strength ($F_{by}$) of AISI 301 (full-hard) stainless steel.

Fig. B2.52 Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of AISI 301 (full-hard) stainless steel.
### 2014 Aluminum Alloys (Sheet & Plate, Extrusions, Forgings)

NOTE: Values in (A) columns are minimum guaranteed values. Values in (B) column will be met or exceeded by 90 percent of material supplied.

Table B2.7 Design Mechanical and Physical Properties of 2014 Aluminum Alloy (Sheet and Plate)

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</thead>
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<td><strong>Thickness, in</strong>:</td>
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<tr>
<td>0.020-0.039</td>
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<tr>
<td>0.040-0.049</td>
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<tr>
<td>0.500-1.000</td>
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<tr>
<td>1.001-1.500</td>
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<tr>
<td>1.501-2.000</td>
</tr>
<tr>
<td>2.001-3.000</td>
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<tr>
<td>3.001-4.000</td>
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<tr>
<td>Basis: A B A B A B A B A A A</td>
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#### Mechanical Properties:

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<th>Property</th>
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<th>67</th>
<th>68</th>
<th>65</th>
<th>63</th>
<th>59</th>
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</thead>
<tbody>
<tr>
<td>$F_{cm}$ ksi</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
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</tr>
<tr>
<td>$F_{cu}$ ksi</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
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<td>ST</td>
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<td>62</td>
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</tr>
<tr>
<td>$F_{un}$ ksi</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>41</td>
<td>42</td>
<td>41</td>
<td>42</td>
<td>41</td>
<td>41</td>
<td>40</td>
<td>41</td>
<td>40</td>
<td>41</td>
<td></td>
</tr>
<tr>
<td>$F_{br,bur}$ ksi</td>
<td>(r/D = 1.5)</td>
<td>(r/D = 2.0)</td>
<td>(r/D = 1.5)</td>
<td>(r/D = 2.0)</td>
<td>(r/D = 1.5)</td>
<td>(r/D = 2.0)</td>
<td>(r/D = 1.5)</td>
<td>(r/D = 2.0)</td>
<td>(r/D = 1.5)</td>
<td>(r/D = 2.0)</td>
<td>(r/D = 1.5)</td>
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<td>101</td>
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<td>98</td>
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<td>98</td>
</tr>
<tr>
<td></td>
<td>129</td>
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<td>133</td>
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<td>129</td>
<td>124</td>
<td>129</td>
<td>124</td>
<td>129</td>
<td>124</td>
</tr>
<tr>
<td>$e$, percent</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
<td>ST</td>
<td>L</td>
<td>T</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$E$, 10^3 psi</td>
<td>10.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_s$, 10^3 psi</td>
<td>10.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$, 10^3 psi</td>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Physical Properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$, lb/in.²</td>
<td>0.101</td>
</tr>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.23 (at 212°F)</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(F)/(F)/ft</td>
<td>90 (at 77°F)</td>
</tr>
<tr>
<td>$x$, 10^-4 in./in./F</td>
<td>12.5 (68°F to 212°F)</td>
</tr>
</tbody>
</table>
Table B2.8 Design Mechanical and Physical Properties of Clad 2014 Aluminum Alloy (Clad Sheet and Plate)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Clad 2014</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sheet and plate</td>
</tr>
<tr>
<td></td>
<td>-T6°</td>
</tr>
<tr>
<td>Condition</td>
<td>0.039</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>A</td>
</tr>
<tr>
<td>Basis</td>
<td></td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$ ksi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>64</td>
</tr>
<tr>
<td>T</td>
<td>63</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$F_{tm}$ ksi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>56</td>
</tr>
<tr>
<td>T</td>
<td>55</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$F_{ty}$ ksi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>56</td>
</tr>
<tr>
<td>T</td>
<td>57</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$ ksi</td>
<td></td>
</tr>
<tr>
<td>(e/D = 1.5)</td>
<td>96</td>
</tr>
<tr>
<td>(e/D = 2.0)</td>
<td>122</td>
</tr>
<tr>
<td>$F_{ty}$ ksi</td>
<td></td>
</tr>
<tr>
<td>(e/D = 1.5)</td>
<td>78</td>
</tr>
<tr>
<td>(e/D = 2.0)</td>
<td>90</td>
</tr>
<tr>
<td>e, percent</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>7</td>
</tr>
<tr>
<td>T</td>
<td>7</td>
</tr>
<tr>
<td>$E$, 10⁹ psi</td>
<td></td>
</tr>
<tr>
<td>10.5</td>
<td></td>
</tr>
<tr>
<td>$E_{ty}$, 10⁹ psi</td>
<td></td>
</tr>
<tr>
<td>10.7</td>
<td></td>
</tr>
<tr>
<td>$G$, 10⁹ psi</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td></td>
</tr>
</tbody>
</table>
Table B2.9 Design Mechanical and Physical Properties of 2014 Aluminum Alloy (Extrusions)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>2014</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Extruded rod, bar, and shapes</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td></td>
</tr>
<tr>
<td>&lt;25</td>
<td>&gt;25, ≥22</td>
</tr>
<tr>
<td>Thickness, in.³</td>
<td>0.125–0.499</td>
</tr>
<tr>
<td>Basis</td>
<td>A A</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{\text{re}}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>60</td>
</tr>
<tr>
<td>$T$</td>
<td>60</td>
</tr>
<tr>
<td>$F_{\text{tp}}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>53</td>
</tr>
<tr>
<td>$T$</td>
<td>53</td>
</tr>
<tr>
<td>$F_{\text{ru}}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>55</td>
</tr>
<tr>
<td>$T$</td>
<td>53</td>
</tr>
<tr>
<td>$F_{\text{bw}}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>35</td>
</tr>
<tr>
<td>$T$</td>
<td>35</td>
</tr>
<tr>
<td>$F_{\text{bw}}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>90</td>
</tr>
<tr>
<td>$T$</td>
<td>114</td>
</tr>
<tr>
<td>$e$, percent</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>7</td>
</tr>
<tr>
<td>$T$</td>
<td>5</td>
</tr>
<tr>
<td>$E$, 10⁶ psi</td>
<td></td>
</tr>
<tr>
<td>$E_t$, 10⁶ psi</td>
<td></td>
</tr>
<tr>
<td>$G$, 10⁶ psi</td>
<td></td>
</tr>
</tbody>
</table>
### 2014 Aluminum Alloys (Sheet & Plate, Extrusions, Forgings) (Cont.)

#### Table B2.10 Design Mechanical and Physical Properties of 2014 Aluminum Alloy (Forgings)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>2014</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Die forings</td>
</tr>
<tr>
<td></td>
<td>Length ≤3 times width</td>
</tr>
<tr>
<td>Form</td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td>-T4</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>≤4 inches</td>
</tr>
<tr>
<td>Cross-sectional area, in.²</td>
<td>≤16</td>
</tr>
<tr>
<td>≤16</td>
<td>&gt;16</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
</tbody>
</table>

### Mechanical properties:

- **σ_{lmax}, ksi**
  - \( L \): 55
  - \( T \): 52
  - \( ST \): 60

- **σ_{up}, ksi**
  - \( L \): 30
  - \( T \): 28
  - \( ST \): 35

- **σ_{up}, ksi**
  - \( L \): 30
  - \( T \): 28
  - \( ST \): 34

- **σ_{max}, ksi**
  - \( L \): 34
  - \( T \): 39
  - \( ST \): 40

- **σ_{brm}, ksi**
  - \((e/D = 1.5)\): 91
  - \((e/D = 2.0)\): 117

- **σ_{brm}, ksi**
  - \((e/D = 1.5)\): 88
  - \((e/D = 2.0)\): 85

- **e, percent**
  - \( L \): 11
  - \( T \): 3
  - \( ST \): 3

- **E, 10¹² psi**
  - 10.5

- **Eₚ, 10¹² psi**
  - 10.7

- **G, 10¹² psi**
  - 4.0
**EFFECT OF TEMPERATURE ON 2014 ALUMINUM ALLOYS**

Fig. B2.53 Effect of temperature on the ultimate strength ($F_{tu}$) of 2014-T6 aluminum alloy (bare and clad sheet 0.020-0.039 in. thick; bare and clad plate 1.501-4.000 in. thick; rolled bar, rod and shapes; hand and die forgings; extruded bar, rod and shapes 0.125-0.749 in. thick with cross-sectional area $\geq 25$ sq. in.).

Fig. B2.54 Effect of temperature on the tensile yield strength ($F_{ty}$) of 2014-T6 aluminum alloy (bare and clad plate 3.001-4.000 in. thick; rolled bar, rod and shapes; hand and die forgings; extruded bar, rod and shapes 0.125-0.499 in. thick with cross-sectional area $\geq 25$ sq. in.).
Fig. B2.55 Effect of temperature on the compressive yield strength (F<sub>c</sub>) of 2014-T6 aluminum alloy (all products except thick extrusions).

Fig. B2.56 Effect of temperature on the ultimate shear strength (F<sub>u</sub>) of 2014-T6 aluminum alloy (all products except thick extrusions).

Table B2.11 Design Mechanical and Physical Properties of 2024 Aluminum Alloy (Sheet and Plate)

<table>
<thead>
<tr>
<th>Alloy Form</th>
<th>Condition</th>
<th>Thickness (in.)</th>
<th>Mechanical properties:</th>
</tr>
</thead>
<tbody>
<tr>
<td>2024</td>
<td></td>
<td>0.020 - 0.050</td>
<td>F&lt;sub&gt;y&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>0.500 - 1.000</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>0.250 - 0.500</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>B</td>
<td>0.84 - 0.86</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>0.90 - 1.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>0.95 - 1.00</td>
</tr>
</tbody>
</table>

Fig. B2.57 Effect of temperature on the ultimate bearing strength (F<sub>ru</sub>) of 2014-T6 aluminum alloy (all products except thick extrusions).

Fig. B2.58 Effect of temperature on the bearing yield strength (F<sub>bry</sub>) of 2014-T6 aluminum alloy (all products except thick extrusions).

Fig. B2.59 Effect of temperature on the tensile and compressive modulus (E and E<sub>c</sub>) of 2014 and 2017 aluminum alloys.

2024 ALUMINUM ALLOY (BARE SHEET & PLATE, EXTRUSIONS, BAR, ROD & WIRE)
### Table B2.12 Design Mechanical and Physical Properties of 2024 Aluminum Alloy (Extrusions)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Form</th>
<th>2024 Extruded bars, rods, and shapes</th>
<th>Heat treated by user</th>
<th>Heat treated cold worked and aged</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-T4*</td>
<td>-T42</td>
<td>-T81</td>
</tr>
<tr>
<td>Condition</td>
<td>Thickness, in.</td>
<td>0.050-0.249</td>
<td>0.250-0.499</td>
<td>0.500-0.749</td>
</tr>
<tr>
<td>Cross-sectional area, in.²</td>
<td>≥25</td>
<td>≥25, ≥32</td>
<td>≥32</td>
<td></td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_{\text{ax}}$, ksi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>57</td>
<td>61</td>
<td>60</td>
<td>62</td>
</tr>
<tr>
<td>$T$</td>
<td>57</td>
<td>61</td>
<td>60</td>
<td>62</td>
</tr>
<tr>
<td>$F_{\text{ay}}$, ksi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>42</td>
<td>47</td>
<td>44</td>
<td>47</td>
</tr>
<tr>
<td>$T$</td>
<td>42</td>
<td>46</td>
<td>43</td>
<td>46</td>
</tr>
<tr>
<td>$F_{\text{az}}$, ksi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>38</td>
<td>41</td>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>$T$</td>
<td>38</td>
<td>41</td>
<td>39</td>
<td>42</td>
</tr>
<tr>
<td>$F_{\text{bx}}$, ksi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(e/D=1.5)$</td>
<td>30</td>
<td>32</td>
<td>32</td>
<td>33</td>
</tr>
<tr>
<td>$(e/D=2.0)$</td>
<td>85</td>
<td>91</td>
<td>85</td>
<td>91</td>
</tr>
<tr>
<td>$F_{\text{by}}$, ksi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(e/D=1.5)$</td>
<td>108</td>
<td>114</td>
<td>108</td>
<td>114</td>
</tr>
<tr>
<td>$(e/D=2.0)$</td>
<td>59</td>
<td>66</td>
<td>60</td>
<td>66</td>
</tr>
<tr>
<td>$e$, percent</td>
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<td>$L$</td>
<td>12</td>
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<td>10</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>$E$, 10⁶ psi</td>
<td>10.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_x$, 10⁶ psi</td>
<td>10.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$, 10⁶ psi</td>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table B2.13 Design Mechanical and Physical Properties of 2024 Aluminum Alloy (Bar, Rod, and Wire); Rolled, Drawn or Cold Finished; Rolled Tubing

<table>
<thead>
<tr>
<th>Alloy</th>
<th>2024</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Bar, rod and wire; rolled, drawn or cold-finished</td>
</tr>
<tr>
<td>Condition</td>
<td>-T4 or -T351</td>
</tr>
<tr>
<td>Cross-sectional area, in.</td>
<td>≤36</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>Up to 1.000</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{L}$, ksi</td>
<td>62</td>
</tr>
<tr>
<td>$F_{T}$, ksi</td>
<td>51</td>
</tr>
<tr>
<td>$F_{w}$, ksi</td>
<td>40</td>
</tr>
<tr>
<td>$F_{e}$, ksi</td>
<td>40</td>
</tr>
<tr>
<td>$F_{w}$, ksi</td>
<td>32</td>
</tr>
<tr>
<td>$F_{e}$, ksi</td>
<td>37</td>
</tr>
<tr>
<td>$F_{w}$, ksi</td>
<td>118</td>
</tr>
<tr>
<td>$F_{e}$, ksi</td>
<td>56</td>
</tr>
<tr>
<td>$F_{w}$, ksi</td>
<td>64</td>
</tr>
<tr>
<td>$F_{e}$, ksi</td>
<td>67</td>
</tr>
<tr>
<td>$\sigma$, percent</td>
<td>10</td>
</tr>
<tr>
<td>$L$</td>
<td>10</td>
</tr>
<tr>
<td>$T$</td>
<td>10</td>
</tr>
<tr>
<td>$E_x$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$E_y$, 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td></td>
</tr>
</tbody>
</table>

### Physical properties:
- $\rho$, lb/in^3: 0.100
- $C$, Btu/(lb)(F): 0.23 (at 212° F)
- $K$, Btu/(hr)(ft)^2(F)/ft: 70 (at 77° F)*
- $\alpha$, 10^-6 in./in./F: 12.6 (68° to 212° F)
<table>
<thead>
<tr>
<th>Condition</th>
<th>Tensile Strength (psi)</th>
<th>Yield Strength (psi)</th>
<th>Elongation in 2 in. (inches)</th>
<th>Reduction in Area (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-T3</td>
<td>60,000</td>
<td>48,000</td>
<td>22</td>
<td>60</td>
</tr>
<tr>
<td>-T4</td>
<td>50,000</td>
<td>38,000</td>
<td>20</td>
<td>65</td>
</tr>
<tr>
<td>-T6</td>
<td>40,000</td>
<td>30,000</td>
<td>18</td>
<td>70</td>
</tr>
<tr>
<td>-T61</td>
<td>40,000</td>
<td>30,000</td>
<td>18</td>
<td>70</td>
</tr>
<tr>
<td>-T8</td>
<td>30,000</td>
<td>24,000</td>
<td>14</td>
<td>75</td>
</tr>
</tbody>
</table>

**Table 32.14 Design, Mechanical and Physical Properties of 7075-T6 Aluminum Alloy (Sheet, Plate, and Forged Stock)**

CLAD 2024 ALUMINUM ALLOY (SHEET, PLATE, FORGED STOCK)

ANALYSIS AND DESIGN OF FLYING VEHICLE STRUCTURES
EFFECT OF TEMPERATURE ON 2024 ALUMINUM ALLOYS

Fig. B2.60 Effect of temperature on the ultimate tensile strength (F_u) of 2024-T3 and 2024-T4 aluminum alloy (all products except extrusions).

Fig. B2.64 Effect of exposure at elevated temperatures on the room-temperature ultimate tensile strength (F_u) of 2024-T3 and 2024-T4 aluminum alloy (all products except thick extrusions).

Fig. B2.68 Effect of temperature on the ultimate bearing strength (F_bru) of clad 2024-T3 and clad 2024-T4 aluminum alloy (sheet).

Fig. B2.61 Effect of temperature on the ultimate tensile strength (F_u) of 2024-T3 and 2024-T4 aluminum alloy (extrusions).

Fig. B2.65 Effect of temperature on the compressive yield strength (F_cy) of clad 2024-T3 and clad 2024-T4 aluminum alloy (sheet).

Fig. B2.69 Effect of temperature on the bearing yield strength (F_b) of clad 2024-T3 and clad 2024-T4 aluminum alloy (sheet).

Fig. B2.62 Effect of temperature on the tensile yield strength (F_y) of 2024-T3 and 2024-T4 aluminum alloy (all products except extrusions).

Fig. B2.66 Effect of elevation at elevated temperatures on the room-temperature tensile yield strength (F_y) of 2024-T3 and 2024-T4 aluminum alloy (all products except thick extrusions).

Fig. B2.70 Effect of temperature on the tensile and compressive modulus (E and E_c) of 2024 aluminum alloy.

Fig. B2.63 Effect of temperature on the tensile yield strength (F_y) of 2024-T3 and 2024-T4 aluminum alloy (extrusions).

Fig. B2.67 Effect of temperature on the ultimate shear strength (F_sh) of clad 2024-T3 and clad 2024-T6 aluminum alloy (sheet).

Fig. B2.71 Effect of temperature on the elongation of 2024-T3 and 2024-T4 aluminum alloy (all products except thick extrusions).
<table>
<thead>
<tr>
<th>Alloy</th>
<th>7075</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Form</strong>:</td>
<td>Sheet and plate</td>
</tr>
<tr>
<td><strong>Condition</strong>:</td>
<td>-T6</td>
</tr>
<tr>
<td><strong>Thickness, in</strong></td>
<td>0.015-0.039</td>
</tr>
<tr>
<td><strong>Basis</strong>:</td>
<td>A</td>
</tr>
<tr>
<td><strong>Mechanical properties</strong>:</td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$, ksi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>76</td>
</tr>
<tr>
<td>T</td>
<td>76</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$F_{tp}$, ksi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>66</td>
</tr>
<tr>
<td>T</td>
<td>65</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$F_{cp}$, ksi</td>
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</tr>
<tr>
<td>L</td>
<td>67</td>
</tr>
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<td>T</td>
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<td>ST</td>
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<td>$F_{ut}$, ksi</td>
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</tr>
<tr>
<td>L</td>
<td>46</td>
</tr>
<tr>
<td>$F_{pu}$, ksi</td>
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<tr>
<td>(e/D = 1.5)</td>
<td>114</td>
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<tr>
<td>(e/D = 2.0)</td>
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<td>$P_{h1}$, ksi</td>
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</tr>
<tr>
<td>(e/D = 1.5)</td>
<td>92</td>
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<tr>
<td>(e/D = 2.0)</td>
<td>106</td>
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<tr>
<td>$\epsilon$, percent</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>7</td>
</tr>
<tr>
<td>T</td>
<td>7</td>
</tr>
<tr>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>$E$, 10^3 psi</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>10.3</td>
</tr>
<tr>
<td>$E_c$, 10^3 psi</td>
<td>10.5</td>
</tr>
<tr>
<td>$G$, 10^3 psi</td>
<td>3.9</td>
</tr>
</tbody>
</table>

**Physical properties**: | | | | | | | | | | | | | | | | |
| $\omega$, lb/in.² | 0.101 |
| C, Btu/(lb)(F) | 0.23 (at 212°F) |
| K, Btu/(hr)(ft²)(F)/ft | 76 (at 77°F) |
| $\epsilon$, 10^-3 in./in (°F) | 12.9 (68° to 212°F) |

*For the stress relieved temper -T611, all values for the -T6 temper apply with the exception of $F_{cp}$. Applicable $F_{cp}$ values are as follows: 2.001-2.500 L 42 2.501-3.000 L 40 0.250-2.000 L 44 0.250-2.000 L 44 2.001-2.500 L 42 2.501-3.000 L 40 0.250-2.000 L 44 See Table 3.111.11.
Table B2.16 Design Mechanical and Physical Properties of Clad 7075 Aluminum Alloy (Sheet and Plate)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Clad 7075</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sheet and plate</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Condition</th>
<th>Thickness, in.</th>
<th>0.015-0.039</th>
<th>0.040-0.062</th>
<th>0.063-0.087</th>
<th>0.188-0.240</th>
<th>0.250-0.499</th>
<th>0.500-1.000</th>
<th>1.001-2.000</th>
<th>2.001-2.500</th>
<th>2.501-3.000</th>
<th>3.001-3.500</th>
<th>3.501-4.000</th>
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</thead>
<tbody>
<tr>
<td>Basis</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
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<td>$T$</td>
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<td>$F_{tp}, ksi$</td>
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<tr>
<td>$S$</td>
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<td>$F_{tp}, ksi$</td>
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<td>64</td>
<td>66</td>
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<td>...</td>
<td>...</td>
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</tr>
<tr>
<td>$F_{um, kn}$</td>
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<td>43</td>
<td>44</td>
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<td>45</td>
<td>46</td>
<td>47</td>
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<td>116</td>
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<tr>
<td>$(e/D = 2.0)$</td>
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<td>139</td>
<td>137</td>
<td>141</td>
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<td>142</td>
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<td>142</td>
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<td>84</td>
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<td>101</td>
<td>104</td>
<td>102</td>
<td>106</td>
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<td>107</td>
<td>98</td>
<td>100</td>
<td>99</td>
<td>102</td>
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<td>$e$, percent</td>
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<td>8</td>
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</tr>
<tr>
<td>$E$, 10^6 psi</td>
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<td>...</td>
<td>10.3</td>
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<td>$E_o$, 10^6 psi</td>
<td>9.5</td>
<td>9.8</td>
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<td>10.0</td>
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<td>$G$, 10^3 psi</td>
<td>9.7</td>
<td>...</td>
<td>...</td>
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<td>10.0</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>$w$, lb/in.</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.23 (at 212°F)</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(F)/F</td>
<td>70 (at 77°F)</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$a$, 10^-4 in./F</td>
<td>12.9 (68° to 212°F)</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$d$ For the stress relieved temper -T66, all values for the -T6 temper apply with the exception of $F_{um}$, Applicable $F_{um}$ values are as follows:

<table>
<thead>
<tr>
<th>Thickness (in.)</th>
<th>Direction of test</th>
<th>$F_{u}$ (A values)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015-0.039</td>
<td>L</td>
<td>62</td>
</tr>
<tr>
<td>0.040-0.062</td>
<td>L</td>
<td>61</td>
</tr>
<tr>
<td>0.063-0.087</td>
<td>L</td>
<td>60</td>
</tr>
<tr>
<td>0.188-0.240</td>
<td>L</td>
<td>65</td>
</tr>
<tr>
<td>0.250-0.499</td>
<td>L</td>
<td>64</td>
</tr>
<tr>
<td>0.500-1.000</td>
<td>L</td>
<td>64</td>
</tr>
<tr>
<td>1.001-2.000</td>
<td>L</td>
<td>64</td>
</tr>
<tr>
<td>2.001-2.500</td>
<td>L</td>
<td>64</td>
</tr>
<tr>
<td>2.501-3.000</td>
<td>L</td>
<td>64</td>
</tr>
</tbody>
</table>

$e$ These values except in the $ST$ direction have been adjusted to include the influence of the $1/32$ in. nominal plating thickness.

$e$ See Table 3.1.1.1.
Table D2.17 Design Mechanical and Physical Properties of 7075 Aluminum Alloy (Extrusions)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>7075</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Extrusions (rod, bars, and shapes)</td>
</tr>
<tr>
<td>Condition</td>
<td>-76*</td>
</tr>
<tr>
<td>Cross-sectional area, in.²</td>
<td></td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>Basis</td>
</tr>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>76</td>
</tr>
<tr>
<td>$T$</td>
<td>70</td>
</tr>
<tr>
<td>$F_{+}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>71</td>
</tr>
<tr>
<td>$T$</td>
<td>71</td>
</tr>
<tr>
<td>$F_{tv}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>71</td>
</tr>
<tr>
<td>$T$</td>
<td>71</td>
</tr>
<tr>
<td>Physical properties:</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{10^6}$ psi</td>
<td></td>
</tr>
<tr>
<td>$E_{tu}$ 10^6 psi</td>
<td></td>
</tr>
<tr>
<td>$G_{10^6}$ psi</td>
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</tr>
<tr>
<td></td>
<td>10.3</td>
</tr>
<tr>
<td>$0.101$</td>
<td></td>
</tr>
<tr>
<td>$0.23$ (at 212°F)</td>
<td></td>
</tr>
<tr>
<td>$76$ (at 77°F)</td>
<td></td>
</tr>
<tr>
<td>$12.9$ (68°F to 212°F)</td>
<td></td>
</tr>
</tbody>
</table>

*For the stress relieved tempers -T6110 and -T6311, all values for the -76 temper apply, with the exception of $F_{tu}$; $L$ values are listed below: $F_{tu}$ (A values)

<table>
<thead>
<tr>
<th>Thickness (in.)</th>
<th>Area (sq. in.)</th>
<th>Direction of Test</th>
<th>$F_{tu}$ (A values)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.040 ≤ T ≤ 0.090</td>
<td>≤ 20</td>
<td>L.</td>
<td>70</td>
</tr>
<tr>
<td>0.090 ≤ T ≤ 0.200</td>
<td>≤ 20</td>
<td>L.</td>
<td>70</td>
</tr>
<tr>
<td>0.200 ≤ T ≤ 0.400</td>
<td>≤ 20</td>
<td>L.</td>
<td>72</td>
</tr>
<tr>
<td>0.400 ≤ T ≤ 0.600</td>
<td>≤ 20</td>
<td>L.</td>
<td>71</td>
</tr>
</tbody>
</table>

*For extrusions with outstanding legs, the load carrying ability of such legs shall be determined on the basis of the properties of the appropriate column corresponding to the leg thickness.
### Table B2.18 Design Mechanical and Physical Properties of 7075 Aluminum Alloy (Hand Forgings and Die Forgings)

<table>
<thead>
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<th>7075</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td></td>
</tr>
<tr>
<td>Hand-forged stock, length</td>
<td></td>
</tr>
<tr>
<td>≥3 times width</td>
<td></td>
</tr>
<tr>
<td>7075</td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td>-T6²</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>≥3</td>
</tr>
<tr>
<td>Cross-sectional area, in.²</td>
<td></td>
</tr>
<tr>
<td>≥16</td>
<td>≥16, ≥36</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mechanical properties:</th>
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</thead>
<tbody>
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<td>$F_{lm}$ ksi</td>
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</tr>
<tr>
<td>$L$</td>
<td>75</td>
</tr>
<tr>
<td>$T$</td>
<td>75</td>
</tr>
<tr>
<td>$ST$</td>
<td>70</td>
</tr>
<tr>
<td>$P_{lm}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>64</td>
</tr>
<tr>
<td>$T$</td>
<td>63</td>
</tr>
<tr>
<td>$ST$</td>
<td>63</td>
</tr>
<tr>
<td>$F_{lm}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>64</td>
</tr>
<tr>
<td>$T$</td>
<td>63</td>
</tr>
<tr>
<td>$ST$</td>
<td>63</td>
</tr>
<tr>
<td>$P_{lm}$ ksi</td>
<td></td>
</tr>
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<td>$T$</td>
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</tr>
<tr>
<td>$ST$</td>
<td>44</td>
</tr>
<tr>
<td>$P_{lm}$ ksi</td>
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</tr>
<tr>
<td>$L$</td>
<td>97</td>
</tr>
<tr>
<td>$T$</td>
<td>95</td>
</tr>
<tr>
<td>$ST$</td>
<td>95</td>
</tr>
<tr>
<td>$P_{lm}$ ksi</td>
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</tr>
<tr>
<td>$L$</td>
<td>90</td>
</tr>
<tr>
<td>$T$</td>
<td>91</td>
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<tr>
<td>$ST$</td>
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</tr>
<tr>
<td>$e$, percent</td>
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<tr>
<td>$L$</td>
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<td>$T$</td>
<td>4</td>
</tr>
<tr>
<td>$ST$</td>
<td>2</td>
</tr>
<tr>
<td>$E$, 10⁶ psi</td>
<td>10.3</td>
</tr>
<tr>
<td>$E_{m}$, 10⁶ psi</td>
<td>10.5</td>
</tr>
<tr>
<td>$G$, 10⁶ psi</td>
<td>3.9</td>
</tr>
</tbody>
</table>
Table B2. 19 Design Mechanical and Physical Properties of 7075 Aluminum Alloy
(Bar, Rod, Wire and Shapes; Rolled, Drawn or Cold-Finished)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>7075</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Bar, rod, wire and shapes, rolled, drawn or cold-finished</td>
</tr>
<tr>
<td>Condition</td>
<td>-T6 or -T651</td>
</tr>
<tr>
<td>Thickness, in</td>
<td>Up to 1.000 * 1.001 2.000 * 3.000 * 4.000 *</td>
</tr>
<tr>
<td>Basis</td>
<td>A A A A</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>77 77 77 77</td>
</tr>
<tr>
<td>$LT$</td>
<td>77 75 72 69</td>
</tr>
<tr>
<td>$F_{tm}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>66 66 66 66</td>
</tr>
<tr>
<td>$LT$</td>
<td>66 66 63 60</td>
</tr>
<tr>
<td>$F_{re}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>64 64 64 64</td>
</tr>
<tr>
<td>$LT$</td>
<td>64</td>
</tr>
<tr>
<td>$F_{ms}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>46 46 46 46</td>
</tr>
<tr>
<td>$LT$</td>
<td></td>
</tr>
<tr>
<td>$F_{mre}$, ksi:</td>
<td>(t/D=1.5) (t/D=2.0)</td>
</tr>
<tr>
<td>(t/D=1.5)</td>
<td>100 100 100 100</td>
</tr>
<tr>
<td>(t/D=2.0)</td>
<td>123 123 123 123</td>
</tr>
<tr>
<td>$F_{res}$, ksi:</td>
<td></td>
</tr>
<tr>
<td>(t/D=1.5)</td>
<td>86 86 86 86</td>
</tr>
<tr>
<td>(t/D=2.0)</td>
<td>92 92 92 92</td>
</tr>
<tr>
<td>$e$, percent:</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>7 7 7 7</td>
</tr>
<tr>
<td>$LT$</td>
<td>4 3 2 1</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>10.3</td>
</tr>
<tr>
<td>$E_{m}$, 10^6 psi</td>
<td>10.5</td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td>3.9</td>
</tr>
</tbody>
</table>
EFFECT OF TEMPERATURE ON 7075 ALUMINUM ALLOYS

Fig. B2.72 Effect of temperature on the ultimate tensile strength ($F_{tu}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.73 Effect of temperature on the tensile yield strength ($F_{ty}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.74 Effect of exposure at elevated temperatures on the room-temperature ultimate tensile strength ($F_{tu}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.75 Effect of exposure at elevated temperatures on the room-temperature tensile yield strength ($F_{ty}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.76 Effect of temperature on the compressive yield strength ($F_{cy}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.77 Effect of temperature on the ultimate shear strength ($F_{su}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.78 Effect of temperature on the ultimate bearing strength ($F_{bru}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.79 Effect of temperature on the bearing yield strength ($F_{bry}$) of 7075-T6 aluminum alloy (all products).

Fig. B2.80 Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of 7075-T6 aluminum alloy.

Fig. B2.81 Effect of exposure at elevated temperatures on the elongation of 7075-T6 aluminum alloy (all products except thick extrusions).
<table>
<thead>
<tr>
<th>Alloy</th>
<th>AZ31B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sheet and plate</td>
</tr>
<tr>
<td>Condition</td>
<td>-O</td>
</tr>
<tr>
<td>Thickness (in.)</td>
<td>0.016-0.060</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{m, ksi}$</td>
<td>52</td>
</tr>
<tr>
<td>$L$</td>
<td>32</td>
</tr>
<tr>
<td>$T$</td>
<td>40</td>
</tr>
<tr>
<td>$F_{e, ksi}$</td>
<td>18</td>
</tr>
<tr>
<td>$L$</td>
<td>18</td>
</tr>
<tr>
<td>$T$</td>
<td>32</td>
</tr>
<tr>
<td>Physical properties:</td>
<td></td>
</tr>
<tr>
<td>$E_{m, psi}$</td>
<td>6.5</td>
</tr>
<tr>
<td>$E_{e, psi}$</td>
<td>6.5</td>
</tr>
</tbody>
</table>

* Estimated. * Transverse $F_{e, ksi}$ allowables are equal to or greater than the longitudinal $F_{e, ksi}$ allowables.
AZ31B MAGNESIUM ALLOY (SHEET & PLATE) (Cont.)

Fig. B2.82 Typical stress-strain and tangent-modulus curves for AZ31B-O magnesium alloy at room temperature (longitudinal).

Fig. B2.83 Effect of temperature on the ultimate tensile strength ($F_{UT}$) of AZ31B-H24 magnesium alloy.

Fig. B2.84 Effect of temperature on the tensile yield strength ($F_{TY}$) of AZ31B-H24 magnesium alloy.

Fig. B2.85 Effect of temperature on the compressive yield strength ($F_{CY}$) of AZ31B-H24 magnesium alloy.

Fig. B2.86 Effect of temperature on the ultimate shear strength ($F_{SU}$) of AZ31B-H24 magnesium alloy.

Fig. B2.87 Effect of temperature on the ultimate bearing strength ($F_{BPR}$) of AZ31B-H24 magnesium alloy.

Fig. B2.88 Effect of temperature on the bearing yield strength ($F_{BYY}$) of AZ31B-H24 magnesium alloy.

Fig. B2.89 Typical stress-strain and tangent-modulus curves for AZ31B-H24 magnesium alloy at room temperature.
<table>
<thead>
<tr>
<th>Alloy</th>
<th>Sheet * and plate *</th>
<th>Sand castings *</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>HK31A</td>
<td></td>
</tr>
<tr>
<td>Condition</td>
<td>-O</td>
<td>-H24</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>0.016-0.250</td>
<td>0.251-0.500</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_{te}$, ksi</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>$F_{tr}$, ksi</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>$F_{re}$, ksi</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>$F_{re}$, ksi</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>$(c/D = 1.5)$</td>
<td>43</td>
<td>46</td>
</tr>
<tr>
<td>$(c/D = 2.0)$</td>
<td>51</td>
<td>54</td>
</tr>
<tr>
<td>$(c/D = 1.5)$</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>$(c/D = 2.0)$</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>$e$, percent</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>6.5</td>
<td></td>
</tr>
<tr>
<td>$E_t$, 10^6 psi</td>
<td>6.5</td>
<td></td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td>2.4</td>
<td></td>
</tr>
<tr>
<td>Physical properties:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega$, lb./in.*</td>
<td>0.0647</td>
<td></td>
</tr>
<tr>
<td>C, BTU/(lb.)/(F)</td>
<td>0.25 (32°F to 212°F)</td>
<td></td>
</tr>
<tr>
<td>K, BTU/(hr.)/(ft.)/(F) ft.</td>
<td>60.0 (at 68°F)</td>
<td></td>
</tr>
<tr>
<td>$\alpha$, 10^-4 in./in./F</td>
<td>15 (68°F to 302°F)</td>
<td></td>
</tr>
</tbody>
</table>

* Properties for sheet and plate are taken parallel to the direction of rolling. Transverse properties are equal to or greater than the longitudinal properties.

* Mechanical properties are based upon the guaranteed tensile properties from separately-cast test bars. The mechanical properties of bars cut from castings may be as low as 70 percent of the tabulated values.

* Reference should be made to the specific requirements of the procuring or certifying agency with regard to the use of the above values in the design of castings.
Fig. B2.90  Effect of temperature on the ultimate tensile strength ($F_u$) of HK31A-H24 magnesium alloy.

Fig. B2.91  Effect of temperature on the tensile yield strength ($F_y$) of HK31A-H24 magnesium alloy.

Fig. B2.92  Effect of exposure at elevated temperatures on the room-temperature ultimate tensile strength ($F_u$) of HK31A-H24 magnesium alloy.

Fig. B2.93  Effect of exposure at elevated temperatures on the room-temperature tensile yield strength ($F_y$) of HK31A-H24 magnesium alloy.

Fig. B2.94  Effect of temperature on the ultimate tensile strength ($F_u$) of HK31A-T8 magnesium alloy (sand casting).

Fig. B2.95  Effect of temperature on the tensile yield strength ($F_y$) of HK31A-T8 magnesium alloy (sand casting).
Table B2.22 Design Mechanical and Physical Properties of AZ61A Magnesium Alloy (Extrusions and Forgings)

<table>
<thead>
<tr>
<th>Form</th>
<th>AZ61A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extruded bar, rod, and</td>
<td></td>
</tr>
<tr>
<td>solid shapes</td>
<td>Extra.</td>
</tr>
<tr>
<td>Extruded hollow shapes</td>
<td>duced</td>
</tr>
<tr>
<td>Extruded tubes</td>
<td></td>
</tr>
<tr>
<td>Forging</td>
<td></td>
</tr>
<tr>
<td><strong>Condition</strong></td>
<td>-F</td>
</tr>
<tr>
<td><strong>Thickness, in.</strong></td>
<td>0.249 0.250-2.499 0.028-0.750</td>
</tr>
<tr>
<td><strong>Basis</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Mechanical properties:</strong></td>
<td></td>
</tr>
<tr>
<td>$F_{th}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>38 39 36 38 38</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{op}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>21 24 16 16 22</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{op}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>14 14 11 11 14</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{m}$ ksi</td>
<td>19 19 12 12 12</td>
</tr>
<tr>
<td>$F_{om}$ ksi</td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>45 45 50 50 60</td>
</tr>
<tr>
<td>$D = 1.5$</td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>28 28 28 28 28</td>
</tr>
<tr>
<td>$D = 2.0$</td>
<td></td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>6.3</td>
</tr>
<tr>
<td>$E_m$, 10^6 psi</td>
<td>6.3</td>
</tr>
<tr>
<td>$G$, 10^3 psi</td>
<td>2.4</td>
</tr>
<tr>
<td>$\omega$, lb/in.$^2$</td>
<td>0.0647</td>
</tr>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.25 (at 78°F)†</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(ft)²/(F)/ft.</td>
<td>46 (212°F to 572°F)</td>
</tr>
<tr>
<td>$\sigma$, 10^4 in./in./F.</td>
<td>14 (65°F to 212°F)</td>
</tr>
</tbody>
</table>

a Properties of extruded bars, rods, shapes, tubes, and forgings are taken parallel to the direction of extrusion or maximum metal flow during fabrication.

† Estimated.
### Table B2.23 Design Mechanical and Physical Properties of AZ80A Magnesium Alloy

(Extrusions and Forgings)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>AZ80A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Extruded bars, rods, and solid shapes</td>
</tr>
<tr>
<td><strong>Condition</strong></td>
<td>-F</td>
</tr>
<tr>
<td><strong>Thicknes, in.</strong></td>
<td>&lt;0.249</td>
</tr>
<tr>
<td><strong>Basis</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Mechanical properties:</strong></td>
<td></td>
</tr>
<tr>
<td>$F_{max}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>43</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{op}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>28</td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{cp}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td></td>
</tr>
<tr>
<td>$F_{min}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$F_{bm}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$(e/D = 1.5)$</td>
<td>48</td>
</tr>
<tr>
<td>$(e/D = 2.0)$</td>
<td>58</td>
</tr>
<tr>
<td>$F_{bpm}$, ksi</td>
<td></td>
</tr>
<tr>
<td>$(e/D = 1.5)$</td>
<td>36</td>
</tr>
<tr>
<td>$(e/D = 2.0)$</td>
<td>40</td>
</tr>
<tr>
<td>$\epsilon$, percent</td>
<td></td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>9.5</td>
</tr>
<tr>
<td>$E_n$, 10^6 psi</td>
<td>6.5</td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td>2.4</td>
</tr>
</tbody>
</table>
Table B2.24 Material Specifications for AZ63A Magnesium Alloy

<table>
<thead>
<tr>
<th>Specification</th>
<th>Type of product</th>
</tr>
</thead>
<tbody>
<tr>
<td>QQ-M-56</td>
<td>Sand castings</td>
</tr>
<tr>
<td>QQ-M-55</td>
<td>Permanent-mold castings</td>
</tr>
</tbody>
</table>

Table 42.3.0(b). Design Mechanical and Physical Properties of AZ63A Magnesium Alloy (Castings)

<table>
<thead>
<tr>
<th>Alloy</th>
<th>AZ63A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sand and permanent-mold castings</td>
</tr>
<tr>
<td>Condition</td>
<td>-F  -T4  -T5  -T6</td>
</tr>
<tr>
<td>Thickness (in.)</td>
<td></td>
</tr>
<tr>
<td>Basis</td>
<td></td>
</tr>
</tbody>
</table>

Mechanical properties:
- $F_{u}$ ksi: 24  34  24  34
- $F_{y}$ ksi: 10  10  11  16
- $F_{tp}$ ksi: 10  10  11  16
- $F_{sw}$ ksi: 16  17  19
- $F_{brw}$ ksi: 36  36  50
- $F_{br}$ ksi: 36  36  50
- $F_{br}$ ksi: 36  36  45
- $e$, percent: 4  7  2  3
- $E$, 10⁷ psi: 6.5
- $E$, 10⁷ psi: 6.5
- $G$, 10⁷ psi: 2.4

Fig. B2.96 Typical stress-strain and tangent-modulus curves for AZ63A-T4 magnesium alloy (sand casting) at room temperature.

Fig. B2.97 Typical stress-strain and tangent-modulus curves for AZ63A-F magnesium alloy (sand casting) at room temperature.
### Table B2.25  Design Mechanical and Physical Properties of 8Mn Titanium Alloy

<table>
<thead>
<tr>
<th>Alloy</th>
<th>8Mn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sheet, plate, and strip</td>
</tr>
<tr>
<td>Condition</td>
<td>Annealed</td>
</tr>
<tr>
<td>Thickness, in</td>
<td></td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
</tbody>
</table>

#### Mechanical Properties:

- $F_{tu}$, ksi  
  - $L$  
  - $T$  
  - $120$  
  - $120$  

- $F_{ty}$, ksi  
  - $L$  
  - $T$  
  - $110$  
  - $110$  

- $F_{sh}$, ksi  
  - $(e/D = 1.5)$  
  - $(e/D = 2.0)$  
  - $170$  
  - $130$  

- $F_{br}$, ksi  
  - $(e/D = 1.5)$  
  - $(e/D = 2.0)$  
  - $10$  
  - $15.5$  
  - $16.0$  

- $e$, percent  
  - $10$  

- $E$, 10³ psi  
  - $15.5$  

- $G$, 10³ psi  
  - $16.0$  

- $v$, 10⁻¹⁴ in./in./F  
  - $0.171$  

- $C$, Btu/(lb)(F)  
  - $0.118$ (at 68°F)  

- $K$, Btu/(hr)(ft)(F)/ft  
  - $6.3$  

- $a$, 10⁻⁴ in./in./F  
  - $4.8$ (at 200°F)  

#### Physical Properties:

- $0.171$  

#### Diagrams:

**Fig. B2.96** Effect of temperature on the ultimate tensile strength ($F_{tu}$) of 8Mn annealed titanium alloy.

**Fig. B2.101** Effect of temperature on the ultimate shear strength ($F_{sh}$) of 8Mn annealed titanium alloy.

**Fig. B2.99** Effect of temperature on the tensile yield strength ($F_{ty}$) of 8Mn annealed titanium alloy.

**Fig. 2.102** Effect of temperature on the ultimate bearing strength ($F_{br}$) of 8Mn annealed titanium alloy.

**Fig. B2.100** Effect of temperature on the compressive yield strength ($F_{cy}$) of 8Mn annealed titanium alloy.

**Fig. 2.103** Effect of temperature on the tensile and compressive modulus ($E$ and $E_c$) of 8Mn titanium alloy.
### Table B2.26 Design Mechanical and Physical Properties of 6Al-4V Titanium Alloy

<table>
<thead>
<tr>
<th>Alloy</th>
<th>6Al-4V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Bar</td>
</tr>
<tr>
<td>Condition</td>
<td>Annealed</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td>$\geq 1.5$</td>
</tr>
<tr>
<td>Basis</td>
<td>A</td>
</tr>
</tbody>
</table>

#### Mechanical properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value, ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{uy}$</td>
<td>130</td>
</tr>
<tr>
<td>$F_{ut}$</td>
<td>130</td>
</tr>
<tr>
<td>$F_{ty}$</td>
<td>120</td>
</tr>
<tr>
<td>$F_{ty}$</td>
<td>120</td>
</tr>
<tr>
<td>$F_{cy}$</td>
<td>126</td>
</tr>
<tr>
<td>$F_{cy}$</td>
<td>126</td>
</tr>
<tr>
<td>$F_{tu}$</td>
<td>80</td>
</tr>
<tr>
<td>$F_{ty}$</td>
<td>76</td>
</tr>
<tr>
<td>$F_{brm}$</td>
<td></td>
</tr>
<tr>
<td>$(e/D = 1.5)$</td>
<td>196</td>
</tr>
<tr>
<td>$(e/D = 2.0)$</td>
<td>248</td>
</tr>
<tr>
<td>$E_{brm}$</td>
<td></td>
</tr>
<tr>
<td>$(e/D = 1.5)$</td>
<td>174</td>
</tr>
<tr>
<td>$(e/D = 2.0)$</td>
<td>205</td>
</tr>
<tr>
<td>$e_c$, percent</td>
<td>10</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>16.0</td>
</tr>
<tr>
<td>$G$, 10^6 psi</td>
<td>16.4</td>
</tr>
</tbody>
</table>

#### Physical properties:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value, (\text{lb/in.}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>0.160</td>
</tr>
<tr>
<td>$C$, Btu/(lb)(F)</td>
<td>0.135 (at 68°F)</td>
</tr>
<tr>
<td>$K$, Btu/(hr)(ft)(F)/ft</td>
<td>3.8 (at 63°F)</td>
</tr>
<tr>
<td>$a$, 10^(-4) in./in./F</td>
<td>4.6 (at 200°F)</td>
</tr>
</tbody>
</table>
### Table B2.27 Design Mechanical and Physical Properties of Inconel X Nickel Alloy

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form</td>
<td>Sheet</td>
</tr>
<tr>
<td>Condition</td>
<td>Precipitation heat-treated</td>
</tr>
<tr>
<td>Thickness, in.</td>
<td></td>
</tr>
<tr>
<td>Mechanical properties:</td>
<td></td>
</tr>
<tr>
<td>$F_{tu}$ ksi</td>
<td>155</td>
</tr>
<tr>
<td>$F_{ty}$ ksi</td>
<td>155</td>
</tr>
<tr>
<td>$F_{ccy}$ ksi</td>
<td>100</td>
</tr>
<tr>
<td>$F_{c}$ ksi</td>
<td>100</td>
</tr>
<tr>
<td>$F_{cu}$ ksi</td>
<td>105</td>
</tr>
<tr>
<td>$F_{cu}$ ksi ($e/D = 1.5$)</td>
<td>108</td>
</tr>
<tr>
<td>$F_{cu}$ ksi ($e/D = 2.0$)</td>
<td>286</td>
</tr>
<tr>
<td>$F_{ccy}$ ksi ($e/D = 1.5$)</td>
<td>186</td>
</tr>
<tr>
<td>$F_{ccy}$ ksi ($e/D = 2.0$)</td>
<td>20</td>
</tr>
<tr>
<td>$E$, 10^6 psi</td>
<td>31.0</td>
</tr>
<tr>
<td>$E_p$, 10^6 psi</td>
<td>31.0</td>
</tr>
<tr>
<td>$G$, 10^3 psi</td>
<td></td>
</tr>
</tbody>
</table>

### Physical properties:

- $\omega$, lb/in.$^3$ : 0.304
- $C$, Btu/(lb)(F) : 0.109
- $K$, Btu/(hr)(ft)$^2$/(F)/ft : 5.7 (30°F to 212°F)
- $\alpha$, 10$^{-4}$ in./in./°F : 6.4 (100°F to 200°F)

---

**Fig. B2.108** Effect of temperature on the ultimate tensile strength ($F_{tu}$) of precipitation heat treated Inconel X nickel alloy.

**Fig. B2.109** Effect of temperature on the tensile yield strength ($F_{ty}$) of precipitation heat treated Inconel X nickel alloy.

**Fig. B2.110** Effect of temperature on the compressive yield strength ($F_{ccy}$) of precipitation heat treated Inconel X nickel alloy.

**Fig. B2.111** Effect of temperature on the ultimate shear strength ($F_{su}$) of precipitation heat treated Inconel X nickel alloy.

**Fig. B2.112** Effect of temperature on the tensile modulus ($E$) of Inconel X nickel alloy.

**Fig. B2.113** Effect of temperature on the ultimate bearing strength ($F_{br}$) of precipitation heat treated Inconel X nickel alloy.

**Fig. B2.114** Effect of temperature on the bearing yield strength ($F_{cbr}$) of precipitation heat treated Inconel X nickel alloy.
PART C
PRACTICAL STRENGTH ANALYSIS &
DESIGN OF STRUCTURAL COMPONENTS

CHAPTER C1

COMBINED STRESSES. THEORY OF YIELD AND ULTIMATE FAILURE.

C1.1 Uniform Stress Condition

Aircraft structures are subjected to many types of external loadings. These loads often cause axial, bending and shearing stresses acting simultaneously. If structures are to be designed satisfactorily, combined stress relationships must be known. Although in practical structures uniform stress distribution is not common, still sufficient accuracy for design purpose is provided by using the stress relationships based on uniform stress assumptions. In deriving these stress relationships, the Greek letter sigma (σ) will represent a stress intensity normal to the surface and thus a tensile or compressive stress and the Greek letter tau (τ) will represent a stress intensity parallel to the surface and thus a shearing stress.

C1.2 Shearing Stresses on Planes at Right Angles.

For equilibrium a resisting couple must exist on top and bottom face of cube. Taking moments about lower left edge of cube:

\[ \tau_x \, dx \, dy \, (dz) - \tau_z \, dz \, dy \, (dx) = 0 \]

hence, \[ \tau_x = \tau_z \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (1) \]

Thus if a shearing unit stress occurs on one plane at a point in a body, a shearing unit stress of same intensity exists on planes at right angles to the first plane.

C1.3 Simple Shear Produces Tensile and Compressive Stresses.

Fig. C1.3 shows an elementary block of unit dimensions subjected to pure shearing stresses.

Fig. C1.4 shows a free body after the block has been cut along a diagonal section.

For equilibrium the sum of the forces along the x-x axis equals zero.

\[ EF_x = -\sigma(1) \cos 45^\circ + 2(\tau_1 \cos 45^\circ) = 0 \]

hence, \[ \sigma = \frac{2(\tau_1 \cos 45^\circ) \cos 45^\circ}{1} = \tau - \quad (2) \]

Therefore when a point in a body is subjected to pure shear stresses of intensity τ, normal stresses of the same intensity as the shear stresses are produced on a plane at 45° with the shearing planes.

C1.4 Principal Stresses

For a body subjected to any combination of stresses 3 mutually perpendicular planes can be found on which the shear stresses are zero. The normal stresses on these planes of zero shear stress are referred to as principal stresses.

C1.5 Shearing Stresses Resulting From Principal Stresses.

In Fig. C1.5 the differential block is subjected to tensile principal stresses \( \sigma_x \) and \( \sigma_z \) and zero principal stress \( \sigma_y \). The block is cut along a diagonal section giving the free body of Fig. C1.6. The stresses on the diagonal section have been resolved into stress components parallel and normal to the section as shown. For equilibrium the summation of the stresses along the axes (1-1) and (2-2) must equal zero.

\[ EF_{1-1} = 0 \]

\[ \sigma_n \, dy \, dx - \sigma_x \, dz \, dy \cos \theta \quad - \quad \sigma_y \, dx \, dy \sin \theta = 0, \]

whence \[ \sigma_n = \sigma_x \, dz \, dy \cos \theta + \sigma_z \, dx \, dy \sin \theta \]

C1.1
Cl. 1.2 COMBINED STRESSES. THEORY OF YIELD AND ULTIMATE FAILURE.

Cl. 5.1 Combined Stress Equations

Fig. Cl. 7 shows a differential block subjected to normal stresses on two planes at right angles to each other and with shearing forces on the same planes. The maximum normal and shearing unit stresses will be determined.

Fig. Cl. 8 shows a free body diagram of a portion cut by a diagonal plane at angle θ as shown.

For equilibrium the sum of the forces in the z and x directions must equal zero.

\[ \Sigma F_z = 0 \]

\[ \sigma_n \ dudy \ cos \ \theta + \tau \ dudy \ sin \ \theta - \sigma_x \ dzdy - \tau_{xz} \ dzdy = 0 \]

\[ \sigma_n \ dudy \ sin \ \theta - \tau \ dudy \ cos \ \theta - \sigma_z \ dydx - \tau_{zx} \ dydx = 0 \]

\[ \Sigma F_x = 0 \]

\[ \sigma_n \ dudy \ sin \ \theta - \tau \ dudy \ cos \ \theta - \sigma_z \ dydx - \tau_{zx} \ dydx = 0 \]

By dividing each equation by \( du \) and noting that

\[ \frac{dzdy}{dudy} = \cos \ \theta \] and \[ \frac{dydx}{dudy} = \sin \ \theta \], we obtain:

\[ (\sigma_n - \sigma_x) \ cos \ \theta + (\tau - \tau_{xx}) \ sin \ \theta = 0 \]
The maximum normal stress \( \sigma_n \) will be maximum when \( \theta \) equals such angle \( \theta' \) as to make \( \tau = 0 \). Thus if \( \tau = 0 \) and \( \theta = \theta' \) in equations (3) and (7), we obtain,

\[
(\sigma_n - \sigma_z) \cos \theta' - (\tau - \tau_{xz}) \cos \theta = 0 \quad - - - - -(7)
\]

\[
(\sigma_n - \sigma_x) \cos \theta' - \tau_{xz} \sin \theta' = 0 \quad - - - - -(8)
\]

\[
(\sigma_n - \sigma_z) \sin \theta' - \tau_{xz} \cos \theta' = 0 \quad - - - - -(9)
\]

In equations (8) and (9) \( \sigma_n \) represents the principal stress. Dividing one equation by another to eliminate \( \theta' \),

\[
\frac{\sigma_n - \sigma_x}{\tau_{xz}} = \frac{\tau_{xz}}{\delta_n - \delta_z}
\]

whence,

\[
\sigma_n^2 - (\sigma_x - \sigma_z) \sigma_n + \sigma_x \sigma_z = \tau_{xz}^2,
\]

or

\[
\sigma_n = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_z}{2})^2 + \tau_{xz}^2} - - - - -(10)
\]

In equation (10), tensile normal stress is plus and compression minus. For maximum \( \sigma_n \) use plus sign before radical and minus sign for minimum \( \sigma_n \).

To find the plane of the principal stresses, the value of \( \theta' \) may be solved for from equations (8) and (9), which gives:

\[
\tan 2 \theta' = \frac{2 \tau_{xz}}{\sigma_x - \sigma_z} - - - - - - - - -(11)
\]

\( \theta' \) is measured from the plane of the largest normal stress \( \sigma_x \) or \( \sigma_z \). The direction of rotation of \( \theta' \) from this plane is best determined by inspection. Thus if only the shearing stresses \( \tau_{xz} \) were acting, the maximum principal stress would be one of the 45° planes, the particular 45° plane being easily determined by inspection of the sense of the shear stresses. Furthermore if only the largest normal stress were acting it would be the maximum principal stress and \( \theta' \) would equal zero. Thus if both \( \sigma \) and \( \tau \) act, the plane of the principal stress will be between the plane on which \( \sigma \) acts and the 45° plane. As stated before \( \sigma \) refers to either \( \sigma_x \) or \( \sigma_z \) whichever is the largest.

Maximum Value of Shearing Stress. \( (\tau_{\text{max}}) \)

The maximum value of \( \tau \) from equation (3) equals,

\[
\tau_{\text{max}} = (\sigma_{n(\text{max})} - \sigma_{n(\text{min})})/2
\]

Substituting the maximum and minimum values of \( \sigma_n \) from (10) in (12), we obtain maximum shearing stress as follows:

\[
\tau_{\text{max}} = \pm \sqrt{(\frac{\sigma_x - \sigma_z}{2})^2 + \tau_{xz}^2} - - - - -(13)
\]

C1.7 Mohr's Circle for Determination of Principal Stresses.

It is sometimes convenient to solve graphically for the principal stresses and the maximum shear stress. Mohr's circle furnishes a graphical solution. (Fig. C1.9a). In the Mohr method, two rectangular axes \( x \) and \( z \) are chosen to represent the normal and shearing stresses respectively. Taking point 0 as the origin lay off to scale the normal stresses \( \sigma_x \) and \( \sigma_z \) equal to OB and OA respectively. If tension, they are laid off to right of point 0 and to the left if compression. From B the shear stress \( \tau_{xz} \) is laid off parallel to OQ and with the sense of the shear stress on the face DC of Fig. C1.9b, thus locating point C. With point E the midpoint of AB as the center and with radius EC describe a circle cutting OB at F and G. AD will equal BC and will represent the shear on face AB of Fig. b. It can be proven that OF and OG are the principal stresses \( \sigma_{n(\text{max})} \) and \( \sigma_{n(\text{min})} \) respectively and EC is the maximum shear stress \( \tau_{xz} \). The principal stresses occur on planes that are parallel to OF and OG. (See Figs. c and d). The maximum shear stress occurs on two sections parallel to CH and CI where HEI is perpendicular to OB. If \( \sigma_x \) should equal zero then 0 would coincide with A.

C1.8 Components of Stress From Principal Stresses by Mohr's Circle.

In certain problems the principal stresses may be known as in Fig. C1.9 and it is desired to find the stress components on other planes designated by angle \( \theta \). In Fig. C1.11 the axes x and z represent the normal and shear stresses respectively. The principal stresses are laid off to scale on ox giving points D and E respectively. Construct a circle with A the midpoint of DE and with diameter ED. Draw angle CAB equal
to 29. It can be proven that OB represents the normal stress on the plane defc of Fig. C1.10, and CB represents the shear stress \( \tau \) on this plane.

![Diagram](image)

Fig. C1.10

\( \sigma_z, \sigma_x \) are principal stresses.

![Diagram](image)

Fig. C1.11

\( \sigma_n \) on plane (defc)

\( \sigma_x \)

\( \sigma_x = 10000 \)

Substituting values,

\[ \sigma_n = \frac{10000 + 0}{2} \pm \sqrt{\left(\frac{10000 + 0}{2}\right)^2 + 5000^2} = 5000 \]

+ 7070 hence, \( \sigma_n(\text{max}) = 5000 + 7070 = 12070 \) psi

\( \sigma_n(\text{min}) = 5000 - 7070 = -2070 \) psi

\[ \tau_{\text{max}} = \frac{1}{2}(\sigma_n(\text{max}) - \sigma_n(\text{min})) \] (Ref. Eq. 12)

\[ = \frac{1}{2}(12070 - (-2070)) = 7070 \) psi

\( \tau_{\text{max}} \) can also be computed by equation (13),

whence,

\[ \tau_{\text{max}} = \pm \sqrt{\left(\frac{10000 + 0}{2}\right)^2 + 5000^2} = \pm 7070 \) psi

\[ \tan 2\theta' = \frac{2 \tau_{\text{max}}}{\sigma_x - \sigma_z} = \frac{2 \times 5000}{10000 - 0} = 1 \]

hence, \( \theta' = 22.5^\circ \).

Example Problem 2.

The maximum normal and shear stresses will be determined for the block loaded as shown in Fig. C1.14.

![Diagram](image)

Fig. C1.14

\( \sigma_x = 10000 \)

\( \sigma_z = 20000 \)

\( \tau_{\text{xx}} = 12000 \)

\( \sigma_x = 10000 \)

\( \tau_{\text{xx}} = 12000 \)

\( \sigma_z = 20000 \)
Fig. C1.15 shows the graphical solution using Mohr's circle. From point O, \( \sigma_x = 10000 \) and \( \sigma_z = -30000 \) are laid off equal to OB and OA respectively. TF equal to 12000 is laid off parallel to OZ at A locating C. With Z the midpoint of AB as the center of a circle of radius EQ a circle is drawn which cuts the ox axis at P and D. The maximum normal and shear stresses are indicated on the figure.

\[ \sigma^3 - (\sigma_x + \sigma_y + \sigma_z) \sigma^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - 2\sigma_y \sigma_z) \sigma - \sigma_x \sigma_y \sigma_z = 0 \]

\[ (A) \]

Fig. C1.16 shows the principal stress system which replaces the system of Fig. C1.16. It can be shown that the maximum shear stress \( \gamma_{\text{max.}} \) is given by

\[ \gamma_{\text{max.}} = \pm \frac{1}{2} (\sigma_1 - \sigma_3) \]

or \( \gamma_{\text{max.}} = \pm \frac{1}{2} (\sigma_2 - \sigma_3) \)

or \( \gamma_{\text{max.}} = \pm \frac{1}{2} (\sigma_3 - \sigma_2) \)

The planes on which these shear stresses act are indicated by the dashed lines in Fig. C1.18, namely, adhf, bdge and cdef. The largest of the shear stresses in equations (15) depends on the magnitude and signs of the principal stresses, remembering that tension is plus and compression is minus when making the substitution in equations (15).

Fig. C1.17

Fig. C1.18

Cl.11 Principal Stresses

The strains under combined stresses are usually expressed as strains in the direction of the principal stresses. Consider a case of simple tension as illustrated in Fig. C1.19. The stress \( \sigma_1 \) causes a lengthening unit strain \( \varepsilon \) in the direction of the stress \( \sigma_1 \), and a shortening unit strain \( \varepsilon' \) in a direction at right angles to the stress \( \sigma_1 \).

The ratio of \( \varepsilon' \) to \( \varepsilon \) is called Poisson's ratio and is usually given the symbol \( \mu \). Thus, \( \mu = \varepsilon' / \varepsilon \).
Since \( \varepsilon = \sigma / E \), we obtain,
\[
\varepsilon' = \mu \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E} \quad \text{(16)}
\]

Now consider the cubical element in Fig. C1.20 subjected to the three principal stresses \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), all being tension. The total unit strain \( \varepsilon' \) in the direction of stress \( \sigma_1 \) will be expressed. Obviously, \( \sigma_1 \) tends to stretch the element in the direction of \( \sigma_1 \), whereas stresses \( \sigma_2 \) and \( \sigma_3 \) tend to shorten the element in the direction of \( \sigma_1 \), hence,
\[
\varepsilon' = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E}, \quad \text{whence}
\]
\[
\varepsilon' = \frac{\sigma_1}{E} - \mu (\sigma_2 + \sigma_3)
\]
and similarly for \( \varepsilon_2 \) and \( \varepsilon_3 \),
\[
\begin{align*}
\varepsilon_2 &= \frac{\sigma_2}{E} - \mu (\sigma_1 + \sigma_3) \\
\varepsilon_3 &= \frac{\sigma_3}{E} - \mu (\sigma_1 + \sigma_2)
\end{align*}
\quad \text{--- (17)}
\]

For a two-dimensional stress system, that is, stresses acting in one plane, \( \sigma_3 = 0 \) and the principal strains become,
\[
\begin{align*}
\varepsilon_1 &= \frac{1}{E} (\sigma_1 - \mu \sigma_2) \\
\varepsilon_2 &= \frac{1}{E} (\sigma_2 - \mu \sigma_1) \\
\varepsilon_3 &= \frac{1}{E} (\sigma_1 + \sigma_2)
\end{align*}
\quad \text{--- (18)}
\]

Equations 17 and 18 give the strains when all the principal stresses are tensile stresses. For compressive principal stresses use a minus sign when substituting the principal stresses in the equations.

C1.12 Elastic Strain Energy

The strain energy in the elastic range for the unit cube in Fig. C1.20 when subjected to combined stresses is equal to the work done by the three gradually applied principal stresses \( \sigma_1, \sigma_2, \) and \( \sigma_3 \). These stresses produce strains equal to \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \) and thus the work done per unit volume equals the strain energy. Thus if \( U \) equals the strain energy, we obtain,
\[
U = \frac{\sigma_1 \varepsilon_1}{2} + \frac{\sigma_2 \varepsilon_2}{2} + \frac{\sigma_3 \varepsilon_3}{2} \quad \text{--- (19)}
\]

The strain energy can be expressed in terms of stress by substituting values of \( \varepsilon \) in terms of \( \sigma \) from equations (17) into equation (19), which gives,
\[
U = \frac{1}{2E} \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \right)
\quad \text{--- (20)}
\]

For a two dimensional stress system, \( \sigma_3 = 0 \) and equation (20) becomes
\[
U = \frac{1}{2E} (\sigma_1^2 - 2\mu \sigma_1 \sigma_2 + \sigma_2^2) \quad \text{--- (21)}
\]


The basic philosophy governing the structural design of a flight vehicle is to develop an adequate light weight structure that will permit the vehicle to accomplish the operations or missions that were established as design requirements. The job of a commercial airliner is to carry passengers and cargo from place to place at the lowest cost. To carry out this job a certain amount of flight and ground maneuvering is required and the loads due to these maneuvers must be carried safely and efficiently by the structure. A military fighter airplane must be maneuvered in flight far more severely to accomplish its desired job as compared to the commercial airliner, thus the flight acceleration factors for the military fighter airplane will be considerably higher than that of the airliner. In other words, every type of flight vehicle will undergo a different load environment, which may be repeated frequently or infrequently during the life of the vehicle. The load environment may involve many factors such as flight maneuvering loads, air gust loads, take off and landing loads, repeated loads, high and low temperature conditions, etc.

Limit Loads. Limit loads are the calculated maximum loads which may be subjected to the flight vehicle in carrying out the job it is designed to accomplish during its life time of use. The term limit was no doubt chosen because every flight vehicle is limited relative to the extent of its operations. A flight vehicle could easily be designed for loads greater than the limit loads, but such extra strength which is not necessary for safety would only increase the weight of the structure and decrease the commercial or military payload or in general be detrimental to the design.

Factor of Safety. Factor of safety can be defined as the ratio considered in structural design of the strength of the structure to the maximum calculated operational loads, that is, the limit loads.
Yield Factor of Safety. This term is defined as the ratio of the yield strength of the structure to the limit load.

Ultimate Factor of Safety. This term is defined as the ratio of the ultimate strength of the structure to the limit load.

Yield Load. This term is defined as the limit load multiplied by the yield factor of safety.

Ultimate Load. This term can be defined as the limit load multiplied by the ultimate factor of safety. This resulting load is often referred to by engineers as the design load, which is misleading because the flight vehicle structure must be designed to satisfy both yield and ultimate failure and either one may be critical.

Yield Margin of Safety. This term usually expressed in percent represents the additional yield strength of the structure over that strength required to carry the limit loads.

\[
\text{Yield Margin of Safety} = \frac{\text{Yield Strength}}{\text{Limit Load}} - 1
\]

Ultimate Margin of Safety. This term usually expressed in percent represents the additional ultimate strength of the structure over that strength required to carry the ultimate loads.

\[
\text{Ultimate Margin of Safety} = \frac{\text{Ultimate Strength}}{\text{Ultimate Load}} - 1
\]

Cl. 14 Required Strength of Flight Structures.

Under Limit Loads:

The flight vehicle structure shall be designed to have sufficient strength to carry simultaneously the limit loads and other accompanying environmental phenomena for each design condition without undergoing excessive elastic or plastic deformation. Since most materials have no definite yield stress, it is common practice to use the unit stress where a .002 inches per inch permanent set exists as the yield strength of the material, and in general this yield stress can be used as the maximum stress under the limit loads unless definitely otherwise specified.

Under Ultimate Loads:

The flight vehicle structure shall be designed to withstand simultaneously the ultimate loads and other accompanying environmental phenomena without failure. In general no factor of safety is applied to the environmental phenomena but only to the limit loads.

Failure of a Structure:

This term in general refers to a state or condition of the structure which renders it incapable of performing its required function. Failure may be due to rupture or collapse or due to excessive deflection or distortion.

Cl. 15 Determination of the Ultimate Strength of a Structural Member Under a Combined Load System. Stress Ratio-Interaction Curve Method.

Since the structural designer of flight vehicles must insure that the ultimate loads can be carried by the structure without failure, it is necessary that reliable methods be used to determine the ultimate strength of a structure. Structural theory as developed to date is in general sufficiently developed to accurately determine the ultimate strength of a structural member under a single type of loading, such as axial tension or compression, pure bending or pure torsion. However, many of the members which compose the structure of a flight vehicle are subjected simultaneously to various combinations of axial, bending and torsional load systems and thus a method must be available to determine the ultimate strength of a structure under combined load systems. A strictly theoretical approach appears too difficult for solution since failure may be due to overall elastic or inelastic buckling, or the local elastic or inelastic instability.

The most satisfactory method developed to date is the so-called stress ratio, interaction curve method, originally developed and presented by Shanley. In this method the stress conditions on the structure are represented by stress ratios, which can be considered as non-dimensional coefficients denoting the fraction of the allowable stress or strength for the member which can be developed under the given conditions of combined loading.

For a single simple stress, the stress ratio can be expressed as,

\[
R = \frac{f}{F} - 1
\]

where \( f \) is the applied stress and \( F \) the allowable stress. The margin of safety in terms of the stress ratio \( R \) can be written,

\[
\text{M.S.} = \frac{1}{R} - 1.0
\]

Load ratios can be used instead of stress ratios and is often more convenient.

For example for axial loading,

\[
R = \frac{P}{P_a}, \text{ where } P = \text{ applied axial load and } P_a \text{ the allowable load.}
\]
For pure bending,

\[ R = \frac{M}{M_a} \text{, where } M = \text{applied bending moment and } M_a = \text{the allowable bending moment.} \]

For pure torsion,

\[ R = \frac{T}{T_a} \text{, where } T = \text{applied torsional moment and } T_a = \text{the allowable torsional moment.} \]

For combined loadings the general conditions for failure are expressed by Shanley as follows:

\[ R_1^x + R_2^y + R_3^z = 1.0 \quad -(24) \]

In this above expression, \( R_1, R_2 \) and \( R_3 \) could refer to compression, bending and shear and the exponents \( x, y, \) and \( z \) give the relationship for combined stresses. The equation states that the failure of a structural member under a combined loading will result only when the sum of the stress ratios is equal to or greater than 1.0.

For some of the simpler combined load systems, the exponents of the stress ratios in equation (24) can be determined by the various well known theories of yield and failure that have been developed. However, in many cases of combined loading and for particular types of structures the exponents in equation (24) must be determined by making actual failure tests of combined load systems.

Since the stress ratio method was presented by Shanley many years ago, much testing has been done and as a result reliable interaction equations with known exponents have been obtained for many types of structural members under the various combined load systems. In a number of the following chapters, the interaction equations which apply will be used in determining the ultimate strength design of structural members.

1. Maximum Principal Stress Theory
2. Maximum Shearing Stress Theory
3. Maximum Strain Theory
4. Total Strain Energy Theory
5. Strain Energy of Distortion Theory
6. Octahedral Shear Stress Theory

The reader may review the explanation and derivation of these 6 theories by referring to such books as listed at the end of this chapter.

Test results indicate that the yield strength at a point in a stressed structure is more accurately defined by theories 5 and 6 followed in turn by theory 2. Since theories 5 and 6 give the same result, they might be considered as the same general theory. In this chapter we will only give the resulting equations as derived by theory 6, since theories 5 and 6 appear to be the theories used in flight vehicle structural design.

Cl.17 The Octahedral Shear Stress Theory.

Since this theory gives the same results as the well known energy of distortion method it is often referred to as the Equivalent Stress Theory. The octahedral shear stress theory may be stated as follows: In elastic action at any point in a body under combined stress action begins only when the octahedral shearing stress becomes equal to 0.47 \( f_y \), where \( f_y \) is the tensile elastic strength of the material as determined from a standard tension test. Since the elastic tensile strength is somewhat indefinite, it is common practice to use the engineering yield strength \( f_{ty} \). In this theory it is assumed that the tensile and compressive yield strengths are the same.

Figs. Cl.21 and Cl.22 illustrate the conditions of equilibrium involving the octahedral shear stress. In Fig. Cl.21, the cube is subjected to the 3 principal stresses as shown. A tetrahedron is cut from the cube and shown in Fig. Cl.22. Three of the sides of this tetrahedron are parallel to the
principal axes, while the normal to the fourth side makes equal angles with the principal axes. The octahedral shear and normal stresses are the resulting stresses on the fourth side.

The equation for the value of the normal octahedral stress is,

\[ f_{\text{oct}} = \frac{1}{\sqrt{2}} (f_1 + f_2 + f_3) \quad \text{--- (25)} \]

The equation for the octahedral shear stress is,

\[ f_{s\text{oct}} = \frac{1}{\sqrt{2}} \sqrt{(f_1-f_2)^2 + (f_2-f_3)^2 + (f_3-f_1)^2} \quad \text{--- (26)} \]

Now the octahedral shear stress is 0.47 of the normal stress.

Let \( \bar{F} \) be the effective axial stress in uniaxial tension or compression which results in the given octahedral shear stress.

\[ \bar{F} = f_{s\text{oct}} / 0.47 = \frac{3}{\sqrt{2}} f_{s\text{oct}} \quad \text{--- (27)} \]

Therefore multiplying Eq. (28) by \( 3/\sqrt{2} \) we obtain for a condition of principal triaxial stresses,

\[ \bar{F} = \frac{3}{\sqrt{2}} \sqrt{(f_1-f_2)^2 + (f_2-f_3)^2 + (f_3-f_1)^2} \quad \text{--- (28)} \]

Let \( \bar{F} \) equal the allowable tensile or compressive stress. If the yield strength is being determined then

Margin of Safety M.S. = \( \frac{\bar{F}}{f} - 1 \) \quad \text{--- (29)}

For a biaxial stress system taking \( f_s = 0 \), we obtain,

\[ \bar{F} = \sqrt{f_1^2 + f_2^2 - f_1 f_2} \quad \text{--- (30)} \]

It is often more convenient to use the x, y and z components of stresses instead of the principal stresses. Fig. C1.23 illustrates the various component stresses.

For a triaxial stress system,

\[ f_s = \frac{3}{\sqrt{2}} \sqrt{(f_x-f_z)^2 + (f_z-f_y)^2 + (f_y-f_x)^2} - 6(f_{sxz} f_{szx} f_{syz})^2 \quad \text{--- (31)} \]

For a biaxial stress system, \( f_y, f_{szx} \),

\[ f_s = 0 \]

\[ \bar{F} = \sqrt{f_1^2 + f_2^2 - f_1 f_2} + 3f_{szx} \quad \text{--- (32)} \]

C1.18 Example Problem 1.

A cylindrical stiffened thin sheet fuselage is fabricated from 2024 aluminum alloy sheet which has a tensile yield stress \( F_{ty} = 40000 \) psi. Find the yield margin of safety under the following limit load conditions.

(1) A limit bending moment produces a bending stress of 37000 psi (tension) at top point of fuselage section. The flexural shear stress is zero at this point.

(2) Same as condition (1) but pressurization of fuselage produces a circumferential tension stress of 8600 psi and a longitudinal tension stress of 4500 psi.

(3) Same as condition (2) but a yawing maneuver of airplane produces a limit torsional shearing stress of 5000 psi in fuselage skin.

SOLUTION: Condition (1)

This is a uniaxial stress condition for point being considered.

Yield M.S. = \( \frac{F_{ty}}{F} - 1 = \frac{40000}{37000} - 1 = .06 \)

SOLUTION: Condition (2)

There are no flexural shear stresses at the fuselage point being considered. Since no torsion is being applied to fuselage no torsional shear stresses exist. The stress system at the point being considered is thus a biaxial stress system and \( f_1 \) and \( f_s \) are principal stresses.

\[ f_1 = 37000 + 4300 = 41300 \text{ psi} \]

\[ f_s = 8600 \text{ psi} \]

From equation (30),

\[ \bar{F} = \sqrt{f_1^2 + f_s^2 - f_1 f_s} = \sqrt{41300^2 + 8600^2 - 41300 \times 8600} \]
whence \( \bar{f} = 37700 \) psi

\[
\text{M.S.} = \frac{F}{\bar{f}} - 1 = \frac{40000}{37700} - 1 = 0.06
\]

**SOLUTION:** Condition (3)

Since a torsional shear stress has now been added, the new stress is still two dimensional, however the principal stresses are not due to the addition of the torsional shear stress.

\( f_x = 41300 \) psi, \( f_y = 3600 \) psi, \( f_z = 8000 \) psi.

Instead of finding the principal stresses and using Eq. (30), we will use the \( f_x \) and \( f_z \) stresses and use Eq. (32)

\[
\bar{f} = \sqrt{f_x^2 + f_z^2 - f_x f_z + 3f_y^2}
\]

\[
= \sqrt{41300^2 + 3600^2 - 41300 \times 3600 + 3 \times 8000^2}
\]

\( \bar{f} = 40200 \) psi. \( \text{M.S.} = \frac{40000}{40200} - 1 = -0.01 \)

Thus yield is indicated since M.S. is negative.

**Example Problem 2.**

A cylindrical pressure vessel is 100 inches in diameter and 1 inch thick. The vessel is made of steel with \( F_{ty} = 42000 \) psi. Determine the internal pressure that will produce yielding.

**SOLUTION:** This applied stress system is biaxial with no flexural or torsional shear.

Let: \( p \) equal internal pressure
\( t \) = wall thickness = 1 in.
\( d \) = diameter = 100 in.

\( f_x = \frac{pd}{2t} \) and \( f_y = \frac{pd}{4t} \)

From Eq. 30

\[
\bar{f} = \sqrt{f_x^2 + f_y^2 - f_x f_y}
\]

The vessel wall is to be stressed to the yield stress of 42000, thus \( \bar{f} = 42000 \).

Hence

\[
(42000)^2 = \left( \frac{pd}{2t} \right)^2 + \left( \frac{pd}{4t} \right)^2 - \left( \frac{pd}{2t} \right) \left( \frac{pd}{4t} \right)
\]

Solving, \( p = 970 \) psi.

**PROBLEMS**

(1) The combined stress loading at a point in a structure is as follows: \( f_x = -1000 \), \( f_y = -2500 \), \( f_z = 2000 \). Determine the magnitude and direction of the principal stresses. Determine the maximum shearing stress. Solve both analytically and graphically.

(2) Same as Problem 1, but change \( f_y \) to 4000 and \( f_z \) to -3000 and \( f_x \) to 2500.

(3) A solid circular shaft is subjected to a limit bending moment of 128200 inch pounds and a torsional moment of 250,000 inch pounds. If diameter is 4 inches and the yield tensile stress is 42,000, what is yield Margin of Safety.

(4) A thin walled cylinder of diameter 6 inches is subjected to an axial tensile load of 15,000 pounds, and a torsional moment of 12,000 inch pounds. What should be the wall thickness if the permissible yield stress is 30,000 psi.

(5) A closed and cylindrical vessel is 15 inches in diameter and a wall thickness of 0.25 inches. The vessel is subjected to an internal pressure of 10,000 psi, and a tensile load of 22,000 pounds. If the yield tensile stress of the material is 75,000 psi, what torsional moment can be added without causing yield.

**REFERENCES:**

- Timoshenko, Strength of Materials.
CHAPTER C2
STRENGTH OF COLUMNS WITH STABLE CROSS-SECTIONS

C2.1 Methods of Column Failure. Column Equations.

In Chapter A18, the theory of the elastic and inelastic instability of the column was presented. The equations from Chapter A18 for a pin end support condition are:

For elastic primary failure,

$$ F_0 = \frac{\pi^2 E}{(L/\rho)^2} \quad - - - - - - - - - - - - (1) $$

For inelastic primary failure,

$$ F_0 = \frac{\pi^2 E_t}{(L/\rho)^2} \quad - - - - - - - - - - - - (2) $$

Where $F_0$ = compressive unit stress at failure = P/A stress.

$E$ = Young’s modulus

$E_t$ = tangent modulus

L = column length

$\rho$ = radius of gyration of cross-section

Fig. C2.1 shows a typical plot of $F_0$ versus $L/\rho$. If the column dimensions are such as to cause it to fail in range CD in Fig. C2.1, the primary failure is due to elastic instability and equation (1) holds. This range of $L/\rho$ values is often referred to by engineers as the long column range.

The range AB in Fig. C2.1 is for a range of $L/\rho$ values of below 20 to 25, and represents a range where failure is due to plastic crushing of the column. In other words, the column is too short to buckle or bow under load but crushes under the high stresses. This column range of stresses is usually referred to as the block compression strength.

A column, however, may fail by local buckling or crippling due to distortion of the column cross-section in its own plane. The horizontal dashed line in Fig. C2.1 represents the condition where the primary column strength is limited by the local weakness. This line moves up or down according to the value of the local weakness. The determination of the column strength when failure is due to local weakness is covered in another chapter.


The column strength is influenced by the end support restraint against rotation and by any lateral supports between the column ends. The letter c is commonly used to indicate the end fixity coefficient, and c = 1.0 for zero end restraint against rotation, which can be produced mechanically by a pin or ball and socket end support fitting. Thus including the end restraint effect equations (1) and (2) can be written,

$$ F_0 = \frac{cn^2 E}{(L/\rho)^2}, \quad F_0 = \frac{cn^2 E_t}{(L/\rho)^2} \quad - - - - - - - - - - - - (3) $$

Let $L' = \text{effective length of the column}$ which equals the length between inflection points of the deflected column under load.

Then $L' = L/\sqrt{c} \quad - - - - - - - - - - - - (4)$

Thus equation (3) can be written as,

$$ F_0 = \frac{\pi^2 E}{(L'/\rho)^2}, \quad F_0 = \frac{\pi^2 E_t}{(L'/\rho)^2} \quad - - - - - - - - - - - - (5) $$

If we let $P = \text{failing or critical load}$, equation (5) can be written as equation (6) by realizing that $P = F_0 A$ and $\rho = \sqrt{I/A}$.

$$ P = \frac{\pi E I}{(L')^2}, \quad P = \frac{\pi E_t I}{(L')^2} \quad - - - - - - - - - - - - (6) $$

Fig. C2.1
C2.3 Design Column Curves for Various Materials.

For routine design purposes it is convenient to have column curves of allowable failing column stress $F_c$ versus the effective slenderness ratio $L'/\rho$. In equation (5) we will assume values of $F_c$, then find the tangent modulus $E_t$ corresponding to this stress and then solve for the term $L'/\rho$. Table C2.1 shows the calculations for 17.7 PH (TH1050) stainless steel sheet at room temperature. The results are then plotted in Fig. C2.7 to give the column strength curve. Similar data was calculated for the material under certain exposure time to different elevated temperatures and the results are also plotted in Fig. C2.7. Figs. C2.3 to C2.15 give column curves for other materials under various temperature conditions. Use of these curves will be made in example problems later in this chapter. The horizontal dashed line is the compressive yield stress. Values above these cut-off lines should be substantiated by tests.

C2.4 Tangent Modulus $E_t$ from Ramberg-Osgood Equation.

The basic Ramberg-Osgood relationship for $E_t$ is given as follows: (See Ref. 1)

$$\frac{E_t}{E} = \frac{1}{1 + \frac{3}{7} n \left( \frac{F}{F_{o,y}} \right)^n}$$

(7)

$E_t$ = tangent modulus of elasticity

$E$ = modulus of elasticity

For definition of other terms see Article Bl.12 of Chapter Bl.

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This equation is plotted in Fig. C2.16. For a given material, $n$, $F_{o,y}$, and $E$ must be known. Then assuming values of $F$, we can find corresponding values of $E_t/E$ from Fig. C2.16. For values of $E$, $F_{o,y}$, and $n$ refer to Table Bl.1 in Chapter Bl.

C2.5 Non-Dimensional Column Curves.

Quite useful non-dimensional column curves have been derived by Cozzone and Malcon (See Ref. 3).

The Euler column equation is

$$P = \frac{\pi^2E_t}{(L'/\rho)^2}$$

which can be written,

$$\frac{E_t}{P} = \frac{(L'/\rho)^2}{\pi^2}$$

(8)

The problem therefore resolves itself into obtaining and expression for $E_t/P$ from the non-dimensional relationship. To do this multiply both sides of equation (7) by $F_{o,y}/P$ and equate to $B^2$.

$$\frac{(E_t/F_{o,y})}{(E/P)} = \frac{1}{F_{o,y}/P} \frac{3}{7} \frac{n}{n}$$

Fig. C2.17 shows a plot of this equation as taken from Ref. 3, and shows $F/P_{o,y}$ versus $B$ for various values of $n$.

The shape of the knee of the stress-strain curve is given by the shape parameter $n$ and the abscissa $B$ incorporates the
Fig. C2.17

B = \sqrt{\frac{F_{o.7}}{F_c}} (L'/P)
particular properties of the material $E_0=\rho/F$.

Inserting value of $P = \frac{F}{E_0} (L'/\rho)^2$ in equation (3),

$$B = \frac{1}{\pi} \sqrt{\frac{F_0 + \pi}{E_0}} \left(\frac{L'}{\rho}\right)$$

or $B = \frac{1}{\pi} \sqrt{\frac{F_0 + \pi}{E_0}} \left(\frac{L'}{\rho}\right)$

The use of the curves in Fig. C2.17 will be illustrated later in the example problem solutions.

C2.5 Strength of Columns with Variable Cross-Section or Moment of Inertia.

To save weight in a built up column or forged column, the member is tapered or is made with a non-uniform cross-section. To find the ultimate strength of such columns, it is usually necessary to use a trial and error method. The general method of solution involving a consideration of column deflection will be illustrated for a case of a long column with uniform cross-section.

Fig. C2.18 shows a pin ended column in a deflected neutral equilibrium position when carrying the ultimate or critical load $P$. Assume that the shape of the deflected column follows a sine curve relationship with the deflection at midpoint equal to unity (see Fig. C2.18).

The equation of the deflected column $y_0 = L_0 \sin \frac{\pi x}{L}$. If $P$ is the end load, the bending moment at any point $M = Py = P_0 \sin \frac{\pi x}{L}$.

By the well known "moment area" principle (see Chapter A7; Art. A7.14), the deflection of a point (A) on the elastic curve away from a tangent to elastic curve (B) equals the first moment of the M/EI diagram between (A) and (B) about (A).

Thus in Fig. C2.18, the deflection of point (O) away from tangent at midpoint c equals unity in our assumed conditions and also equals the first moment of the area of the M/EI diagram between (0) and (C) about (O). (Fig. C2.19).

The value of the ordinate for M/EI diagram at any point $x$ from 0 is $\frac{P}{EI} \sin \frac{\pi x}{L}$. The total area under the curve is,

$$area = \frac{P}{EI} \int_0^L \sin \frac{\pi x}{L} \, dx = \frac{P}{EI} \left[ -\frac{L}{\pi} \cos \frac{\pi x}{L} \right]_0^L$$

$$= \frac{P}{EI} \left[ \left(\frac{L}{\pi} \cos \frac{L}{\pi} \right) - \left(\frac{L}{\pi} \cos \frac{0}{\pi} \right) \right] = \frac{P}{EI} \left(\frac{L}{\pi} + \frac{L}{\pi} \right)$$

hence area $= \frac{2PL}{EI}$ and half area $= \frac{PL}{EI}$

The center of gravity of the half area is

$$A \bar{x} = \int_0^{L/2} \frac{PL}{EI} \bar{x} = \frac{P}{EI} \int_0^{L/2} \sin \frac{\pi x}{L} \, dx ,$$

$$\frac{L}{\pi} \bar{x} = \int_0^{L/2} \sin \frac{\pi x}{L} \, dx$$

Integrating this simple expression and solving for $x$ we obtain:

$$\bar{x} = \frac{L}{\pi}$$

Taking moments about point (O) of the M/EI diagram between 0 and C about (O):

$$y_0 = 1 = \frac{PL}{\pi EI} \cdot \frac{L}{\pi} = \frac{PL}{\pi^2 EI} \text{ hence, } P = \frac{\pi^2 EI}{L}$$

which is the Euler equation, and thus the assumed sine curve was the proper one for the deflected elastic curve of the column.

Suppose that the elastic curve of deflected column had been assumed as a parabola with unit deflection at midpoint. Fig. C2.20 shows the M/EI diagram.

The area of one-half the diagram $= \frac{1}{2} \left(\frac{2}{3} \right) \frac{PL}{EI} = \frac{PL}{3EI}$

Taking moments about 0 of the area between 0 and C;

$$y_0 = 1 = \frac{PL}{3EI} \cdot \frac{5}{16} L = \frac{5PL}{48EI} \text{ hence,}$$
C.9 Solve for load P by writing and expression for the deflection at the center point which equals unity. This is done by using the moment area principle as was done in the previous example problem involving a column with uniform section.

In the above outlined procedure, E has been assumed constant or, in other words, the column failure is elastic or failing stresses are below the proportional limit stress of the material. The practical problem usually involves a slenderness ratio where failure is due to inelastic bending and thus E is not constant. For this case, a trial and error method of solution is necessary using the tangent modulus of elasticity which varies with stress in the inelastic stress range.

C.8 Design Column Curves for Columns with Non-Uniform Cross-Section.

Figs. C.2.21 and C.2.22 give curves for rapid solution of two types of stepped columns. Figs. C.2.23 and C.2.24 gives curves for the rapid solution of two forms of tapered columns. Use of these curves will be illustrated later in this chapter.

C.7 Column Fixity Coefficients c for Use with Columns with Elastic Side Restraints and Known End Bending Restraint.

Figs. C.2.25 and C.2.26 give curves for finding fixity coefficient c for columns with one and two elastic lateral restraints and Fig. C.2.27 gives curves for finding c when restraining moments at column ends are known. Use of these various curves will be illustrated later.

C.8 Selection of Materials for Elevated Temperature Conditions.

Light weight is an important requirement in aerospace structural design. For columns that fail in the inelastic range of stresses, a comparison of the \( F_{cy} \) weight ratio of materials gives a fairly good picture of the efficiency of compression members when subjected to elevated temperature conditions. In this ratio \( F_{cy} \) is the yield stress at the particular temperature and \( w \) is the weight per cu. inch of the material. Fig. C.2.28 shows a plot of \( F_{cy}/w \) for temperature ranges up to 600°F. with 1/2 hour time exposure for several important aerospace materials.

C.9 Example Problems.

Problem 1.

Fig. C.2.29 shows a forged (I) section member 30 inches long, which is to be used as
C2.12

STRENGTH OF COLUMNS WITH STABLE CROSS-SECTIONS

Fig. C2.28

(1) AISI Steel. \( F_{tu} = 180,000 \)
(2) 17-7 PH. Stainless Steel. \( F_{tu} = 210,000 \)
(3) 7075-T6 Alum. Alloy
(4) AZ31B Magnesium Alloy
(5) 6AL-4V Titanium Alloy

![Graph]

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a compression member. Find the ultimate strength of the member if made from the following materials and subjected to the given temperature and time conditions.

Case 1. Material 7079-T6 Alum. Alloy hand forging and room temperature.

Case 2. Same as Case 1, but subjected 1/2 hour to a temperature of 300°F.

Case 3. Same as Case 2, but for 600°F.

Case 4. Material 17-4 PH stainless steel, hand forging at room temperature.

Solution:

Since the column may fail by bending about either the X or Y axes, the column strength for bending about each of these axes will be calculated. Since the column strength is a function of the radius of gyration of the cross-section, the first step in the solution will be the calculation of \( I_X \) and \( I_Y \), from which \( \rho_X \) and \( \rho_Y \) can be found.

Calculating \( I_X \): In Fig. C2.30 the section will be first considered a solid rectangle \( 2.5 \times 2.75 \) and then the properties of portions (1) and (2) will be subtracted.

\( I_X (\text{rectangle}) = \frac{1}{12} \times 2.5 \times 2.75^3 = 4.32 \)

Portions (1) and (2)

\( I_X = \frac{1}{12} \times 1.5 \times 1.25^4 - 4 \times \frac{1.25 \times 0.25 \times 1.293^2}{12} = -1.29 \)

\( (I_X, \text{ of (2) about its x centroidal axis is negligible}) \)

\( I_X = 4.32 - 1.29 = 3.03 \text{ in.}^4 \)

\( \rho_X = \sqrt{\frac{I_X}{A}} \), Area \( A = 2.5 \times 2.75 - 2 \times 0.75 \times 1.25 - 4 \times 0.25 \times 0.625 = 4.375 \text{ sq. in.} \)

\( \rho_X = \sqrt{\frac{3.03}{4.375}} = 0.63 \text{ in.} \)

Calculation of \( I_Y \):

\( I_Y (\text{solid}) = \frac{1}{12} \times 2.75 \times 2.5^3 = 3.58 \)

Portion (1) = \(-0.19 \times 0.75 \times 0.375\times(2^{-1}-1.13 \times 0.75^3/12) = 1.52 \)

Portion (2) = \(-0.26 \times 0.38 \times 0.33^3\times4 - 4 \times 0.25 \times 1.25^3/36 \) = \(-0.488 \)

\( I_Y = 3.58 - 1.52 - 0.488 = 1.58 \text{ in.}^4 \)

\( \rho_Y = \sqrt{\frac{I_Y}{A}} = \sqrt{\frac{1.58}{4.375}} = 0.60 \)

Column strength is considerably influenced by the end restraint conditions. For failure by bending about the x-x axis, the end restraint against rotation is zero as the single fitting bolt has an axis parallel to the x-x axis and thus c the fixity coefficient is 1. For failure by bending about the y-y axis we have end restraint which will depend on the rigidity of the bolt and the adjacent fitting and structure. For this example problem, this restraint will be such as to make the end fixity coefficient \( c = 1.5 \).
For failure about x-x axis,
\[ L' = \frac{L}{\sqrt{\frac{f}{c}}} = \frac{30}{\sqrt{1.5}} = 30, \quad L'/\rho_x = 30/3.33 = 36 \]

For failure about y-y axis,
\[ L' = \frac{30}{\sqrt{1.5}} = 24.6, \quad L'/\rho_y = 24.6/0.60 = 41 \]

Therefore failure is critical for bending about y-y axis, with \( L'/\rho = 41 \).

Case 1. The material is 7079-T6 Alum. Alloy hand forging. Fig. C2.14 gives the failing stress \( F_c \) for this material plotted against the \( L'/\rho \) ratio. Thus using \( L'/\rho = 41 \) and the room temperature curve, we read \( F_c = 50500 \) psi. Thus the failing load \( F = F_cA = 50500 \times 4.375 = 220,000 \) lbs.

Case 2. Using the 300°C curve in Fig. C2.14 for the same \( L'/\rho \) value, we read \( F_c = 40,400 \), and thus \( F = 40,400 \times 4.375 = 177,000 \).

Case 3. Using the 600°F curve, \( F_c \) reads 6100 and thus \( F = 6100 \times 4.375 = 26700 \) lbs. Thus subjecting this member to a temperature of 600°F for 1/2 hour reduces its strength from 220,000 to 26,700 lbs., which means that Alum. Alloy is a poor material for carrying loads under such temperatures since the reduction in strength is quite large.

Case 4. Material 17-4 PH stainless steel forging. Fig. C2.8 gives the column curves for this material. For \( L'/\rho = 41 \) and using the room temperature curve we read \( F_c = 135,200 \) and thus \( F = 135,200 \times 4.375 = 531,000 \) lbs.

C2.10 Solution Without Using Column Curves.

When primary bending failure occurs at stresses above the proportional limit stress, the failing stress is given by equation (5) which is,
\[ F_c = \frac{n^2E_b}{(L'/\rho)^2} \]

Since \( E_b \) is the tangent modulus of elasticity, it varies with \( F_c \), and thus the relation of \( E_b \) to \( F_c \) must be known before the equation can be solved. To plot column curves for all materials in their many manufactured forms plus the various temperature conditions would require several hundred individual column charts. The use of such curves can be avoided if we know several values or parameters regarding the material as presented by Ramsburg and Osgood and expanded by Cozzone and Melcon (see Arts. C2.4 and C2.5) for use in column design.

Thus we make use of the curves in Fig. C2.17.

Case 1. Material 7079-T6 Alum. Alloy forging. Table B1.1 of Chapter B1 summarizes certain material properties. The properties needed to use Fig. C2.17 are the shape factor \( n \), the modulus \( E_b \) and the stress \( F_c \). Referring to Table B1.1, we find that \( n = 26, E_b = 10,500,000 \) and \( F_c = 59,500 \).

The horizontal scale in Fig. C2.17 involves the parameter,
\[ B = \frac{1}{n} \sqrt{\frac{E_b}{F_c}} = \frac{E_b}{F_c} (L'/\rho) \]

Substituting:
\[ B = \frac{1}{29} \sqrt{\frac{59,500}{50500 \times 41}} = 1.01 \]

Using Fig. C2.17 with 1.01 on bottom scale and projecting vertically upward to \( n = 26 \) curve and then horizontal to scale at left side of chart we read \( F_c/F_{c,r} = .940 \).

Then \( F_c = 59,500 \times .940 = 56,100 \), as compared to 50,500 in the previous solution using Fig. C2.14.

Case 2. From Table B1.1 for this material subjected to a temperature of 600°F for 1/2 hour, we find \( n = 29, E_b = 9,400,000 \) and \( F_c,r = 46,500 \).

Then \( B = \frac{1}{29} \sqrt{\frac{46,500}{9,400,000 \times 41}} = .917 \)

From Fig. C2.17 for \( B = .917 \) and \( n = 29 \), we read \( F_c/F_{c,r} = .98 \), thus \( F_c = 46,500 \times .98 = 45,900 \) as compared to 40,400 in the previous solution.

EXAMPLE PROBLEM 2.

Fig. C2.31 shows an extruded (I) section. A member composed of this section is 32 inches long. The member is braced laterally in the x direction, thus failure will occur by bending about x-x axis. The member is pin ended and thus \( n = 1 \). The material is 7075-T6 Extrusion. The problem is to find the failing stress \( F_c \) under room temperature conditions. Fig. C2.31

This (I) section corresponds to Section 15 in Table A3.15 in Chapter A3. Reference
to this table gives,

\[ A = 0.554 \text{ sq. in.} \quad \sigma_x = 0.618 \]

\[ L' = L, \sin c = 1, \quad L'/\sigma_x = 32/0.618 = 51.7 \]

Fig. C2.11 gives the column curves for this material. For \( L'/\rho = 51.7 \) and room temperature we read \( F_C = 38,500 \text{ psi} \).

Solution by using Fig. C2.17,

From Table B1.1 for this material we find \( n = 16.6, \quad E_C = 10,500,000 \) and \( F_{o,r} = 72,000 \).

Then, \( B = \frac{1}{n} \sqrt{\frac{72,000}{10,500,000} (51.7)} = 1.36 \)

From Fig. C2.17 for \( B = 1.36 \) and \( n = 16.6 \), we read \( F_C/F_{o,r} = 0.537 \), hence \( F_C = 0.537 \times 72,000 = 38,600 \).

Consider the member is subjected to a temperature of 450°F for 1/2 hour.

From Fig. C2.11, \( F_C = 24,400 \text{ psi} \)

Using Fig. C2.17:-

From Table B1.1, \( n = 16.6, \quad E_C = 7,800,000 \) and \( F_{o,r} = 29,000 \).

Then \( B = \frac{1}{n} \sqrt{\frac{29,000}{7,800,000} (51.7)} = 1.00 \)

From Fig. C2.17 we find \( F_C/F_{o,r} = 0.74 \)

Then \( F_C = 0.74 \times 29,000 = 21,450 \text{ psi} \)

A very common aluminum alloy in aircraft construction is 2014-76 extrusions. Let it be required to determine the allowable stress \( F_C \) for our member when made of this material.

Since we have not presented column curves for this material, we will use Fig. C2.17.

From Table B1.1, for our material, we find \( n = 18.5, \quad E_C = 10,700,000 \) and \( F_{o,r} = 53,000 \).

Then \( B = \frac{1}{n} \sqrt{\frac{53,000}{10,700,000} (51.7)} = 1.16 \)

From Fig. C2.17 for \( B = 1.16 \) and \( n = 18.5 \), we read \( F_C/F_{o,r} = 0.71 \), hence \( F_C = 0.71 \times 53,000 = 37,560 \).

The result shows that the 2014-76 material gave a failing stress of 37,560 as compared to 36,900 for the 7075-76 material which has a F oy of 70,000 as compared to \( F_{o,y} = 53,000 \) for the 2014-76 material. The reason for the 7075 material not showing much higher column failing stress \( F_C \) over that for the 2014 alloy is due to the fact that the stress existing under a L'/\rho value of 61.7 is near the proportional limit stress or \( E_C \) is not much different than \( E_C \), the elastic modulus.

To illustrate a situation where the 7075 material becomes more efficient in comparison to the 2014 alloy, let us assume that our member has a rigid connection at its end which will develop an end restraint equivalent to a fixity coefficient \( c = 2 \).

Then \( L'/\rho = \frac{32}{\sqrt{2}} = 22.6 \) and \( L'/\rho = \frac{32}{0.618} = 36.7 \).

For the 7075 material from Fig. C2.11, \( F_C = 58,300 \text{ psi} \)

For the 2014 material, we use Fig. C2.17

\[ B = \frac{1}{n} \sqrt{\frac{53,000}{10,700,000} (36.7)} = 0.823 \]

From Fig. C2.17 for \( B = 0.823 \) and \( n = 18.5 \), we read \( F_C/F_{o,r} = 0.87 \), when \( F_C = 0.87 \times 53,000 = 46,100 \text{ as compared to 58,300 for the 7075 material, thus } 7075 \text{ material would permit lighter weight of required structural material.} \)

The student should realize that if the stress range is such as to make \( E_T = E_C \), then the bending failure is elastic instead of inelastic and equation (5), using Young's modulus of elasticity \( E_C \), can be solved directly without resort to column curves or a consideration of \( E_T \), since \( E_T \) is equal to \( E_C \).

The student should realize that equation (5) is for strength under primary column failure due to bending as a whole and not due to local buckling or crippling of the member or by twisting failure. The subject of column design when local failure is involved is covered in a later chapter.

In example problem 2, we have assumed that local crippling is not critical, which calculation will show is true as explained and covered in a later chapter.

C2.11 Strength of Stepped Column.

The use of curves in Fig. C2.22 will be illustrated by the solution for the strength of two stepped columns in order to illustrate both elastic and inelastic failure of such columns.

Case I. Elastic failure.

Fig. C2.32 shows a double stepped pin ended column. The member is machined from a 1 inch diameter extruded rod made from
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

7075-T6 material. The problem is to find the maximum compressive load this member will carry.

\[
\begin{array}{c|c|c}
\text{Portion 1} & \text{Portion 2} & \text{Portion 1} \\
\hline
\frac{3}{4}'' \text{ Dia.} & \frac{1}{4}'' \text{ Dia.} & \text{Portion 2} \\
\hline
b=15'' & a=30'' & b=15'' \\
L=60'' & L=60'' & \\
\end{array}
\]

**Fig. C2.32**

**PORTION 1**

\[
A_1 = 0.7854 \text{ in.}^2 \quad A_s = 0.4418 \\
I_1 = 0.0491 \text{ in.}^4 \quad I_s = 0.0155 \\
E_c = 10,500,000 \quad E_s = 10,500,000
\]

From Fig. C2.22, \( P_{cr} = \frac{B(EI_s)}{L^2} \). This is the Euler equation for failure under elastic bending. If the ratio a/L equals 1 or a uniform section, B becomes \( \pi \) or 10 as shown in Fig. C2.22. The curves in Fig. C2.22 apply only to elastic failure. Since the member in Fig. C2.32 is rather slender we will assume the failure is elastic and then check this assumption.

\[
\frac{E_1}{E_s} = \frac{10,500,000 \times 0.0491}{10,500,000 \times 0.0155} = 3.17, \quad \frac{a}{L} = \frac{30}{60} = 0.5
\]

From Fig. C2.22 for a/L = 0.5 and \( EI_1/EL_s = 3.17 \) we read B = 7.0.

Whence, \( P_{cr} = \frac{B EI_1}{L^2} = \frac{7 \times 10,500,000 \times 0.0491}{60^2} = 1000 \text{ lb.} \)

The stresses in each portion are,

\[
\begin{align*}
f_1 &= \frac{1000}{0.7854} = 1280 \text{ psi} \\
f_s &= \frac{1000}{0.4418} = 2270 \text{ psi}
\end{align*}
\]

These compressive stresses are below the proportional limit stress of the material so \( E_c \) is constant and our solution is correct.

**Case 2. Inelastic Failure.**

The column has been shortened to the dimensions as shown in Fig. C2.33. The diameters and material remain the same as in Case 1.

\[
\begin{array}{c|c|c}
\text{Portion 1} & \text{Portion 2} & \text{Portion 2} \\
\hline
\text{b=3''} & \text{a=6''} & \text{b=3''} \\
L=12'' & L=12'' & \\
\end{array}
\]

**Fig. C2.33**

This is a relatively short column so the failing stress should fall in the inelastic range where \( E \) is not constant, therefore the solution is a trial and error procedure. We will base our first guess or trial on an average \( L/\rho \) value.

\[
\begin{align*}
p & \text{ for portion 1 is 0.25 inches} \\
p & \text{ for portion 2 is 0.1875} \\
\text{Average } p &= (6 \times 0.25 + 6 \times 0.1875) / 12 = 0.22
\]

Then \( L/\rho = 12/0.22 = 54.5, \) use 55.

**Fig. C2.11** is a column curve for 7075-T6 Alum. Alloy extruded material. With \( L/\rho = 55 \), we read allowable stress \( P_c = 33,500 \text{ psi.} \) Therefore

\[
\begin{align*}
P &= P_c \times A = 33,500 \times 0.7854 = 26,300 \text{ lb.} \\
f_1 &= 33,500 \quad \text{and } f_s = 26,300 / 0.4418 = 59,500
\end{align*}
\]

The stress \( f \) in portion 2 is above the proportional limit stress so a plasticity correction must be made in using the curves in Fig. C2.22.

Referring to Table B1.1 in Chapter B1, we find the following values for 7075-T6 extrusions: \( n = 16.6, F_{y}, = 72,000, \) \( E_c = 10,500,000. \)

The tangent modulus \( E_t \) will be found for the stresses \( f_1 \) and \( f_s. \)

**For Portion 1**, \( f_1/f_{y} = 33,500/72,000 = 0.465 \)

Referring to Fig. C2.16 and using 0.465 and \( n = 16.6, \) we read \( E_t/E = 1.0, \) thus \( E_t = E \) and thus no plasticity correction for Portion 1.

**For Portion 2**, \( f_s/f_{y} = 59,500/72,000 = 0.826 \)

From Fig. C2.16, we obtain \( E_t/E = 0.675 \) whence, \( E_t = 0.675 \times 10,500,000 = 7,090,000. \)

\[
\begin{align*}
EI_1 &= 10,500,000 \times 0.0491 \\
EI_s &= 7,090,000 \times 0.0155 = 4.7
\end{align*}
\]

From Fig. C2.22 for a/L = 0.5, we obtain \( B = 5.6. \)

Then \( P_{cr} = \frac{B EI_1}{L^2} = \frac{5.5 \times 10,500,000 \times 0.0491}{144} = 20,000 \text{ lb.} \)
Our guessed strength was 26,300 lb. Our guessed strength and calculated strength must be the same so we must try again.

**Trial 2.** Assume a critical load \( P = 23500 \) lb.

\[
f_1 = 23500 \div 0.7954 = 29300
\]

\[
f_2 = 23500 \div 0.4418 = 53100
\]

**Portion 1.** \( f_f/F_0 = 29900/72000 = 0.415
\)

From Fig. C2.16 for \( n = 16.6 \), we read \( E_t/E = 1.0 \).

**Portion 2.** \( f_2/F_0 = 53100/72000 = 0.738
\)

From Fig. C2.16, \( E_t/E = 0.90 \), whence

\[
E_t = 0.90 \times 10,500,000 = 9,450,000.
\]

\[
\frac{EI_1}{EI} = \frac{10,500,000 \times 0.0491}{9,450,000 \times 0.0155} = 3.52.
\]

From Fig. C2.22 for \( a/L = 0.5 \), we read \( B = 6.62 \).

\[
The \text{Pcr} = \frac{B \cdot EI_1}{L^2} = 6.62 \times 10,500,000 \times 0.0491 = 23,650 \text{ lb}.
\]

This practically checks the assumed value, thus the answer is between 23,500 and 23,650 and if further accuracy is desired another trial should be carried through.

The other types of columns with non-uniform cross-sections as shown in Figs. C2.21, C2.22 and C2.24 are solved in a similar manner. These charts are to be used only with pin ended columns. The end fixity coefficient \( c \) for tapered columns is not the same as for uniform section columns.

**C2.12 Column Strength With Known End Restraining Moment.**

Fig. C2.27 shows curves for finding the end fixity coefficient \( c \) for two conditions of known end bending restraint.

To illustrate the use of these curves, a simple problem will be solved.

Fig. C2.34 shows a 3-bay welded steel tubular truss. The problem is to determine the allowable compressive stress for member AB. This strength is influenced by the fixity existing at ends A and B. The diameter and wall thickness of each tube in the truss is shown on the figure. The material is AISI steel, \( F_{ty} = 50,000, F_{ty} = 70,000, E = 29,000,000. \)

![Fig. C2.34](image)

The member AB is welded to three adjacent tubes at joints (A) and (B). Since these tubes are the same at joints (A) and (B), the fixity at the ends (A) and (B) of member AB is the same.

Referring to Fig. C2.27, the term \( \mu \) is defined as the bending restraint coefficient—spring constant expressed as inch pounds per radian. In Fig. A, the moment \( M \) required to rotate (A) through 1 radian when far end is fixed is \( 4EI/L \).

For derivation of this value refer to Art. All.4 of Chapter All. If the far end \( (B) \) is pinned in (Fig. A), a moment \( M = 3EI/L \) will rotate end (A) through one radian. To be slightly conservative, we will assume the far ends of members coming into joints (A) and (B) as pinned. Thus \( \mu = 3EI/L \). The sum of \( \mu \) = \( 3EI/L \) will be computed for the 3 members which form the support of member AB at end (A).

Member AC: \( I = 0.03867, I/L = 0.001289 \)

Member AB: \( I = 0.02775, I/L = 0.00071 \)

Member AF: \( I = 0.02402, I/L = 0.000982 \)

\[
\mu = \frac{3EI}{L} = 3(0.001289 + 0.00071 + 0.000982) = 0.029,000,000
\]

\[
\mu = 258,000
\]

In Fig. C2.27 we need the term \( \mu L/EI \). The \( L/EI \) refers to member AB. Thus \( \mu L/EI = (258,000 \times 30)/29,000,000 \times 0.0367 = 7.28 \).

We use the upper curve in Fig. C2.27 since restraint at both ends of member AB is the same. Thus for \( \mu L/EI = 7.28 \), we read end fixity coefficient \( c = 2.58 \).

Then \( L' = L/\sqrt{c} = 30/\sqrt{2.58} = 18.6 \).

\[
\rho \text{ for member } = 0.422 \text{ inches.}
\]

\[
L'/\rho = 18.6/0.422 = 44.0
\]

From column curve in Fig. C2.3, we read
allowable failing stress to be \( F_0 = 55,200 \) psi.

If the far ends of the connecting members were assumed fixed instead of pinned, then \( \mu = 425/L \), or we can multiply previous value of 7.28 by 4/3, which gives 9.7 which, used in Fig. C2.27, gives \( c = 2.80 \).

\[
L'/\rho = \sqrt{2.9 \times 0.422} = 42.5.
\]

Then from Fig. C2.3, \( F_0 = 56,600 \) psi. Since the far ends are less than fixed, the assumption that far ends are pinned gives fairly accurate results.

In a truss structure all members are carrying axial loads and axial loads effect the ability of members to resist rotation of their ends. Art. A11.12 of Chapter A11 explains how to take account of the effect of axial load upon the stiffness of a member as required in calculating the end restraint coefficient \( \mu \).


Figs. C2.28 and C2.29 provide curves for finding the end fixity coefficient \( c \) to take care of elastic lateral supports at points midway between the column ends.

To illustrate the use of these charts, a round bar 0.5 inches in diameter and 24 inches long is braced laterally as shown in Fig. C2.35. The bar is made of AISI Steel, heat treated to \( F_{tu} = 125,000 \). The spring constant for the lateral support is 775 lbs. per inch.

\[
\text{Moment of Inertia of } 1/2 \text{ rod} = 0.003066,
\]

Radius of Gyration = \( 0.125 \) inches.

\[
q = \frac{x^4}{12} = \frac{775 \times 24^3}{29,000,000 \times 0.003066} = 120
\]

From Fig. C2.25 for \( x/L = 10/24 = 0.416 \) and \( q = 120 \), we find \( c = 2.92 \).

Then \( L' = L/\sqrt{c} = 24/\sqrt{2.92} = 14.08 \)

\[
L'/\rho = 14.08/0.125 = 113
\]

\[
F_0 = \frac{n^8E}{(L'/\rho)^8} = \frac{n^8 \times 29,000,000}{(113)^8} = 22,500 \text{ psi}
\]

If the stress is above the proportional limit stress for the material, then the trial and error approach must be used as illustrated in the problem dealing with a tapered column.

C2.14 Problems.

(1) 6061-T6 Aluminum Alloy sheet, heat-treated and aged has the following properties:

(a) Under room temperature: \( F_{0,1} = 35,000 \) psi, \( E_0 = 11,000,000 \) psi, and \( n = 31 \)

(b) For 1/2 exposure at 300°F: \( F_{0,3} = 29,000 \), \( E_0 = 9,500,000 \) and \( n = 26 \).

For the above two cases (a) and (b), determine \( E_0 \) (tangent modulus values) from Fig. C2.16 and then calculate and plot column curves for these 2 material conditions.

(2) Fig. C2.36 shows the cross-section of a compression member. Calculate the failing compressive load under the following cases:

Case 1. \( L = 25 \) inches.

Material AISI Steel 4140, \( F_{tu} = 180,000 \). Take end fixity coefficient \( c = 1 \) for bending about \( x-x \) axis and 1.5 about axis \( y-y \).

(3) Same as Problem (2) but member is subjected to a temperature of 350°F for 1/2 hour.

(4) Two extruded channel sections identical to Section No. 50 in Table A3.11 in Chapter A3, are riveted back to back and used as a column member. If member is 26 inches long and end fixity is \( C = 1 \) and material is 7075-T6 extrusion, what is the failing compressive load.

If member is fastened rigidly to adjacent structure which provides a fixity \( c = 2 \), what will be the failing load.

(5) Consider the column in problem (4) is made from 2014-T6 Aluminum Alloy extrusion. Find failing load.

(6) The pin ended single stepped column as shown in Fig. C2.37 is made of AISI-4130 normalized steel, \( F_{tu} = 90,000 \), \( F_{0,y} = 70,000 \). Determine the maximum compressive load member will carry.
(7) Same as Problem (5) but member is exposed 1/2 hour to a temperature of 500°F.

(8) Same as Problem (5) but change dimension (a) to 10 inches, and L to 14.28 inches.

(9) Find the failing compressive load for the doubly stepped column in Fig. C2.38 if member is made from 7079-T6 hand forging.

(10) Same as Problem (7) but change dimensions a = 6", b = 4", L = 14".

(11) The cylindrical tapered member in Fig. C2.39 is used as a compression member. If member is made from AISI Steel 4140, Ftu = 125,000, what is the failing load.

https://example.com/fig-c2-39.png

Fig. C2.39

(12) Same as Problem (7) but change dimensions to a = 6", b = 4.0", L = 14 inches.

References:

(1) NACA Technical Note 902.
(2) Non-dimensional Bunking Curves, by Cozzone Z. Melcon, Jr. of Aero. Sciences, October, 1946.
(3) Chart from Lockheed Aircraft Structures Manual.
CHAPTER C3

YIELD AND ULTIMATE STRENGTH IN BENDING

C3.1 Introduction.

Members subjected to bending alone or in combination with axial and torsional loads are quite common in flight vehicle structures. The limit design loads on a structural member must be carried without permanent distortion and the ultimate design loads must be carried without rupture or failure. The well known bending stress equation \( f_0 = M_0 / I \), assumes a linear variation of stress with strain or, in other words, the equation holds for stresses below the proportional limit stress or, in general, the elastic range. Failure of a member in bending, unless there is local weakness, does not occur at stresses in the elastic range but occurs at stresses in the inelastic range. Since the ultimate strength of a member is needed to compare against the ultimate design load to be carried, a theory or procedure is necessary which will accurately determine the ultimate and yield strength of a member in bending.

C3.2 Basic Approach to Finding the Bending Strength of Members.

The problem is to determine the internal resisting moment of a beam section when subjected to stresses which fall in the inelastic range of stresses. This stress can be taken as the ultimate tensile or compressive stress of the material or limited to some stress or deformation in the inelastic range. To obtain the true internal resisting moment, we must know how the normal tension and compressive stress varies over the cross-section. The stress-strain curve for the material provides the source for obtaining the true stress picture. If a material has a different shape in the tensile and compressive inelastic zones, the neutral axes does not coincide with the centroidal axis, thus adding some difficulty to an analysis method. The analysis procedure for determining the true internal resisting moment is best explained by an example solution.

C3.3 Bending Strength of a Solid Round Bar.

Fig. C3.1a shows the cross-section of a round solid bar made of aluminum alloy. The stress-strain curve up to a unit strain of .010 in. per inch is given in Fig. C3.2. Note that the shape of the curve in the inelastic zone is not the same for both tension and compression. In this example solution, we will find the internal resisting moment when we limit the unit strain at the extreme edge on the compressive side of the beam section to 0.010. Now plane sections remain plane after bending in both elastic and inelastic stress conditions when member is in pure bending. We will guess the neutral axis as located 0.0375 inches above the centroidal axis as shown in Fig. C3.1b. Having assumed the maximum unit strain as .010, we can draw the strain diagram of Fig. C3.1b. We now divide the cross-section in Fig. C3.1a into 20 narrow horizontal strips. Having the strain curve in Fig. C3.1b, we can find the unit...
strain at the midpoint of each strip. With the strain known on each strip, the stress existing can be found by use of the stress-strain curve in Fig. C3.2. The total load on each strip then equals the stress times the area of the strip. The internal resisting moment equals the summation of the load on each strip times the distance from the strip to the neutral axis.

Table C3.1 shows the detail calculations. If the neutral axis has been selected in the correct position, the values in column (6) of the table should add up to zero since total tension must equal the total compression on the beam cross-section. The small discrepancy of 740 pounds in the summation of column (6) is not enough to change the location of the neutral axis or the total internal resisting moment appreciably. Column (7) gives the total internal resisting moment as 56735 in. lbs. when the strain is limited to the 0.01 strain as previously discussed. The stress at this strain from Fig. C3.2 is 49000 psi. Using this stress in the well known beam formula \( M = \frac{F}{c} \), we obtain \( M = 49000 \times 0.075 = 38450 \), which is much less than the true value of 56735.

<table>
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<th>Strip No.</th>
<th>Strip Area &quot;A&quot;</th>
<th>( y )</th>
<th>( \varepsilon )</th>
<th>Unit Stress ( \sigma )</th>
<th>( F = \sigma A )</th>
<th>Res. Moment ( M = Fr )</th>
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<td>0.935</td>
<td>0.0687</td>
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Since it is desirable to use the beam formula in finding bending stresses due to a given bending moment, and also for determining the true internal resisting moment of a beam section, structural design engineers make use of a fictitious falling bending stress \( f_0 \), which is referred to as a modulus of rupture stress in pure bending. Then the ultimate bending moment that can be developed by a given beam cross-section and a given material is \( M = f_0 I / C \). Design curves for finding \( f_0 \), the modulus of rupture, are given later in this chapter.

Since there are many flight vehicle materials and all kinds of shapes used in structural members, the basic approach for solution as illustrated in Table C3.1 becomes very time consuming. Design engineers always search for simplified methods which give sufficient accuracy. Thus Cozzzone (Ref. 1) has developed a simplified procedure for finding the modulus of yield or rupturing bending stress \( f_0 \). The method is widely used in the aerospace industry in structural design.

C3.4 The Cozzzone Simplified Procedure.

The Cozzzone method in its simplest form assumes a symmetrical rectangular beam section and the same shape of the stress-strain curve in both tension and compression. Fig. C3.3 represents the true bending stress variation over the beam cross-section when failure occurs. Cozzzone now replaces this true curve by a trapezoidal stress variation as shown in Fig. C3.4. The stress \( f_0 \) is a fictitious stress which is assumed to exist at the neutral axis or at zero strain.

![Image](Fig. C3.3)

![Image](Fig. C3.4)

![Image](Fig. C3.5)

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The value of \( f_0 \) is determined by making the requirement that the internal moment of the true stress system must equal the moment of the assumed trapezoidal stress system which results from the assumed stress-strain curve as shown in Fig. C3.5. Fig. C3.6 shows the trapezoidal stress pattern drawn to a larger scale and showing only one half of the symmetrical beam section. The trapezoidal stress pattern has been divided into a rectangle (r) and a triangle (b) as shown in the figure.

Let \( M_b \) = total internal resisting moment.
\[ m_r = \text{internal moment developed by portion (r)}. \]
\[ m_b = \text{internal moment developed by portion (b)}. \]

Then \( M_b = m_r + m_b \)

Since \( f_b \) varies linearly from zero to \( f_b \), the stress is elastic and thus the beam equation holds, or
\[ m_b = f_b L / c \]

for entire beam section. The stress variation on portion (r) is constant or rectangular, thus
\[ m_r = f_0 \int_0^c y \, dy = Q \]

Then \( m_r = f_0 L / 2 \) for entire beam section.

But \( f_b = f_m - f_0 \) (from Fig. C3.6)

Thus, \( M_b = (f_m - f_0) \frac{L}{2} + 2f_0Q \), or
\[ M_b = f_m \frac{L}{2} - f_0 \frac{L}{2} - 2f_0Q \]

\[ \frac{M_b}{L} = f_m - f_0 \left( \frac{2Q}{I/c} - 1 \right) \]

Let \( k = \frac{2Q}{I/c} \)

\[ k = \frac{2Q}{I/c} \]

(2)

\( k \) is a beam section shape factor.

Let \( F_b = M_b / I \), then from equation (1)
\[ F_b = f_m - f_0 (k - 1) \]

(3)

\( F_b \) is a fictitious Mo/I stress or the modulus of rupture for a particular cross-section at a given maximum stress level.

The values of \( k \) vary between 1 and 2.0. If calculated value of \( k \) is greater than 2 use 2.0. Fig. C3.7 shows the value of the shape factor \( k \) for several typical shapes. Fig. C3.8 shows curves for the rapid determination of the \( k \) factor for 3 common beam sections.

![Fig. C3.7 k Factor for Some Typical Shapes](image)

![Fig. C3.8 Section Factor K for I, \( \square \), and C Sections (Ref. 2)](image)

C3.5 Design Curves for Finding Modulus of Rupture (\( F_b \)).

The modulus of rupture \( F_b \) may be a yield modulus, that is, in equation (3) the value of \( f_m \) is equal to the yield stress of the material. It may also be the ultimate modulus of rupture, in which case the value of \( f_m \) in equation (3) equals the ultimate strength of the material. The modulus of rupture may be limited to a stress between the yield and ultimate stress of the material because of local crippling or by excessive distortion. Regardless of what value is used for \( f_m \) in equation (3), the corresponding value of \( f_0 \) must be known before the value of \( F_b \) can be determined. Figs. C3.9 to C3.26 give strain curves for various material and the corresponding \( f_0 \) curve. The use of these two curves permit the determination of \( F_b \) if the \( k \) shape factor for the particular beam section being considered is known. In deriving the values of \( f_0 \), the following assumptions are made.

1. The stress-strain curve is assumed the same in tension and compression.
2. The neutral axis is assumed to coincide with the centroidal axis.
3. During bending plane sections remain plane.
4. The cross-section is not subject to local or torsional instability.
(5) Beam-column, curvature and shear lag effects are considered negligible.

C3.6 General Accuracy of Method.

(1) It is exact for a rectangular section under pure bending with moment vector parallel to a principal axis.

(2) For double symmetric sections under pure bending and moment vector parallel to a principal axis, the accuracy should be within 5 percent.

(3) Single symmetric sections will vary from practically exact to definitely unconservative (moment vector normal to axis of symmetry).

(4) For sections subject to combined bending and axial load, the results will vary from practically exact to conservative.

(5) For unsymmetrical bending, with and without axial load, the results will vary from practically exact to conservative.

C3.7 Example Problems in Finding Bending Strength.

EXAMPLE PROBLEM 1.

A rectangular beam section is 0.25 inches wide and 1.5 inches deep. What yield and ultimate bending moment will the section develop when made from 7075-T6 extruded aluminum alloy.

Solution: The modulus of bending stress is given by equation (3),

\[
F_b = \frac{f_m}{I/c} = \frac{2}{3} \times 0.75 \times 0.25 \times 0.375 \times \frac{1.5 \times (1/1.75)}{0.0288} = 1.408
\]

\[
k = \frac{20}{12} \times 0.25 \times 1.5 \times (1/1.75) = 0.0388
\]

\[k = 1.50\]

The value of k could also be found in Fig. C3.7.

Material is 7075-T6 aluminum alloy. From Fig. C3.17, \(F_{tu} = 75000, F_{ty} = 65000\).

To find the yield bending strength, the value of \(f_m\) in equation (3), the maximum stress permitted on the most remote fiber is 65000, the yield stress of the material. To find \(f_o\), we go to Fig. C3.17 and find the point on the stress-strain curve that corresponds to a stress of 65000. This point is projected vertically downward to intersect the curve \(f_o\). This point is then projected to the stress scale at the edge of the chart to give the value of \(f_o\). The value of \(f_o\) from this chart operation gives \(f_o = 29000\). Then from equation (3)

\[F_{byield} = 65000 \times 29000 (1.5 - 1) = 78500 \text{ psi.} \]

Then yield bending moment \(M_{yp} = F_b I/c\)

\[M_{yp} = 75000 \times 0.0388 = 7460 \text{ in. lb.}\]

For finding the ultimate resisting bending, we use \(F_{tu}\) which is 75000 as the value of \(f_m\) in equation (3). Again going to Fig. C3.17 to stress of 75000 on stress-strain curve and the vertically down to \(f_o\) curve, we obtain \(f_o = 70500\).

Then \(F_{b(ult)} = 75000 + 70500 (1.5-1) = 110250 \text{ psi}\)

Then \(M_{ult} = F_b I/c = 110250 \times 0.0388 = 10370 \text{ in. lb.}\)

Let us assume that is is desired to limit the strain in the extreme fiber to 0.03 inches per inch. What would be the bending moment developed under this limitation.

From Fig. C3.17 for a unit strain of 0.03 the corresponding stress from the stress strain curve is 74700 and the \(f_o\) stress is 61200.

Then \(F_b = 74700 + 61200 (1.5-1) = 105300\)

Then \(M = F_b I/c = 105300 \times 0.0388 = 9900 \text{ in. lb.}\)

EXAMPLE PROBLEM 2.

The symmetrical I beam section in Fig. (a) is subjected to an ultimate design pure bending moment \(M = 14000 \text{ in. lb.}\). What is the margin of safety if the beam is made of magnesium forging A261A which has \(F_{tu} = 38000\) and \(F_{ty} = 22000\).

\[
\frac{I_x}{\frac{12}{12}} = 1.375 \times 2^2 = 1.25 \times 1.75
\]

\[= 1.25 \times 1.75
\]

\[= 0.916 - 0.558 = 0.358
\]

\[
\frac{I_x}{c} = 0.358/1 = 0.358
\]

\[Q = 1.375 \times 0.125 \times 0.9375 + 0.875 \times 0.125 \times 0.4375 = 0.509
\]

\[k = \frac{20}{12} \times 0.209 = 1.17
\]

\[k = 0.358
\]

\[k could have been from Fig. C3.3 when using m = 0.125/1.375 = 0.091, and n = 0.125/1.75 = 0.0715.\]
From Fig. C3.19, for $f_m = F_{tu} = 38000$ we find in projecting vertically downward to $f_0$ curve gives $f_0 = 23700$. Then $subt. in$ equation (3)

$$F_b = 38000 + 23700 \times (1.17 - 1) = 40770$$

$$Mult = \frac{F_b}{c} = 40770 \times \frac{0.358}{14000} = 1.05 \text{ lb.}$$

$$Margin \ of \ Safety = \frac{Mult}{M} - 1 = \frac{15000}{14000} - 1 = .07$$

Assume we desire the stress intensity at a point 0.5 inches from neutral axis if full bending strength was developed. From Fig. C3.19, the unit strain when stress is 38000 is 0.35. Then since plane sections remain plane after bending, the unit strain at point .50 inch from neutral axis is (.5/1) (.35) = .0175. From Fig. C5.19, the stress existing at this strain is $f = 31000$ psi. A linear variation of stress as used in the flexural equation would give half the maximum stress or $38000/2 = 19000$ psi as against the true stress of 31000.

Curves for finding $F_b$. $F_b = f_m + f_0 (k - 1)$. 

333.
Curves for finding $F_b$.  

$$F_b = f_m + f_o (k - 1).$$
EXAMPLE PROBLEM 3. Unsymmetrical Section.

Fig. (b) shows a tee beam section, symmetrical about the vertical axis. If the material is 17-4 PH stainless steel, what ultimate bending moment will be developed if bottom portion is the tension flange.

The neutral axis will first be determined.

\[
y = \frac{\Sigma y\ A}{\Sigma A} = \frac{2 \times 0.1 \times 1 + 1.4 \times 0.1 \times 1.95}{2 \times 1 + 1.4 \times 1} = 1.39
\]

Fig. (c) shows the unit strain picture. The lower edge of the beam section is strained to the maximum value of 0.035 as shown on the stress-strain curve in Fig. C3.31. Since plane sections remain plane the unit strain \(\varepsilon_a\) at the upper edge of section is \(\varepsilon_a = 0.035 \times 0.61/1.39 = 0.0154\).

Solution 1

Equation (3) was derived for a symmetrical section about the neutral axis. The equation involves finding the resisting moment developed by one half the beam section and multiplying by 2. This is permissible since the unit strain at both top and bottom edges is the same. In this solution we will continue to use equation (3). To do this it is necessary to make symmetrical sections for the portion on each side of the neutral axis of the entire section.

Label the beam portion below the neutral axis as (1) and that above by (2).

Portion 1. Fig. (d) shows how the lower portion (1) is made a symmetrical section about neutral axes by adding the dashed portion. The internal bending resistance will be found for this entire section in Fig. (d). One half of this amount will then be the true moment developed by portion (1).

\[
I = \frac{1}{12} bh^2 = \frac{1}{12} \times 0.1 \times 2.78^2 = 0.178
\]

\[
I_1/c_1 = 0.178/1.39 = 0.128
\]

\[
Q_1 = 1.39 \times 0.1 \times 0.695 = 0.0965
\]

\[
2Q_1 = 0.193
\]

\[
k_1 = 2Q_1/I_1/c_1 = 0.193/0.128 = 1.5
\]

From Fig. C3.22, \(F_{tu} = 180000\) which equals \(f_m\).

\[
f_0\text{ from curve } = 156000
\]

Then \(F_{b1} = f_m + f_0 (k - 1)\)

\[
= 150000 + 156000 (1.5 - 1) = 258000
\]

\[
M = (F_{b1} I_1/c_1)^{1/2} = 258000 \times 0.128 \times 0.5 = 16550 \text{ in. lb.}
\]

The factor 1/2 is due to the fact that portion (1) is only one half the beam section in Fig. (d).

Portion 2. Fig. (e) shows the developed symmetrical section for the upper portion (2) of the beam section.

\[
I_2 = \frac{1}{12} \times 1.5 \times 1.22^2 - \frac{1}{12} \times 1.4 \times 1.02^2 = 0.104
\]
\[ I_{a}/c_{a} = 0.104/0.61 = 0.1704 \]
\[ Q_{a} = 0.61 \times 1 \times 0.305 + 1.4 \times 0.1 \times 0.56 = 0.097 \]
\[ 2Q_{a} = 0.194 \]
\[ k_{a} = 0.194/0.1704 = 1.14 \]

The stress for a unit strain of .0154 from the stress strain curve in Fig. C3.22 is 172000, and \( f_{0} = 129400 \).

Then \( F_{b} = 172000 + 129400 \times (1.14-1) = 190100 \)

\[ M_{a} = \frac{1}{2} \left( F_{b} I_{a}/c_{a} \right) = 190100 \times 0.1704 = 16200 \]

Total resisting moment = \( M_{1} + M_{a} = 16550 + 16200 = 32750 \text{ in. lbs.} \)

**Solution 2:**

Instead of making each portion a symmetrical section as was done in solution (1) and dividing the results by two, we will find the internal bending resistance of each portion as is when bending about the neutral axis of the entire section. Equation (3) now becomes for each portion of beam section,

\[ F_{b} = f_{m} + f_{0} (k - 1) - - - 3' \]

where \( k_{i} = \frac{Q_{i}}{I_{i}/c_{i}} \), \( k_{a} = \frac{Q_{a}}{I_{a}/c_{a}} \)

The section modulus of each portion refers to neutral axis of entire beam section. Fig. (g) shows lower portion (1).

\[ I_{1} = \frac{1}{12} bc_{1}^{3} = \frac{1}{3} \times 1 \times 1.39^{3} = 0.0865 \]

\[ I_{1}/c_{1} = 0.0865/1.39 = 0.0645 \]

\[ Q_{1} = 1.39 \times 1 \times 0.695 = 0.965 \]

\[ k_{1} = Q_{1}/I_{1}/c_{1} = 0.0645/0.0865 = 1.50 \]

From equation (3'), \( F_{b} = 180000 + 156000 \times (1.50 - 1) = 258000 \)

\[ M_{1} = F_{b} x I_{1}/c_{1} = 258000 \times 0.0645 = 16620 \]

Fig. (h) shows upper portion (2).

\[ I_{a} = \frac{1}{12} x 1.5 \times 0.61^{3} - \frac{1}{12} \times 1.4 \times 0.56^{3} = 0.0514 \]

\[ I_{a}/c_{a} = 0.0514/0.61 = 0.0842, \]

\[ Q_{a} = 0.61 \times 1 \times 0.305 + 1.4 \times 1 \times 0.56 = 0.0971 \]

\[ k_{a} = 0.0971/0.0842 = 1.15 \]

As explained in solution (1), for \( \varepsilon = 0.0154 \), \( f_{b} = 172000 \text{ and } f_{0} = 129400 \).

\[ F_{b} = 172000 + 129400 \times (1.15-1) = 191400 \]

\[ M_{a} = 191400 \times 0.0842 = 16100 \]

\[ M_{total} = M_{1} + M_{a} = 16620 + 16100 = 32720 \text{ in. lbs.} \]

**EXAMPLE PROBLEM 4.** Fig. C3.24 shows an unsymmetrical I beam section. The material is 7079-T6 aluminum alloy die forging. The upper portion is in bending compression. It will be assumed that the compressive crippling stress for the outstanding upper legs of the section is 65000 psi. (The theory and method of calculating crippling compressive strength is given in another chapter.) The ultimate design bending moment is 16500 in. lb. Find M.S.

**Solution:**

\[ y = 2x0.1x1.4x1.1x0.65 + 0.65x0.3x0.05 = 1.178 \]

The maximum compressive stress permitted is 65000 psi. From Fig. C3.22, using the stress-strain curve, we obtain a unit strain of .009 for this stress. The unit strain at the bottom edge of section is \( \varepsilon_{0} = 0.009 \times \frac{1.178}{0.822} = 0.0129 \) as shown in Fig. C3.25. From Fig. C3.23 this strain causes a stress of 67500 psi.

Upper portion: (See Fig. 1)

\[ I_{NA} = \frac{1}{3} \times 1.5 \times 0.622^{3} - \frac{1}{3} \times 1.4 \times 0.722^{3} = 0.102 \]
\[ I/c = 0.102/0.822 = 0.124 \]
\[ q = 0.822 \times 0.1 \times 0.411 + 1.4 \times 0.1 \times 0.772 = 0.1418 \]
\[ k = 0.1418/0.124 = 1.142 \]
\[ f_{\text{max}} = 65000. \quad \text{From Fig. C3.23, } f_0 = 30000 \]
Then \[ f_b = 65000 + 30000 (1.142 - 1) = 69260 \]
\[ m_i = f_b I/c = 69260 \times 0.124 = 8580 \text{ in. lb.} \]

**Lower Portion.**

\[ \text{INA} = \frac{1}{3} \times 0.75 \times 1.178^3 \]
\[ = \frac{1}{3} \times 0.65 \times 1.078^3 \]
\[ \text{INA} = 0.137, \quad I/c = 0.137/1.178 = 0.1164 \]
\[ k = 0.1425/0.1164 = 1.222 \]
\[ f_{\text{max}} = 67500 \]

From Fig. C3.23, \[ f_0 = 44500 \]
\[ f_b = 67500 + 44500 (1.222 - 1) = 77730 \text{ psi} \]
\[ m_s = 0.1164 \times 77730 = 9000 \text{ in. lb.} \]

Total internal allowable resisting moment \[ m_a = m_i + m_s \]
on \[ M_a = 8580 + 9000 = 17580 \text{ in. lb.} \]

\[ R_b = \frac{M}{M_a} = \frac{77730}{17580} = 0.44 \text{ (Load ratio)} \]

\[ \text{M.S.} = \frac{1}{R_b} - 1 = \frac{1}{0.44} - 1 = 0.08 \]

\section*{C3.9 Section with One Axis of Symmetry with Moment Vector not Parallel to Either Axis.}

Since the symmetrical axis is a principal axis, the procedure in this case is the same as for the double symmetric case.

\[ R_b = R_{bX} + R_{by} \]

\[ \text{M.S.} = \frac{1}{R_b} - 1 \]

\section*{C3.10 Unsymmetrical Section with No Axis of Symmetry.}

Fig. C3.27 shows an unsymmetrical section subjected to the applied moment vector \( M \).

For this case the procedure is as follows:

1. Determine principal axes location by equation,
\[ \tan \theta = \frac{2I_{xy}}{I_y - I_x} \]
where \( x \) and \( y \) are centroidal axes, \( I_x \) and \( I_y \) are moments of inertia about these axes and \( I_{xy} \) the product of inertia.

2. Resolve the given moment \( M \) into components \( M_{xp} \) and \( M_{yp} \).

3. Follow the same procedure as before.

4. The stress ratio \( R_b = R_{bxp} + R_{byp} \)

\[ \text{M.S.} = \frac{1}{R_b} - 1 \]
C3.11 Alternate More Exact Method for Complex Bending.

A beam section when resisting a pure external bending moment bends about an axis that is called the neutral axis. No matter what the shape of the beam cross-section for any given external moment, there is an axis about which bending takes place. The general case involves an unsymmetrical beam cross-section and material which has different stress-strain curves in compression and tension in the inelastic range. The neutral axis therefore does not pass through the centroid of the cross-section and thus the method of solution is a trial and error approach. The solution procedure is outlined in Chapter A19, Article A19.17, and therefore will not be repeated here. Also the chapter dealing with the design of beams with non-buckling webs explains and illustrates how the ultimate bending resistance of an entire beam section is determined.

C3.12 Strength Under Combined Bending and Flexural Shear.

The previous part of this chapter has dealt with the determination of the strength of a beam section in pure bending. The usual beam design problem involves flexural shear with bending. In finding the true internal resisting moment, the Cozzone simplified method derives a trapezoidal bending stress distribution which will produce the same internal resisting moment as the true internal bending stress system. A triangular stress system is then derived which will also give the true bending moment.

Now the equation for flexural shear stress for a triangular bending stress distribution is

\[ f_s = \frac{VQ}{It} \]  \hspace{1cm} (A)

Thus to use equation (A) for a trapezoidal bending stress, a correction factor (C) must be applied or equation (A) becomes

\[ f_s = \frac{CVQ}{It} \]  \hspace{1cm} (B)

To illustrate how the correction factor (C) can be determined, the I beam section used in example problem (2) will be used.

We will assume the ultimate design moment of 14000 in. lbs. is produced by a load of 600 lbs. acting on a cantilever beam at a point 22.50 inches from the fixed end of the beam. Thus the beam section at the support is subjected to bending moment of 14000 and a shear \( V = 600 \text{ lbs.} \). The problem is to find the margin of safety under this combined loading.

For pure bending only the stress ratio is

\[ R_b = \frac{M}{M_a} = \frac{14000}{15000} = 0.933 \]  \hspace{1cm} (the value of 15000 is obtained from example problem 2).

The stress ratio in shear is \( R_s = \frac{f_s}{f_{su}} \), where \( f_s \) is the flexural shear stress and \( f_{su} \) the ultimate shear stress of the material. The problem therefore is to find the value of \( f_s \).

The equivalent trapezoidal and triangular bending stress distribution will be determined for the design bending moment of 14000 in. lbs.

For a triangular stress variation, \( F_b = \frac{M_c}{I} = \frac{14000}{0.358} = 39150 \text{ psi} \).

From example problem (2) the shape factor \( k \) was 1.17. On Fig. C3.19, the curve for \( k = 1.17 \) has been plotted. Starting with the \( F_b \) stress of 39150 at the left scale, run horizontal to an intersection with the \( k = 1.17 \) curve, the projecting vertically downward to intersections with the stress-strain curve and the \( f_s \) curve to give 35800 and 19700 for \( f_s \). The stress results are shown graphically in Fig. C3.28a and Fig. C3.28b.

The flexural shear stress is a function of the rate of change of the bending stress. Thus we can obtain a shear correction factor \( C \) by comparing the bending stresses in the two stress distribution diagram.

The shear stress is maximum at the neutral axis in this particular problem. The total normal force on the cross-section of beam above the neutral axis equals the stress times the area.

For simplification, the beam section will be divided into the two portions labeled (a) and (b).
For Case 1:-

Load on portion (a) is,

\[
\frac{19700 \times 1.375 \times 0.125}{2} = 3380 \text{ lb.}
\]

\[
\frac{14100 \times 1.375 \times 0.125}{2} = 2600 \text{ lb.}
\]

Portion (b)

\[
19700 \times 0.875 \times 0.125 = 2155
\]

\[
14100 \times 0.5 \times 0.875 \times 0.125 = 771
\]

Total Force = 8906 lbs.

For Case 2:-

Portion (a)

\[
\frac{39150 + 34250}{2} \times 1.375 \times 0.125 = 6300
\]

Portion (b)

\[
34250 \times 0.5 \times 0.875 \times 0.125 = 1674
\]

Thus the correction factor is,

\[
C = \frac{8906}{3174} = 1.09
\]

Then \( f_g \) at neutral axis is,

\[
f_g = \frac{CVQ}{It} = \frac{1.09 \times 600 \times 0.209}{0.365 \times 0.125} = 3080 \text{ psf.}
\]

The ultimate shear stress for this particular magnesium material is 19000 psf. (See chapter on material properties.)

Stress Ratio \( R_s = \frac{f_g}{f_{sw}} = \frac{3060}{19000} = 0.161 \)

The interaction equation for combined bending and flexural shear is

\[
R_s + R_b = 1
\]

whence, Margin of Safety = \( \frac{1}{\sqrt{R_s + R_b}} - 1 \)

or, M.S. = \( \frac{1}{\sqrt{0.333^2 + 0.161^2}} - 1 = 0.05 \)

Thus the effect of the flexural shear stress was to reduce the margin of safety of .07 in pure bending to .05 in the combined stress action.

As further calculation of the shear correction factor \( C \), its value for the shear stress at the upper edge of portion (b) of the cross-section will be determined. The correction factor for this point is found by comparing the loads on portion (a) for Case 1 and Case 2 stress patterns.

Case 1. Load on (a) = 3380 + 2600 = 5980

Case 2. Load on (a) = 6300.

Hence, \( C = \frac{5980}{6300} = 0.95 \)

C3.13 Strength Under Combined Bending Flexural Shear and Axial Compression.

The subject of the ultimate strength design under combined loads is treated in detail in a later chapter.

A conservative interaction equation for combined bending, shear and axial load is,

\[
(R_a + R_b)^2 + R_s^2 = 1
\]

or M.S. = \( \frac{1}{\sqrt{(R_a + R_b)^2 + R_s^2}} - 1 \)

\( R_b \) includes effect of secondary bending moment due to axial load times deflection.

C3.14 Further Values of \( f_m \) and \( f_0 \).

Table C3.2 gives the yield and ultimate values of \( f_m \) and \( f_0 \) for a number of other materials common to the aerospace field of structures. The yield and ultimate modulus in bending \( f_b \) is found by substituting in the equation.

\[
f_b = f_m + f_0 (k - 1)
\]

<table>
<thead>
<tr>
<th>Material</th>
<th>Size</th>
<th>( f_m ) or ( f_0 ) (k - 1)</th>
<th>Yield</th>
<th>Ultimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2014-T6 Alum. Al. Forgings</td>
<td>4 in.</td>
<td>52</td>
<td>52</td>
<td>97.5</td>
</tr>
<tr>
<td>6061-T6 Alum. Al. Sheet</td>
<td>0.03</td>
<td>35</td>
<td>40</td>
<td>14.3</td>
</tr>
<tr>
<td>7075-T6 Alum. Al. Forgings</td>
<td>0.20</td>
<td>65</td>
<td>70</td>
<td>70.0</td>
</tr>
<tr>
<td>7075-T6 Alum. Al. Hand Forgings(L)</td>
<td>0.08</td>
<td>60</td>
<td>71</td>
<td>67.0</td>
</tr>
<tr>
<td>A-2121 Magnesium Al. Extr. (Long)</td>
<td>0.25</td>
<td>21</td>
<td>20</td>
<td>20.7</td>
</tr>
<tr>
<td>2014-0 Magnesium Al. Sheet</td>
<td>0.36</td>
<td>18</td>
<td>20</td>
<td>20.6</td>
</tr>
<tr>
<td>AISI Alloy Steel (Normalized)</td>
<td>0.188</td>
<td>70</td>
<td>80</td>
<td>83.1</td>
</tr>
<tr>
<td>AISI Alloy Steel (Normalized)</td>
<td>0.186</td>
<td>75</td>
<td>85</td>
<td>86.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>AISI Alloy Steel (Heat Treated)</td>
<td>0.06</td>
<td>100</td>
<td>100</td>
<td>100.0</td>
</tr>
<tr>
<td>17-7PH Stainless Steel</td>
<td>0.05</td>
<td>150</td>
<td>150</td>
<td>150.0</td>
</tr>
<tr>
<td>PHS-7- Mo (RHS50) Stainless Steel</td>
<td>0.03</td>
<td>200</td>
<td>200</td>
<td>200.0</td>
</tr>
<tr>
<td>Ti-6Al-4V Titanium Alloy</td>
<td>0.02</td>
<td>110</td>
<td>110</td>
<td>110.0</td>
</tr>
</tbody>
</table>

PROBLEMS

(1) A round tube 1-1/2 inches in diameter has a wall thickness of .086 inches. It is made of aluminum alloy whose stress-strain curve is shown in Fig. C3.2. If the maximum unit strain in
(2) Same as Problem (1) but use a square tube with 1-1/4 inch outside dimension and .081 inch wall thickness.

Use the Cozzone method for solving the following problems.

(3) Find the ultimate bending moment that each of the following beam sections will develop when bending about the principal axis and made from each of the following materials.

(a) 7075-T6 Aluminum Alloy Extrusion.
Ftu = 75000, Fty = 65000.

(b) Ti-6Al-4V Titanium Alloy.
Ftu = 130,000, Fty = 120,000.

(c) AISI Alloy Steel, heat treated.
Ftu = 150,000, Fty = 132,000.

(4) A simply supported beam has a span of 24 inches. Depth of beam limited to 2 inches. It must carry an ultimate load of 4000 lbs. located at midpoint of beam. Material is 7075-T6 aluminum alloy extrusion. Design an I shaped section to carry this load. Neglect areas of corner fillets that would be used in extruded shapes.

REFERENCES:
CHAPTER C4

STRENGTH AND DESIGN OF ROUND, STREAMLINE, OVAL AND SQUARE TUBING IN TENSION, COMPRESSION, BENDING, TORSION AND COMBINED LOADINGS.

C4.1 Introduction.

Before the advent of the stressed-skin structure for aircraft, or during the period when fuselage and wing were fabric covered, round, oval and square tubing were used in designing the major structure of the fuselage and wing. If the wing and tail were externally braced, streamline tubing was used. The development of the metal covered structure eliminated the use of tubing in fuselage and wing design, however, tubing continued to be used for landing gear structure, engine mounts, control systems, fixed equipment such as passenger seats, etc.

With the opening of the space age, tubing as a structural unit in space vehicles is again being widely used because drag in space is not an important factor. Round tubing is the best shape for transmitting torsional forces and thus widely used in control systems. Round and square tubing permit simple connection or end fitting design. The metals industry has made available a large number of diameter and wall thicknesses and thus the structural designer has a large number of sizes to select from.

C4.2 Design for Tension.

In general the strength design requirements are that the limit loads must be carried without exceeding the tensile yield stress \( F_{ty} \) of the material and the ultimate design load which is equal to the limit load times a factor of safety must be carried without failure, which means the tensile stress cannot exceed the \( F_{tu} \) of the material. In general for aircraft, the factor of safety is 1.5. For unmanned missiles and space vehicles the factor of safety may be as low as 1.2. Since the ratio of \( F_{tu} \) to \( F_{ty} \) for materials varies widely, sometimes the yield under limit loads is more critical than failure under the ultimate design loads, thus the student should always be sure he has the critical situation.

Since elevated temperature and time of exposure affect the yield and ultimate strength of materials, the problem of material selection relative to light weight becomes an important design factor. Fig. C4.1 shows a plot of the \( F_{tu}/w \) ratio versus elevated temperatures up to 800 degrees, where \( w \) is the density of the material. The tube materials are AISI alloy steel and 2024 aluminum alloy. Observation of Fig. C4.1 shows that for temperatures below 350°F, aluminum alloy is lighter unless the steel is heat-treated to \( F_{tu} = 180,000 \) or above. Above 350°F, the ultimate strength of aluminum alloy falls off rapidly, but steel continues its rather uniform decrease in tensile strength.

The graph explains why aluminum alloy cannot be used entirely for the surface of supersonic airplanes flying at speeds around 2000 miles per hour, as aerodynamic heating would produce surface temperatures in the region where the strength of aluminum alloy decreases rapidly.

![Fig. C4.1](image)

Practically every structural tubular member in a flight vehicle structure must be fastened or attached to another adjacent member. The connection can be made by using some sort of end fitting which is fastened to the tube by rivets or bolts or by welding. If rivets or bolts are used, holes are cut in the tube walls which means the tube is weakened since tube area is cut away, thus net area on any tube cross-section must be used in calculating the tube tensile strength. If welding is used in the connection, the welding heat causes grain growth in the tube material adjacent to the weld area, which decreases the tube strength, thus a welding correction factor must be used in calculating the ultimate tensile strength. Designing for tension will be illustrated later in this chapter.
C4.3 Design for Compression.

The strength of members with stable cross-sections when acting as columns can be calculated by Euler's equation if the bending failure is elastic, or E is constant (eq. C4.1) and for inelastic bending failure, Euler's equation with the tangent modulus E_t replacing E (eq. C4.2) checks test results. (The student should refer to Chapters A18 and C2 for theory on column strength.)

\[ F_C = \frac{n^2 E}{(L'/\rho)^2} \quad \text{--- (C4.1)} \]

\[ F_C = \frac{n^2 E_t}{(L'/\rho)^2} \quad \text{--- (C4.2)} \]

L' is the effective length and equals \( L/\sqrt{\phi} \), where \( \phi \) is the column end fixity coefficient.

Long and Short Columns

For many years the problem or subject of inelastic column strength or failure was treated almost entirely from a consideration of test results. That is, sufficient tests were made to establish the shape of the failure stress curve in the region where the failure was at stresses above the proportional limit stress of the material. Mathematical curves were then derived to fit the test results. Engineers referred to the columns which failed by inelastic bending as short columns, and thus referred to the equations that fit the test data as short column equations. The columns that failed by elastic bending were then referred to as long columns. The test curve for long columns would follow the Euler column equation (C4.1) and thus tests were not necessary to establish allowable failing stresses in the so-called long column range. Thus over the years short column equations based on test results have been presented by official government agencies for use in structural design. The official publication for the aerospace field is the Military Handbook MIL-HDBK (Ref. 1).

C4.4 Column Formulas for Round Steel Tubes.

From (Ref. 1) the basic short column equations are:

\[ F_C = F_{Co} \left[ 1 - 0.385 \frac{(L'/\rho) n^2 \sqrt{E/F_{Co}}}{m} \right] \quad \text{--- (C4.3)} \]

\[ F_C = F_{Co} \left[ 1 - 0.353 \frac{(L'/\rho) n}{\sqrt{E/F_{Co}}} \right] \quad \text{--- (C4.4)} \]

Where \( F_{Co} \) is the column yield stress (upper limit of column stress for primary failure). It can be determined from test results by extending the short column curve to a point corresponding to zero length, ignoring any tendency of the test curve to rise rapidly for very short lengths where failure is by block compression. Table C4.1 shows the resulting short and long column equations after values of \( F_{Co} \) and \( E \) have been substituted in equations C4.3 and C4.4 and \( E \) in the Euler equation. The column headed transitional \( L'/\rho \) represents the value of \( L'/\rho \) where failure change from inelastic to elastic failure or, in other words, it is the dividing point between the so-called long and short column range. Thus if the equations are used, the \( L'/\rho \) value must be known in order to select the proper equation.

C4.5 Column Formulas for Aluminum Alloy Tubes.

From (Ref. 1), the basic short column equations for aluminum alloys are:

\[ F_C = F_{Co} \left[ 1 - 0.385 \frac{(L'/\rho) n^2 \sqrt{E/F_{Co}}}{m} \right] \quad \text{--- (C4.5)} \]

\[ F_C = F_{Co} \left[ 1 - 0.353 \frac{(L'/\rho) n}{\sqrt{E/F_{Co}}} \right] \quad \text{--- (C4.6)} \]

\[ F_C = F_{Co} \left[ 1 - 0.272 \frac{(L'/\rho) n^2 \sqrt{E/F_{Co}}}{m} \right] \quad \text{--- (C4.7)} \]

For long columns:

\[ F_C = \frac{n^2 E/(L'/\rho)^2}{-\text{--- (C4.8)}} \]

The equations for determining \( F_{Co} \) are given in Table C4.2 (from Ref. 1). The table also indicates which of the three short column equations to use for the various aluminum alloy materials.

To illustrate the use of Table C4.2, the column formula for 2024-T3 aluminum alloy tubing will be derived:

From Chapter B2, we find the following strength properties for 2024-T3 tubing,

\[ F_{tu} = 64000 \quad F_{cy} = 42000 \]

From Table C4.2, the equation for \( F_{Co} \) is,

\[ F_{Co} = F_{cy} \left( 1 + \frac{\sqrt{F_{cy}}}{1000} \right) \quad \text{substituting,} \]

\[ F_{Co} = 42000 \left( 1 + \frac{\sqrt{42}}{1000} = 42000 \left( 1 + .205 \right) = 50600 \right) \]

From Table C4.2, we note that short column equation C4.5 applies.

Substituting in this equation,

\[ F_C = 50600 \left[ 1 - 0.395 \frac{(L'/\rho) n^2 \sqrt{10,500,000/50600}}{m} \right] \]

or \[ F_C = 50600 - 431 L'/\rho \quad \text{--- (C4.9)} \]

C4.6 Column Formulas for Magnesium Alloys.

From (Ref. 1) the following short column equations for various magnesium alloy materials.
Table C4.1 Column Formulas for Round Steel Tubes

<table>
<thead>
<tr>
<th>Material</th>
<th>Ftu, ksi</th>
<th>Fty, ksi</th>
<th>Fco, ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1025</td>
<td>55</td>
<td>36</td>
<td>35</td>
</tr>
<tr>
<td>4150</td>
<td>95</td>
<td>78(£)</td>
<td>79.5</td>
</tr>
<tr>
<td>Heat-treated alloy</td>
<td>125</td>
<td>103</td>
<td>113</td>
</tr>
<tr>
<td>steel</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heat-treated alloy</td>
<td>150</td>
<td>132</td>
<td>145</td>
</tr>
<tr>
<td>steel</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heat-treated alloy</td>
<td>180</td>
<td>153</td>
<td>179</td>
</tr>
<tr>
<td>steel</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The equations given in Table C4.1 are for solid columns only and are not applicable to long columns. The equations for long columns are given in Table C4.4, and the equations for short columns are given in Table C4.3.

C4.7 Short Column Equations for Other Materials.

For other metals for which short column equations are not available, the use of Euler’s equation, using the tangent modulus E, can be used (eq. C4.2). Refer to Chapter C2 for information on how to construct column strength curves using this equation.

C4.8 Column Failure Due to Local Failure.

The equations as presented give the allowable stress due to failure by bending of the column as a whole and the action is elastic or inelastic instability of the column as a whole. As the slenderness ratio L/δ gets smaller, the Fc stress increases. Now if the diameter of the tube is relatively large and the wall thickness relatively small or, in other words, if the diameter/thickness (D/t) ratio is large, failure will result by local crippling or crushing of the tube wall and this local failure stress is usually represented by the symbol Fcc. The values of Fcc in general have been determined by tests (see design charts for Fcc versus D/t ratio).

C4.9 Design Column Charts.

In design, column strength charts are a great timesaver as compared to substituting in the various column equations, thus a number of column charts are presented in this chapter to facilitate the strength check of columns and the strength design of columns. Fig. C4.2 is a chart of L/δ versus Fc for heat treated round alloy steel tubing. Fig. C4.3 is a similar type of chart for aluminum alloy round tubing. Fig. C4.4 gives column charts for magnesium alloy materials. All three charts are taken from (Ref. 1). Figs. C4.5 and C4.6 represent a further simplification for the design of steel and aluminum round tubing.

C4.10 Section Properties of Round Tubing.

Table C4.2 gives the section properties of round tubing. A tube is designated by giving its outside wall diameter (D) and its wall thickness (t). Thus a 2-1/4 - .056 means a tube with 2-1/4 inch outside diameter and a wall thickness in inches of .056. Since a tube is symmetrical about any axis, the polar moment of inertia, which is needed in torsion problems, equal twice the rectangular moment of inertia as given in Table C4.3. For weight comparison, the weight of steel and aluminum tubing is
### Table C4.2 Column Formulas for Aluminum Alloys

| Alloy and Temper | Product                          | $F_{se}$ | Short Columns | Transitional $L'/ho$ | Long Columns |
|------------------|----------------------------------|----------|---------------|------------------------|--------------|
| 2014-T3, T4, T451 | Sheet and Plate*; Rolled Rod, Bar and Shapes; Drawn Tube | $F_{se}(1 + \sqrt{F_{se}/1000})$ | Equation C4.5 | 1.732$\sqrt{E/F_{se}}$ | Equation C4.8 |
| 2024-T3, T351, T36, T4, T42 | All Products | $F_{se}(1 + \sqrt{F_{se}/1335})$ | Equation C4.6 | 1.346$\sqrt{E/F_{se}}$ | Equation C4.8 |
| 6061-T4, T451, T4510, T4511 | Sand and Permanent Mold Castings | $F_{se}(1 + \sqrt{F_{se}/2000})$ | Equation C4.7 | 1.224$\sqrt{E/F_{se}}$ | Equation C4.8 |

*Includes clad as well as bare sheet and plate.

Transitional $L'/\rho$ is that above which the columns are "long" and below which they are "short".
given in the last two columns of the table.

C4.11 Some General Facts in Tubing Design.

1. For a given area, the larger the tube diameter, the greater the column strength if failure due to local crippling is not critical.

2. The higher the D/t ratio of tube the lower the crippling or local failure strength.

3. If columns fall within the long column category, the use of higher strength alloy steel or aluminum alloy will not increase strength of column since E is practically constant for all chrome-moly steel alloys and likewise for all aluminum alloys. Failure is due to elastic buckling of the column as a whole and is therefore a function only of I, L' and E.

4. The column end restraint affects the necessary tube size. Consult the design requirements of the Army, Navy, and C.A.A. in this matter. In general with welded steel tubular trusses a coefficient of C = 2 is permissible except for engine mount and Macella structures. For trusses with riveted joints a value of not over 1.5 is generally permissible.

5. The student should realize that practical limitations such as clearance requirements may determine the diameter of the tube instead of strength-weight considerations. Thus design can consist of checking the tubes available under the given restrictions.

C4.12 Effect of Welding of Steel Tubes Upon the Tension and Column Strength.

Since welding affects the grain structure of the tube material adjacent to the weld, tests show the strength of the material adjacent to the weld is decreased as compared to the unwelded material. If a tapered weld is used, the effect of the weld is decreased. Table C4.4 shows the allowable stresses in tension to use when tension loads are carried.

In short columns, the primary column failing stress may be greater than the local crippling strength of the tube adjacent to the weld at the end of the tube. This local failing stress due to welding is referred to as the weld cut-off stress and the column compressive stress \( F_c \) should not exceed this value. This cut-off weld stress is shown by the horizontal lines in Fig. C4.2 and C4.5.

---

**Table C4.4**

<table>
<thead>
<tr>
<th>Type of Weld</th>
<th>Normalized Tube Welded</th>
<th>Welded after HT or Norm. after Weld</th>
<th>HT after Welding</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Tapered Welds of 30° or Less</em></td>
<td>.947 F( \text{tu} )</td>
<td>90,000 psi</td>
<td>.90 F( \text{tu} )</td>
</tr>
<tr>
<td>All others</td>
<td>.841 F( \text{tu} )</td>
<td>80,000 psi</td>
<td>.80 F( \text{tu} )</td>
</tr>
</tbody>
</table>

*Note: Gussets or plate inserts considered 0° "taper" with \( \xi \).

**For (X-4130) Special. comparable, values to the F\( \text{tu} \) equal to 90,000 and 80,000, are stresses 94,500 and 84,100 psi, respectively.*

Ref. Anc-5

---

C4.13 Illustrative Problems in Strength Checking and Design of Round Steel Tubes as Columns and Tension Members.

**Problem 1**

Tube size = 1-1/2" - .058, Length \( L = 30 \) in. 
End fixity coefficient \( C = 1 \).
Material: Alloy steel, F\( \text{tu} \) = 95000. The tube is welded at ends. 
Ultimate design loads are: \( P = -14,500 \) lbs. compression, and \( P = 18500 \) lbs. tension. 
Required: Margin of Safety (M.S.)

**Solution:** The compressive (M.S.) will be determined first. As the simplest solution, we can use the column curves in Fig. C4.5. For a length of 30 and \( C = 1 \), from the upper right chart we project upward to the intersection with the 1-1/2 diameter tube and then horizontally to the left hand scale to read the column strength of 14800 lbs. which we will call the allowable falling \( F_a \).

\[ \text{M.S.} = \frac{F_a}{P} - 1 = \frac{14800}{14500} - 1 = 0.02 \]

The tube strength could also be found by using Fig. C4.2 as follows:

\[ L' = \frac{L}{\sqrt{C}} = \frac{30}{\sqrt{1}} = 30 \]

\[ \frac{L'}{\rho} = \frac{30}{5.102} = 58.7. \quad \rho \text{ is found from Table C4.3 as well as the tube area .2628 sq. in. Using 58.7 for } L'/\rho \text{ on lower scale and projecting upward to the } F_{tu} = 95000 \text{ curve, which is the lower curve, and then horizontally to left hand scale we read } F_c = 56500 \text{ psi.} \]

Whence, \( F_a = F_{cA} = 56500 \times .2628 = 14850 \text{ lb.} \)

The solution obviously could be made by substituting in the short column equation for steel having \( F_{tu} = 95000, \) or
Fig. C4.2 Allowable column stress for heat-treated alloy-steel round tubing.

Fig. C4.3 2024-T3 and 6061-T6 round aluminum alloy tubing.

Fig. C4.4 Allowable column stresses for magnesium-alloy columns.
Fig. C4.5
STRENGTH OF CHROME MOLY ROUND STEEL TUBES

\[ F_T = 75,000 \text{ PSI} \quad F_{TU} = 95,000 \text{ PSI} \]
<table>
<thead>
<tr>
<th>Dia. Gage</th>
<th>A</th>
<th></th>
<th>I</th>
<th>I/Y</th>
<th>D/t</th>
<th>Weight/Lb/100 ins.</th>
<th>Steel</th>
<th>Dural</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>0.022</td>
<td>0.01578</td>
<td>0.00103</td>
<td>0.000825</td>
<td>11.38</td>
<td>0.14</td>
<td>0.165</td>
<td></td>
</tr>
<tr>
<td>2/8</td>
<td>0.028</td>
<td>0.03063</td>
<td>0.01231</td>
<td>0.00486</td>
<td>13.39</td>
<td>0.16</td>
<td>0.31</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>0.040</td>
<td>0.03739</td>
<td>0.01698</td>
<td>0.00388</td>
<td>19.72</td>
<td>0.08</td>
<td>0.36</td>
<td></td>
</tr>
<tr>
<td>5/8</td>
<td>0.064</td>
<td>0.06018</td>
<td>0.01186</td>
<td>0.00823</td>
<td>7.85</td>
<td>0.16</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>3/4</td>
<td>0.125</td>
<td>0.07523</td>
<td>0.03113</td>
<td>0.00539</td>
<td>14.28</td>
<td>0.16</td>
<td>0.52</td>
<td></td>
</tr>
<tr>
<td>7/8</td>
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*AN STANDARD TUBING*
STRENGTH & DESIGN OF ROUND, STREAMLINE, OVAL AND SQUARE TUBING
IN TENSION, COMPRESSION, BENDING, TORSION AND COMBINED LOADINGS

C4.10

\[
F_c = 79500 - 51.9 \left( L'/\rho \right)^{1.4} \\
= 79500 - 51.9 \left( 58.7 \right)^{1.4} = 56500 \text{ psi}
\]

The short column equation applies since, as shown in Table C4.1, the transitional L'/\rho is 91 and the value for our tube is 58.7.

**Tensile Strength**

Since the tube is welded, the tube material adjacent to the weld is weakened. The weld correction values are given in Table C4.4. We will assume a weld other than tapered. Let \( F_{Pa} \) = allowable or failing tensile strength of tube.

\[
F_{Pa} = F_{tu} \text{ (weld factor) (area of tube)} = 95000 \times 0.341 \times 0.2828 = 21000 \text{ lbs.}
\]

M.S. = \( F_{Pa}/L' = 21000/18500 = 1 = 0.13 \), thus compression is critical.

**PROBLEM 2**

**Case 1.** Tube size 1-1/4 - .049, \( L = 40 \) in. \( c = 1 \)
Material: Alloy steel, \( F_{tu} = 95000 \)
Find ultimate compressive load it will carry.

Solution: From Fig. C4.5, \( F_{Pa} = 6000 \) lbs.

**Case 2.** If tube was heat treated to \( F_{tu} = 150,000 \), what compressive load would it carry.

Solution: Fig. C4.5 cannot be used since \( F_{tu} = 150,000 \), thus we will use Fig. C4.2.

\[
L' = L/\sqrt{c} = 40/\sqrt{4} = 4. \text{ From Table C4.3, } \rho = 0.425 \text{ and area (A) = } 0.1849.
\]

\[
L'/\rho = 40/0.425 = 94. \text{ From Fig. C4.2, using the } 150,000 \text{ curve, we find } F_c = 32500.
\]

Then \( F_{Pa} = F_{CA} = 32500 \times 0.1849 = 6000 \) lb. Thus heat treating the tube from 95000 to 150,000 for \( L'/\rho \) did not increase the column strength. For a \( L'/\rho = 94 \), it is a long column and failure is elastic and \( E \) is constant.

The strength could also be calculated by Euler’s equation from Table C4.1.

\[
F_c = 286,000,000/(L'/\rho)^2
\]

\[
= 286,000,000/(94)^2 = 32500 \text{ psi, the same as previously calculated.}
\]

**Case 3.** Same as Case 1, but assume tube is welded to several other tubes at its end and that the end fixity developed is \( c = 2 \).

\[
L' = L/\sqrt{c} = 40/\sqrt{2} = 28.4
\]

In Fig. C4.5 we can use \( L = 40 \) and \( c = 2 \) scale at bottom of chart or use \( c = 1 \) scale and \( L' = 28.4 \). Reading the chart we obtain \( F_{Pa} = 29200 \) lbs. Thus the c = 2 fixity increased the strength of the tube from 6000 to 9200.

**Case 4.** Same as Case 3 but heat treated to \( F_{tu} = 150,000 \) after welding.

\[
L'/\rho = 28.4/0.425 = 66.8
\]

From Fig. C4.2 using 150,000 curve, we read \( F_c = 55000 \), whence

\[
F_{Pa} = F_{CA} = 63000 \times 0.1849 = 11650 \text{ lb.}
\]

In this case heat treating produced additional strength, whereas in Case 2 it did not. The reason for this is that failure occurs in the inelastic stress range and heat treating raises the material properties in the inelastic range. The end fixity changed the column from a so-called long column to a short column.

The strength could be found also by substituting in the short column equal for 150,000 steel as given in Table C4.1,

\[
F_c = 145000 - 18.56 \left( L'/\rho \right)^2
\]

\[
= 145000 - 18.56 (66.8)^2 = 63000 \text{ psi.}
\]

**PROBLEM 3**

**Case 1.** Tube size 2 - .065, \( L = 24 \), \( c = 1.5 \)
Material \( F_{tu} = 95000 \). Welded at ends.
Ultimate design load = 25000 lbs. What is M.S.

Solution: \( L' = L/\sqrt{c} = 24/\sqrt{1.5} = 19.7 \)

From Fig. C4.5 for \( L = 19.7 \) on \( c = 1 \) scale, we project upward to the 2 inch tube and note that it intersects the horizontal weld cut-off line which gives an allowable column load at left scale of \( F_a = 26700 \) lb. Failure in this case is local crippling adjacent to welds at the tube ends.

M.S. = \( F_{Pa}/F_a = 26700/25000 = 1 = 0.07 \)

**Case 2.** Assume tube is heat treated to \( F_{tu} = 125000 \) after welding. What is tube strength.

\[
L'/\rho = 19.7/0.6845 = 28.8
\]

Using Fig. C4.2 with \( L'/\rho = 28.8 \) and projecting up to 125000 curve, we again note that horizontal weld cut-off line is intersected.
giving $P_0 = 95000$, whence $P_a = 95000 \times 0.3561 = 33750$ lbs.

If the tube had not been welded at ends the dashed part of the column curve could have been used, thus giving additional strength.

**Problem 4**

Fig. C4.7 shows a steel tubular engine mount structure for a 1050 H.P. radial engine. The ultimate design tension and compressive load in each member as determined from a stress analysis for the various flying and landing conditions are shown in ( ) adjacent to each member. The true length $L$ of each member is also shown. Using chrome-moly steel tubes, $F_{tu} = 95000$, select tube sizes for the given loads. It is common practice to assume the column and fixity $c = 1$ for engine mount members, since the mount is subjected to considerable vibration and the true rigidity given by the engine mount ring is difficult to accurately determine.

![Diagram](image)

Consider member (3). Ultimate design load = 9250. Referring to the column charts of Fig. C4.5, we find for $c = 1$ and $L = 31.4$ the following tube sizes for a strength near 9250 lbs.

1-1/2 - .035, $P_a = 9850$ (weak). (Note--$P_a =$ allowable load).

1-3/8 - .049, $P_a = -10350$, weight = 5.78 lb./100", M.S. = (10350/9250) = 1 = 0.12

1-1/4 - .058, $P_a = -10000$, weight = 6.15, M.S. = 10000/9250) = 1 = .08

Thus use 1-3/8 - .049 since it is the lightest as well as the strongest.

Consider member (4), Load = 5470, $L = 30$, $c = 1$

From Fig. C4.6:

1-1/8 - .049, $P_a = -7100$, wt. = 4.68, M.S. = .30
1-1/4 - .055, $P_a = -6500$, wt. = 3.78, M.S. = .19
1 - .058, $P_a = -6000$, wt. = 4.86, M.S. = .10

The results show that 1-1/4 - .035 is the lightest. Since there is danger in welding .035 thickness to the other heavier tube gauges particularly the engine mount ring which is usually relatively heavy for this size engine, a minimum tube thickness of .049 will be used, hence the 1-1/8 - .049 tube will be selected.

Consider Member (2)

Design loads 11650 tension and 4250 compression. Since the tension load appears critical, the tube will be designed for the tension load and then checked for the compressive load. The $F_{tu}$ of the material equals 95000 psi. Since the engine mount in a welded structure, the strength of the tube adjacent to the end welds must be reduced to .841 x 95000 = 80000 psi (see Table C4.4).

Hence tube area required = 11650/80000 = 0.146 sq. in. From Table C4.3, which gives the section properties of round tubes, we select the following sizes:

1 - .049, Area = .146, M.S. = (.146/1.146) = 1 = 0
1 - .058, Area = .172, M.S. = (.172/1.146) = 1 = .19
1-1/8 - .049, Area = .166, M.S. = .14
1-3/8 - .035, Area = .147, M.S. = 0

To obtain a reasonable margin of safety, the 1-1/8 - .049 will be selected.

Many structural designers prefer to have large margins of safety on engine mount members as considerable trouble has been encountered in the failure or cracking of engine mount members.

The strength of the 1-1/8 - .049 tube as a column for length = 31.4 and $c = 1$ equals -6700 lbs. from Fig. C4.5 which gives a margin of safety of (6700/4250) = 1 = .58 on the maximum compressive load. The student should select a tube size for member (1).

C4.14 Illustrative Problems Using Aluminum Alloy and Magnesium Round Tubes as Columns and Tension Members:

In general alloy steel round tubes must be heat treated to around 180,000 to 200,000 ultimate tension strength before they can compare favorably with aluminum round tubes on a material weight basis. However, aluminum alloy as used for tubes cannot be welded satisfactorily and thus in a truss structure the end connections involving riveted and bolted connections add weight and design difficulties as compared to welded connections in steel trusses.
STRENGTH & DESIGN OF ROUND, STREAMLINE, OVAL AND SQUARE TUBING
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C4.12

PROBLEM 1

Case 1. Tube size $L = .049$ round.

$L = 24, c = 1$, Material 2024-T3
Find failing compressive load.

Solution: The column curves in Fig. C4.6 are
slightly conservative because the equation used
was slightly different from the equation now
specified in (Ref. 1).

Use $L = 24$, we read for 1-049 tube a
failing load of 2800 lb.

As a second solution, we will use Fig.
C4.3: $L' = L/\sqrt{c} = 24/\sqrt{1.5} = 24$. $L'/P =
24/.3367 = 71.3$. From Fig. C4.3, we read
$P_a = 20000$. Then $P_a = P_{oc} = 20000 \times \frac{1464}{2800} =
28001$.

The answer could be obtained by substitu-
ting in equation C4.9,

\[ P_o = 50600 - 431 L'/\rho = 50600 - 431 (71.3) = 19900 \]

$P_a = 19900 \times \frac{1464}{2800} = 28001$.

A column may also fail by local crushing
or crippling of the tube wall, thus the
compressive stress $P_{oc}$ should be determined to
see if it is less than the primary bending
failing stress for the column.

For our tube the diameter over thickness
ratio $D/t = 1.0/.049 = 20.40$. Values of $D/t$
are given in Table C4.3.

Referring to the small chart in the upper
right hand corner of Fig. C4.3, we find for a
$D/t$ of 20.4 that $P_{oc} = 47500$ psi. Since this
stress is greater than the bending failing
column stress of 20,000, it is not critical.

Case 2. Same as Case 1 but use $c = 1.5$ and
change material to 6061-T6 aluminum
alloy.

$L' = L/\sqrt{c} = 24/\sqrt{1.5} = 19.7, L'/\rho = 19.7/.3367 =
53.5$

From Fig. C4.3, $P_o = 22500$

Whence $P_a = 22500 \times \frac{1464}{2800} = 23000$ lb.

$P_{oc}$ for $D/t = 20.4$ from Fig. C4.3 is 38500 (not
critical).

Case 3. Same as Case 2 but change material to
magnesium alloy, $P_{oc} = 10,000$.

For $L'/\rho = 53.5$ and using lower curve on
Fig. C4.4, we read $P_o = 7600$. Then $P_a = P_{oc} =
7600 \times \frac{1464}{2800} = 3110$ lb.

C4.15 Strength of Streamline Tubing.

If a round tube is exposed to the air-
stream, the air drag is about 15 times greater
than if it were given a streamlined shape, thus
streamline tubes are used when the member is
exposed to the airstream.

Streamline tubes are drawn from round
tubes. In designating a streamline tube, the
round tube from which it was made is used and
then the fineness ratio is also given. The
fineness ratio is the ratio $L/D$, which dimen-
sions are shown in Fig. C4.8. The most common
fineness ratio used is 2.5 to 1. Table C4.4.
shows the section properties of streamline
tubing having a fineness ratio of 2.5 to 1.

Figs. C4.9 and C4.10 give curves for finding
the column failing stress $P_o$ and the local
compressive stress $P_{oc}$.

Fig. C4.8

PROBLEM 1. Case 1.

A streamline tube made from a basic round
tube of 2-1/2 - .065 size has a fineness ratio of
2.5 to 1. The length $L$ is 30 in. Take
c = 1. Material is alloy steel $F_{st} = 75000$.
Find the ultimate compressive load the member
will carry.

Solution: From Table C4.4 for 2-1/2 - .065
size we find the following section properties:
Area ($A$) = .4972, $p$ (major axis) = .5137 in.
Then $L' = L/\sqrt{c} = 30/\sqrt{1.5} = 30$, and $L'/\rho =
30/.5137 = 58.5$
$D/t$ value for tube = 2.5/.065 = 38.5

From Fig. C4.9 for $L'/\rho = 58.5$ and $D/t =
38.5$, we read $P_o = 46500$ psi. For $D/t = 38.5$
and reading from small chart in upper right
hand corner of Fig. C4.9, we read $P_{oc} = 65000$.
Thus $P_o$ is critical and $P_a = 46500 \times .4972 =
23000$ lb.

Case 2. Same as Case 1 but change material
to 2024-T6 aluminum alloy.

For this material we use Fig. C4.10.
For $L'/\rho = 58.5$, we read $P_o = 26000$ psi.
For $D/t = 38.5$, we read $P_{oc} = 37500$ (not
critical). Thus $P_a = 26000 \times .4972 = 12900$ lb.

C4.16 Strength of Oval and Square Shaped Tubes
in Compression.

Tables C4.6 and C4.7 give the section
properties for square and oval shaped tubes
respectively. For the design of these shaped
Table C.4.4 Section Properties of Streamline Tubing (Fineness Ratio 2.5 to 1)

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<th>D/O</th>
<th>Wall</th>
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<th>M/12</th>
<th>Area, in.²</th>
<th>M/12</th>
<th>Area, in.²</th>
<th>M/12</th>
<th>Wall, in.</th>
<th>Area, in.²</th>
<th>M/12</th>
<th>Area, in.²</th>
<th>M/12</th>
<th>Wall, in.</th>
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<td>0.115 0.06</td>
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<td>0.076 0.03</td>
<td>0.066 0.02</td>
<td>0.060 0.01</td>
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<td>0.016 0.01</td>
<td>0.014 0.01</td>
<td>0.010 0.01</td>
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<td>0.005 0.01</td>
<td>0.005 0.01</td>
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</table>

Fig. C.4.10 Streamline 2024-T3 Tubing

Fig. C.4.9 Allowable Column and Crushing Stresses for Chromolybdenum Streamline Tubing, $F_{T}=75000$ psi

Note: Higher values of allowable stress can be used in short column range if substantiated by tests.
<table>
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<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>0.095</td>
<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>0.095</td>
<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>0.095</td>
<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>0.095</td>
<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
<tr>
<td>2</td>
<td>08</td>
<td>0.095</td>
<td>0.176</td>
<td>0.008</td>
<td>0.013</td>
<td>0.025</td>
</tr>
</tbody>
</table>

**Squares Table C.4.6: Section Properties of Streamline Tubing**

**Ovals Table C.4.7: Section Properties of Oval Tubing**
tubes the primary column strength can be found by using the curves in Figs. C4.9 and C4.10. The crushing stress $F_{cc}$ for oval shaped tubes can conservatively be taken as that for streamline tubes as given in Figs. C4.9 and C4.10. For square tubes the local crushing stress $F_{cc}$ can be taken as the crippling stress of a flat plate. For this stress refer to the chapter which covers the buckling and crippling stress of flat plates with various widths, thickness and boundary edge conditions.

ULTIMATE BENDING STRENGTH OF ROUND TUBES

C4.17 Charts for Finding Modulus of Rupture Stress.

Chapter C3 was concerned with the theory and methods of determining the ultimate yield and failing stress of a section in pure bending. It was concerned with finding a fictitious stress $F_{b}$ which, when substituted in the well known beam formula $M = F_{b}I/c$, would give the value of the bending moment which would cause failure.

The same procedure as was used in Chapter C3 can be used to find the modulus of rupture stress ($F_{b}$) for round tubes. However, since round tubes have been a standard and available structural member for many years, much testing has been done on tubes and as a result rather complete design curves are available for finding the modulus of rupture in bending ($F_{b}$) for round tubes. Figs. C4.11 to C4.14 inclusive give curves for finding modulus of rupture $F_{b}$ round tubes when fabricated from alloy steels, aluminum alloys, magnesium alloys and titanium alloy.

C4.18 Problems involving Bending Strength of Tubes.

PROBLEM 1

A 1-1/4 - .056 round tube is used as a simply supported beam with the supports at the ends. The span or length of the beam is 24 inches. It carries a uniform distributed load $w$ in pounds per inch. Find the value of $w$ to cause the tube to fail in bending if the tube is made from the following materials: - alloy steel $F_{tu} = 95000$, 6061-T6 aluminum alloy, and 6Al-4V titanium.

Solution: The maximum bending moment occurs at the midpoint of the span and equals $wL^2/8 = w \times 24^2/8 = 72 \times w$ in. lb.

The section properties for the tube are $D/t = 21.55$, and $I/y = .06187$ obtained from Table C4.3.

The beam equation involving the modulus of rupture $F_{b}$ is $M = F_{b}I/y$. Substituting 72 in for $M$, we obtain:

$$w = \frac{F_{b}I}{72y}$$

Consider the alloy steel, $F_{tu} = 95000$. From Fig. C4.11 for $D/t = 21.55$, we read $F_{b} = 117000$. Then $w = (117000 \times .06187)/72 = 100.2$ lb. per inch. Consider tube material as 6061-T6 which has a $F_{tu} = 46200$ psi.

From Fig. C4.12 for $D/t = 21.55$, we read that $F_{b}/F_{tu} = 1.06$. Thus $F_{b} = 46200 \times 1.06 = 44500$.

Then $w = (44500 \times .06187)/72 = 38.1$.

Tube material 6Al-4V titanium.

From Fig. C4.14 for $D/t = 21.55$, we read $F_{b} = 191600$. Then $w = (191600 \times .06187)/72 = 118.7$ lbs. per in.

PROBLEM 2

A beam simply supported at its ends has a span of 30 inches. The ultimate design load consists of two equal loads of 2000 lbs. each. The beam is symmetrically loaded with each load located 12 inches from the ends.

Select the lightest round tube when made from the following material: (1) alloy steel $F_{tu} = 220000$, (2) 7075-T6 aluminum alloy. Compare the resulting tube weights.

Solution: The maximum bending moment is constant and occurs between the load points.

$$M = 2000 \times 12 = 24000 \text{ in. lb.}$$
Fig. C4.12 Bending modulus of rupture for aluminum-alloy round tubing.

Fig. C4.13 Bending modulus of rupture of magnesium-alloy round tubing.

Fig. C4.11 Bending modulus of rupture for round alloy-steel tubing.
Since the allowable or failing bending stress is a function of $D/t$, and since we do
not have a tube size, the design or solution
procedure is by trial and error.

Observation of the modulus of rupture
curves show that as $D/t$ increases $F_b$ decreases.
This is due to the fact that failure in bending
is a local failure and the thinner the wall and
larger the diameter, the lower the buckling or
crushing stress. However, the larger the $D/t$
value the greater the section modulus $I/y$ of
the tube, which means increasing bending
resistance. Thus we have two influences which
act oppositely relative to effecting the bending
strength.

There are many ways of guessing a tube
size for checking purposes. In this example
problem we will assume two values for $D/t$ and
see what $I/y$ would calculate to be. The two
values of $D/t$ will be 45 and 25.

Consider the material alloy steel $F_{tu}=220000$:.

For $D/t = 45$ from Fig.C4.11, $F_b = 232000$
Then $I/y = M/F_b = 24000/232000 = 0.103$
For $D/t = 25$, $F_b = 266000$
Then $I/y = 24000/266000 = 0.089$

Therefore we will refer to Table C4.3 and
select tubes that have an $I/y$ value near the
0.089 to 0.103 range and then find their true
bending strength. Table (A) shows the
selection and the necessary calculations,
using Fig. C4.11.

<table>
<thead>
<tr>
<th>Tube Size</th>
<th>I/y</th>
<th>Area</th>
<th>D/t</th>
<th>$F_b$</th>
<th>$M=F_bI/y$</th>
<th>M.S. =</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-7/8 -.035</td>
<td>.09136</td>
<td>.2023</td>
<td>53.6</td>
<td>227000</td>
<td>20800</td>
<td>-0.13</td>
</tr>
<tr>
<td>1-3/4 -.049</td>
<td>.1083</td>
<td>.2618</td>
<td>35.7</td>
<td>246000</td>
<td>25400</td>
<td>+0.10</td>
</tr>
<tr>
<td>1-5/8 -.049</td>
<td>.0928</td>
<td>.2426</td>
<td>33.15</td>
<td>250000</td>
<td>23200</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

The lightest available tube with a
positive margin of safety is $1-3/4 -.049$ and its
weight for a 30 inch length is $30 \times .2618$
the $0.293 = 2.22$ lbs.

Consider the tube made from 7075-76 aluminum
alloy material which has a $F_{tu} = 77000$.

From Fig. C4.12:–

For $D/t = 60$, $F_b/F_{tu} = .54$, thus $F_b = .54 \times$
$77000 = 64700$

$I/y = M/F_b = 24000/64700 = .37$

For $D/t = 30$, $F_b/F_{tu} = 1.045$, thus $F_b = 1.045 \times$
$77000 = 80500$

$I/y = 24000/80500 = .30$

Therefore we will select a tube from Table
C4.3 that has a $I/y$ value in the region of .30
to .37.

Try 2-3/4 - .058. $I/y = .3233$, $D/t = 47.4$. From
Fig. C4.12, $F_b/F_{tu} = 0.90$. Then $F_b = .90$
$\times 77000 = 69300$ psi. Then $M_a = F_b I/y = 69300 \times .3233 = 22400$. This is less than the design
bending moment of 24000 so this tube is weak.

Try 3-068. $I/y = .3868$, $D/t = 51.7$
$F_b/F_{tu} = .895$, $F_b = .868 \times 77000 = 68000$
$M_a = F_b I/y = 68000 \times .3868 = 26300$
M.S. = (26300/24000)-1 = .09.

A study of other tubes in Table C4.3
shows that no other tube would be lighter in
weight.

Tube weight = $30 \times .5361 \times .101 = 1.70$
lbs., as against 2.22 lbs. for the alloy steel
heat treated to 220,000. Thus aluminum alloy
tubes from a weight standpoint usually yield
results better than most materials. This
conclusion applies to only low temperatures,
below 250°F, as aluminum alloys lose strength
rapidly for temperatures above 250°C to 300°F.

The student should calculate the lightest
titanium tube and the lightest magnesium tube
using Figs. C4.14 and C4.13 respectively and
compare the weight results with the steel and
aluminum as found above.

ULTIMATE TORSIONAL
STRENGTH OF ROUND TUBES
C4.19 Torsional Modulus of Rupture

In Article A6.2 of Chapter A6, the torsion
formula for circular sections, $f_s = T\pi/J$, was
derived. This equation assumes the maximum
shear stress on the cross-section of a round
bar or tube does not exceed the proportional
limit of the material, or the stress variation
is linear as shown in Fig. C4.15 and this
situation exists under the flight vehicle limit
loads. Before a round bar made of ductile
thus the internal shear stress distribution is similar to that indicated in Fig. C4.16. The shear stress-strain curve is similar in shape to the tension stress-strain curve and is equal to approximately 0.6 of the ordinates. Thus to find the ultimate internal torsional resisting moment, we can divide the cross-section into a number of circular elements. From the stress curve in Fig. C4.16, the stress at the midpoints of the circular elements can be found. Multiplying the area of the element by this stress times the distance of the element from the center axis gives the moment developed, and adding up the moment of the shear force on all the circular elements gives the total allowable or ultimate internal resisting moment $T_a$.

If this moment $T_a$ is to be used in the torsion equation, we must replace the true stress curve in Fig. C4.16 by a triangular stress curve with maximum value $F_{st}$ which will produce the same internal resisting moment as the true stress curve. Thus

$$T_a = F_{st} J/r, \text{ or } F_{st} = T_a r/J \quad \text{(C4.10)}$$

$F_{st}$ is called the torsional modulus of rupture stress. It is a fictitious stress, being higher than the ultimate shear stress of the material. Tubes subjected to torsion usually fail by inelastic or plastic instability or by elastic instability if the D/t ratio is large. To obtain reliable values of $F_{st}$, resort is usually made to tests, and since the round tube or rod is the most efficient and most available shape, much testing has been done over the years thus design curves are readily available for round tubes of the most common flight vehicle materials.

C4.20 Torsional Modulus of Rupture Curves.

Figs. C4.17 to C4.24 inclusive present curves for finding the modulus of rupture stress $F_{st}$ for steel alloys from $F_{tu} = 95000$ to $250,000$ psi. It should be noted that the torsional strength is influenced by the $D/t$ and the $L/D$ values of the tube. Figs. C4.25 to C4.30 inclusive give curves for the various aluminum alloys and Fig. C4.31 gives information for one magnesium alloy. All of these curves were taken from (Ref. 1).

The curves are based on a theoretical investigation by Lee and Ades (see Ref. 2), and have been found to be in good agreement with experimental results.

C4.21 Problems Illustrating Use of Torsional Modulus of Rupture Curves.

Problem 1. A 15/8 - .065 round tube is 16.25 inches long. What ultimate torsional moment will it develop if made of the following materials: (1) alloy steel $F_{tu} = 95000$, (2) aluminum alloy 2024-T3.

Solution: $D/t = 25, J/r = 2.3 (1/y) = 2 \times 0.11948 = 0.2390$. $L/D = 16.25/1.625 = 10$. For alloy steel, $F_{tu} = 95000$, we use Fig. C4.17 where for $D/t = 25$ and $L/D = 10$, we read $F_{st}/1000 = 47.3$. Thus $F_{st} = 47300$.

Then $T_a = F_{st} J/r = 47300 \times 0.2390 = 11300$ in-lb.

For 2024-T3 aluminum alloy, we refer to Fig. C4.26 where we read $F_{st} = 28000$.

Then $T_a = F_{st} J/r = 28000 \times 0.2390 = 6630$ in-lb.

Problem 2.

A round tube 10 inches long is to carry an ultimate torsional moment of 7000 in-lb. Select the lightest tube if made from aluminum alloy 2024-T4 and alloy steel $F_{tu} = 200,000$ and compare the resulting weight of each.

Solution: Since the modulus of rupture depends on $D/t$ and $L/D$ and since the tube size is unknown, we will use the trial and check approach. The design calculations can conveniently be made in table form as follows:

<table>
<thead>
<tr>
<th>Trial Tube Size</th>
<th>Area</th>
<th>$D/t$</th>
<th>$J/r$</th>
<th>$L/D$</th>
<th>$F_{st}$</th>
<th>$T_a = \frac{F_{st} J}{r}$</th>
<th>M.S. = $\frac{T_a}{F_{tu}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alum. Alloy 2024-T4 (Fig. C4.27)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1-1/2 - .065</td>
<td>.2930</td>
<td>23.05</td>
<td>2016</td>
<td>6.7</td>
<td>30000</td>
<td>9050</td>
<td>.14</td>
</tr>
<tr>
<td>1-3/4 - .058</td>
<td>.3063</td>
<td>30.20</td>
<td>2535</td>
<td>5.7</td>
<td>28000</td>
<td>7080</td>
<td>.01</td>
</tr>
<tr>
<td>1-7/8 - .049</td>
<td>.2811</td>
<td>38.25</td>
<td>2500</td>
<td>5.3</td>
<td>28000</td>
<td>6500</td>
<td>.07</td>
</tr>
<tr>
<td>2.0 - .049</td>
<td>.3003</td>
<td>40.30</td>
<td>2860</td>
<td>5.0</td>
<td>26000</td>
<td>7440</td>
<td>.06</td>
</tr>
<tr>
<td>Alum. Steel $F_{tu} = 200,000$ (Fig. C4.21)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 - .035</td>
<td>.1061</td>
<td>28.56</td>
<td>.0495</td>
<td>10</td>
<td>98000</td>
<td>4850</td>
<td>.31</td>
</tr>
<tr>
<td>1 - .049</td>
<td>.1461</td>
<td>20.40</td>
<td>.0664</td>
<td>10</td>
<td>106000</td>
<td>7050</td>
<td>.01</td>
</tr>
<tr>
<td>1-1/4 - .035</td>
<td>.1336</td>
<td>35.70</td>
<td>.0790</td>
<td>8</td>
<td>94000</td>
<td>7430</td>
<td>.06</td>
</tr>
</tbody>
</table>
Fig. C4.19 Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 150$ ksi.

Fig. C4.20 Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 180$ ksi.

Fig. C4.17 Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 95$ ksi.

Fig. C4.18 Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 125$ ksi.
Fig. C4.21  Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 200$ ksi.

Fig. C4.22  Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 220$ ksi.

Fig. C4.23  Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 240$ ksi.

Fig. C4.24  Torsional modulus of rupture -
alloy steels heat treated to $F_{tu} = 260$ ksi.
The lightest aluminum tube with a positive margin of safety is 2 - .049. The weight of a 10 in. length = .3032 x .101 x 10 = .303 lbs. The lightest steel tube is 1-1/4 - .035 and its weight is .1336 x .262 x 10 = .376 lbs. Although the steel is heat treated to $F_{tu} = 200,000$, it still is heavier than the aluminum alloy tube.

**STRENGTH OF ROUND TUBES UNDER COMBINED LOADINGS**

Reference should be made to Articale C1.15 of Chapter C1 for general explanation of stress-ratios and interaction equations as used in determining the ultimate strength of structural members under combined loadings.

C4.22 Combined Bending & Compression.

In tubes subjected to combined bending and compression, the stress due to compression is uniform over the cross-section whereas the bending stress is not uniform over the cross-section. The following stress ratio equation is possibly somewhat conservative but is recommended by (Ref. 1).

\[ \frac{R_c + R_b'}{R_b} = 1 \quad \text{(C4.11)} \]

which can be written

\[ \frac{f_c + f_b'}{f_b} = 1.0 \quad \text{(C4.12)} \]

where $f_b'$ equals the maximum bending stress including the secondary moments due to axial load times beam deflection

- $f_b' =$ Bending modulus of rupture stress
- $f_c =$ Compressive column axial stress
- $R_c =$ Allowable column stress

Equation C4.12 could also be written,

\[ \frac{P}{M} + \frac{M}{R_a} = 1.0 \quad \text{(C4.13)} \]

Where $P =$ column load
- $R_a =$ allowable column load at failure
- $M =$ bending moment
- $M_a =$ allowable ultimate bending moment

Fig. C4.32 shows a plot of equation C4.11.

The margin of safety equation is,

\[ M.S. = \frac{1}{\frac{R_c + R_b'}{R_b}} - 1 \quad \text{(C4.14)} \]

Fig. C4.32 shows margin of safety curves for estimating closely the M.S. values if equation C4.14 is not used.

**Fig. C4.32**

Combined Bending & Compression

\[ R_c + R_b' = 1 \]

**C4.23 Illustrative Problem Involving Combined Bending and Compression.**

Fig. C4.33 shows a round 2024-T3 tube acting as a beam-column. It is supported at pin A and by the inclined strut BC at point C. Let it be required to determine the margin of safety when carrying a uniform lateral load of 10 lb./in.

To find the axial load in tube AC, take moments about point (B).

\[ ZM_B = -40 \times 10 \times 20 + 20 R_{AH} = 0, \text{ hence } R_{AH} = 400 \text{ lb.} \]

Thus axial load in tube is 400 lb. compression.

The column strength of a 2024-T3 1-1/4 - .049 round tube 40 inches long with end fixity $C = 1$ is obtained from Fig. C4.8 and equals -2100 lb., hence stress ratio

\[ R_c = \frac{\text{Column load}}{\text{Column strength}} = \frac{-400}{-2100} = 0.191 \]

The bending moment will be maximum at the center of span because of symmetry of loading and the value of the moment is obtained from the following equation which includes the secondary moments due to deflection. (Reference table A10.1 of chapter A10).

\[ M_{max} = wL^2 (1 - \text{sec} \frac{1}{24}) \quad \text{(A)} \]

\[ j = \sqrt{\frac{5I}{F}} = \sqrt{\frac{10,300,000 \times 0.0354}{400}} = \sqrt{927} = 29.3 \]
bending and tension on round tubes, which is widely used, is

\[ R_b + R_t^{1/4} = 1 \quad \text{(C4.15)} \]

This equation is plotted in Fig. C4.34. The stress ratio \( R_t = F_t/F_{tu} \). The figure is based on \( D/t = 10 \) and in general is conservative for other \( D/t \) ratios.

From Table A10.2 of Chapter A10 cos of \( .663 = .7757 \).

hence \( \frac{1}{\cos L/2} = 1/1.7757 = 1.29 \)

Substituting in equation (A)

\[ M_{\text{MAX.}} = 9600 \times (1 - 1.29) = - 2490 \text{ in. lb.} \]

From Fig. C5.2, the modulus of rupture \( F_b \) for a 2024-T3 round tube which has \( D/t \) value of 25.5 equals 64000 psi. Therefore the bending strength \( M = F_b I/y = 64000 \times .0534 = 3420 \text{ in.lbs.} \)

Stress ratio in bending \( R_b = \frac{2490}{3420} = .728 \)

\[ R_C + R_b = .191 + .728 = .917 \] which shows that member is not weak.

Margin of safety M.S. = \( \frac{1}{R_c + R_b} - 1 = \frac{1}{.191 + .728} - 1 = .09 \)

The margin of safety could be read directly from curves in Fig. C4.32.

The student should notice that the maximum bending moment of 2490 in. lb. is 24 percent greater than the primary moment which equals \( wL^2/8 = 10 \times 40^2/8 = 2000 \). The lateral deflection at the midpoint of the tube thus equals \( 490/400 = 1.22 \text{ inch.} \)

The secondary moments due to lateral deflection do not vary linearly, so if design loads were increased the calculation of the maximum bending should be repeated instead of assuming that the moment would increase directly as the applied load to the beam.

C4.23 Combined Bending and Tension.

The interaction equation for combined bending and tension from (Ref. 1) is,

\[ R_b^a + R_{at} = 1 \quad \text{(C4.16)} \]

This equation is plotted in Fig. C4.35 for M.S. = 0. Curves showing various M.S. values
for various values of $R_b$ and $R_{st}$ are also shown on the figure.

The expression for Margin of Safety is,

$$M.S. = \frac{1}{\sqrt{R_b + R_{st}}} - 1$$

(C4.17)

**ILLUSTRATIVE PROBLEM 1**

A 1-1/2" .068 round steel tube ($F_{tu} = 125000$) is 30 inches long. It is subjected to an ultimate design bending moment of 10,000 in. lbs. and a torsional moment of 5000 in. lbs. Find the Margin of Safety.

Solution: $D/t = 25.85$, $I/y = .0912$, $L/D = 20$

To find $F_b$, we refer to Fig. C4.11 where for $D/t = 25.85$ and $F_{tu} = 125000$ we read $F_b = 148000$. Then $M_a = F_bI/y = 148000 \times .0912 = 13500$ in. lb.

Then $R_b = \frac{N}{M_a} = \frac{10000}{13500} = .74$

To find $F_{st}$, we refer to Fig. C4.18 where for $D/t = 25.85$ and $L/D = 20$ we read $F_{st}/1000$ to be 61, whence $F_{st} = 61000$.

$R_{st} = \frac{T}{T_a} = \frac{6000}{11120} = .54$

$$M.S. = \frac{1}{\sqrt{R_b + R_{st}}} - 1$$

$$= \frac{1}{\sqrt{.74^2 + .54^2}} - 1 = \frac{1}{.917} - 1 = .09$$

thus the ultimate strength of the tube has 9 percent Margin of Safety under the combined loading.

In a design problem which involves a trial and error procedure, using an equivalent torsional moment $T_e$ which will produce the same torsional stress as produced by the combined bending and torsional loads is quite useful in shortening the trial and error procedure.

Let $f_s(\max) = T_e r/2I$  \hspace{1cm} (C3.18)

$$f_s(\max) \text{ also equals } \sqrt{f_s^2 + (f_b/2)^2}$$  \hspace{1cm} (C4.19)

Also $f_s = T_r/2I$, and $f_b = Mr/I$

Substituting these values in C4.18,

$$f_s(\max) = \frac{T_e}{2I} \left[ \frac{M}{1 + (T/M)^2} \right]$$

(C4.20)

Equating C4.20 and C4.18 and solving for $T_e$,

$$T_e = M\sqrt{1 + (T/M)^2}$$

(C4.21)

Having the value of $T_e$, select tube sizes that will develop this torsional moment $T_e$ as was done in Problem 2 of Art. C4.21. These sizes are then checked for combined bending and torsion as illustrated above in the example problem.

**C4.24 Ultimate Strength in Combined Compression, Bending and Torsion.**

The interaction equation for combined compression, bending and torsion from (Ref. 1) is,

$$R_C + R_{st} = (1 - R_o)^2$$

(C4.22)

Fig. C4.36 and 37 show a plot of this equation. The expression for the Margin of Safety is,

$$M.S. = \frac{1}{R_C + \sqrt{R_b^2 + R_{st}^2}} - 1$$

(C4.22)

To illustrate the use of the interaction curves, let us assume the following values for the three stress ratios:

$$R_C = .333, \ R_b = .20, \ R_{st} = .20$$

Then $R_{st}/R_C = .20/.333 = .60$.

In Fig. C4.36 locate point (a) at the intersection of $R_C = .333$ and $R_b = .20$. Since the intersection point (a) lies inside the $R_{st}/R_C = .60$ curve, we know that a positive margin of safety exists. A line is now drawn through (o) and (a) and extended to an intersection with the $R_{st}/R_C = .60$ curve at point (b). Projecting vertically downward to $R_b$ scale, we read $R_b = .538$. Then M.S. = (.538/.333) - 1 = .62.
Projecting horizontally from point (u) to R_b scale, we read R_b = .325. Then M.S. = (.325/.200) - 1 = .52; which checks value previously found. If the ratio of R_fy/R_c is greater than one, we use the curves in Fig. C4.37 and use the ratio R_c/R_st.

Substituting in equation C4.23,

\[ M.S. = \frac{1}{.333 + \sqrt{.20^2 + .20^2}} = 1 = .52 \]

as given from use of curves.

\[ \frac{L}{2J} = \frac{25}{2 \times 12.43} = \frac{25}{24.85} = 1.005 \]

\[ \sec \frac{L}{2J} = \frac{1}{\cos \frac{L}{2J}} = \frac{1}{.5361} = 1.87 \] (See Table A10.2)

hence,

\[ M_{\text{max.}} = 10 \times 154.7 \left(1 - 1.87 \right) = -1347 \text{ in. lb.} \]

The column strength for a 25 inch length and end fixity c = 1 can be read from Fig. C4.6 and equals 3700 lb. Then R_c = P/P_a = 1600/3700 = .432.

To find the ultimate bending strength, we refer to Fig. C4.12, where for D/t = 1.125/.049 = 22.96, we read F_b/F_y = 1.04. Then F_b = 1.04 \times 65000 = 664500. Thus M_a = F_b \times y = 664500 \times .0427 = 2750 \text{ in. lb.}

Therefore \( R_{\text{f}} = \frac{1347}{2750} = .487 \).

To find the ultimate torsional strength we refer to Fig. C4.26 where for D/t = 22.96 and estimating location of L/D = 25/1.125 = 22.2 line, we read F_s = 27500.

Then M_a = F_s \times J/\pi = 27500 \times .0427 \times 2 = 2350

whence R_st = \( \frac{T}{\text{T}_a} = \frac{650}{2350} = .276 \)

\[ \text{M.S.} = \frac{1}{R_c + \sqrt{R_f^2 + R_s^2}} - 1 \]

\[ = \frac{1}{.432 + \sqrt{.487^2 + .276^2}} - 1 = \frac{1}{.992} - 1 = .01 \]

C4.25 Ultimate Strength in Combined Bending and Flexural Shear.

The interaction curve from this type of combined loading from (Ref. 2) is,

\[ R_b + R_s = 1 \] (C4.24)

\[ \text{M.S.} = \frac{1}{\sqrt{R_b + R_s}} - 1 \] (C4.25)

The allowable flexural or transverse shear stress is taken as 1.2 times the allowable torsional stress of the tube (F_st).

ILLUSTRATIVE PROBLEM

A 1-1/8 - .049 round tube made from 2024-T3 aluminum alloy carries the ultimate design loads as shown in Fig. C4.38. Find the margin of safety under the combined loading.

\[ \begin{align*}
  & 1600 \quad w = 10\, \text{lb/in.} \\
  & 25^o \quad L = 0.049 \quad 2024-T3 \quad \text{RD. Tube} \\
  & 1600\# \quad \text{1-1/8} \\
  & \text{Fig. C4.38}
\end{align*} \]

Solution:

The maximum bending moment due to symmetry will occur at midpoint of tube. For a beam column carrying a uniform side load with no end moments, the maximum moment is given by the following expression. (See Chapter A10, Table A10.1).

\[ M_{\text{max.}} = wJ^2 \left(1 - \sec \frac{L}{2J}\right) \]

\[ J = \sqrt{\frac{EI}{P}} = \sqrt{\frac{10,300,000 \times .0240}{1600}} = \sqrt{154.7} = 12.43 \]

A 1-1/4 - .066 round aluminum alloy 2024-T3 tube is used as a simple beam and carries an ultimate design load of 600 lb. as shown in Fig. C4.39.
The critical section is adjacent to the midpoint of beam where shear and bending forces are largest.

The ultimate bending strength can be found from Fig. C4.26 where for D/t = 21.60 we read \( F_b/F_{tu} = 1.07 \). Then \( F_b = 62000 \times 1.07 = 66300 \) and therefore \( M_b = F_b l/2y = 66300 \times 0.06187 = 4100 \). Then \( R_b = M/M_b = 3750/4100 = .915 \).

The allowable shear stress \( F_s \) can be taken as 1.2 \( F_{st} \) (Ref. 2).

To find \( F_{st} \), we refer to Fig. C4.26 where for \( D/t = 21.6 \) and \( L/D = 20 \), we read \( F_{st} = 28400 \). Then \( F_s = 1.2 \times 28400 = 34000 \).

The maximum shear stress in a round tube which occurs at the centerline axis is given by the equation,

\[
f_s = \frac{4V}{3A} \left(1 + \frac{dD}{D^2}ight)
\]

where \( V \) = vertical beam shear load
\( A \) = tube area
\( D \) = outside diameter
\( d \) = inside diameter

An approximate formula is

\[
f_s = 2\frac{V}{A}, \text{ for } D/t = 10 \text{ error less than 1%}.
\]

Using the approximate equation,

\[
f_s = \frac{2 \times 300}{.2172} = 2780 \text{ psi}.
\]

\[
R_s = \frac{F_s}{F_{su}} = \frac{2780}{34000} = .081.
\]

\[
\text{M.S.} = \frac{1}{\sqrt{R_s^2} + R_s} = \frac{1}{\sqrt{.081^2 + .081^2}} = .09.
\]

The effect of the shear stress is less than 1 percent. In using \( F_s = 1.2 F_{st} \), if the result is greater than \( F_{su} \) for the material, use \( F_{su} \) for \( F_s \).

C4.26 Ultimate Strength in Combined Compression, Bending, Flexural Shear and Torsion.

The interaction equation for this combined loading as presented in Ref. 2 is,

\[
R_c + R_{st} + \sqrt{R_t^2 + R_s^2} = 1 \quad \text{(C4.26)}
\]

Interaction curves for this equation are given in Fig. C4.40, where \( R_s \), the stress ratio for flexural shear can be found as explained in the previous article.

C4.27 Ultimate Strength in Combined Tension and Torsion.

The interaction equation for this type of loading as presented in Ref. 3 is,

\[
R_t^* + R_{st} = 1 \quad \text{(C4.27)}
\]

Where \( R_t = F_t/F_{tu} \)

\[
\text{M.S.} = \frac{1}{\sqrt{R_t^* + R_{st}}} \quad \text{(C4.28)}
\]

C4.28 Ultimate Strength in Combined Tension, Torsion and Internal Pressure \( p \) in psi.

Ref. 3 gives the following interaction equation,
(3) If the truss of Fig. 2 is heat treated to 180000 psi after welding, how much weight could be saved over the results obtained in problem (2).

(4) Same as problem (2) but change material to 2024-T3 aluminum alloy round tubes. Members to be fastened together by rivets and gusset plates. For design of tension members assume that rivet or bolt holes cutout 10% of tube area.

(5) Fig. 3 shows a typical tubular engine mount structure. The engine is supported at points A and B. For design purposes assume that engine torque is reacted 60% at A and 40% at B. Tube material is steel $F_{tu} = 95000$ psi. Use $C = 1$ for all members. Determine tube sizes for the following design conditions:

Condition I Vertical Load factor = 10  (down)
Thrust Load factor = 2  (forward)
Engine Torque Load factor = 2

Condition II (Same as I except vertical load factor is 5 up.

General Data: Weight of power plant installation = 440 lb.

Maximum engine thrust = 400 lb. Engine N.P. = 120 at 2000 R.P.M.

(6) The loads shown in Fig. 4 are to be transmitted to the support at the left, or in other words, a cantilever structure.

The problem is to design the lightest truss configuration using round tubes of alloy steel $F_{tu} = 95000$ and welded together at the truss joints. Use $C = 1.5$ for end fixity of all members. There are
no restrictions on type of truss or arrangement of members, however, the goal is the lightest truss. Omit consideration of weight of any gusset plates at truss joints.

(7) Same as problem (6), but instead of a cantilever truss use a simply supported truss with supports at points (A) and (B).

(8) Fig. 5 shows a front beam and front lift strut in an externally braced monoplane. The wing beam and lift strut are in the same vertical plane. The ultimate design loads on the beam for the critical conditions are \( w = 50 \text{ lb./in.} \) and \( w = -30 \text{ lbs. per inch} \). Minus means load is acting down.

(a) Design a streamline tube to act as the lift strut. Material is 2024-T3 aluminum alloy.

(b) Same as (a) but made from alloy steel \( F_{ty} = 75000 \). Compare the weights of the two designs.

![Fig. 5](image)

(9) Fig. 6 illustrates the strut and wire bracing for attaching float to fuselage of a seaplane. Determine the necessary sizes for the streamline struts AC and BD for the following load conditions.

Condition 1. \( V = -32000 \text{ lbs.} \), \( H = 8000 \text{ lbs.} \)
Condition 2. \( V = -8000 \text{ lbs.} \), \( H = -28000 \text{ lbs.} \)

Material 2024-T3 aluminum alloy. Use \( C = 1 \).

(10) Tube size 2 - .065 round. \( L = 44 \text{ in.} \), \( C = 1.5 \). Material alloy steel \( F_{tu} = 95000 \), welded at ends. Design ultimate loads equal 22000 lb. compression and 28000 lbs. tension. Find margin of safety.

(11) Same as Problem 10 but heat treated to \( F_{tu} = 150000 \) after welding.

(12) The ultimate design load if 20,000 lbs. compression. \( L = 30 \text{ in.} \). Use \( C = 1 \).

Design the lightest round tube from the following materials and compare their weights.

(a) Aluminum alloy 2024-T3
(b) Alloy steel \( F_{tu} = 180,000 \)
(c) Magnesium alloy \( F_{ty} = 10,000 \)

(13) Same design load as in problem (10) but design the lightest streamline tube from 2024-T3 aluminum alloy material.

(14) A round tube is to carry an ultimate pure bending moment of 14000 in. lbs. Select the lightest tube size from the following materials and compare their weights.

(a) Alloy steel \( F_{tu} = 240,000 \), (b) 2024-T3 aluminum alloy, (c) Magnesium alloy \( F_{ty} = 30000 \), (d) Titanium 6AL-4V alloy.

(15) A round tube 20 inches long is to carry an ultimate torsional moment of 15000 in. lb. Select the lightest tube size from the following materials and compare their weights.

(a) Alloy steel \( F_{tu} = 130,000 \), (b) Aluminum alloy 2024-T3, (c) Magnesium alloy \( F_{tu} = 36000 \).

(16) Determine the lightest 2024-T3 aluminum alloy round tube 10 inches long to carry a combined bending and torsional design load of 4500 and 3000 lb. in. respectively.

(17) Same as Problem 16, but change material to alloy steel \( F_{tu} = 85000 \).

(18) A 1-1/2 - .065 2024-T3 round tube 50 inches long is used as a beam-column. The distributed load on beam is 12 lb. per inch and the axial load is 700 lbs. What is the M.S. under these loads.

(19) If the tube in problem (18) was also subjected to a torsional moment of 1400 in. lb., what would be the M.S.

References

(2) Lockheed Report 2077.
CHAPTER C5
BUCKLING STRENGTH OF FLAT SHEET IN COMPRESSION, SHEAR, BENDING AND UNDER COMBINED STRESS SYSTEMS

C5.1 Introduction.

Chapter A18, Part 2, introduced the student to the theoretical approach to the problem of determining the buckling equation for flat sheet in compression with various edge or boundary conditions. A similar theoretical approach has been made for other load systems, such as shear and bending, thus the buckling equations for flat sheet have been available for many years. This chapter will summarize these equations and provide design charts for practical use in designing sheet and plate structures. Most of the material in this chapter is taken from (Ref. 1), NACA Technical Note 3781—Part I, "Buckling of Flat Plates" by Gerard and Becker. This report is a comprehensive study and summary of practically all important theoretical and experimental work published before 1957. The report is especially useful to structural design engineers.

C5.2 Equation for Elastic Buckling Strength of Flat Sheet in Compression.

From Chapter A18, the equation for the elastic instability of flat sheet in compression is,

\[ \sigma_{cr} = \frac{n^2 k_c E}{12(1-\nu^2)} \left( \frac{t}{b} \right)^3 \]  \hspace{1cm} (C5.1)

Where \( k_c \) = buckling coefficient which depends on edge boundary conditions and sheet aspect ratio (a/b)
E = modulus of elasticity
\( \nu \) = elastic Poisson's ratio
b = short dimension of plate or loaded edge
t = sheet thickness

C5.3 Buckling Coefficient \( k_c \)

Fig. C5.1 shows the change in buckled shape as the boundary conditions are changed on the unloaded edges from free to restrained.

In Fig. (a) the sides are free, thus sheet acts as a column. In Fig. (b) one side is restrained and the other side free, and such a restrained sheet is referred to as a flange. In Fig. (c) both sides are restrained and this restrained element is referred to as a plate.

Fig. C5.1 (Ref. 1) Transition from column to plate as supports are added along unloaded edges. Note changes in buckle configurations.

Fig. C5.2 gives curves for finding the buckling coefficient \( k_c \) for various boundary or edge conditions and a/b ratio of the sheet.

The letter C on edge means clamped or fixed against rotation. The letter F means a free edge and SS means simply supported or hinged. Fig. C5.3 shows curves for \( k_c \) for various degrees of restraint (z) along the sides of the sheet panel, where \( z \) is the ratio of rotational rigidity of the plate edge stiffener to the rotational rigidity of the plate.

Fig. C5.4 shows curves for \( k_c \) for a flange that has one edge free and the other with various degrees of edge restraint. Fig. C5.5 illustrates where the compressive stress varies linearly over the length of the sheet, a typical case being the sheet panels on the upper side of a cantilever wing under normal flight condition.

Fig. C5.6 gives the \( k_c \) factor for a long sheet panel with two extremes of edge stiffener, namely a zee stiffener which is a torsionally weak stiffener and a hat section
BUCKLING STRENGTH OF FLAT SHEET IN COMPRESSION, SHEAR, BENDING AND UNDER COMBINED STRESS SYSTEMS

Fig. C5.2 (Ref. 1) Compressive-buckling coefficients for flat rectangular plates.

Fig. C5.3 (Ref. 1) Compressive-buckling-stress coefficient of plates as a function of a/b for various amounts of edge rotational restraint.

Fig. C5.4 (Ref. 1) Compressive-buckling-stress coefficient of flanges as a function of a/b for various amounts of edge rotational restraint.

Fig. C5.5 (Ref. 1) Average compressive-buckling-stress coefficient for rectangular flat plate of constant thickness with linearly varying axial load:

\[ f_{av} = \frac{k_{av} \pi^2}{12(1 - \nu^2)} \]
which is a closed section and, therefore, a relatively torsionally strong stiffener. Fig. C5.6a gives the compression buckling coefficients \( k_c \) for isosceles triangular plates.

![Fig. C5.6 (Ref. 1) Compressive-buckling coefficient for long rectangular stiffened panels as a function of b/t and stiffener torsional rigidity.](image)

Substituting in Eq. C5.1,

\[
\sigma_{cr} = \frac{\pi^2 \times 4.0 \times 10,700,000}{12 \left(1 - \frac{3}{5}\right)} \left(\frac{C_t}{5}\right)^4 = 2480 \text{ psi.}
\]

This stress is below the proportional limit stress for the material, thus equation C5.1 applies and needs no plasticity correction.

C5.4 Equation for Inelastic Buckling Strength of Flat Sheet in Compression.

If the buckling or instability occurs at a stress in the inelastic or plastic stress range, then \( E \) and \( \nu \) are not the same as for elastic buckling, thus a plasticity correction factor is required and equation C5.1 is written,

\[
\sigma_{cr} = \frac{\eta \pi^2 k_c E}{12(1-\nu^2)} \left(\frac{t}{b}\right)^4 \quad - - - - - - - - - - (C5.2)
\]

where \( \eta \) is the plasticity reduction factor and equals \( \sigma_{cr} \) plastic/\( \sigma_{cr} \) elastic.

The values of \( k_c \) and \( \nu_c \) are always the same since the coefficient \( \eta \) contains all changes in those terms resulting from inelastic behavior.

A tremendous amount of theoretical and experimental work has been done relative to the value of the so-called plasticity correction factor. Possibly the first values used by design engineers were \( \eta = E_t / E \) or \( \eta = E_{sec} / E \). Whatever the expression for \( \eta \) it must involve a measure of the stiffness of the material in the inelastic stress range and since the stress-strain relation in the plastic range is non-linear, a replap must be made to the stress-strain curve to obtain a plasticity correction factor. This complication is greatly simplified by using the Ramberg and Osgood equations for the stress-strain curve which involves 3 simple parameters. (The reader should refer to Chapter B1 for information on the Ramberg-Osgood equations.) Thus using the Ramberg-Osgood parameters (Ref. 1) presents Figs. C5.7 and C5.8 for finding the compressive buckling stress for flat sheet panels with various boundary conditions for both elastic and inelastic buckling or instability.

C5.5 Simple Problems to Illustrate Use of Curves in Figs. C5.7 and C5.8.

- The sketch shows a 3 x 9 inch sheet panel. The sides are simply supported. The material is aluminum alloy 2024-T3. The thickness is .094". \( E = 10,700,000. \)
BUCKLING STRENGTH OF FLAT SHEET IN COMPRESSION, SHEAR, BENDING AND UNDER COMBINED STRESS SYSTEMS

Fig. C5.7 Chart of Nondimensional Compressive Buckling Stress for Long Hinged Flanges. \( \eta = \frac{(E_g/E)(1 - \nu^2)}{(1 - \nu^2)^2} \).

Fig. C5.8 Chart of Nondimensional Compressive Buckling Stress for Long Clamped Flanges and for Supported Plates with Edge Rotational Restraint. \( \eta = \frac{(E_g/2E)\left\{1 + 0.5 \left[1 + (3E/E_g)\right]^{1/2}\right\}(1 - \nu^2)^2}{(1 - \nu^2)^2} \).
\( \nu_e = 0.3 \). Find the buckling stress \( \sigma_{cr} \).

Solution: We use Fig. C5.8 since it covers the boundary conditions of our problem. The parameter for bottom scale is,

\[
\frac{k_c}{12} = \left( \frac{t}{r} \right)^2 \frac{4 \pi^2}{\nu_e (1 - \nu_e)} \sigma_e. \tag{A}
\]

For \( a/b = 9/3 = 3 \), we find \( k_c \) from curve (c) of Fig. C5.2 equals 4.0.

The use of Fig. C5.8 involves the use of \( \sigma_e \) and \( n \) the Hamberg-Osgood parameters. Referring to Table B1.1 of Chapter B1, we find for 2024-T3 aluminum alloy that \( \sigma_e = 30,000 \) and the shape factor \( n = 11.5 \).

Substituting in (A):

\[
\frac{4 \pi^2 \times 10,000,000}{12 (1 - .03^2)} = 39,000
\]

From Fig. C5.8 using .98 on bottom scale and \( n = 11.5 \) curve, we read on left hand scale that \( \sigma_{cr}/\sigma_e \approx .94 \).

Then \( \sigma_{cr} = 39,000 \times .94 = 36,960 \text{ psi} \).

If we neglected any plasticity effect, then we would use equation C5.2 with \( \eta = 1.0 \), or,

\[
\sigma_{cr} = \frac{\pi^2 \times 4.0 \times 10,000,000}{12 (1 - .03^2)} = 38,400 \text{ psi}
\]

Whereas the actual buckling stress was 36,960, or in this case the plasticity correction factor is 328/324 = .954.

The sheet thickness used in this example of .094 is relatively large. If we change the sheet thickness to .061 inches the results would be practically no correction within the accuracy of reading the curves, and the buckling stress \( \sigma_{cr} \) would calculate to be 11,200 psi, which is below the proportional limit stress and thus no plasticity correction.

C5.6 Cladding Reduction Factors.

Aluminum alloy sheet is available with a thin covering of practically pure aluminum and is widely used in aircraft structures. Such material is referred to as alclad or clad aluminum alloy. The mechanical strength properties of this clad material is considerably lower than the core material. Since the clad is located at the extreme fibers of the alclad sheet, it is located where the strains attain their highest value when buckling takes place. Fig. C5.9 shows make up of an alclad sheet and Fig. C5.10 shows the stress-strain curves for cladding, core and alclad combinations.
BUCKLING UNDER SHEAR LOADS

C5.7 Buckling of Flat Rectangular Plates
Under Shear Loads.

The critical elastic shear buckling stress for flat plates with various boundary conditions is given by the following equation:

\[ \tau_{cr} = \frac{\pi^2 k_s E}{12 (1 - \nu_s^2)} \left( \frac{b}{h} \right)^2 \]  \hspace{1cm} (C5.4)

where \( b \) is always the shorter dimension of the plate as all edges carry shear. \( k_s \) is the shear buckling coefficient and is plotted as a function of the plate aspect ratio \( a/b \) in Fig. C5.11 for simply supported edges and clamped edges.

If buckling occurs at a stress above the proportional limit stress, a plasticity correction must be included and equation C5.4 becomes

\[ \tau_{cr} = \frac{\eta_s \pi^2 k_s E}{12 (1 - \nu_s^2)} \left( \frac{b}{h} \right)^2 \]  \hspace{1cm} (C5.5)

Test results compare favorably with the results of equation C5.5 if \( \eta_s = G_s/G \) where \( G \) is the shear modulus and \( G_s \) the shear secant modulus as obtained from a shear stress-strain diagram for the material.

A long rectangular plate subjected to pure shear produces internal compressive stresses on planes at 45 degrees with the plate edges and thus these compressive stresses cause the long panel to buckle in patterns at an angle to the plate edges as illustrated in Fig. C5.12, and the buckle patterns have a half wave length of 1.25b.

![Fig. C5.12 (Ref. 7)]

Fig. C5.12 is a chart of non-dimensional shear buckling stress for panels with various edge rotational restraint. This chart is similar to the chart in Figs. C5.7 and C5.8 in that the values \( \sigma_{cr} \), and \( n \) must be known for the material before the chart can be used to find the shear buckling stress.

BUCKLING UNDER BENDING LOADS

C5.8 Buckling of Flat Plates Under Bending Loads.

The equation for bending instability of flat plates in bending is the same as for compression and shear except the buckling coefficient \( k_b \) is different from \( k_c \) or \( k_s \). When a plate in bending buckles, it involves relatively short wave length buckles equal to 2/3 b for long plates with simply supported edges (see Fig. C5.14). Thus the smaller buckle patterns cause the buckling coefficient \( k_b \) to be larger than \( k_c \) or \( k_s \).

![Fig. C5.14 (Ref. 7) Bending Buckle Patterns]

For bending elastic buckling the equation is,

\[ \sigma_{cr} = \frac{\pi^2 k_b E}{12 (1 - \nu_b^2)} \left( \frac{b}{h} \right)^2 \]  \hspace{1cm} (C5.6)

For bending inelastic buckling,

\[ \sigma_{cr} = \frac{\eta_b \pi^2 k_b E}{12 (1 - \nu_b^2)} \left( \frac{b}{h} \right)^2 \]  \hspace{1cm} (C5.7)

where \( k_b \) is the buckling coefficient and is obtained from Fig. C5.15 for various \( a/b \) ratios and edge restraint against rotation. In the \( a/b \) ratio the loaded edge is (b).

The plasticity reduction factor can be obtained from Fig. C5.6 using simply supported edges.

BUCKLING OF FLAT SHEETS UNDER COMBINED LOADS

The practical design case involving the use of thin sheets usually involves a combined load system, thus the calculation of the buckling strength of flat sheets under combined stress systems is necessary. The approach used involves the use of inter-action equations or curves (see Chapter C1, Art. C1.15 for explanation of inter-action equations).

C5.9 Combined Bending and Longitudinal Compression.

The interaction equation that has been widely used for combined bending and longi-
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

Fig. C5.13 (Ref. 1) Chart of Nondimensional Shear Buckling Stress for Panels With Edge Rotational Restraint. \( \eta = (\frac{E}{\text{E}_0}) \frac{(1 - \nu^2)}{(1 - \nu^2)} \).

Fig. C5.11 (Ref. 1) Shear-Buckling-Stress Coefficient of Plates as a Function of \( a/b \) for Clamped and Hinged Edges.

Fig. C5.15 Bending-Buckling Coefficient of Plates as a Function of \( a/b \) for Various Amounts of Edge Rotational Restraint.
BUCKLING STRENGTH OF FLAT SHEET IN COMPRESSION, SHEAR, BENDING AND UNDER COMBINED STRESS SYSTEMS

C5.8

Tudinal compression is,

\[ R_b^{1.78} + R_c = 1 \]  \hspace{1cm} (C5.8)

This equation was originally presented in Ref. 2 and the interaction curve from plotting this equation is found in many of the structures manuals of aerospace companies.

Fig. C5.15 is a plot of eq. C5.8. It also shows curves for various margin of safety values.

![Fig. C5.15 Combined Bending & Long. Compression](image)

\[ R_b^{1.78} + R_c = 1 \]

0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1

C5.10 Combined Bending & Shear.

The interaction equation for this combined loading (Ref. 1 & 2) is,

\[ R_b^* + R_s^* = 1 \]  \hspace{1cm} (C5.9)

The expression for margin of safety is,

\[ M.S. = \frac{1}{\sqrt{R_b^* + R_s^*}} - 1 \]  \hspace{1cm} (C5.10)

Fig. C5.16 is a plot of equation C5.9. Curves showing various M.S. values are also shown. \( R_s \) is the stress ratio due to torsional shear stress and \( R_{st} \) is the stress ratio for transverse or flexural shear stress.

C5.11 Combined Shear and Longitudinal Direct Stress. (Tension or Compression.)

The interaction equation is (Ref. 3,4).

\[ R_L + R_s^* = 1.0 \]  \hspace{1cm} (C5.11)

\[ M.S. = \frac{2}{R_L + \sqrt{R_L^* + 4R_s^*}} - 1 \]  \hspace{1cm} (C5.12)

Fig. C5.17 is a plot of equation C5.11. If the direct stress is tension, it is included on the figure as negative compression using the compression allowable.

C5.12 Combined Compression, Bending & Shear.

From Ref. 5, the conditions for buckling are represented by the interaction curves of Fig. C5.18. This figure tells whether the sheet will buckle or not but will not give the margin of safety. Given the ratios \( R_c \), \( R_s \) and \( R_b^* \), if the value of the \( R_c \) curve defined by the given value of \( R_b \) and \( R_s \) is greater numerically than the given value of \( R_c \), then the panel will buckle.

![Fig. C5.18 (Ref. 5)](image)
**Fig. C5.17**
Combined Shear & Long, Direct Stress
\( R_L - R_S = 1.0 \)

**Fig. C5.19 (Ref. 5)** Combined Compression, Bending and Shear.
The margin of safety of elastically buckled flat panels may be determined from Fig. C5.19. The dashed lines indicate a typical application where \( R_c = .161 \), \( R_s = .23 \), and \( R_p = .38 \). Point 1 is first determined for the specific value of \( R_s \) and \( R_p \). The dashed diagonal line from the origin 0 through point 1, intersecting the related \( R_s/R_p \) curve at point 2, yields the allowable shear and bending stresses for the desired margin of safety calculations. (Note when \( R_c \) is less than \( R_s \) use the right half of the figure; in other cases use the left half).

C5.13 Illustrative Problems.

In general a structural component composed of stiffened sheet panels will not fail when buckling of the sheet panels occurs since the stiffening units can usually continue to carry more loading before they fail. However, there are many design situations which require that initial buckling of sheet panels satisfy certain design specifications. For example, the top skin on a low wing passenger airplane should not buckle under accelerations due to air gusts which occur in normal everyday flying thus preventing passengers from observing wing skin buckling in normal flying conditions. Another example would be that no buckling of fuselage skin panels should occur while airplane is on ground with full load aboard in order to prevent public from observing buckling of fuselage skin. In many airplanes, fuel tanks are built integral with the wing or fuselage, thus to eliminate the chances of leakage developing, it is best to design no buckling of sheet panels that bound the fuel tanks occur in flying and landing conditions. In some cases aerodynamic or rigidity requirements may dictate no buckling of sheet panels. To insure that buckling will not occur under certain load requirements, it is good practice to be conservative in selecting or calculating the boundary restraints of the sheet panels.

Problem 1.

Fig. C5.20 shows a portion of a cantilever wing composed of sheet, stiffeners and ribs. The problem is to determine whether skin panels marked (A), (B) and (C) will buckle under the various given load cases. The sheet material is aluminum alloy 2024-T3.

Load Case 1:

\[ P_1 = 700 \text{ lb.}, \quad P_s = 0, \quad P_p = 0 \]

With only loads \( P_1 \) acting, the one cell stiffened cantilever beam is subjected to a compressive axial load of \( 2 \times 700 = 1400 \text{ lb.} \). Since the \( P_s \) loads are not acting through the centroid of the cross-section, a bending moment is produced about the x-x axis equal to \( 3.7 = 5170 \text{ in. lb.} = M_x \), where 3.7 is distance from load \( P_s \) to x-x axis.

Area of Zee Strut = .18
Area of Corner Member = 0.25 sq. in.

The sheet thicknesses, stiffener areas and all necessary dimensions are shown on Fig. C5.20. The total cross-sectional area of beam section including all skin and stringers is 3.73 sq. in. The moment of inertia about x-x centroidal axis calculates to be 49.30 in.4.

Since the beam section is symmetrical, the top panels A, B and C are subjected to the same stress under the \( P_1 \) load system.

Compressive stress due to transferring loads \( P_1 \) to centroid of beam cross-section is,

\[ f_c = \frac{2P_1}{\text{area}} = \frac{1400}{3.73} = 375 \text{ psi} \]

Compressive stress due to constant bending moment of 5170 in. lbs. is,

\[ f_c = \frac{M_x/L_x}{\text{I}_x} = \frac{5170 \times 4.233}{49.30} = 444 \text{ psi} \]

Total \( f_c = 375 + 444 = 819 \text{ psi} \).

The skin panels are subjected to compression as shown in Fig. a. The boundary edge conditions given by the longitudinal stiffeners
and the rib flanges will be conservatively assumed as simply supported. \( F_{cr} \) is same as \( \sigma_{cr} \)

\[
F_{cr} = \frac{\pi^2 k_c E}{12 (1 - \nu_e^2)} \left( \frac{t}{b} \right)^3
\]

(See Eq. C5.1)

\[
a/b \text{ of skin panel} = 15/5 = 3
\]

From Fig. C5.2 for Case C, we read \( k_c = 4 \).

\[
F_{cr} = \frac{\pi^2 \times 4.0 \times 10,700,000 \text{ psi} \times (0.035)^2}{12 (1 - 0.3)} = 1900 \text{ psi}
\]

Since \( F_{cr} \) is the buckling stress is less than the applied stress \( f_c \), the panels will not buckle.

M.S. = \( F_{cr} / f_c \) - 1 = (1900/819) - 1 = 1.32

**Load Case 2.**

\( P_1 = 700 \text{ lb.}, P_2 = 500, P_3 = 0 \)

The two loads \( P_2 \) and \( P_3 \) acting in opposite directions produce a couple or a torsional moment of 500 x 16.5 = 8250 in. lb. on the beam structure, which means we have added a pure shear stress system to the compressive stress system of Case 1 loading.

The shear stress in the top panels A, B and C is,

\[ f_s = T/2At = 8250/2 \times 138 \times 0.035 = 554 \text{ psi.} \]

(Where \( A \) is the cell inclosed area)

The shear buckling stress is

\[
F_{s,cr} = \frac{\pi^2 k_s E}{12 (1 - \nu_e^2)} \left( \frac{t}{b} \right)^3 \quad (\text{See Eq. C5.4})
\]

\[
a/b = 15/5 = 3. \text{ From Fig. C5.11, for hinged or simply supported edges, we read} \ k_s = 5.8.
\]

\[
F_{s,cr} = \frac{\pi^2 \times 5.8 \times 10,700,000 \text{ psi} \times (0.035)^2}{12 (1 - 0.3)} = 2760 \text{ psi}
\]

The shear panels are now loaded in combined compression and shear so the interaction equation must be used. From Art. C5.12 the interaction equation is \( R_c + R_s = 1 \).

\[
R_c = f_c / F_{cr} = 819/1900 = 0.431
\]

\[
R_s = f_s / F_{s,cr} = 554/2760 = 0.309
\]

The \( R_c + R_s = 0.431 + 0.309 = 0.74. \text{ Since the result is less than 1.0, no buckling occurs.} \]

\[
\text{The M.S.} = \frac{2}{R_c + \sqrt{R_c^2 + 4R_s}} - 1 = \frac{2}{0.431 + \sqrt{0.431^2 + 4 \times 0.309}} - 1 = 0.69
\]

**Load Case 3.**

\( P_1 = 700, P_2 = 500, P_3 = 100 \text{ lb.} \)

The two loads \( P_2 \) produce bending and flexural shear on the beam. The bending moment produces a different and compressive stress on the three shear panels since the bending moment is not constant over the panel moment. To simplify we will take average bending moment on the panel.

\[ M_{x,av} = 200 \times 52.5 = 10500 \text{ in. lb.} \]

\[ f_s \text{ due to this bending} = M_{x,av}/I_x = 10500 \times \frac{4.253}{49.3} = 903 \text{ psi.} \]

Total \( f_c = 819 + 903 = 1722 \text{ psi.} \)

\[ R_c = f_c / F_{cr} = 1722/1900 = 0.906 \]

The two loads \( P_3 \) produce a traverse shear load \( V = 200 \text{ lb.} \). The flexural shear stress must be added to the torsional shear stress as found in Case 2 loading.

Due to symmetry of beam section and \( P_2 \) loading the shear flow \( q \) at midpoint of beam panel (B) is zero. We will thus start at this point and go clockwise around cell. The shear flow equation (see Chapter A15) is,

\[ q = \frac{V}{I_x} \]

\[
q = \frac{-200}{49.3} = 4.05 \text{ ZZA}
\]

\[
q_1 = 0 \text{ (Refer to Fig. b)}
\]

\[
q_{a1} = -4.05 \times 2.5 \times 0.035 = 4.23 = -1.50
\]

\[
q_{a2} = -1.50 - 4.05 \times 1.3 \times 3.69 = -4.20
\]

\[
q_{a3} = -4.20 - 4.05 \times 5 \times 0.035 = 7.20
\]

\[
q_3 = -7.20 - 4.05 \times 2.5 \times 3.69 = 10.94
\]

\[
q_{a3} = -10.94 - 4.05 \times 0.051 \times 3.69 \times 3.69/2 = -12.34
\]

(See Fig. b for plot of shear flow)
The shear flow \( q \) on panel (A) varies from 4.20 to 7.20 or the average \( q = (4.2 + 7.2)/2 = 5.7 \). Thus the average shear stress is 5.7/.035 = 163 psi. It is in the same direction as the torsional shear flow and thus is additive.

Total \( f_p = 163 + 854 = 917 \) psi

\[ \frac{R_c}{f_p} = \frac{917}{2760} = .332 \]

\[ R_c + R_p = 1, \text{ Subt.} = .906 + .332^2 = 1.015, \text{ since the result is greater than 1.0, initial buckling has started. The margin of safety is slightly negative and equals,} \]

\[ M.S. = \frac{2}{.906 + \sqrt{.906^2 + 4 \times .332^2}} - 1 = -.01 \]

In this example problem, the panels were assumed simply supported, which is conservative. Reference to Fig. C5.6 shows that \( R_g \) could be assumed higher as the panel is riveted to a zee shaped stringer which has some torsional resistance and thus panel is not free to rotate at its boundaries.

Panel (C) is less critical because the flexural shear is acting opposite to the torsional shear stress, thus \( f_g \) total = 545 - 163 = 682 psi. The \( R_g = 682/2760 = .246 \).

\[ R_c + R_g = .906 + .246^2 = .956. \text{ Since the result is less than 1.0, panel will not buckle.} \]

Panel (B) carries a small shear flow, being zero at center of panel and increasing uniformly to 1.5 lb. per inch at the edges, and flowing in opposite directions from the centerline. Thus transverse shear will have negligible effect. Thus \( R_g = 854/2760 = .309 \).

\[ R_c + R_g = .906 + .309^2 = 1.00, \text{ or panel} \]

(8) is on the verge of buckling under the assumptions made in the solution.

PROBLEMS

(4) A sheet panel 5" x 12.5" x .051" has all edges simply supported. The panel is subjected to combined compression and shear loads which produce the following stresses:

\[ f_c = 2400 \text{ psi, applied normal to 5" side.} \]

\[ f_s = 2800 \text{ psi. Will the sheet buckle under the given load system if made of aluminum alloy 2024-T3 material. What is the margin of safety.} \]

(5) If the material in problem (4) is changed to alloy steel \( F_{tu} = 65000 \text{ psi, what would be the margin of safety. If sheet was heat treated to} \]

\( F_{tu} = 180,000, \text{ what would be the M.S.} \)

(6) A 3" x 12" x .040" sheet panel is subjected to the following combined stresses.

\[ f_c = 3000, f_s = 10000, f_g = 8000. \text{ The} f_c \text{ and} f_s \text{ stresses are normal to the 3" side. If sides are simply supported, will panel buckle if made of} \]

\( 7075-T6 \text{ aluminum alloy. What is M.S.} \)

What will be the M.S. if material is changed to Titanium T1-2Mn.

References:


(2) ANC-5 Amendment 2. Aug. 1946.

(3) NACA ARR-No. L6A05

(4) NACA ARR-No. 3K13

(5) ANC-5 Revision of 1942

General References on Theory

(6) NACA Tech. Note 3781.


(11) "Buckling of Metal Structures" by Bleigh. McGraw-Hill Co.
CHAPTER C6
LOCAL BUCKLING STRESS FOR COMPOSITE SHAPES

C6.1 Introduction.

Thin flat sheet is inefficient for carrying compressive loads because the buckling stresses are relatively low. However, this weakness or fault can be greatly improved by forming the flat sheet into composite shapes such as angles, channels, zees, etc. Most of the many composite shapes can also be made by the extruding process. Formed or extruded members are widely used in Flight Vehicle Structures, thus methods of calculating the compressive strength of such members is necessary.

C6.2 Compressive Buckling Stress for Equal Flanged Elements.

The simplest equal flanged member that can be formed is the angle shape. Other shapes with equal flanges are the T section and the cruciform section as shown in Fig. C6.1.

These sections can be considered as a group of long flanges, as illustrated, for the angle section in Fig. C6.2. Since the flanges which make up the section are equal in size, each flange will buckle at the same stress. Therefore each flange cannot restrain the other and thus it can be assumed that each flange is simply supported along the flange junction as illustrated in Fig. C6.2.

From Equation C5.1 of Chapter C5, the buckling compressive stress for a long flange is,

\[ \sigma_{cr} = \frac{\pi^2 K_c E}{12(1-b^4/b^4)} \left( \frac{L}{b} \right)^2 \]

From Fig. C5.2 of Chapter C5, \( K_c = 0.43 \), then

\[ \sigma_{cr} = \frac{0.43 \pi^2 E}{12(1-0.34)} \left( \frac{L}{b} \right)^2 = 0.386 \frac{E}{b} \left( \frac{L}{b} \right)^2 \]

If the buckling stresses are above the proportional limit stress, use Fig. C5.7 in Chapter C5 to take care of the plasticity effect.

For formed angles, the flange width \( b \) extends to centerline of adjacent leg, but for extruded angles, the width \( b \) extends to inside edge of the adjacent flange or leg.

C6.3 Compressive Buckling Stress for Simple Flange-Web Elements.

The most common flange-web structural shapes are channels, zees, and hat sections. A flange has one unloaded edge free, whereas a web has no free unloaded edge and thus has an unknown restraint on the boundary between the web and the flange. Fig. C6.3 shows the breakdown of a Z section into two flange and one web plate elements.

The buckling strength of the web and flange elements depends on the boundary restraint between the two elements. If this restraint which is unknown could be found in terms of a known rotational restraint \( \varepsilon \) as presented in Chapter C5, the buckling coefficients could be found from charts in Chapter C5. Having the
buckling stress for each element, the critical buckling stress will be the smaller of the two. The buckling load based on the buckling stress is not the failure load as more load can be taken by the material in the corner regions before local failure of crippling takes place. The subject of local crippling of formed and extruded shapes is covered in Chapter C7.

Using the moment distribution method or a step by step analysis procedure, several research studies have determined the restraint factors between web and flange elements for simple shapes like channels, Z, H, square tubes and formulated design charts for such shapes. (Refs. 1 to 5 inclusive.)

C6.4 Design Charts for Local Buckling Stresses of Some Composite Web-Flange Shapes.

Figs. C6.4 to C6.7 inclusive give charts for determining the local buckling stress of channel, Z, H, square tube and hat shaped sections. For formed sections, the width b extends to centerline of adjacent element and for extruded sections the width b extends to inside edge of adjacent element.

C6.5 Problems Illustrating Use of Charts.

**PROBLEM 1.**

The Z section in Fig. (a) is formed from aluminum alloy 2024-T3 sheet. What compressive stress will start local buckling of an element of the member.

Solution:

\[ b_w = 1.5 - 0.064 = 1.436 \]
\[ b_r = 0.75 - 0.032 = 0.718 \]
\[ b_r/b_w = 0.718/1.436 = 0.50 \]
\[ t_w/t_f = 0.064/0.064 = 1.0 \]

From Fig. C6.4, we read \( k_w = 2.9 \)

\[ \sigma_{cr} = \frac{k_w n^* E}{12(1-b_r/b_w)} (t_w)^* \]

\[ \sigma_{cr} = \frac{2.9 \times 10,700,000 \times 0.064}{12(1-0.5^2)} = 56100 \text{ psi} \]

This stress is above the proportional limit stress of the material, thus a plasticity correction must be made. The buckling occurs on the flange.

From Table Bl.1 of Chapter Bl, we obtain for 2024-T3 aluminum alloy: \( \sigma_{o.*} = 39000 \) and \( n = 11.5 \).

For the plasticity correction of shapes covered in Fig. C6.4 (Ref. 4), the plasticity correction for a flange free on one edge can be used with accuracy. Thus we can use chart in Fig. C6.7 of Chapter C5 to correct for plasticity effects.

The parameter for bottom scale of Fig. C5.7 is equation (A) divided by \( \sigma_{o.*} \), or \( 56100/39000 = 1.44 \). Using this value and the \( n = 11.5 \) curve, we read from Fig. C5.7 that \( \sigma_{cr}/\sigma_{o.*} = 1.02 \).

Therefore the local buckling stress is

\[ \sigma_{cr} = 39000 \times 1.02 = 39800 \text{ psi} \]

**PROBLEM 2.**

If the member in Problem 1 is subjected to a 300°F temperature for 2 hours duration, what would be the local buckling stress.

From Table Bl.1 for this temperature condition,

\[ \sigma_{o.*} = 35700, \ E_o = 10,300,000, \ n = 15 \]

\[ \frac{k_w n^* E}{12(1-b_r/b_w)} (t_w)^* = \frac{2.9 \times 10^4 \times 10,300,000}{12(1-0.5^2)} (35700) \]

\[ \frac{0.064}{1.436} = 1.51 \]

Using this value on bottom scale of Fig. C5.7 and \( n \) curve is 15, we read \( \sigma_{cr}/\sigma_{o.*} = 1.03 \). Thus \( \sigma_{cr} = 1.03 \times 35700 = 36800 \text{ psi} \).

**PROBLEM 3.**

Same as Problem 1, but change material to Titanium Ti-80M Sheet.

From Table Bl.1 we obtain for this material:

\[ E_o = 15,500,000, \ \sigma_{o.*} = 113500, \ n = 13.7 \]

\[ \sigma_{cr} = \frac{2.9 \times 10^4 \times 15,500,000 \times 0.064}{12(1-0.5^2)} = 81200 \text{ psi} \]

This stress is near the proportional limit stress so plasticity correction should be small if any.

\[ \sigma_{cr}/\sigma_{o.*} = 81200/113500 = 0.68 \]

From Fig. C5.7, using \( n = 13.7 \) curve, we read \( \sigma_{cr}/\sigma_{o.*} = 0.88 \). Thus \( \sigma_{cr} = 113500 \times 0.88 = 91300 \), thus no plasticity correction.

**PROBLEM 4.**

The rectangular tube has the dimensions as shown in Fig. (b). It is extruded from aluminum alloy 2014-T6. Determine the local compressive buckling stress.
Fig. C6.4 (Ref. 2) Channel- and Z-section stiffeners.

\[ \sigma_{cr} = \frac{k_w n^2 E}{12 (1 - V_e^2)} \frac{t_w}{b_w^2} \]

Fig. C6.5 (Ref. 2) H-section stiffeners.

\[ \sigma_{cr} = \frac{k_w n^2 E}{12 (1 - V_e^2)} \frac{t_w}{b_w^2} \]

Fig. C6.6 (Ref. 2) Rectangular-tube-section stiffeners.

\[ \sigma_{cr} = \frac{k_b n^2 E}{12 (1 - V_e^2)} \left( \frac{t_b}{b} \right)^2 \]

Fig. C6.7 (Ref. 5) Buckling stress for hat-section stiffeners.

\[ t = t_f = t_w = t_T; \quad \sigma_{cr} = \frac{k_w n^2 E}{12 (1 - V_e^2)} \frac{t}{b_w^2 \sqrt{t}} \]  
(Data of Ref. 12.)
C8.4 LOCAL BUCKLING STRESS FOR COMPOSITE SHAPES

Solution:
\[ b = 1 - .08 = .92 \]
\[ h = 2 - .08 = 1.92 \]
\[ b/h = .92/1.92 = .479 \]
\[ t_b/t_h = 1.0 \]

Fig. (b)

From Fig. Cl.6, we read \( k_h = 5.2 \).

\[ \sigma_{cr} = \frac{5.2 \pi^2 \times 10,700,000 \times .04}{12(1-.3^2)} \times .82 = 21900 \text{ psi}. \]

As shown in Fig. Cl.6, buckling occurs on the side of the tube. The computed buckling stress is below the proportional limit stress, thus no plasticity correction.

PROBLEM 5.

Same as Problem 4 but change the thickness of the side to .072, but leave the b side .04 in thickness.

Solution:
\[ b = 1 - .144 = .856 \]
\[ h = 2 - .08 = 1.92 \]
\[ b/h = .856/1.92 = .444 \]
\[ t_b/t_h = .04/.072 = .555 \]

From Fig. C8.6, \( k_h = 4.3 \).

\[ \sigma_{cr} = \frac{4.3 \pi^2 \times 10,700,000 \times .072}{12(1-.3^2)} \times .82 = 58600 \text{ psi}. \]

This stress is above the proportional limit stress thus a plasticity correction is necessary. (Ref. 4) gives no value for a plasticity correction but recommends the correction for a clamped long flange which is a slightly conservative correction. This plasticity correction should also be used for the hat shape as shown in Fig. C5.7.

Thus Fig. C5.8 of Chapter C5 can be used to correct for effect of plasticity.

From Table 31.1, for our material \( \sigma_{y} = 53000 \) and \( n = 18.5 \).

The value of the parameter for bottom scale of Fig. C5.8 is \( \sigma_{cr}/\sigma_{y} = 58600/53000 = 1.10 \).

From Fig. C5.8, using \( n = 18.5 \) curve, we read \( \sigma_{cr}/\sigma_{y} = .91 \). Therefore \( \sigma_{cr} = 53000 \times .91 = 48100 \).

Thus by changing the long side of tube from .04 to .072, the buckling stress was increased from 21900 to 48100 psi.

In the design of rectangular tubes, the designer should select the tube thicknesses for both long and short sides so that buckling occurs on both sides, thus giving the lightest section for buckling strength.

As pointed out previously in this chapter, the load on the member which causes local buckling is not the falling or maximum load for a short length of the member. This local falling or crippling stress is treated in the next chapter. Since the buckling stress may fall in the inelastic stress range, the buckles will not entirely disappear when the load is removed. Since limit or applied loads must be carried without permanent distortion, it is thus important to know when local buckling starts. For those missile and space vehicles that carry no humans, the factor of safety on limit loads is considerably less than for aircraft, thus the spread between local buckling and local falling strength becomes important in design.

C8.5 Buckling of Stiffened Flat Sheets Under Longitudinal Compression.

In supersonic aircraft, it is important that the skin also be designed so that the wing, not buckling under flight conditions since a buckled surface could affect the aerodynamic characteristics of the airflow around the wing, thus it is important to know how the skin or its stiffening units initially buckle in order to design so that such buckling will not occur under flight conditions.

Gallagher and Boughan (Ref. 6) and Boughan and Beaub (Ref. 7) determined the local buckling coefficients for idealized web, Z and T stiffened plates. The results of their studies are shown in Figures C8.6 to C8.12 and are discussed in Ref. 4). The initial local buckling stress for plate or stiffener is given by the equation:

\[ \sigma_{cr} = \frac{k_b \pi^2 \times 5}{12(1-\nu_b^2)} \times (b_b)^2 \tag{8} \]

If the buckling stress is above the proportional limit stress of the material, correct for plasticity effect by using Fig. C5.8 of Chapter C5.

Problem Illustrating Use of Charts.

Fig. C shows a plate with idealized Z section stiffeners. The material is 2024-T3

![Diagram of plate with idealized Z section stiffeners]
Web stiffeners. $0.5 < t_w/t_b < 2.0$

Fig. C6.8 (Ref. 7) Compressive-local-buckling coefficients for infinitely wide idealized stiffened flat plates.

$$\sigma_{cr} = \frac{k_g n^2 E}{12 (1 - \nu_e^2) b_y}$$

Fig. C6.9 (Ref. 6) Z-section stiffeners. $t_w/t_b = 0.50$ and 0.79.

Fig. C6.10 (Ref. 6) Z-section stiffeners. $t_w/t_b = 0.63$ and 1.0.

Fig. C6.11 (Ref. 7) T-section stiffeners. $b_y/b_w > 10; b_y/b_b > 0.25, t_w/t_f = 1.0$.

Fig. C6.12 (Ref. 7) T-section stiffeners. $t_w/t_f = 0.7, b_y/b_l > 10; b_w/b_b > 0.25$. 

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aluminum alloy. Determine the initial buckling stress under longitudinal compression.

\[ \frac{bf}{bw} = \frac{0.5}{1.5} = 0.333, \quad \frac{bw}{bs} = \frac{1.5}{4.0} = 0.375, \]

\[ \frac{tw}{bs} = \frac{0.025}{0.125} = 0.2 \]

Using the above three values and referring to Fig. C5.9, we read the buckling coefficient \( ks \) to be 4.2. Substituting in equation (3)

\[ \sigma_{cr} = \frac{1.2 \pi^2 \times 10,700.000 \times (0.25)^2}{12(1-0.3^2)} = 39000 \text{ psi.} \]

This stress is no doubt above the proportional limit stress so a check for plasticity effect will be made. For this effect we use Fig. C5.8 of Chapter C5.

For our 2024-T3 material, we find from Table B1.1 of Chapter B1, that \( \sigma_{cr}/\sigma_{pl} = 33600 \) and the shape parameter \( n = 11.5 \).

The bottom scale parameter on Fig. C5.8 is equation (B) divided by \( \sigma_{cr} \), thus it equals 39000/33600 = 1.17. Using this value and the \( n = 11.5 \) curve on Fig. C5.3, we read on left side scale that \( \sigma_{cr}/\sigma_{pl} = 0.36 \). Therefore the buckling stress \( \sigma_{cr} = 0.36 \times 39000 = 33600 \text{ psi.} \)

References


(7) Boughan and Bash: Charts for Calculating the Critical Compressive Stress for Local Instability of Idealized T Stiffened Panels. NACA WRL-204, 1944.
CHAPTER C7
CRIPPLING STRENGTH OF COMPOSITE SHAPES AND
SHEET-STIFFENER PANELS IN COMPRESSION

SHEET EFFECTIVE WIDTHS.

C.7.1 Introduction.

Chapter C6 was concerned with the local buckling stress of composite sections when loaded in compression. Tests of short lengths of sections composed of flange-plate elements often show that after the section has buckled locally, the unit still has the ability to carry a greater load before failure occurs. In other words, the local buckling and local failure loads are not the same. For cases where local buckling occurs at low stress, the crippling or failing stress will be higher. When local buckling occurs at high stress such as .7 to .8 $F_{Cy}$, buckling and crippling stress are practically the same. Fig. C7.1 illustrates the stress distribution on the cross-section after local buckling has occurred but prior to local crippling or failure.

Fig. C7.1

As the load on the section is increased, the buckles on the flat portions get larger but most of the increasing load is transferred to the much stiffer corner regions until the stress intensity reaches a high enough value to cause sufficient deformation to cause failure.

A theoretical solution for the local crippling stress for all types of shapes has not been developed as the boundary restraint between flange and plate elements is unknown and also the manner in which the stress builds up in the corner regions is not well understood. Consequently, the methods of solution are semi-empirical in character, and the results of such methods have been sufficiently proven by tests. Two methods of calculating crippling stresses will be presented in this chapter.

C.7.2 METHOD 1. THE ANGLE METHOD, or the Needham Method.

This method which will be referred to as the angle method or the Needham method was presented in (Ref. 1). In this method the member section is divided into equal or unequal angles as illustrated in Fig. C7.2. The strength of these angle elements can be established by theory or tests. The ultimate strength or failing strength can then be found by adding up the strengths of the angle elements that make up the composite section.

Needham made a large number of tests on angle and channel sections. From a study of these test results as well as other published test data on channels, square and rectangular tubes, etc., he arrived at the following equation for the crippling or failing stress of angle sections.

\[
\frac{F_{CS}}{F_{Cy}} = \frac{C_e}{S_e} \left( \frac{t}{b'} \right)^{0.76}
\]

(C7.1)

where,

- \(F_{CS}\) = crippling stress (psi)
- \(F_{Cy}\) = compression yield stress (psi)
- \(S_e\) = Young’s modulus of elasticity in compression (psi)
- \(b'/t\) = equivalent b/t of section = \(\frac{(a + b)}{ct}\)
- \(C_e\) = coefficient that depends on the degree of edge support along the edges of contiguous angle units. Specifically they are:
  - \(C_e = 0.316\) (two edges free)
  - \(C_e = 0.342\) (one edge free)
  - \(C_e = 0.366\) (no edge free)

The crippling stress for angles, channels, zees and rectangular tubes can be determined directly from use of equation C7.1. The crippling load on an angle unit is then,

\[
P_{CS} = F_{CS} A
\]

(C7.2)

where \(A\) is the area of the angle.

The crippling stress of other formed
structural shapes can be determined by dividing
the shape into a series of angle units and
computing the crippling loads for these
individual angle units by use of equations C7.1
and C7.2. The weighted crippling stress for
the entire section is obtained from the
following equation:

\[ F_{cs} = \frac{2 (\text{crippling loads of angles})}{2 (\text{area of angles})} \quad (C7.3) \]

C7.3 Design Curves.

Fig. C7.3 gives curves for determining the
crippling stress of angle units as per equation
C7.1, and Fig. C7.4 gives curves for determining the
crippling loads for angle units. Using
these curves and equation C7.3, the crippling
stress of composite shapes other than angles,
channels, zees and rectangular tubes can readily
be calculated.

Illustrative problems, using this method,
will be given later and the results compared
with method 2. Crippling stresses for
composite sections should be limited to the
values given in Table C7.1 unless substantiated
by test results.

C7.4 METHOD 2. For Crippling Stress Calculation.
(The Gerard Method).

Introduction: References 2 and 3 gives
the results of a very comprehensive study by
Gerard on the subject of crippling stresses.
From a thorough study of published theoretical
studies and most available test or experimental
results, Gerard has developed and presented a
more generalized or broader semi-empirical
method of determining crippling stresses. In
one sense it is generalization or broader
application of the Needham method which was
presented as method 1. The student and
practicing structures engineer should refer to
the above references for a complete discussion
of how the resulting crippling stress equations
were obtained and how these check the extensive
test results. In this short chapter we can
only present the resulting equations, design
curves for same and example problems in the use
of the information, in the determination of
crippling stresses.

C7.5 Stresses and Displacements of Flat Plates
After Buckling Under Conditions of Uniform
End Shortening.

Fig. C7.5 shows a picture of the resulting
stress distribution on flat plates after
buckling under conditions of uniform end
shortening as determined by Coan in (Ref. 4).
The Gerard method recognizes the effect of
distortion of the free unloaded edges upon the
failing strength of the member section.

\[ \frac{F_{cs}}{F_{cy}} = 0.56 \left( \frac{gt^3}{A} \right) \left( \frac{E}{F_{cy}} \right)^{1/2} \quad \text{--- (C7.4)} \]

For sections with straight unloaded edges
such as plates, tee, cruciform and H sections,
the following equation for crippling stress
applies within ± 5 percent limits.

\[ \frac{F_{cs}}{F_{cy}} = 0.67 \left( \frac{gt^3}{A} \right) \left( \frac{E}{F_{cy}} \right)^{1/2} \quad \text{--- (C7.5)} \]

For 2 corner sections, Z, J, and channel
sections, the following equation applies within
± 10 percent limits.

\[ \frac{F_{cs}}{F_{cy}} = 3.2 \left( \frac{t}{A} \right) \left( \frac{E}{F_{cy}} \right)^{1/2} \quad \text{--- (C7.6)} \]

Where:
- \( F_{cs} \) = crippling stress for section (psf)
- \( F_{cy} \) = compressive yield stress (psf)
- \( t \) = element thickness (inches)
- \( A \) = section area (in.²)
- \( E \) = Young's modulus of elasticity
- \( g \) = number of flanges which compose the
  composite section, plus the number of
  cuts necessary to divide the
  section into a series of flanges.

The cut-off or maximum crippling stress \( F_{cs} \)
for a composite section should be limited to the
(Note: $P_{cc}$ same as $P_{cb}$)

Fig. C7.4 Dimensionless Crippling Load vs. $b'/t$ (Ref. 1)

(Not: $F_{cc}$ same as $F_{cb}$)

Fig. C7.3 Dimensionless Crippling Stress vs. $b'/t$ (Ref. 1)
C7.4 CRIPPLING STRENGTH OF COMPOSITE SHAPES AND SHEET-STIFFENER PANELS IN COMPRESSION

C7.7 Correction for Cladding.

Since many formed sections are made from clad sheets, the clad covering acts to reduce the value of buckling stress and thus a correction factor \( \overline{\eta} \) must be used to take care of this reduction in strength. This correction from (Ref. 3) is,

\[
\overline{\eta} = \left[ 1 + 3 \left( \frac{\sigma_{\text{cl}}}{\sigma_{\text{cr}}} \right) f \right] / (1 + 3f)
\]

where,

\( \sigma_{\text{cl}} \) = cladding yield stress
\( \sigma_{\text{cr}} \) = buckling stress
\( f \) = ratio of total cladding thickness to total thickness. \( f = 0.10 \) for clad 2024-T3 and .08 for clad 7075-T6.

C7.8 Maximum Values for Crippling Stresses.

The cut-off or maximum crippling stress for a composite section should be limited to the values in Table C7.1 unless test results are obtained to substantiate the use of higher crippling stresses.

<table>
<thead>
<tr>
<th>Type of Section</th>
<th>Max. Fcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angles</td>
<td>.7 ( F_{\text{CY}} )</td>
</tr>
<tr>
<td>V Groove Plates</td>
<td>( F_{\text{CY}} )</td>
</tr>
<tr>
<td>Multi-Corner Sections, Including Tubes</td>
<td>.8 ( F_{\text{CY}} )</td>
</tr>
<tr>
<td>Stiffened Panels</td>
<td>( F_{\text{CY}} )</td>
</tr>
<tr>
<td>Tee, Cruciform and H Sections</td>
<td>.8 ( F_{\text{CY}} )</td>
</tr>
<tr>
<td>2 Corner Sections, Zee, J, Channels</td>
<td>.9 ( F_{\text{CY}} )</td>
</tr>
</tbody>
</table>

Fig. C7.6 Method of cutting simple elements to determine \( g \).

values in Table C7.1 unless higher values can be substantiated by test results. The cut-off values given in Table C7.1 are no doubt slightly conservative. Design curves for equations C7.4, 5 and 6 are given in Figs. C7.7, C7.8 and C7.9.
Fig. C7.8
Curve for Plates, Tees, Cruci Form and H Sections.

\[ F_{cs} = 0.87 \left( \frac{A}{t^2} \frac{F_{cy}}{A} \right)^{1/3} \]

See Table C7.1 for Cut-off Values

---

Fig. C7.9
Crippling Stress \( F_{cs} \) for 2 Corner Sections, Z, J and Channel Sections.

\[ F_{cs} = 3.2 \left[ \frac{A}{t^2} \frac{F_{cy}}{A} \right]^{1/3} \]

See Table C7.1 for Cut-off Values
C7.9 Restraint Produced by Lips and Bulbs.

Quite often in formed sections, the flange element which has a free edge is rather small in width as illustrated in Fig. a. Also for extruded sections, a bulb is often used as illustrated in Figs. b. The question then arises, is the lip or bulb sufficiently large enough to provide a simple support to the adjacent plate element. Since the compressive buckling coefficient for a plate element is 4.0 and 0.43 for a flange element, the use of a small lip or bulb can increase the value of the coefficient considerably above 0.43 and thus produce a more efficient load carrying element. The problem of determining the dimensions of a lip or bulb to give at least a simply supported edge condition to the adjacent plate element has been investigated theoretically by Windenburg (Ref. 5). The results of his studies gives the following design criterion.

\[ 2.73 \frac{b_l}{t} - \frac{A_l}{b_l t} = 5 \quad \text{(C7.8)} \]

Where \( I_L \) and \( A_L \) are the moment of inertia and area of the lip or bulb respectively. (See Fig. C7.10).

\[ (a) \quad (b) \]

Fig. C7.10

From Fig. C7.10a for the lip, \( A_L = b_L t \), and \( I_L = t b_L^3 / 3 \).

In substituting these values in equation (C7.8), the dimensions of the lip are expressed as

\[ 0.910 \frac{b_l}{t} = \frac{b_L t}{b_l t} = 5 \frac{b}{t} \quad \text{(C7.9)} \]

To determine \( b_l \) and \( t \), an additional requirement is specified, namely, that the buckling stress of the lip must be greater or equal to the buckling stress of the adjacent plate element.

From Chapter C6, the compressive buckling coefficient using \( b_0 = 0.388 \) is 0.388 for a flange element and 3.617 for a plate element. Therefore,

\[ 0.388 \frac{E_0 (b_L)_{cyc}}{b_L t} = 3.617 \frac{E_0 (b_P)_{cyc}}{b_P t} \quad \text{(C7.10)} \]

From equations C7.9 and C7.10, the following relationship is obtained,

\[ \frac{b_L}{t} \geq 0.328 \frac{b}{t} \quad \text{(C7.11)} \]

Fig. C7.11 shows the results as a curve.

\[ \text{Fig. C7.11 Minimum lip dimensions required for flange to buckle as simply supported plate (Ref. 5).} \]

In extruded sections, a circular bulb is often used to stiffen a free edge as illustrated in Fig. C7.10b. The moment of inertia of the bulb area about the centerline of the plate element is,

\[ I = \frac{m b^4}{64} + \frac{m D^4}{4} \left( \frac{D-L}{2} \right) \]

As for the case of the lip, the buckling stress of the bulb must be greater or equal to the buckling stress of the adjacent plate element, which gives,

\[ \frac{b}{t} \geq 1.6 \left( \frac{D}{t} \right)^4 - 0.374 \left( \frac{D}{t} \right)^2 \quad \text{(C7.12)} \]

Fig. C7.12 shows design curve representing the above equation.

\[ \text{Fig. C7.12 Minimum bulb dimensions required for buckle as simply supported plate (Ref. 5).} \]
C7.10 Illustrative Problems in Calculating Crippling Stresses.

Problem 1. Find the crippling stress for the equal leg angle shown in Fig. a. The material is aluminum alloy 2024-T3.

Solution Method 1.

Material properties are:

\[ F_{Cv} = 40,000, \quad E_c = 10,700,000 \]

For this method, we use Fig. C7.3. The parameter for bottom scale of Fig. C7.3 is \((a + b)/2t\), where \((a)\) and \((b)\) are leg lengths measured to centerline of adjacent leg of angle. For our case \(a = b = 1 - 0.025 = 0.975\). Thus \((a + b)/2t = 1.95/0.1 = 19.5\).

From Fig. C7.3 using 19.5 on lower scale, and the curve for two edges free, we read on the left hand scale that \(F_{Cs}/F_{Cv} = 0.0330\). Since we have only one angle, the crippling stress \(F_{Cs} = F_{Cc} = 0.0330 \times 40,000 \times 10,700,000 = 2160\) psi.

Solution by Method 3 (Gerard Method)

For angle sections we use equation C7.4.

\[ \frac{A}{E t^2} \left( \frac{F_{Cv}}{E} \right)^{1/3} \]

where \(A\) = section area and \(g\) equals the number of flanges plus cuts, or \(g = 2\) for an angle section. Substituting,

\[ \frac{0.025}{2 \times 0.05^2} \left( \frac{40,000}{10,700,000} \right)^{1/3} = 1.138 \]

Using this value in Fig. C7.7, we read \(F_{Cs}/F_{Cv} = 0.50\). Therefore \(F_{Cs} = 0.50 \times 40,000 = 20,000\) psi.

Problem 2. Same as Problem 1, but change material to aluminum alloy 7075-T6.

\[ F_{Cy} = 67,000, \quad E_c = 10,500,000 \]

Solution Method 1.

\[ F_{Cs} = F_{Cc} = 0.0330 \times 67,000 \times 10,500,000 = 28,000\] psi

Solution by Method 2.

\[ \frac{0.025}{2 \times 0.05^2} \left( \frac{67,000}{10,500,000} \right)^{1/3} = 1.485 \]

Solution Method 1.

Problem 3. Same as Problem 1 but change material to Titanium T1-8Bi. \(F_{Cy} = 110,000, \quad E_c = 15,500,000.\)

Solution Method 1.

\[ F_{Cs} = F_{Cc} = 0.0330 \times 110,000 \times 15,500,000 = 43,000\] psi

Solution Method 2.

\[ \frac{0.025}{2 \times 0.05^2} \left( \frac{110,000}{15,500,000} \right)^{1/3} = 1.565 \]

From Fig. C7.7, \(F_{Cs}/F_{Cy} = 0.38\), whence \(F_{Cs} = 0.38 \times 110,000 = 41,300\).

Problem 4.

Find the crippling stress for the channel section shown in Fig. b if the material is aluminum alloy 2024-T3. \(F_{Cy} = 40,000, \quad E_c = 10,700,000.\)

Solution Method 1.

As shown in Fig. c, the channel is composed of 2 equal angle units (1) and (2). Since they are the same size, we need only calculate the failing stress for one angle.

\[ \frac{a + b}{2t} = \frac{725 + 725}{15} = 4.5 \]

From Fig. C7.3 for \(b'/t = 14.5\), we read \(F_{Cc}/F_{Cy} = 0.045\) (for one edge free).

Then \(F_{Cs} = F_{Cc} = 0.045 \times 40,000 \times 10,700,000 = 28,400\) psi

Solution Method 2.

For a channel section we use equation C7.6 which is plotted on Fig. C7.9. The parameter for bottom scale of Fig. C7.9 is,

\[ \frac{A}{tB} \left( \frac{F_{Cv}}{E} \right)^{1/3} = \frac{137}{0.05^2} \left( \frac{10,700,000}{40,000} \right)^{1/3} = 8.50 \]

From Fig. C7.9 \(F_{Cs}/F_{Cy} = 0.65\), whence,

\(F_{Cs} = 0.65 \times 40,000 = 26,000\) psi.

Problem 5. Same as Problem 4 but change material to aluminum alloy 7075-T6. \(F_{Cy} = 67,000, \quad E_c = 10,500,000.\)

Solution by Method 1.

\[ F_{Cs} = 0.045 \times \sqrt{67,000 \times 10,500,000} = 38,200\] psi

Solution by Method 2.

\[ \frac{137}{0.05^2} \left( \frac{67,000}{10,500,000} \right)^{1/3} = 10.17 \]
From Fig. C7.9, \( \frac{F_{os}}{F_{cy}} = .57 \), whence \( F_{os} = .57 \times 67,000 = 38,200 \) psi.

Problem 6.

Find the crippling stress for the square tube as shown in Fig. d. Material is 2024-T3 aluminum alloy. \( F_{cy} = 40,000 \), \( E_c = 10,700,000 \).

Solution by Method 1.

The square tube is considered as made up of 4 equal angles with no edge free.

\[ b'/t = \frac{(a+b)}{2t} = \frac{1.95}{0.1} = 19.5 \]

From Fig. C7.3, using upper curve, we obtain \( F_{cc}/\sqrt{F_{cy}} = .0392 \). Whence,

\[ F_{os} = F_{cc} \times \sqrt{40,000 \times 10,700,000} = 25,500 \text{ psi} \]

Solution by Method 2.

Area \( A = .373 \). \( g = \) number of cuts plus flanges or \( 4 + 3 = 12 \).

For rectangular tubes we use equation C7.4 or Fig. C7.7.

\[ A \times \frac{F_{cy}^{1/3}}{E_c^{1/3}} = (\frac{.373}{12 \times .05^2}) \times (\frac{40,000}{10,700,000})^{1/3} = 0.758 \]

From Fig. C7.7, \( F_{os}/F_{cy} = .70 \)

Therefore \( F_{os} = 40,000 \times .70 = 28,000 \) psi.

Problem 7. Same as Problem 6, but change material to magnesium HK31A-O Sheet, subjected to a temperature of 300°F for 1/2 hour.

Solution by Method 1.

From Table B1.1 of Chapter B1, \( F_{cy} = 11,100 \), \( E_c = 6,160,000 \).

\( F_{os} = .0392 \times 11,100 \times 6,160,000 = 10,250 \) psi

Solution by Method 2.

\[ A \times \frac{F_{cy}^{1/3}}{E_c^{1/3}} = (\frac{.373}{12 \times .05^2}) \times \frac{11,100}{6,160,000}^{1/3} = .528 \]

From Fig. C7.7, \( F_{os}/F_{cy} = .96 \)

\( F_{os} = .96 \times 11,100 = 10,600 \) psi

From Table C7.1, the cut-off or maximum crippling for rectangular tubes is .8 \( F_{cy} \) or .8 \( \times 11,100 = 8900 \) psi. Thus unless tests substantiate higher values, the crippling stress should be taken as 8900 psi.

Problem 8. Same as Problem 6, but change material to stainless steel 17-7PH (TH1050), \( F_{cy} = 162,000 \), \( E_c = 29,000,000 \)

Method 1.

\( F_{os} = .0392 \times \frac{162,000 \times 29,000,000}{12 \times .05^2} = 25,000 \) psi

Method 2.

\[ (\frac{.373}{12 \times .05^2}) \times \frac{162,000}{29,000,000}^{1/3} = .95 \]

From Fig. C7.7, \( F_{os}/F_{cy} = .595 \). \( F_{os} = 162,000 \times .595 = 95,200 \) psi.

Problem 9.

Determine the crippling stress for the formed section shown in Fig. e if material is aluminum alloy 2024-T3. \( F_{cy} = 40,000 \), \( E_c = 10,700,000 \).

Solution by Method 1 (Needham).

The section is divided into 6 angle units by the dashed lines in Fig. e. They are numbered (1) to (3) since we have symmetry.

The procedure will be to find the failing load for each angle and add up the total for the 6 angle units. The crippling stress will then equal this total load divided by the section area.

**Angle unit (1) (One edge free)**

\[ b'/t = \frac{(a+b)}{2t} = [\frac{(.375 - .02) + (.5 - .02)}{.05}] / .05 = 10.44 \]

From Fig. C7.3, \( F_{cc}/\sqrt{F_{cy}} = .09 \)

Whence, \( F_{cc} = .06 \times \sqrt{40,000 \times 10,700,000} = 39,300 \) psi.
Area of angle (1) = 0.0309 = A

\[ F_{cc} = 0.0309 \times 39,300 = 1215 \text{ lb.} \]

Angle unit (2) Area = 0.0459 (no edge free)

\[ b'/t = (a + b)/2t = (0.48 + 0.73)/0.08 = 15.1 \]

From Fig. C7.3 \[ F_{cc}/F_{cy} = 0.0475 \]

\[ F_{cc} = \sqrt{40,000 \times 10,700,000} \times 0.0475 = 31,100 \text{ psi} \]

\[ F_{cc} = 31,100 \times 0.0459 = 1428 \text{ lb.} \]

Angle unit (3) Area = 0.0609 (no edge free)

\[ b'/t = (0.73 + 0.605)/0.08 = 16.7 \]

From Fig. C7.3, \[ F_{cc}/F_{cy} = 0.046 \]

\[ F_{cc} = 0.046 \times \sqrt{40,000 \times 10,700,000} = 30,200 \text{ psi} \]

\[ F_{cc} = 0.0609 \times 30,200 = 1540 \text{ lb.} \]

\[ F_{os} = 2F_{cc}/\text{area} \]

\[ = (2 \times 1215 + 2 \times 1428 + 2 \times 1540)/0.255 = 32,800 \text{ psi} \]

Solution by Method 2 (Gerard)

\[ A = \frac{F_{cy}}{E_{c}} \]

\[ g = \text{number of flanges plus number of cuts} = 12 + 5 = 17. \]

Substituting in the above term,

\[ \left( \frac{0.255}{17} \times 0.04 \right)^{\frac{1}{17} \times 0.04} = 0.573 \]

From Fig. C7.7, \[ F_{os}/F_{cy} = 0.895, \text{ whence } F_{os} = 40,000 \times 0.895 = 35,600 \text{ psi.} \]

From Table C7.1, it is recommended for multi-corner sections that \( F_{os} \) maximum be limited to 0.8 \( F_{cy} \) unless tests can prove higher values.

\[ F_{os,\text{max}} = 0.8 \times 40,000 = 32,000 \text{ psi.} \]

Since this is less than the above calculated values, it should be used.

Problem 10. Find the crippling stress for the extruded bulb angle shown in Fig. f if material is 2014-75 extruded aluminum alloy.

This particular bulb section is taken from Table A3.16 of Chapter A3 as Section No. 2. The area from that table is 0.113 sq. in.

Solution by Method 2 (Gerard)

For this material \( F_{cy} = 53,000 \), \( E_{c} = 10,700,000 \).

The first question that arises is the bulb size sufficient to give an end stiffness to the (a) leg so that the bulb may be equivalent to the normal corner.

In Fig. f, \( b' = 0.78 \), hence \( b'/t = 15 \)

Referring to Fig. C7.12, we observe that for a \( b'/t \) value of 15 we need a \( D/t \) ratio of at least 3.8. The \( D/t \) value for our bulb angle is \( (7/32)/0.05 = 4.4 \), thus bulb has sufficient stiffness to develop a corner. The next question that arises should the bulb angle still be classed as an angle section for which equation C7.4 applies or be classed as a channel or 2 corner section with the bulb acting as a short thick leg of the channel. For this case, equation C7.6 would apply.

The crippling stress will be calculated by both equations.

By equation C7.4 or Fig. C7.7:

If bulb is considered as a full corner then \( g = 4 \) flanges plus 1 cut = 5.

\[ A = \frac{F_{cy}}{E_{c}} \]

\[ = \left( \frac{0.113}{5 \times 0.05} \right) \left( \frac{53,000}{10,700,000} \right)^{1/17} = 0.635 \]

From Fig. C7.7, \[ F_{os}/F_{cy} = 0.82, \text{ hence } F_{os} = 0.82 \times 53000 = 43600 \]

By equation C7.6 or Fig. C7.9,

\[ A = \frac{F_{cy}}{E_{c}} \]

\[ = \left( \frac{0.113}{7 \times 0.05} \right) \left( \frac{53,000}{10,700,000} \right)^{1/17} = 0.70 \]

From Fig. C7.9, \[ F_{os}/F_{cy} = 0.7, \text{ hence } F_{os} = 0.7 \times 53000 = 37100 \]

Possibly the best estimate of the crippling stress would be the average of the two above results or 40,300 psi.

In Table C7.1, the so-called cut-off stress for angles is 0.7 \( F_{cy} \) and channels 0.9 \( F_{cy} \). If we use the average value of 0.8 \( F_{cy} \), it gives \( F_{os} = 0.8 \times 53,000 = 42,400 \) as maximum permissible because of limited test results on bulb angles.

For the case where the bulb or lip does not develop the stiffness necessary to assume a full corner, then the bulb is only
considered as an additional flange and the g count would be four instead of 5, thus reducing the crippling stress.

**EFFECTIVE SHEET WIDTHS**

C7.11 Introduction.

The previous discussion in this chapter has dealt with the crippling stress of formed or extruded sections when acting alone, that is not fastened to any other structure along its length. However, the major structure of aerospace vehicles, such as the wing or body, involves a sheet covering which is strengthened by attached or integral fabricated stiffeners such as angles, tees, etc. Since the sheet and stiffeners must deform together, the sheet will therefore carry compressive load and to neglect this load carrying capacity of the sheet would be too conservative in aerospace structural design where weight saving is very important.

C7.12 Sheet Effective Widths.

Fig. C7.12a illustrates a continuous flat thin plate fastened to stiffener and the entire unit is subjected to a uniform compressive load. Up to the buckling strength of the sheet the compressive stress distribution is uniform over both stiffeners and sheet as in (Fig. b) assuming same material for sheet and stiffeners. As the load is increased the sheet buckles between the stiffeners and does not carry a greater stress than the buckling stress. However as the stiffeners are approached, the skin being stabilized by the stiffeners to which it is attached can take a higher stress and immediately over the stiffeners the sheet can take the same stress as the ultimate strength of the stiffener, assuming that the sheet has a continuous connection to the stiffener. Fig. c shows the general stress distribution after the sheet has buckled. This distribution depends on the degree of restraint provided by the stiffeners and the panel dimension.

Various theoretical studies (Ref. 6) have been made to determine this stress distribution after buckling. In general they lead to long and complicated equations. To provide a simple basis for design purposes, an attempt has been made to find an effective width of sheet w which would be considered as taking a uniform stress (Fig. d) which would give the same total sheet strength as the sheet under the true non-uniform stress distribution of Fig. c.

The question of sheet effective stress has been considered by many individuals. The names of VonKarman, Sechler, Timoshenko, Newall, Frankland, Margure, Fischel, Gerard, and many more are closely associated with the present knowledge on effective sheet width.

Fig. b Sheet stress distribution before buckling

Fig. c Equivalent sheet effective width

Fig. a Sheet-stiffener panel

Fig. C7.13

From Chapter C5, the buckling compressive stress of a sheet panel is,

\[ F_{cr} = \frac{k_{cr}E}{12(1-\nu_0^2)} \left(\frac{t}{b}\right)^2 \]  

(C7.13)

If we assume that the stiffener to which the sheet is attached provides a boundary restraint equal to a simple support, then \( k_c = 4.0 \), and if Poisson's ratio \( \nu_0 \) is taken as 0.3, then equation C7.13 reduces to,

\[ F_{cr} = 3.60E(t/b)^2 \]  

(C7.14)

The Von-Karman-Sechler method as first proposed consisted of solving equation C7.14 for a width \( w \) in place of \( b \), when \( F_{cr} \) was equal to the yield stress of the material since experiments had shown that the ultimate strength of a sheet simply supported at the edges was independent of the width of the sheet. Thus equation C7.14 changes to,

\[ F_{cy} = 3.60E(t/w)^2 \]  

whence,

\[ w = 1.90t \sqrt{F_{cy}} \]  

(C7.15)

Since the crippling or local failing stress of a stiffener can exceed the yield strength of the material, equation C7.15 was later changed by replacing \( F_{cy} \) by the stress in the stringer \( F_{ST} \), thus giving,
\[ w = 1.90 t \sqrt{E/F_{st}} \]  
\( \quad \quad \quad \quad (C7.16) \)

Some early experiments by Newell indicated the constant 1.90 was too high and for light stringers a value of 1.7 was more realistic, thus 1.7 has been widely used in the industry.

If we assume the stiffness of the stiffener and its attachment to the sheet as developing a fixed or clamped edge condition for the sheet, then

\[ F_{cr} = 6.35E(t/b)^2 \quad \text{or} \quad w = 2.52t \sqrt{E/F_{st}} \]

For general design purposes, it is felt that 1.9 or equation C7.15 is appropriate for determining the effective width \( w \). If stiffener is relatively light, use 1.7. Fig. C7.14 illustrates the effective width for sheet-stiffener units which are fastened together by a single attachment line for each flange of the stiffener.

The crippling stress is determined for the stiffener alone. This stress is then used in equation C7.15 to determine the effective width \( w \). The total area then equals the stiffener area plus the area of the effective sheet width \( w \). The radius of gyration should include the effect of the effective skin area.

Fig. C7.15 illustrates the case where stiffeners are fastened to sheet by two rows of rivets on each stiffener flange. In this case, the rivet lines are so close together that the effective width \( w \) for each rivet line would overlap considerably. A common practice in industry for such cases is to use the effective width for one rivet line attachment as per equation C7.16 to represent sheet width to go with each stiffener flange. However, in calculating the crippling stress of the stiffener alone, the stiffener flange which is attached to sheet is considered as having a thickness equal to \( 3/4 \) the sum of the flange thickness plus the sheet thickness.

The effective sheet width as calculated by equation C7.16 assumes that no inter-rivet buckling of the sheet occurs or, in other words, the rivet or spot welds are close enough together to prevent local buckling of the sheet between rivets when the sheet is carrying the stiffener crippling stress. The subject of inter-rivet sheet buckling is discussed later in this chapter.

Fig. C7.16 illustrates a procedure to follow for determining the effective width \( w \) when sheet and stiffener are integral in manufacture.

For Case 1, find the crippling stress for the tee section alone, assuming the vertical stem of the tee has both ends simply supported. For value of \( t \) in equation C7.15, use \( (t_s + t_f)/2 \). The effective stiffener area equals the area of the tee plus the area of the sheet of width \( w \).

For Case 2, determine the crippling stress for the I section acting alone. Calculate \( w/2 \) from equation C7.15 to include as effective sheet area. The column properties should include I section plus effective sheet.

C7.12 Effective Width \( w_e \) for Sheet with One Edge Free.

In normal sheet-stiffener construction, the sheet usually ends on a stiffener and thus we have a free edge condition for the sheet as illustrated in Fig. C7.16a. The sheet ends at a distance \( b' \) from the rivet line. For a sheet free on one edge, the buckling coefficient in equation C7.13 is 0.43, thus equation C7.13 reduces to,

\[ F_{cr} = .397E(t/b')^2, \quad \text{and replacing } b' \text{ by } w, \quad \text{we obtain,} \]

\[ w_e = .62t \sqrt{E/F_{st}} \]  
\( \quad \quad \quad \quad (C7.17) \)
Then the total effective sheet width for this end stiffener would thus equal \( w_1 + w/2 \).

**C.7.13 Effective Width When Sheet and Stiffener Have Different Material Properties.**

In practical sheet-stiffener construction it is common to use extruded stiffeners which have different material strength properties in the inelastic stress range as compared to the sheet to which the stiffener is attached. For example, in Fig. C.7.17 the stiffener material could have the stress-strain curve represented by curve (1) and the sheet to which it is attached by the curve (2). Now when the stiffener is stressed to point (B), the sheet directly adjacent to the stiffener attachment line must undergo the same strain as the stiffener and thus the stress in the sheet will be that given at point (A) in Fig. C.7.17. This difference in stress will influence the effective width \( w \). Correction for this condition can be made in equation C.7.17 by multiplying it by \( F_{sh}/F_{st} \), which gives

\[
w = 1.90t(F_{sh}/F_{st}) \left( \frac{E}{P_{st}} \right) \quad \text{(C.7.18)}
\]

Where \( F_{sh} \) is the stiffener stress and \( F_{st} \) is the sheet stress existing at the same strain as existing for the stiffener and obtained from a stress-strain curve of the sheet material.

**Fig. C.7.17**

For a rather complete study and comparison of the various effective widths theories as compared, see article by Gerard (Ref. 7). Equation C.7.18 is in general conservative for higher \( b/t \) ratios.

**C.7.14 Inter-Rivet Buckling Stress.**

The effective sheet area is considered to act monolithically with the stiffener. However, if the rivets or spot welds that fasten the sheet to the stiffener are spaced too far apart, the sheet will buckle between the rivets before the crippling stress of the stiffener is reached, which means the sheet is less effective in helping the stiffener carry a subjected compressive load. Thus, in general, to save structural weight, structural designers select rivet spacings that will prevent inter-rivet buckling of the sheet. In general, the rivet spacing along the stiffeners in the upper surface of the wing will be closer together than on the bottom surface of the wing since the design compressive loads on the top surface are considerably larger than those on the bottom surface.

The following method is widely used by engineers concerned with aerospace structures relative to calculating inter-rivet buckling stresses. It is assumed that the sheet between adjacent rivets acts as a column with fixed ends.

The general column equation from Chapter C2 for stable cross-sections is,

\[
F_c = \frac{Cn^2E_t}{(L/p)^2} \quad \text{--- (C.7.19)}
\]

Where \( C \) is the end fixity coefficient and varies from a value of 1 for a pin end support to 4 for a fixed end support.

The effective column length \( L' = L/\sqrt{C} \), thus equation C.7.19 can be written

\[
F_c = \frac{\pi^2E_t}{(L'/p)^2} \quad \text{--- (C.7.20)}
\]

Let \( p \) the rivet spacing be considered the column length \( L \). Assume a unit of sheet 1 inch wide and \( t \) its thickness. Then moment of inertia of cross section \( = 1 \times t^2/12 \), and area \( A = 1 \times t \). Then radius of gyration \( \rho = 0.29t \). Then substituting in equation C.7.20 to obtain the inter-rivet buckling stress \( F_{ir} \),

\[
F_{ir} = \frac{\pi^2E_t}{(\sqrt{C}/0.29t)^2} \quad \text{--- (C.7.21)}
\]

For clamped ends \( C = 4 \), thus

\[
F_{ir} = \frac{\pi^2E_t}{(p/0.58t)^2} \quad \text{--- (C.7.22)}
\]

To plot this equation, the tangent modulus \( E_t \) for the material must be known. However, we can use the various column curves in Chapter C2 which show a plot of \( F_c \) versus \( L'/p \) and in equation C.7.22 the term \( p/0.58t \) corresponds to \( L'/p \).

The fixity coefficient \( C = 4 \) can be used for flat head rivets. For spot welds it should be decreased to 3.5. For the Brazier rivet type use \( C = 3 \) and for counter-sunk or dimpled rivets use \( C = 1 \).
Figs. C7.18 and C7.19 show a plot of equation C7.22 for aluminum alloy materials.

If the inter-rivet buckling stress calculates to be more than the crippling stress of the stiffener, then the effective sheet area can be added to the stiffener area to obtain the total effective area. This total effective area times the stiffener crippling stress will give the crippling load for the total sheet-stiffener unit.

When the sheet between rivets buckles before the crippling stress of the stiffener is reached, the sheet in the buckled state has the ability to approximately hold this stress as the stiffener continues to take load until it reaches the stiffener crippling stress. This buckling sheet strength can be taken advantage of by reducing the effective sheet area. Thus effective sheet width equals,

\[ W_{\text{corrected}} = W_{\text{fr}} / F_{\text{of}} \quad \ldots \quad \text{(C7.23)} \]

The area of the corrected effective sheet is then added to the area of the stiffener. The crippling load then equals the crippling stress of the stiffener times the total area.

The use of sheet effective widths in finding the moment of inertia of a wing or fuselage cross-section is a widely used procedure in the analysis for bending stresses in conventional wing and fuselage construction. Reference should be made to article A19.13 of Chapter A19 and article A20.3 of Chapter A20 for practical illustrations in the use of effective widths.

C7.15 Illustrative Problem Involving
Effective Sheet.

Conventional airplane wing construction is illustrated in Fig. C7.20. The wing is covered with sheet, generally referred to as skin, and this skin is stiffened by attaching formed or extruded shapes referred to as skin stiffeners or skin stringers. A typical wing section involves one or several interior straight webs and to tie these webs to the skin, a stringer often referred to as the web flange member, is required to facilitate this connection. Fig. C7.21 is a detail of the flange member and the connection at point (1) in Fig. C7.20.

The stiffener or flange member is an extrusion of 7075-T6 aluminum alloy. The skin and web sheets are 7075-T6 aluminum alloy. The skin is fastened to stiffener by two rows of 1/3 inch diameter rivets of the Brazier head type, spaced 7/8 inch apart. The web is attached to stiffener by one row of 3/16 diameter rivets spaced 1 inch apart. The problem is to determine the crippling stress of the stiffener, the effective skin area and the total compressive load that the unit can carry at the failure point. Since the stiffener is braced laterally by the web and the skin, column bending action is prevented and thus the crippling strength is the true resulting strength of this corner member under longitudinal compression. (Additional stresses are produced on these corner members if web buckles under shear stresses and web diagonal tension forces are acting. This subject is treated in the chapter on semi-tension field beams.)

Solution:

Area of stiffener = 0.34 sq. in.
For 7075-T6 extrusion, \( F_{\text{cy}} = 70,000 \), \( E_{\text{c}} = 10,000,000 \).

The Gerard method will be used in calculating the crippling stress. Equation C7.4, or the design curve of Fig. C7.7, applies to this multi-corner shape. The lower scale parameter for Fig. C7.7 is,

\[ \frac{A}{g^{2.5}} \left( \frac{F_{\text{cy}}}{E_{\text{c}}} \right)^{1/3} = \frac{0.24}{6 \times 0.72} \left( \frac{70,000}{10,000,000} \right)^{1/3} = 0.03 \]

Using this value, we read from Fig. C7.7 that \( F_{\text{og}} / F_{\text{cy}} = 0.82 \). Thus \( F_{\text{og}} = 0.82 \times 70,000 = 57,400 \) psi. The \( g \) value of 0.8 was determined as shown in Fig. (a).

Effective Sheet Widths:

\[ \text{Equation C7.16 will be used to determine the sheet effective widths.} \]

For the skin \( t = .05 \).
Material is 7075-T6 aluminum alloy. \( E_{\text{c}} = 10,000,000 \).

\[ W = 1.9t \sqrt{2/F_{\text{og}}} = 1.9 \times 0.5 \sqrt{10,000,000/57,400} = 1.28 \text{ in.} \]

Thus a piece of sheet 1.28/2 = .64 wide acts to each side of the rivet centerline. Observation
of the dimensions in Fig. C7.21 shows the effective width with each skin rivet line would overlap slightly thus we use only the width between rivet line (see Fig. b).

For the web \( t = 0.064 \). Since the web has a free edge, the effective width calculation will be in two steps.

\[
W = \frac{1.6}{2} \times 0.064 \sqrt{10,500,000/57,400} = 0.32
\]

From equation C7.18, \( \sqrt{E/\rho} = 0.62 \times \frac{0.064 \sqrt{10,500,000/57,400}}{0.32} = 0.62 \times \frac{0.064 \sqrt{10,500,000/57,400}}{0.32} = 0.65 \) inch.

Fig. (b) shows the effective sheet width as calculated. Total effective sheet area = 1.28 x 0.5 = 0.78 x 0.5 + (0.55 + 0.52) x 0.064 = 0.86. Total area = 0.135 + 0.24 = 0.435.

The total compressive load that entire unit can carry before failure is then equal to \( A_{F_t} = 0.435 \times 57,400 = 25,000 \) lbs.

This result assumes that no inter-rivet buckling occurs under the stress of 57,400 psi in the sheet between rivets.

The skin rivets are Brazier head type spaced 7/8 inch apart or \( p = 7/8 \).

As discussed under inter-rivet buckling, the end coefficient \( c \) for this type of rivet should be less than 4 or assumed as 3.

Fig. C7.18 gives the inter-rivet buckling stress versus the p/t ratio. This chart is based on a clamped end condition or \( C = 4 \). Since \( C = 3 \) will be used for the Brazier type rivet, we correct the p/t ratio by the ratio \( \sqrt{4} / \sqrt{3} = 1.16 \).

The corrected p/t = 1.16 x 0.375/0.05 = 20.3

From Fig. C7.18 using curve (8) which is our material, we read \( F_{Ir} = 50,000 \) psi, thus skin will not buckle between rivets as crippling stress is 57,400 psi.

The web rivets are of the flat head type and \( C = 4 \) can be used. Spacing is 1 inch. Hence p/t = 1/0.064 = 15.6 and from Fig. C7.18, curve 3, we read \( F_{Ir} = 64,600 \), which is considerably more than the unit crippling stress.

In wing construction, the skin rivets are usually of the flush surface type, either countersunk or dimpled. If we make the skin rivets of the countersunk type, the end fixity coefficient must be reduced to 1 to be safe. Then the corrected p/t ratio to use with Fig. C7.19 would be \( \sqrt{h} / \sqrt{1} \ p/t = 2 \times 0.375/0.05 = 35 \). From Fig. C7.18 and curve 8, \( F_{Ir} = 23,000 \) which is far below the calculated crippling stress, thus the rivet spacing would have to be reduced. Use 9/16 inch spacing corrected p/t = 2 x 0.5625/0.08 = 22. From Fig. C7.18, \( F_{Ir} = 57,400 \) psi, which happens to be the crippling stress and therefore satisfactory.

C7.16 Failing Strength of Short Sheet-Stiffener Panels in Compression.

Gerard (Refs. 2, 3) from a comprehensive study of test results on short sheet-stiffener panels in compression, has shown that his equation C7.4, or Fig. C7.7, can be used to give the local monolithic crippling stress for sheet panels stiffened by Z, Y and H at shaped stiffeners. The method of calculating the value of the g factor is illustrated in Fig. C7.22. Fig. C7.23 is a photograph showing the crippling type of failure for a short panel involving the Z shape stiffener.

C7.17 Failure by Inter-Rivet Buckling.

Howland (Ref. 8) assumed that the sheet acts as a wide column which is clamped at its ends and whose length is equal to the rivet spacing. The inter-rivet buckling stress equation is then,

\[
F_{Ir} = \frac{c_{n} \eta \bar{F}}{12} \left(1 - \frac{t_{s}}{L_{e}}\right) \rho \quad (C7.24)
\]

The end fixity coefficient \( c \) is taken as 4 for flat head rivets and reduced for other types as previously explained for equation C7.21.

\( \eta \) is the plasticity correction factor

\( \bar{F} \) is the cladding correction factor

\( L_{e} \) is Poisson's ratio (use 0.30)

\( t_{s} \) = sheet thickness, inches.

\( p \) = rivet spacing or pitch in inches.

For non-clad materials the curves of Fig. C7.24 can be used. This figure is the same as Fig. C5.8 of Chapter C5. For the clad correction see Table C5.1 of Chapter C5.

C7.18 Failure of Short Panels by Sheet Wrinkling.

In a riveted sheet-stiffener panel, if the rivet spacing is relatively large, the sheet will buckle between rivets, such as illustrated in the photograph of Fig. C7.25. This inter-rivet buckling stress was discussed in the previous
Crippling Strength of Composite Shapes and Sheet-Stiffener Panels in Compression

(c) Hat-stiffened panel.

(b) Z-stiffened panel.

(a) Y-stiffened panel.

Fig. C7.22 Method of cutting stiffened panels to determine g.

Fig. C7.23 A 24S-T aluminum-alloy Y-stiffened panel (on the left) and its 75S-T counterpart after failure.
article. This sheet buckling does not deform the flange of the stiffener to which the sheet is attached. However, if the rivet or spot weld spacing is such as to prevent inter-rivet buckling of the sheet, then failure often occurs by a larger wrinkling of the sheet as illustrated in Fig. C7.26. The larger wrinkle shape subjects to flange of the stiffener to which the sheet is attached to lateral forces and thus the stiffener flange often deforms with the sheet wrinkle shape. This deforming of the stiffener flange produces stresses on the stiffener web, thus wrinkling failure is a combination of sheet and stiffener failure. The action of the wrinkling sheet to deform the stiffener flange places tension loads on the rivets, thus rivet design enters into the failure strength of sheet-stiffener panels under compression.

![Fig. C7.25](image-url)  
**Fig. C7.25**  
Inter-Rivet Buckling

![Fig. C7.26](image-url)  
**Fig. C7.26**  
Wrinkling Failure

Several persons have studied this wrinkling or forced crippling of riveted panels (Refs. 9 and 10). A rather recent study was carried out by Semonian and Peterson (Ref. 11), which is reviewed and simplified somewhat by Gerard in (Ref. 2). The results as given in (Refs. 11 and 2) used to calculate the wrinkling stress.

### C7.19 Equation for Wrinkling Failure Stress $F_w$.

From Ref. 2 we obtain,

$$F_w = \frac{k_w \pi \eta \eta \xi}{12} \left(1 - \nu_s^2\right) \left(b_s\right)^n$$  \hspace{1cm} (C7.25)

$k_w$, the wrinkling coefficient is obtained from Fig. C7.27. This coefficient is a function of the effective rivet offset $f$ which is obtained from Fig. C7.28. Having determined $k_w$ from Fig. C7.27, equation C7.25 can be solved by use of Fig. C7.24.

### C7.20 Rivet Criterion for Wrinkling Failure.

A criterion for the rivet pitch found from test data which results in a wrinkling mode failure is,

$$p/b_s = 1.27/k_w^{1/2}$$  \hspace{1cm} (C7.26)

The lateral force required to make the stringer attachment flange conform to the wrinkled sheet, loads the rivet in tension. An approximate criterion for rivet strength from Ref. 2 is,

$$S_r > \frac{0.7}{\xi_{st}} \frac{t_s}{\xi_c} \left(\frac{d_s}{b_s}\right)^{1/2}$$  \hspace{1cm} (C7.27)

The tensile strength of the rivet $S_r$ is defined in terms of the shank area and it may be associated with either shank failure or pulling of the countersunk head of the rivet through the sheet.

For aluminum alloy 2117-T4 rivets whose tensile strength is $s = 57$ ksi, the criteria are:

$$s = 57 \text{ ksi}, \quad d_s/t_{av} \geq 1.57$$
$$s = \frac{130}{d_s/t_{av}} - \frac{160}{(d_s/t_{av})^{3/4}}, \quad d_s/t_{av} < 1.57$$  \hspace{1cm} (C7.28)

where $t_{av}$ is the average of sheet and stiffener thickness in inches. The effective diameter $d_e$ is the diameter for a rivet made from 2117-T4 material.

The effective diameter of a rivet of another material is,

$$d_e/d = \left(S_r/S\right)^{1/4}$$  \hspace{1cm} (C7.29)

where $S_r$ is the tensile strength of a rivet defined as maximum tensile load divided by shank area in ksi units.

### C7.21 Problem 1. Illustrating Calculation of Short Panel Failing Strength.

Fig. C7.29 shows a sheet-stiffener panel composed of formed 2 stiffeners. The material is aluminum alloy 2024-T3. $F_{yw} = 40,000$. $F_{t_r} = 33,000$, $n = 16.5$, $E_s = 10,700,000$. The problem is to determine the compressive failure strength of a short length of this panel unit.

![Fig. C7.29](image-url)

**Fig. C7.29**

**General Panel Data:**

$t_w = 0.064$  
$b_w = 2.457$  
$b_A = 0.583$  
$t_s = 0.064$  
$b_r = 0.905$  
$b_0 = 0.243$  
$b_s = 2.00$
Fig. C7.24
Curves for finding $F_{IR}$ as per eq. C7.24 and $F_w$ as per eq. C7.25.

$$\frac{3\pi^2 E}{12 (1 - \nu_e^2)} F_{0, \gamma} \left( \frac{t_e}{D} \right)^3$$ or $$\frac{3\pi^2 E}{12 (1 - \nu_e^2)} F_{0, \gamma} \left( \frac{t_e}{D} \right)^3$$

Fig. C7.27 Experimentally determined coefficients for failure in wrinkling mode. (Ref. 2)

Fig. C7.28 (Ref. 2) Experimentally determined values of effective rivet offset.
whence, \( \beta_w / \tau_w = 38, \quad \beta_p / \tau_w = 0.372 \)

\( \beta_A / \tau_w = 9.27, \quad \beta_s / \tau_w = 5.36 \)

The rivets are 3/32 diameter Brazier head type AN456, 2117-T3 material spaced at 0.75 inches.

Area of 2 stiffener = 0.252 in.\(^2\)

Area of skin for \( \beta_S = 2 \) inches = 2 \( \times \) 0.064

Total area of sheet and stiffener = 0.380 in.\(^2\).

Crippling Stress of Stiffener Acting Along \( \tau_{cs} (ST) \).

Since we have a 2 type of stiffener, equation C7.6 and Fig. C7.9 applies. The lower scale parameter in Fig. C7.9 is,

\[
\left( \frac{A}{E} \right) \left( \frac{F_{cy}}{F_{cs}} \right)^{1/2} = \left( \frac{0.252}{0.380} \right) \left( \frac{10,700,000}{10,700,000} \right)^{1/2} = 9.36
\]

From Fig. C7.9, \( F_{cs} / F_{cy} = 0.6 \). Hence, \( F_{cs} = 0.6 \times 40,000 = 24,000 \) psi = \( F_{cs} (ST) \).

Crippling Stress of Panel Considered as a Monolithic Limit. \( F_{cs} (M) \).

Equation C7.4 or Fig. C7.7 applies for monolithic failure of sheet-stiffener panels.

The lower scale parameter on Fig. C7.7 is,

\[
\left( \frac{A}{E} \right) \left( \frac{F_{cy}}{F_{cs}} \right)^{1/2} = \left( \frac{0.252}{0.380} \right) \left( \frac{10,700,000}{10,700,000} \right)^{1/2} = 9.73
\]

\( z = 7.83 \) (see Fig. C7.22b).

From Fig. C7.7, we read \( F_{cs} / F_{cy} = 0.725 \), hence \( F_{cs} = 40,000 \times 0.725 = 28,900 \) psi = \( F_{cs} (M) \).

Inter-Rivet Buckling Stress (\( F_{ir} \)).

The rivet type is Brazier head and the spacing \( p \) is 3/4 inch.

Equation C7.24 applies and Fig. C7.24 is used to solve the equation. The lower scale parameter in Fig. C7.24 is,

\[
\frac{C \beta_s}{12 \left( \frac{1 - \nu_s^2}{E_s} \right) \nu_s} \left( \frac{E_s}{\rho} \right)^{1/2}
\]

For Brazier head rivet \( C = 3 \), \( \nu = 0.3 \).

Substituting:

\[
\frac{3 \beta_s}{12 \left( \frac{1 - 0.3^2}{(330,000)} \right) 0.964} = 5.45
\]

The shape parameter \( n \) for our material is 11.5. Reference to Fig. C7.24 for a value of 5.45 on bottom scale which is off the scale, we estimate the \( F_{ir} / F_{a} \) as above 1.1. Thus \( F_{ir} = 39,000 \times 1.1 = 43,000 \). This value is far about the stiffener or panel crippling stress as previously calculated so inter-rivet buckling is not at all critical.

Failure by Shear or Face Wrinkling. \( (F_w) \).

The wrinkling failure stress by equation C7.25 is,

\[
F_w = \frac{K_w \beta_s \eta E}{12 (1 - \nu_s^2)} \left( \frac{E_s}{\rho} \right)^{1/2}
\]

To determine value of \( K_w \), we use curves in Figs. C7.28 and C7.27.

\[
p/d = 0.075/0.064 = 8, \quad \beta_S / \tau_S = 5.36
\]

From Fig. C7.28, we read \( f / \tau_S = 6.5 \), whence \( f = 0.064 \times 6.5 = 0.416 \)

\[
f / \tau_W = 0.416 \times 0.252 = 0.17
\]

\( \beta_w / \tau_W = 5.36 \times 0.064 = 36 \) (2/0.064 = 1.21

From Fig. C7.27, we read \( K_w = 4.4 \).

To solve equation for \( F_w \) we use Fig. C7.24. The lower scale parameter is,

\[
K_w \beta_s \eta \left( \frac{E_s}{\rho} \right)^{1/2} = \frac{4.4 \beta_s \eta \times 10,700,000,000}{12 (1 - 0.3^2) (39,000) (0.964)} = 1.15
\]

For \( n = 11.5 \), we read from Fig. C7.24 that \( F_w / F_{a} = 0.9 \), whence \( F_w = 0.9 \times 39,000 = 35,100 \) psi. Thus wrinkling failure is not critical as \( F_w \) is larger than \( F_{cs} (M) \) and \( F_{cs} (ST) \).

The results show that the crippling stress for the stiffener alone of 24,000 psi is the smallest value, or the stiffener is unstable as it fails first. The entire panel unit will not reach its failing strength when stiffener stress is 24,000 because the skin wrinkling stress \( f_{w} \) is higher. An approximation suggested in (Ref. 2) is to assume stiffeners carry the same stress as the skin up to \( F_{cs} (ST) \) and beyond this the stiffener carries no additional load. Thus the panel failing stress \( F(p) \) can be calculated from the following equation.

\[
F_p = \frac{F_{w} \beta_s \eta \left( \frac{E_s}{\rho} \right)^{1/2}}{12 (1 - 0.3^2) (330,000) 0.964} = 27,800 \text{ psi}
\]

The total load carried by each stiffener plus its sheet is 27,800 x 3.90 = 106,000 lbs.

The failing strength of the riveted panel cannot exceed the monolithic panel failing stress \( F_{cs} (M) \), which was 29,000 psi for our panel or
greater than the calculated failing stress of 27,800 psi.

Check of Rivet Strength.

From expression C7.26

\[ \frac{p}{b_s} < 1.27/k_w^{1/2} \]
\[ \frac{0.75}{2.5} = 0.75 < 1.27/(4.4)^{1/2} = 0.503 (satisfactory) \]

The criterion for required rivet strength to make the stiffener flange follow the wrinkled sheet is from C7.27

\[ S_T > \frac{0.7}{E S T \cdot \frac{b_s}{d} \cdot \frac{L}{d} \cdot (F_w)^2} \]
\[ S_T = \left( \frac{0.7}{10,700,000} \times \left( \frac{2}{3} \times \frac{0.75}{3} \right) \times 35,100 \right)^{1/2} \]
\[ S_T = 13.7 \text{ psi.} \]

From expression C7.28

\[ S = 57 \text{ ksi for } d_k/t_{AV} = 1.67 \]

For given panel \[ d_k/t_{AV} = 3/32/0.064 = 1.46 \]
thus rivets have plenty of tensile strength to produce a wrinkling failure.

C7.22 General Design Limitations to Prevent Secondary Failure in Sheet-Stiffener Panels.

Sheet stiffener units can be designed as columns if the secondary failure such as inter-rivet buckling and face wrinkling are avoided. The following design rules referring to Fig. (A) will usually avoid these secondary weaknesses.

1. \[ \frac{t_f}{t_s} > 0.5 \] — Promotes overall crippling
2. \[ 0.4 > b_x/b_w = 0.5 \] — Rolling versus local buckling
3. Make \[ b_0 \] as small as possible — Face wrinkling
4. \[ p/b_s = 0.5 \] — — Inter-rivet buckling

(5) \[ p/D = 3 \] — — — — Face wrinkling
(6) Tensile strength of rivet or spot weld attachment per inch should be \( \geq 0.05 F_{cy} t_s \) in order to prevent failure in wrinkling.
(7) As a rough guide do not use bent up stringers if \( b_s/t_s = 30 \) in order to prevent face wrinkling weakness.

C7.23 Y Stiffened Sheet Panels.

A Y shape cannot be formed from sheet, thus it must be extruded. To make the Y shape efficient, the various parts usually have a different thickness. Furthermore, the extruded material has different mechanical properties in the inelastic stress range as compared to rolled sheet that is used for the panel, thus these effects must be considered in calculating the crippling stress of the stiffener and the complete panel unit.

The effective thickness \( ar{t}_w \) of the stiffener is determined by the following equation (Ref. 3):

\[ \bar{t}_w = \text{Z b}_1 t_1/Z b_1 \] — — — — — — — — — (C7.29)

where \( b_1 \) and \( t_1 \) refer to the length and thickness, respectively, of the cross-section elements.

When the yield stress \( F_{cy} \) of stiffener and sheet are different and effective \( F_{cy} \) can be estimated as follows (Ref. 3):

\[ F_{cy} = \left( F_{cy(SH)} + F_{cy(ST)} \left( \frac{\bar{t}_w}{t_s} - 1 \right) \right) \bar{t}_w/t_s \] — — — — — — — — — (C7.30)

The monolithic crippling stress for the sheet stiffener panel can be calculated from equation C7.4 or by the curve in Fig. C7.7. Equation C7.4 is

\[ F_{cy} = 0.56 \left[ \frac{(t_s^*/A)(E/F_{cy})^{1/2}}{t_s} \right] \] — — — — — — — — — — (C7.30)

The constant 0.56 in this equation applies for Y and hat-stiffened panels when the ratio \( t_w/t_s = 1.0 \). For other ratios the correction of this constant which is referred to as \( \beta_x \) in Gerard's basic equation can be obtained from curve in Fig. C7.30.

![Fig. C7.30 \( \beta_x \) correction for \( t_w/t_s \) effect on Y- or hat-stiffened panels.](image-url)
C7.24 Example Problem Y Stiffened Panel.

The compressive monolithic failing stress of a Y stiffened panel, as illustrated in Fig. C7.22a, will be calculated by the Gerard method. Fig. C7.31 shows details of the panel unit. The stiffener is extruded from 2014-T6 aluminum alloy for which $F_{Cy} = 53000$ and $E_c = 10,700,000$. The skin or panel sheet is aluminum alloy 7075-T6 for which $F_{Cy} = 67,000$ and $E_c = 10,500,000$.

![Diagram of Y stiffened panel](image)

Fig. C7.31

Stiffener area = 0.538 in.²
Sheet area = 4.21 x 0.064 = 0.270 in.²
Total area (A) per stiffener unit = 0.808 in.²
Radius of gyration of stiffener alone = 1.123 in.

Solution:

Since the elements of the Y stiffener have different thicknesses, the effective $t_w$ by equation C7.29 is needed.

$$t_w = \frac{\Sigma b t_i}{EB_i} = \frac{1.111 \times 1.333 + 1.432 \times 0.064 + 3.2 \times 0.064 + 1.188 \times 0.064}{1.111 + 1.432 + 3.2 + 1.188} \approx 0.763 \text{ inches}$$

or $\bar{t}_w = 0.763$ inches

Since stiffener and sheet have different material properties, an effective $F_{Cy}$ from equation C7.30 will be calculated.

$$F_{Cy} = \left( \frac{70,000 + 53000}{0.0763} \right) \left( \frac{0.0763}{0.064} \right)^{1/2} \approx 64700$$

The curve in Fig. C7.7 will be used to solve the equation for $F_{Cs(M)}$. The lower scale parameter for Fig. C7.7 modified by $\bar{t}_w$ and $F_{Cy}$ is,

$$\frac{A}{g \times t} \left( \frac{F_{Cy}}{E_c} \right)^{1/2}$$

The value of $g$ from Fig. C7.22a is 16.83 as the average value for a $\delta$ stiffener panel unit. $E_c$ will be taken as 10,600,000 which is the average $E$ for stiffener and sheet. Substituting in the above parameter:

$$\frac{0.808}{15.35 \times 0.0763 \times 0.064} \left( \frac{64,700}{10,500,000} \right)^{1/2} \approx 0.686$$

From Fig. C7.7 we read

$$F_{Cy}/F_{Cy} = 0.76$$

Thus the monolithic crippling a failing stress $F_{Cs(M)} = F_{Cy} \times 0.76 = 64,700 \times 0.76 = 48,200$ psi.

The curve as plotted in Fig. C7.7 is for a $E/E_s$ ratio of 1.0. The ratio for our panel is $0.0763/0.064 = 1.19$. The correction factor from Fig. C7.30 is 1.03. Therefore $F_{Cs(M)} = 1.03 \times 48200 = 49,600$ psi.

Load carried by one stiffener-sheet unit

$$= 0.808 \times 4960 = 40100 \text{ lb}$$

C7.25 Column Curve for Members With Unstable Cross-Sections.

Chapter C2 dealt with the column strength of members with stable cross-sections. For example, if we took a round tube with relatively heavy wall thickness and tested various lengths in compression to obtain the failing stress and then plotted these stress values $F_c$ against the slenderness ratio $L/\rho$ of the member, the test results would closely follow the curves ABFC in Fig. C7.32. The type of failure would be elastic overall bending instability at stresses between points A and $\delta$ and inelastic bending instability at stresses for most of the range BA. Euler's equation as shown in Fig. C7.32 can be used to determine the failing stress for both elastic and inelastic bending instability with the tangent modulus $E_t$ being used in the inelastic stress range.

Now suppose we test various lengths of a member composed of the same material as used for obtaining curve ABC, but use a member with an open cross-section, such as a channel, hat section, etc., with relatively small material thickness. The test results for such members would often follow a curve similar to DEF in Fig. C7.32. Thus it is obvious that Euler's equation cannot be used in the range, as the true failing stresses are far less than that given by the Euler column equation. A short length of the member with $L/\rho$ less than 20 will fail at practically the same stress, thus the failing stress for lengths up to $L/\rho = 20$ will be practically the same and this stress has been given the name of crippling stress $F_{cs}$ and the previous portion of this chapter has been con...
Concerned with methods of calculating this local crippling or failing stress. At point (F) the elastic buckling of some part of the cross-section begins. Between stress points F to E the action for the member involves both overall elastic bending instability plus local buckling which becomes more extensive as the stress increases. The portion EF of the column strength curve is often referred to as the transition range. At present no reliable theoretical theory has been developed for determining the failing stresses in this transition range, thus resort is made to semi-empirical methods which have been checked against test results and found to give reasonably close results.

**C7.26 Methods Used for Determining the Column Failing Stress in the Transition Region.**

**METHOD 1. JOHNSON-EULER EQUATION.**

Possibly the first method used in calculating the column failing stress $F_C$ in the transition range EF in Fig. C7.32 was the well known Johnson-Euler equation which involves the crippling stress. The equation is,

$$F_C = F_{cs} - \frac{F_{cs}^2}{4n^2E} \left(\frac{L'}{\rho}\right)^2$$

where, $F_C =$ column failing stress (psi)

$F_{cs} =$ crippling stress, assumed to occur at $L'/\rho = 0$, where $L' = L/\sqrt{E}$.  

The equation gives a parabolic curve starting from the crippling stress at $L'/\rho = 0$, and becomes tangent to the Euler curve at a stress value equal to one half the crippling stress. Fig. C7.33 shows a plot of equation C7.31 for aluminum alloy material for various values of the crippling stress which is the $F_C$ stress at $L'/\rho = 0$.

This method is quite simple to use as the only additional calculation required is the crippling stress of the column section which is obtained by methods previously explained in this chapter. The other methods usually involve the buckling stress as well as the crippling stress. Since the crippling stress is normally constant below $L'/\rho = 20$, the assumption that $F_{cs}$ is zero at $L'/\rho = 0$ is slightly conservative.

**METHOD 2.**

The following method or slight variations of it appears in the structural design manuals of a number of aerospace companies. The method or procedure for determining the column failing stresses in the so-called transition range involves the use of the basic column curve for stable cross-sections. The procedure can best be explained by reference to Fig. C7.34.

The curve ABC is the Euler column curve for a column with a stable cross-section and for a given material. It involves using the tangent modulus $E_t$ in the inelastic stress range. The following steps are taken to determine the column curve for the so-called transition range:-
(1) Locate point (O) on the basic column curve by drawing a horizontal line through an $F_O$ value equal to $F_{CS}$, the yield stress of the particular material being used.

(2) Draw a horizontal line starting at point D. Point D is at an $F_O$ value equal to the $(F_{CS})$ crippling stress for the column section being considered. Point (E) on this line is determined by projecting vertically downward from point (O).

(3) Locate point (F) at a value of $F_C = 0.9 F_{CR}$ where $F_{CR}$ is the buckling stress for the cross-section. Draw a horizontal line through point (F) to intersect the column curve at point (G).

(4) Connect points E and G with a straight line. The line E3 then represents the column failing stress $F_C$ for values of $L'/p$ between points E and G.

This method requires the use of the column curve for stable sections and the determination of the buckling stress and thus requires more calculations than Method 1. It also requires a graphical construction.

**METHOD 3.**

Another method that is widely used also uses a parabolic curve to represent the column strength in the transition range (see Ref. 12).

The parabolic approximation has the following form:

$$\frac{F_C}{F_{CS}} = 1 - \left(1 - \frac{F_{CR}}{F_{CS}}\right) \frac{F_{CR}}{F_E} \quad \text{(C7.32)}$$

where

- $F_C$ is the column failing stress.
- $F_{CS}$ is the crippling stress.
- $F_{CR}$ is buckling stress for the column cross-section.
- $F_E$ is the Euler column stress for the particular column being considered as found from equation $F_E = \pi^2 E/(L'/p)^2$.

The equation applies for $F_C > F_{CR}$. For cases where $F_{CR} > F_{DL}$ where $F_{DL}$ is the proportional limit stress for the material use $F_{DL}$ instead of $F_{CR}$ in equation C7.32.
C. 27 Example Problems Involving the Finding of the Column Strength of Columns With Unstable Cross-Sections.

PROBLEM 1.

A rectangular tube 21 inches long has the cross-section as shown in Fig. (a). The material is aluminum alloy sheet 7075-T6, with \(F_{C1} = 67,000\) and \(E = 10,500,000\). If the member has a pinned end condition, what is the column failing stress?

Solution. The area of cross-section = .234 in.\(^2\). The least radius of gyration is .042 inches.

Before the column failing stress \(F_c\) for the 21 inch length can be found, the crippling stress \(F_{cs}\) and the buckling stress \(F_{cr}\) must be determined.

The crippling stress will be calculated by the Gerad Method. The parameter for use with Fig. C.7.7 is,

\[
\frac{A}{F_c} \frac{F_{cys}}{F_c} = \frac{.234}{12 \times .04^4 (10,500,000)^{1/4}} = 0.976
\]

From Fig. C.7.7, \(F_{cs}/F_{cys} = .57\), hence \(F_{cs} = .57 \times 67,000 = 38,200\) psi.

The initial buckling stress \(F_{cr}\) will be determined by the theory of Chapter C6.

\[
b = 1 - .06 = .94, \quad h = 2 - .08 = 1.92
\]

\[
b/h = .92/1.96 = .479, \quad t_b/t_h = 1.0
\]

From Fig. C.9.6 of Chapter C6, \(K_b = 5.2\)

\[
F_{cr} = \frac{K_b \pi^4 E}{12 (1 - \nu^2)} \left(\frac{b}{h}\right)^2
\]

\[
F_{cr} = \frac{5.2 \times 10,500,000}{12 (1 - .04^2)} \left(\frac{.94}{1.92}\right)^2 = 21,500 \text{ psi.}
\]

Column Strength by Method 1 (Johnson-Euler eq.)

\[
L = 21 \text{ inches} \quad L' = L/\sqrt{3} = 21/\sqrt{3} = 21
\]

\[
L'/\rho = 21/142 = 50
\]

The Johnson-Euler equation is,

\[
F_c = F_{cs} \frac{L'}{\rho} \left(\frac{L'}{\rho}\right)^2, \quad \text{substituting,}
\]

\[
F_c = 38,200 \times 21/142 \times (50)^4 = 29,400 \text{ psi}
\]

This value could be read directly from Fig. C.7.35 for a value of \(L'/\rho = 50\) and a value of \(F_c = F_{cs} = 38,200\) at \(L'/\rho = 0\).

Column Strength by Method 2

This method requires a graphical construction which involves the column curve for the given material and for a stable cross-section.

Fig. C.7.35 shows the column curve for our material for a column with stable cross-section. It is identical to Fig. C.2.10 of Chapter C2 for the room temperature condition, except the curve in Fig. C.7.35 has been drawn to a smaller vertical scale.

The graphical construction to obtain the column curve in the region between the crippling stress and the buckling stress is as follows:

At a column stress \(F_c = F_{cys} = 67,000\), draw a horizontal line to intersect the basic column curve at point (G). Draw a horizontal line from point (B) which equals the crippling stress of 38,200 psi. Locate point (2) by projecting vertically downward from point (0). Locate point (G) at an \(F_c\) stress equal to \(9\times 21,500 = 193,500\) psi. Connect points (E) and (G) by a straight line. This line represents the column failing curve for the member of our problem for \(L'/\rho\) values between points (E) and (G). Taking \(L'/\rho = 50\), we protect upward to \(\Theta\) line and then horizontally to scale at left to read \(F_c = 27,700\) psi.

Column Strength by Method 3

In this method equation C.7.32 is used.

\[
\frac{F_c}{F_{cs}} = 1 - (1 - F_{for}/F_{cs})(F_{cr}/F_{cs}) - - - - - (A)
\]

\[
F_{cr} = 21,500, \quad F_{cs} = 38,200
\]

C. 23 for a value of \(L'/\rho = 50\) and a value of \(F_c = F_{cs} = 38,200\) at \(L'/\rho = 0\).
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

Substituting in Johnson-Euler equation,
\[ F_C = \frac{30,600 \times 30,600}{4 \times 10^9 \times 500,000 \times 50} = 25,260 \text{ psi} \]

Column Strength by Method 2

On Fig. C7.35 point E' is located at 
\[ F_C = F_{gs} = 30,600 \text{ psi} \]
Point G is located at \[ L' = 0.8 \times 20,900 = 16,720 \text{ psi} \]
The line E'G' is the column curve. For \[ L'/L = 45.8 \text{ psi} \]
we project upward to line E'O' and then horizontally to left scale to read \[ F_C = 25,000 \text{ psi} \]

Column Strength by Method 3

To find \[ F_{gs} \], use Fig. C7.35 with \[ L'/L = 45.8 \text{ psi} \]
and basic Euler curve, gives a value of \[ F_{gs} = 47,000 \text{ psi} \]

Substituting in equation (A):

\[ \frac{30,600}{30,600} = 1 \times \left( 1 - \frac{20,900}{30,600} \right) \times \left( 20,900 \times 47,000 \right) \]
\[ F_{gs} = 30,600 \times 0.411 = 26,300 \text{ psi} \]

C7.28 Column Strength of Stiffener
With Effective Sheet.

Column members are often attached to thin sheet by rivets or spot welding. If the rivet spacing is such as to prevent inter-rivet buckling, then the sheet will assist the stiffener to carry a compressive load and to neglect the sheet would be too conservative.

If the attached sheet is relatively thin, that is, less than the stiffener thickness, the method of using the effective sheet width as a part of the column area is widely used by structural designers and will be used in this example solution.

Example Problems.

For an example problem, we will assume that the Z stiffener in Problem 2 is one of several stiffeners riveted to a sheet of .025 thickness and of the same material as the stiffener.

Solution.

The rivet or spot weld spacing is made such as to prevent inter-rivet buckling. Thus the column area will be as shown in Fig. c, namely, the stiffener area plus the area of the sheet for the effective width w. Since the effective sheet width w is a function of the stiffener stress and since the stiffener
stress is a function of the radius of gyration, the design procedure is of the trial and error category.

**First Trial.**

Assume the effective sheet width is based on column strength of 2 stiffener acting alone. The average column failing stress by the 3 methods in the Problem 2 solution was (25,860 + 25,000 + 25,300)/3 = 25,700 psi.

The effective width equation to be used is,

\[ w = 1.9 \times \sqrt{\frac{E}{F_{ST}}} \]

\[ = 1.9 \times 0.025 \times \sqrt{10,500,000/25,700} = .965 \text{in.} \]

Effective sheet area = 0.965 x .025 = .0242

Area of stiffener = \( A_o = .117 \)

Total area = .1412

Adding the effective sheet to the stiffener will change the radius of gyration. Mr. R. J. White (Ref. 13) has developed equation C7.33 which gives the variation in the radius of gyration in terms of known variables for any stiffener cross-section. Since the failing stress of a column is directly proportional to the radius of gyration squared, equation C7.33 can be equated to the ratio of the column stresses.

\[ \left( \frac{\rho}{\rho_0} \right)^2 = 1 + \left[ 1 + \left( \frac{S}{\rho_0} \right)^2 \right] \frac{w}{A_o} \]

\[ \frac{F_c}{F_{ST}} = 2 \times (\rho/\rho_0)^2 \]

where, \( \rho_0 \) = radius of gyration of stiffener alone.

\( \rho \) = radius of gyration of sheet and stiffener.

\( S \) = distance from centerline of sheet to neutral axis of stiffener.

\( t \) = sheet thickness. \( w \) = sheet effective width.

Equation C7.33 has been put in curve form as shown in Fig. C7.36, which will now be used to compute the radius of gyration \( \rho_0 \) for the stiffener plus effective sheet.

\[ S = .75 + .0125 = .7625 \]

\( \rho_0 \) for stiffener alone = .535

\[ S/\rho_0 = .7625/.535 = 1.425 \]

\( w = .965 \)

\[ \frac{w}{A_o} = .965 \times .025 = .0242 \]

From Fig. C7.36 for the above values of \( S/\rho_0 \) and \( w/A_o \), we obtain

\[ (\rho/\rho_0)^2 = .775 \]

whence \( \rho^2 = .535^2 \times .775 \), which gives \( \rho = .471 \text{in.} \)

Then \( L'/\rho = 24.5/.471 = 52 \).

Use Method 1 for column strength:-

\[ F_c = 30,600 - \frac{30,600^2}{4 \times 10,500,000} (52)^2 = 24,500 \text{psi} \]

**Fig. C7.36 (Ref. 13)**

Curves for the determination of

\[ \left( \frac{\rho}{\rho_0} \right)^2 = \frac{F_c}{F_{ST}} \]
The revised effective width based on this stiffener stress is,

\[ w = 1.9 \times 0.025 \sqrt{10,500,000/24,500} = 0.98 \text{ in.} \]

This value is only .02 inch more than the value of .96 previously, thus the effect on the radius of gyration \( p \) will be negligible. The column failing stress for the sheet stiffener combination is therefore 24,500 psi, and the compressive failing load would be 24,500 x 1.1412 = 34.60 lbs.

C7.28 Sheet-Stiffener Panels With Relatively Heavy Sheet Thickness.

For Z, Y and hat shaped stiffener-sheet panels as previously discussed relative to local failing stress, use entire sheet and stiffener area in computing radius of gyration or, in other words, due not use effective sheet widths but use entire sheet as effective.

PROBLEMS

1. Find the crippling stress for the angle sections of Figs. (a) and (b) when formed from the following materials. Use both Needham and Gerard Methods. (1) 2024-T3 aluminum alloy. (2) 7075-T6 aluminum alloy. (3) 17-7PH (TH10050) stainless steel. (4) Same as (3) but subjected to an elevated temperature of 750\(^\circ\)F for 1/2 hour duration.

2. Find the crippling stress for the two channel shapes of Figs. c and d when formed of following materials. (1) AISI 4130 steel, heat-treated to \( F_p = 180,000 \). (2) Same as (1) but subjected to an elevated temperature of 500\(^\circ\)F for 1/2 duration. (3) Ti-8\%H Titanium. (4) 7075-T6 aluminum alloy.

3. Find the crippling stress of the rectangular tubes in Figs. e and f when formed from following materials. (1) 2024-T3 aluminum alloy. (2) HK31A-0 magnesium. (3) Ti-8\%H Titanium.

4. Same as (2) but subjected to 300\(^\circ\)F for 1/2 hour duration.

5. Find the crippling stress for sections of Figs. g and h when formed from 2024-T3 and 7075-T6 aluminum alloy material.

6. Fig. 1 shows a corner member, in a stiffened wing section. The skin is fastened to the stiffener by one row of 1/8 diameter rivets at 3/4 inch spacing. The web is fastened to stiffener by 2 staggered rows of rivets, with rivet spacing in each of 1-1/8 inch, and rivets are of the flat head type. Material is 2024-T3 aluminum alloy. Also use 7075-T6 aluminum material.

Find crippling stress for stiffener. Will inter-rivet buckling occur. Find effective sheet widths and total failing load unit will carry.

7. Determine the compressive failing strength of a short Z stiffened sheet panel unit such as that illustrated in Fig. C7.29. The various dimensions are as follows: Refer to Fig. C7.29 for meaning of symbols.

\[
\begin{align*}
& t_w = 0.072 \quad b_w = 2.50 \\
& t_s = 0.072 \quad b_A = 0.625 \\
& b_s = 2.50 \quad b_0 = 0.375
\end{align*}
\]
Stiffener and sheet material is 2024-T3 aluminum alloy. Rivets are 1/8 diameter Brazier Head type 2117-T3 material and spaced 7/8 inch.

(8) Same as (7) but change material to 7075-T6 aluminum alloy.

(9) Same as (7) but change material to Ti-6Al-4V Titanium.

(10) In the Example Problem on a Y stiffened panel as given in Art. C7.25, change sheet thickness from .064 to .081 and calculate the resulting panel strength.

(11) The hat stiffener in Fig. g of Problem 4 is one of several stiffeners riveted to skin of thickness .032 and of the same material as the stiffener. If the length of the panel is 20 inches, what will be the column failing stress if end fixity c = 1.5. Also for c = 2.0. Use method involving effective sheet widths.

(12) Same as Problem 11 but use Z stiffener of Fig. h of Problem 4.

References


CHAPTER C8
BUCKLING STRENGTH OF MONOCOQUE CYLINDERS

This chapter presents information on the buckling strength of circular cylinders under compressive, bending and torsional loads acting separately and in combination, without and with internal pressure. Some information on the buckling strength of conical cylinders is presented.

C8.1 Introduction.

Before the advent of the high speed aircraft and particularly the missile and space vehicle, the use of the unstiffened cylinder or monocoque type of structure was quite limited. However, the arrival of the space age has caused the thin walled cylinder to become important in the design of missiles and space structures.

The classical small deflection theory which has proved adequate for determining the buckling strength of flat sheet structures as covered in Chapter A18, has not proved adequate for determining the buckling strength of thin walled cylinders or curved sheet panels since the theory gives results much too high when compared with the experimental or test results.

A more recent large deflection theory has shown closer agreement with experimental results, however, the theory requires a prior knowledge of cylinder imperfections. As a result of the theoretical limitations to date, the strength design of such structures is based primarily on best fit curves for experimental or test results, using theoretical parameters and curves when they appear applicable.

A missile structure in handling operations and flight maneuvers is subjected to tensile, compressive, bending and torsional load systems. The cylinders may be pressurized or unpressurized. The structural design of such structures thus requires a knowledge of the buckling strength under the various load systems, acting separately and in combination. This chapter will be confined to giving test data and design curves for the buckling strength of cylinders under the various types of stress systems.

C8.2 Buckling of Monocoque Circular Cylinders Under Axial Compression.

The term monocoque cylinder means a thin walled cylinder without longitudinal skin stiffeners or transverse intermediate frames attached to cylinder skin.

Investigation by the author of the procedure used by a number of aerospace companies for the strength design of monocoque structures indicated that the design curves given in (Ref. 1) were widely used. It thus seems appropriate to give some design material from (Ref. 1).

Cylinders are usually given four classifications relative to length:-

(1) Short Cylinders. Short cylinders tend to behave as flat plate columns (cylinders with infinite radius). They develop buckles in a sinusoidal wave form similar to flat plate buckling. The end fixity has considerable influence.

(2) Intermediate or Transition Length of Cylinders. Buckling involves a mixed pattern combining sinusoidal and diamond shapes. The effect of end fixity and cylinder length are of nominal importance.

(3) Long Cylinders. Such cylinders buckle in a diamond shaped pattern, and the length and end conditions are not of much importance.

(4) Very Long Cylinders. Such cylinders buckle by over-all column Instability or act as a Euler type column.

Fig. C8.1 shows a photograph of the buckle failure of an intermediate length cylinder under axial compression.

The name of Donnell is prominent in the development of a theory for the buckling strength of cylinders. From (Ref. 2), Donnell's eight order differential equation can be used to provide a small-deflection theoretical solution to the behavior of cylinders in the short to long length range. A solution of Donnell's equation by Batdorf (Ref. 3) gives the critical stress in terms of the buckling coefficient $K_c$, which for simply supported cylinder ends is defined by the equation:

$$K_c = \left( \frac{m^2 + \beta^2}{m^2} \right)^{1/2} \left( \frac{12Z^2}{n^2 (m^2 + \beta^2)} \right)$$

where,

$m = \text{number of half waves in longitudinal direction}$

---

C8.1
Fig. C8.1c Type of Buckling Failure Under Pure Bending Load. No Internal Pressure. (Ref. 9.)

Fig. C8.1d Type of Buckling Failure Under Pure Bending Load. No Internal Pressure. (Ref. 9.)
\[ \beta = \frac{L}{\lambda} \]

\[ \lambda = \text{half wave length of buckles in circumferential direction.} \]

The compressive buckling stress is given by,

\[ \sigma_c \frac{t}{K_o} \left[ \frac{(t/L)^2}{(1 - \nu^2)} \right]^{1/2} = \Phi \] \hspace{1cm} \text{C8.2}

where, \( t \) = wall thickness, \( L \) is the cylinder length and \( \nu \) is Poisson's ratio. The term \( \Phi \) is the plasticity correction factor and equals 1.0 for elastic buckling.

Minimization of Eq. C8.1 with respect to the parameter \( (m^2 + \beta^2)^{1/2} \), gives the critical buckling coefficient for long cylinders in the following form:

\[ K_c = \left( \frac{4(1 - \nu^2)}{(1 - 0.3^2)} \right) \] \hspace{1cm} \text{C8.3}

where \( Z = (L^2)/(rt^2) \) \hspace{1cm} \text{C8.4}

Substitution of Eq. C8.3 into Eq. C8.2 reduces to the well known classical equation

\[ \sigma_c = \frac{C}{r(t/r)} \] \hspace{1cm} \text{C8.5}

where, \( C = 1/\sqrt{3(1 - 0.3^2)} = 0.605 \) for \( \nu = 0.3 \).

The buckling coefficient for simply supported end conditions in the transition range can be determined by substituting the limiting values of \( \beta = 0 \) and \( m = 1 \) in Eq. C8.1. A similar solution for cylinders with fixed edges deviates from the solution for simply supported cylinders only in the flat plate and transition ranges.

Figs. C8.2 to C8.5 inclusive taken from Reference 1, show the theoretical curve which shows the buckling coefficient as a function of the geometrical parameter \( Z \). Theoretically, the short cylinder range would occur at \( Z = 0 \). The values of \( K_c \) of 1.12 and 4.12 for \( Z = 1 \) correspond quite closely to the buckling coefficients of simply supported and fixed ended plate columns. The long cylinder behavior is represented by a 45 degree sloping straight line portion of the curve. The curve connecting the short and long cylinder ranges is referred to as the transition range.

Figs. C8.2 to C8.5 show the plot of extensive test data and a 90 percent probability curve derived by the author of (Ref. 1) by a statistical approach. Fig. C8.6 from (Ref. 1) shows a plot of much test data on a logarithmic chart of \( r/t \) versus \( K_c \) for \( Z = 10,000 \). A best fit curve and 90 and 99 percent probability derived curves are also seen as well as the theoretical curve for \( C = 0.605 \).

Observation shows that the theoretical results are far above the test values. Fig. C8.7 (from Ref. 1) is a set of data that shows curves of \( K_c \) versus \( Z \) for various \( r/t \) values and for 90 percent probability.

C8.3 Additional Convenient Design Charts for Determining Compressive Buckling Stress.

Figs. C8.8a and C8.8b are more convenient design charts as the buckling stress \( F_{cbr} \) can be read from the chart, thus avoiding the calculations involved in using curves in Fig. C8.7. Fig. C8.8a is for 99 percent probability and 95 percent confidence level and Fig. C8.8b for 90 percent probability and 95 percent confidence level. The curves are based on tests of steel and aluminum alloy cylinders only. The accuracy of these curves when used for other materials has not been substantiated by tests. The 99 percent probability curves are recommended as design allowable for structures whose failure would be highly critical. The 90 percent curves can be used for less critical structures.

C8.4 Plasticity Correction.

The plasticity correction factor in the long range \( (L^2)/(rt) = 100 \) is given by the curves in Fig. C8.9 taken from (Ref. 4). In general most practical cylinders in aerospace structures will fall in the long cylinder range.

![Diagram](image_url)  
Fig. C8.9 (Ref. 4) Non-dimensional buckling chart for axially compressed long circular cylinders. \( \Phi = \frac{(K_o - K_o) + V_o}{(1 - \nu^2)} \) \hspace{1cm} \text{Applicable when } L^2/rt = 100.

C8.5 Buckling of Monocoque Circular Cylinders Under Axial Load and Internal Pressure.

Experiments conducted many years ago definitely showed that the compressive buckling strength of monocoque cylinders was increased if internal pressure was added to the closed cylinder. Since weight saving is very important in missile design, the use of pressurized
Fig. C8.2 $r/t = 100$ to 500.

Fig. C8.3 $r/t = 500$ to 1,000.

Fig. C8.4 $r/t = 1,000$ to 2,000.

Fig. C8.5 $r/t$ over 2,000.

Fig. C8.2-C8.5 (Ref. 1) Compressive Buckling Coefficients for Unpressurized Circular Cylinders.

Fig. C8.6 (Ref. 1) Compressive Buckling Coefficients as a Function of $r/t$.

Fig. C8.7 (Ref. 1) Compressive Buckling Stress Coefficients for Unpressurized Circular Cylinders. (90 Percent Probability)
Fig. C8.8a

UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS
IN AXIAL COMPRESSION
(Clamped Ends)

(Taken by Permission from General Dynamics - Fort Worth Structures Manual)

NOTE:
Most of the test data for these curves fall within cylinder dimensions of \( L/r = 1.0 \) and \( 300 = r/t = 1000 \). Caution should be used when curves for \( L/r > 2 \) are used.

Fig. C8.8a
Fig. C8.8b

UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS
IN AXIAL COMPRESSION
(Clamped Ends)

(Taken by Permission from General Dynamics - Fort Worth Structures Manual)

NOTE:
Most of the test data for these curves fall within cylinder
dimensions of L/r ≈ 1.0 and
300 = σ/t = 1000. Caution should be used when curves
for L/r ≈ 3 are used.

3σ% PROBABILITY CURVES
Confidence Level = 99%
cylinders in missile structures has become a common type of missile structural design. The famous Atlas missile was one of the first to use a pressurized monocoque type of structure.

The major reason for the large discrepancy between the actual test strength and the theoretical strength by the linear small deflection theory that is generally accepted is that the discrepancy is due to geometrical imperfections and the associated stress concentrations. Now large internal pressures should smooth out such imperfections and approach a perfect cylinder and thus the resulting buckling strength should approach that given by the linear small deflection theory. However, much of the available test data for pressurized cylinders gives values below that given by the linear small deflection theory. (See Fig. C8.1a for buckling action of pressurized cylinder.)

One of the first experimental and theoretical investigations of the effect of internal pressure upon the buckling compressive stress of monocoque cylinders was by Lo, Crate and Schwartz (Ref. 5). They analyzed the problem of long pressurized cylinders using an extension of the large deflection theory of Von Karman and Tsien (Ref. 6). Plotting their results in terms of the non-dimensional parameters \( \frac{p}{E} (r/t)^2 \) and \( \frac{(p_{cr}/E)(r/t)}{\pi} \), they found that the buckling coefficient \( C \) increased from the Tsien value (Ref. 7) of 0.375 at zero internal pressure to the maximum classical value of 0.606 at \( \frac{(p_{cr}/E)(r/t)}{\pi} = 0.169 \). Fig. C8.10 taken from (Ref. 1) shows the large deflection theoretical curve. Also shown are the experimental values obtained in (Ref. 5) as well as those obtained by the investigators in (Ref. 1) and by other investigators in (Ref. 8). From Fig. C8.10 it is apparent that large discrepancies exist between the theoretical predicted values and the experimental values. Lo, Crate and Schwartz suggested that better correlation with test results could be obtained if the increment in the buckling stress parameter \( (\Delta F_{cr}/E)(r/t) \) were plotted as a function of the pressure parameter as shown in Fig. C8.11. The increase in the critical stress \( \Delta F_{cr} \) due to the internal pressure directly represents the beneficial effect of the internal pressure. The total critical stress is thus obtained by adding the critical stress for the unpressurized cylinder to the increase in the critical stress due to the internal pressure. In order to plot the test points in Fig. C8.11, it was necessary to determine the unpressurized critical buckling stress for each test. The 90 percent probability values from Fig. C8.7 were used. As shown in Fig. C8.11, the general trend of the test data agrees fairly well with the theoretical curve. The investigations in Ref. 1 have shown a best fit curve and a 90 percent probability curve obtained by a statistical approach and this 90 percent curve in Fig. C8.11 is recommended as a design curve for taking into account the effect of internal pressure.

**BUCKLING OF MONOCOQUE CIRCULAR CYLINDERS UNDER PURE BENDING**

C8.6 Introduction.

Flight vehicles are subjected to forces in flight and in ground operations that cause bending action on the structure, thus it is necessary to know the bending strength of cylinders. Two rather extremes have been used in past design practices. One design assumption takes the value of the bending buckling stress as 1.3 times the buckling stress under axial compression. The other assumption is to assume the bending buckling...
stress is equal to the axial compressive buckling stress. The first assumption is considered by some designers as somewhat unconservative while the second assumption is no doubt somewhat conservative.

It is relatively recent (Ref. 10) that a small deflection approach has been completely solved for a cylinder in bending. Tests of cylinders in bending show that the theoretical result is lower than the test results but higher than the buckling stress in axial compression. No large deflection analysis which involves a consideration of initial imperfections of the cylinder has been formulated to date for the buckling stress in bending. Since the stress in bending varies from zero at the neutral axis to a maximum at the most remote element, the lower probability of imperfections occurring within the smaller highest stressed region would lead one to conclude that higher buckling stresses in bending, as compared to the buckling stress in axial compression, should be expected.

C8.7 Available Design Curves for Bending Based on Experimental Results.

The same investigators (Suer, Harris, Skeness and Benjamin) that carried out tests on cylinder in axial compression (Ref. 1), have also carried out an extensive investigation of the buckling strength of monocoque cylindrical vessels in pure bending (see Ref. 9).

As originally developed by Flugge (Ref. 11) for long cylinders, the buckling stress in bending is expressed as:

\[ \sigma_{b} = C_b \frac{t}{r} \]

The theoretical value of the bending buckling coefficient as found by Flugge was about 30 percent higher than the corresponding classical buckling coefficient of 0.605 in axial compression.

Fig. C8.12 (from Ref. 9) gives a plot of considerable test data and a plot of \( C_b \) versus \( r/t \). A best fit curve, a 90 percent probability curve and a 99 percent probability curve, are shown. The dashed curve is a plot of the 90 percent probability curve as previously given in Fig. C8.6 for buckling in axial compression, thus giving a comparison between bending and compressive buckling strengths. As indicated by the plotted test points, the test data above an \( r/t \) value of 1500 is quite limited, thus the accuracy of the curves is somewhat unknown.

Figs. C8.13 and C8.13a give convenient design curves for finding the bending buckling stress based on 95 percent probability and 90 percent probability and confidence level of 95 percent. These curves are from the structures manual of the General Dynamics Corp. (Fort Worth.) Their manual states that most of the test data upon which the curves are based fall within the range of cylinder dimensions \( 0.25 < L/r < 5 \), and \( 25 < r/t < 1500 \), and the curves are based on tests of steel, aluminum and brass cylinders only.

C8.8 Buckling Strength of Circular Cylinders in Bending with Internal Pressure.

The published information on the buckling strength of circular cylinders in bending with internal pressure is very limited and the status of theoretical studies to date leave much unknown regarding this subject.

Reference 9 gives the results of a series of tests of circular cylinders in bending with internal pressure. Fig. C8.14 is taken from that published report. In Fig. C8.14 the experimental data are plotted in terms of the increment \( \Delta C_b \) to the buckling coefficient \( C_b \). The increase in the buckling stress coefficient \( \Delta C_b \) represents the beneficial effect of internal pressure. The total value of the buckling coefficient is obtained by adding the buckling coefficient for unpressurized cylinders to the increase in the buckling coefficient due to the internal pressure. In order to plot the data, it was first necessary to determine the unpressurized buckling coefficient for each specimen. The 90 percent probability design curve of Fig. C8.12 was used for this purpose.

The direct benefit of lateral internal pressure to the stability of cylinders in bending is indicated by those specimens with no net axial stress (the balanced specimens) represented by the circular symbols. At large values of the pressure parameter, the additional benefit of the axial pretention is clearly demonstrated by the large increase in \( \Delta C_b \) of the pretensioned
Fig. C8.13
UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS
IN BENDING
(Taken by Permission from
General Dynamics - Fort Worth
Structures Manual)

NOTE:
Based on test data for
cylinder sizes of
L/r = 5 and
200 = r/t = 1500.

99% PROBABILITY CURVES
Confidence Level = 99%

Dashed curves are extrapolated
into untested regions

Fig. C813
Fig. C8.13a
UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS IN BENDING
(From General Dynamics - Fort Worth Structures Manual)

NOTE:
Based on test data for cylinder sizes of
\( L = L_r = \frac{1}{2} \) and
\( 300 = \frac{t}{r} = 1500. \)

99% PROBABILITY CURVES
Confidence Level = 99%

Dashed curves are extrapolated into untested regions

Fig. C8.13a
specimens, represented by the triangular symbols, over that for balanced specimens. The limiting value of the increase in the buckling stress coefficient for pretensioned pressurized cylinders at very high values of the pressure parameter is given by the line $F_{cr} = \frac{pr}{2t}.

The analysis of the pressurized cylinder data was achieved by selecting a best fit curve for those specimens in which the axial pretension was balanced. This curve (shown in Fig. C8.14) was selected by the investigators as best indicating the general trend of the experimental data. At large values of the pressure parameter, the curve is drawn to approach an asymptote agreement between the best fit curve and experimental data is apparent in Fig. C8.14. A statistical analysis of the test data was performed for the specimens with no axial pretension to establish the 90 percent probability design curve shown in Fig. C8.14. Because data were available only from the tests made by the investigators, they indicate the sample may not be representative and a lower probability curve should perhaps be used for design purposes. The data was not considered sufficient to permit a statistical analysis of the pretensioned test data, and therefore they suggest a lower bound curve be used in the design of pretensioned cylinder. Additional tests are needed too for unpressurized cylinders with $r/t$ ratios greater than 1500 to verify the shape of the design curve.

Fig. C8.14 Increase in Bending Buckling Stress Coefficients Due to Internal Pressure.

**SBUCKLING OF MONOCOQUE CIRCULAR CYLINDERS UNDER EXTERNAL PRESSURE**

**C8.9 External Hydrostatic Pressure.**

Under this type of loading, the cylinder shell is placed in circumferential compressive stress equal to twice the longitudinal compressive stress. The buckling compressive stress for this type of loading is given by the following equation (Ref. 12):

$$F_{cr} = \frac{k_p \pi^2 E}{12 (1 - \nu^2)} \left(\frac{t}{L}\right)^2 - - - - - - - - - C8.7$$

Values of the buckling coefficient $k_p$ are given in Fig. C8.15. Equation C8.7 is for buckling stresses below the proportional limit stress of the material.

**C8.10 External Radial Pressure.**

Under an inward acting radial pressure only the circumferential compressive stress produced is $f_c = \frac{pr}{t}$ where $p$ is the pressure.

The buckling stress under this type of loading from (Ref. 12) is,

$$F_{cr} = \frac{k_r \pi^2 E}{12 (1 - \nu^2)} \left(\frac{t}{L}\right)^2 - - - - - - - - - C8.8$$

Values of the buckling coefficient $k_r$ are given in Fig. C8.16. Equation C8.8 is for buckling stresses within the proportional limit stress of the material.

**C8.11 Buckling of Monocoque Circular Cylinders Under Pure Torsion.**

Fig. C8.17 (from Ref. 12) shows the results of tests of thin walled circular cylinders under pure torsion. The theoretical curve in Fig. C8.17 is due to the work of Batdorf, Stein and Schindelouis (Ref. 15). Their theoretical investigation utilized a modified form of the single equilibrium form of Donnell (Ref. 2) and by use of Galerkins Method obtained the curve shown in Fig. C8.17. This theoretical curve falls above the test results and thus for safety a lowered curve should be used for design purposes. Fig. C8.18 shows a design curve which appears in the structural design manuals of a number of aerospace companies.

The torsional buckling stress is given by the equation:

$$F_{cr} = \frac{k_t \pi^2 E}{12 (1 - \nu^2)} \left(\frac{t}{L}\right)^2 - - - - - - - - - C8.9$$

Fig. C8.19 gives the value of the torsion buckling coefficient $k_t$ and applies for buckling below the proportional limit stress.

To correct for plasticity effect when buckling stress is above the proportional limit stress, the non-dimensional chart of Fig. C8.19 can be used. Figs. C8.20 and C8.21 give other convenient design curves involving buckling stresses for 90 and 99 percent probability.
**Fig. C8.15** Buckling Under External Hydrostatic Pressure. \[ F_{cr} = \frac{k_p 
abla^2 E}{12 (1 - \nu_e^2) L^3}. \]

\[ Z_L = \frac{L^4}{t^2} \sqrt{1 - \nu_e^2} \]

**Fig. C8.16** Buckling Under External Radial Pressure. \[ F_{cr} = \frac{k_r \nabla^2 E}{12 (1 - \nu_e^2) L^3}. \]

\[ Z_L = \frac{L^4}{t^2} \sqrt{1 - \nu_e^2} \]
Fig. C8.17 Comparison of Test Data and Theory for Simply Supported Circular Cylinders in Torsion.

\[ F_{scT} = \frac{k_t \pi^3 E}{12 (1 - \nu^2)} \left( \frac{L}{t} \right)^3; \quad Z_L = \frac{L^3}{rt} (1 - \nu^2)^{1/2} \]

Fig. C8.18 Buckling of Simply Supported Circular Cylinders in Torsion or Transverse Shear

\[ Z_L = \frac{L^3}{rt} \sqrt{1 - \nu^2} \]
C8.12 Buckling Under Transverse Shear.

Shear stresses are also produced under bending due to transverse loads. These shear stresses are maximum at the neutral axis and zero at the most remote portion of the cylinder wall, whereas the torsional shear stress is uniform over the entire cylinder wall. Limited tests indicate a higher buckling shear stress under a transverse shear loading as compared to the torsional buckling stress. A general procedure in industry is to increase the shear buckling stress under torsion by using 1.25 times $k_t$. Thus in Fig. C8.16 find $k_t$ for buckling under torsion and then multiply it by 1.25 in using Equation C8.9 to find buckling stress under transverse shear.

C8.13 Buckling of Circular Cylinders Under Pure Torsion With Internal Pressure.

Internal pressure places the cylinder walls in tension, thus the torsional buckling stress is increased as torsional buckling is due to the compressive stresses that are produced under shear forces.

Hopkins and Brown (Ref. 13), using Donnell's equation, calculated the effect of internal pressure on the buckling stress of circular cylinders in torsion and the results were in fair agreement with test results.

Crate, Batdorf and Baab (Ref. 14), utilized an empirical interaction equation to fit test data. The derived interaction equation was:

$$R_t \times R_p = 1 \quad \text{C8.10}$$

where,

$R_t$ = ratio of applied torsion shear stress

$R_p$ = ratio of allowable torsion shear stress

$R_t$ = ratio of applied internal pressure

external hydrostatic buckling pressure

Note that $R_p$ has a negative sign. The value of the external hydrostatic buckling pressure can be determined by use of Fig. C8.15. Fig. C8.22 shows a plot from Equation C8.10 and its comparison with test data.

C8.14 Buckling of Circular Cylinders Under Transverse Shear and Internal Pressure.

For this load system, the derived interaction equation is similar to Eq. C8.10.

$$R_t + R_p = 1 \quad \text{C8.11}$$

where,

$R_t$ = applied transverse shear stress

allowable transverse shear stress

$R_p$ is same as explained in Article C8.13.
Fig. C8.20

UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS IN TORSION

(From General Dynamics - Fort Worth Structures Manual)

NOTE:
Curves based on test data within range of cylinder dimensions:
\[ \sigma = \frac{M}{I} = \frac{R}{L} \left( \frac{r}{a} \right)^{3/2} \]
where,
\[ N = (1 - \frac{v}{3}) \left( \frac{r}{a} \right)^{3/2} \frac{L}{R} \]

99% PROBABILITY CURVES
Confidence Level = 99%
Fig. C8.21
UNPRESSURIZED, UNSTIFFENED, CIRCULAR CYLINDERS
IN TORSION
(From General Dynamics - Fort Worth
Structures Manual)

NOTE:
Curves based on test data within range of cylinder
dimensions:

\[ f = H = 3 \left( \frac{r}{t} \right)^4 \]

where,

\[ H = \left( 1 - \nu_a^2 \right) A \frac{t^2}{2} \]

90% PROBABILITY CURVES
Confidence Level = 99.6

\[ \frac{L}{t} = \]

0.125
0.25
0.5
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0

Fig. C8.21
C8.15 Buckling of Circular Cylinders Under Combined Load Systems.

It is very seldom that a circular cylinder used in an aerospace structure such as a missile is subjected to a load system causing only one internal stress system such as axial stress, bending stress, and shearing stress. Therefore it is necessary to be able to safely determine the buckling strength under the practical cases of combined stress systems. This problem is handled by the use of interaction equations.

Table C8.1 summarizes the interaction equations that appear in the structures design manuals of several aerospace companies. These equations no doubt have been proven reliable by checking against test data. Fig. C8.23 gives a plot of three interaction curves. These curves are useful in quickly observing whether cylinder is weak and to determine margins of safety.

C8.16 Illustrative Problems for Finding the Buckling Strength of Circular Monocoque Cylinders.

Problem 1. Axial Compressive Strength.

A circular cylinder has a radius \( r = 50 \) inches, a length \( L = 75 \) inches, and a wall thickness \( t \) of .05 inches. The material is aluminum alloy 2024-T3, for which \( E = 10,700,000 \) psi. What compressive load will it
carry using design curves based on 90 percent probability.

Solution:

\[ \frac{r}{t} = 50/0.05 = 1000, \quad L/r = 75/50 = 1.5 \]

\[ F_{cr} = \frac{k_c \pi^2 E}{12 (1 - \nu_e^2)} \left( \frac{r}{t} \right) \text{, (See Eq. C8.2)} \]

To find the buckling coefficient \( k_c \), we use Fig. C9.7.

\[ Z = \frac{L^2}{rt} \sqrt{1 - \nu_e^2} \]

\[ Z = \frac{75}{50} \times \frac{0.05}{1 - 0.3} = 2140 \]

From Fig. C9.7 using \( Z = 2140 \) and \( r/t = 1000 \), we read \( k_c = 230 \), then

\[ F_{cr} = \frac{230 \pi^2 E \times 10,700,000}{12 (1 - 0.3)} \cdot \left( \frac{0.05}{75} \right) = 1210 \text{ psi} \]

The buckling axial compressive load

\[ P = 2mr \quad \text{for} \quad 2m \times 50 \times 0.05 \times 1205 = 12950 \text{ lb.} \]

If we use the design curves of Fig. C8.6b based on 90 percent probability and 99 percent confidence, we read for \( r/t = 1000 \) and \( L/r = 1.5 \), that \( F_{cr}/E = 0.000121 \). Thus \( F_{cr} = 10,700,000 \times .000121 = 1295 \text{ psi.} \) If we multiply this value by the 95 confidence value, we obtain 1230 psi which is practically the same as obtained above using Fig. C9.7.

\[ P_a = 1235 \times 2m \times 0.05 \times 50 = 20360 \text{ lbs.} \]

If we require 99 percent probability and 95 percent confidence, we use Fig. C8.8a. For \( r/t = 1000 \) and \( L/r = 1.5 \), we read \( F_{cr}/E = 0.000082 \) or \( F_{cr} = 10,700,000 \times .000082 = 877 \text{ psi, which gives an axial buckling load of 13,700 lbs.} \) Thus requiring a 99 percent probability decreases the buckling load considerably.

Problem 2. Bending Strength

The same cylinder as used in Problem 1 will be used in this example problem. What bending moment will buckle this cylinder under 90 percent probability.

Solution. The curve in Fig. C8.12 will be used. For \( r/t = 1000 \) and \( L/r = 1.5 \), we read from Fig. C8.12 that the bending buckling coefficient \( C_b = 1.2 \).

\[ F_{bc} = C_b E t/r \quad \text{(See Eq. C8.5)} \]

\[ = 1.2 \times 10,700,000 \times 0.05/50 = 1710 \text{ psi} \]

The bending moment developed at this buckling stress is,

\[ M = F_{bc} \frac{L}{r} = F_{bc} r \frac{r}{t} \]

\[ = 1710 \pi \times 50 \times 0.05 = 670,000 \text{ in. lb.} \]

If we use Fig. C8.13a based on 90 percent probability and 95 percent confidence, we read for \( r/t = 1000 \) and \( L/r = 1.5 \) that \( F_{bc}/E \) is .000160. Thus \( F_{bc} = 10,700,000 \times .000160 = 1710 \text{ psi.} \) Since it is difficult to read Fig. C8.12, it is recommended that Fig. C8.13a be used in design.

If we require 99 percent probability, we use Fig. C8.13 and obtain \( F_{bc}/E = .000092 \), which gives \( F_{bc} = 10,700,000 \times .000092 = 985 \text{ psi as against 1710 for 90 percent probability.} \)

Problem 3. Torsional Strength

Same cylinder as in Problem 1. What torsional moment will this cylinder develop.

Solution. Using design curve in Fig. C8.18

\[ Z = \frac{L^2}{rt} \sqrt{1 - \nu_e^2} = \frac{75}{50} \times 0.05 \sqrt{1 - 0.3} = 2140 \]

For this value of \( Z \), we read the torsional buckling coefficient \( k_t \) from Fig. C8.18 to be 170.

\[ F_{stcr} = \frac{k_t \pi^2 E}{12 (1 - \nu_e^2)} \frac{r}{L} \quad \text{(See Eq. C8.9)} \]

\[ F_{stcr} = \frac{170 \pi^2 \times 10,700,000}{12 (1 - 0.3)} \left( \frac{0.05}{75} \right) = 735 \text{ psi} \]

The torsional moment developed by this buckling stress is,

\[ T = F_{stcr} J/r \quad J/r = 2m^3 r/t = 2m^3 t \]

hence,

\[ T = F_{stcr} 2m^2 t \]

\[ T = 735 \times 2 \times 50^2 \times 0.05 = 578000 \text{ in.lbs.} \]

The result will also be calculated using Figs. C8.20 and C8.21 based on 99 percent and 90 percent probability respectively and 95 percent confidence. Using Fig. C3.21:-

For \( r/t = 1000 \) and \( L/r = 1.5 \), we read \( F_{st}/E = 0.000082 \). Then \( F_{st} = 0.000082 \times 10,700,000 \times 877 \text{ psi.} \) \( T = 877 \times 2 \pi r \times 50^2 \times 0.05 = 668000 \text{ in.lbs.} \)

Using Fig. C8.20 which is for 99 percent proba-
bility, we read \( \frac{F_t}{E} = 0.000060 \) which gives 
\( F_t = 542 \text{ psi} \) which, in turn, gives allowable
torque \( T = 504000 \text{ in. lbs.} \).

Problem 4. Combined Compression and Bending.

The cylinder of Problem 1 is subjected to
an axial compressive load \( P = 10700 \text{ lbs.} \) and
a bending moment \( M = 282,000 \text{ in. lbs.} \). What is
the margin of safety under this combined load-
ing for 90 percent probability and 95 percent
confidence.

Solution. The interaction equation for this
type of Combined loading from Table C8.1 is,

\[
\frac{R_c}{R_b} = \frac{P}{P_a}. \quad (P_a \text{ from Problem 1 solution is} \ 20350 \text{ lb.})
\]

\[
R_c = 10700/20350 = .527
\]

\[
R_b = M/M_a \quad (\text{From Problem 2 results:} \ M_a = 670,000 \text{ in.lbs.})
\]

\[
R_b = 282,000/670,000 = .421
\]

\[
R_c + R_b = .527 + .421 = .948. \text{ Since the result is}
\]

less than 1.0, we have a small margin of
safety.

M.S. = \[
\frac{1}{R_c + R_b} - 1 = \frac{1}{.948} - 1 = .055
\]

Problem 5. Combined Compression, Bending and
Torsion.

Same cylinder as in Problem 1. The load-
ing is the same as Problem 4 plus a torsional
moment \( T \) of 89300 in. lb. Find the M.S. under
this combined loading for 90% probability.

Solution.

The interaction equation for this com-
bined load system from Table C8.1 is,

\[
\frac{R_c + R_b + R_{st}}{R_c + R_b} = 1 \quad \text{---------} \quad (A)
\]

From Problem 4, \( R_c = .527, R_b = .421, \)
\( R_{st} = T/T_a. \) (From Problem 3 results: \( T_a = 688,000 \text{ in. lbd.} \))

\[
R_{st} = 89300/688,000 = .13
\]

Subst. in Eq. (A):

\[
.527 + .421 + .13 = .965, \text{ (since this result is}
\]

less than 1.0, we have a small margin of
safety. From Table C8.1,

\[
\text{M.S.} = \frac{2}{R_c + R_b + \sqrt{(R_c + R_b)^2 + 4R_{st}^2}} - 1
\]

\[
= \frac{.527 + .421 + \sqrt{(.527^2 + .421^2 + 4(.13^2))}}}{.527 + .421 + \sqrt{(.527^2 + .421^2 + 4(.13^2))}} - 1 = .035
\]

Problem 6. Combined Compression, Bending,
Torsion and Transverse Shear.

The same cylinder as in Problem 1 is sub-
jected to the following loads:

\[
P = 10700 \text{ lbs. compression, } M = 282,000 \text{ in.lbs.,}
\]

\[
T = 715000 \text{ in.lb., } V = 1750 \text{ lb., (transverse}
\]

external shear). Will the cylinder carry this
combined loading under 90 percent probability.

Solution:

From Table C8.1 the interaction equation
for this combined loading is,

\[
\frac{R_c + R_{st}}{R_c + R_b} = 1 \quad \text{---------} \quad (B)
\]

\[
R_c = 10700/20350 = .527
\]

\[
R_b = 282,000/670,000 = .421
\]

\[
R_{st} = 715000/688,000 = .104
\]

The equation for shear stress due to
transverse shear load \( V \) is,

\[
f_s = \frac{V}{I \times r^2} \quad \text{---------} \quad \text{area} = \pi r^2
\]

\[
I = \pi r t \times .6366r = 2 r^3 \quad \text{(c.g.) 6366r}
\]

\[
f_s = \frac{V}{\pi r t (2 r^2)} = \frac{V}{\pi r t}
\]

\[
f_s = 1750/\pi \times 80 \times .06 = 228 \text{ psi.}
\]

From Article C8.12 we use buckling co-
efficient for transverse shear to be 1.25 times
that for torsional buckling.

From Problem 3 the torsional buckling stress
calculated to be \( F_{st} = 875 \text{ psi.} \) Therefore the
transverse shear buckling stress is 1.25 x 875
\( = 1095 \text{ psi, } F_s = \)

\[
R_s = f_s/F_s = 228/1095 = .209
\]

Substituting in Equation (B),

\[
.527 + .104^2 + \sqrt{.209^2 + .421^2} = .975. \text{ This}
\]

result is less than 1.0 so a small margin of
safety exists.

Problem 7. Combined Compression, Bending and
Transverse Shear.

Same cylinder and loads as in Problem 6,
but without the torsional moment load.

Solution: The interaction equation is,
Buckling Strength of Monocoque Cylinders

Using the same cylinder as in previous problems, the combined loading is:

\[ M = 478,000 \text{ in. lbs., } T = 358,000 \text{ in. lb.} \]

will the cylinder buckle under this loading.

Solution:

\[ R_b = 470,000 / 370,000 = .701 \]

\[ R_{st} = 358,000 / 588,000 = .602 \]

The interaction equation from Table C8.1 is,

\[ R_b^{1.3} + R_{st}^{1.3} = 1 
\]

\[ .701^{1.3} + .602^{1.3} = .89 \text{ less than 1.0, therefore cylinder will not buckle.} \]

The margin of safety can be determined graphically by using Fig. C8.21 as follows:

Find point (a) as indicated on Fig. C8.21 using \( R_b = .701 \) and \( R_{st} = .602 \). Draw line from origin 0 through point (a) and extend it to intersect interaction curve at point (b). Projecting downward and horizontally from point (b), we read \( R_b = .78 \) and \( R_{st} = .58 \).

The factor of utilization \( U = .70 / .78 = \)

\[ .856 = .995 \text{. Therefore Margin of Safety} = (1/U) - 1 = (1/.995) - 1 = .02 \]

Problem 2. Combined Compression and Torsion.

Same cylinder as in previous problems.

The loads are \( P = 12900 \text{ lbs.} \) \( T = 358000 \text{ in. lbs.} \) What is M.S.

Solution: The interaction equation from Table C5.1 is,

\[ R_c + R_{st}^{1.3} = 1 \]

\[ R_c = 129000 / 20350 = .634 \]

\[ R_{st} = 358000 / 288000 = .520 \]

\[ .634 + .520^{1.3} = .904 \text{ (less than 1.0, therefore no buckling.)} \]

\[ M.S. = (2 - .634^{1.3} + 4 x .520^{1.3})^{1/3} - 1 = .08 \]

C8.17 Problems Involving Internal Pressure with External Loadings.

The same cylinder and material as was used in Problems 1 to 9 will be used in the following example problems, namely, \( r = 50, t = .06, L = 75, E = 10.7 \times 10^6 \).

Problem 10. Axial Compression with Internal Pressure.

The cylinder is subjected to an axial compressive load of 50000 lbs. and an internal pressure of 5 lbs. per sq. in. What is the margin of safety under this combined load system with 90 percent probability.

Solution. From Problem 1, the buckling compressive stress \( F_{cr} \) with no internal pressure was 1255 psi. To obtain the increase in the compressive due to the internal pressure, we use Fig. C8.11. The horizontal scale parameter is,

\[ \frac{F}{E} = \frac{5}{10,700,000} \times .06 = 4.97 \]

From Fig. C8.11, we read on vertical scale,

\[ \frac{\Delta F_{cr}}{\eta} = .21, \text{ whence} \]

\[ \Delta F_{cr} = (\eta \times .21 \times 10,700,000 \times .06) / 50 = 2250 \eta \text{ psi} \]

Since the stress is below the proportional limit stress the plasticity correction is zero or \( \eta = 1.0 \). Thus \( \Delta F_{cr} = 2250 \text{ psi.} \)

Then the total buckling stress with the internal pressure is \( 1255 + 2250 = 3505 \text{ psi.} \)

Now the internal pressure produces a longitudinal tensile stress in the cylinder wall. This tensile stress is,

\[ F_t = pr / 2t = 5 \times 50 / 2 \times .05 = 2500 \text{ psi} \]

This tensile stress must be cancelled before walls can be subjected to a compressive stress due to an external compressive axial load. The allowable failing load \( P_a \) thus equals,

\[ P_a = 2mt (F_{cr} + \Delta F_{cr} + F_t) \]

\[ = 2 \times 50 \times .05 (3545 + 2250) = 65000 \text{ lb.} \]

M.S. = \( (P_a / P) - 1 = (96000/50000) - 1 = .90 \)

This result shows the tremendous effect of internal pressure in increasing the compressive strength of a thin walled unstiffened circular cylinder. With no internal pressure, the failing compressive load as calculated in
Problem 1. Bending with Internal Pressure.

If the same cylinder as used in the preceding example problems is subjected to a bending moment of 3,000,000 in. lbs. when internal pressure is 5 psi, what is the M.S. use 90 percent probability.

Solution. For bending without internal pressure, the bending buckling stress $F_{bcr}$ was found in Problem 2 to be 1710 psi. The increase in this bending buckling stress due to internal pressure is obtained from the curves in Fig. C8.14. The horizontal scale parameter is,

$$\frac{F}{(\frac{P}{t})^a} = \frac{5}{10,700,000} \left(\frac{50}{.05}\right)^a = 4.67$$

From Fig. C8.14, we read

$$\frac{F_{cr}}{E} = \frac{.51}{.05}$$

whence, $F_{bcr} = (\frac{.51 \times 10,700,000 \times .05}{50} = 5350 \text{ psi}$.

As calculated in Problem 11, the internal pressure of 5 psi produces a longitudinal tensile stress in the cylinder wall equal to $F_t = 2300 \text{ psi}$. The allowable bending moment that cylinder can develop is,

$$M_a = F_t (\text{total}) \frac{L}{r} = F_{t \text{total}} \times \pi r^4 t$$

$$F_{t \text{total}} = F_{bcr} + F_{cr} + F_t = 1710 + 5350 + 2300 = 9360 \text{ psi}$$

$$M_a = 9360 \times \pi \times 50^4 \times .05 = 3750000 \text{ in. lbs}.$$  

M.S. = $(M_a/M) - 1 = (3750000/3000000) - 1 = .25$

Again note the tremendous effect of internal pressure on the bending strength. With no internal pressure, the failing bending moment was 970000 as compared to 3750000 with 5 psi internal pressure.

Problem 11. Combined Compression, Bending and Internal Pressure.

The same cylinder will be subjected to an axial load $P = 510000 \text{ lbs}$., a bending moment $M = 14750000 \text{ in. lbs}$., and an internal pressure of 5 psi. Use 90 percent probability. What is the margin of safety under these load conditions.

Solution:

Interaction equation is, $R_a + R_b = 1$

From Problem 10, the allowable load $P_a$ for compression plus 5 psi, internal pressure is 95000 lbs.

$$R_c = P_a \frac{P_t}{P_a} = \frac{510000/95000}{500} = .536$$

From Problem 11, the allowable bending moment $M_a$ with an internal pressure is 3,750,000 in. lbs.

$$R_b = M_a / M = 14750000/3750000 = .392$$

$$R_a + R_b = .536 + .392 = .928 \text{ (less than 1, thus will not buckle)}$$

M.S. = $(1/1.928) - 1 = .08$

Problem 12. Torsion with Internal Pressure.

If the same cylinder is subjected to an internal pressure of 5 psi, what pure torsional moment $T_a$ will it carry without buckling.

Solution:

With no internal pressure, the torsional shearing buckling stress as calculated in Problem 3 was 876 psi. for 90 percent probability.

The increase in shear buckling stress due to the internal pressure will be determined by use of interaction curve in Fig. C8.22. To use this curve, the external hydrostatic pressure (pext.) to cause buckling of the cylinder must be determined. This buckling stress $F_{cr}$ is determined by equation C8.7, which is,

$$F_{cr} = \frac{k_p \pi^2 t^4}{12 (1 - \nu_e^2)}$$

The buckling coefficient $k_p$ is found by use of Fig. C5.15.

$$Z = \ln^2 (1 - \nu_e^4) / 2 = 2.140 \text{ (See Problem 1)}$$

From Fig. C5.15, we read $k_p = 60$

$$F_{cr} = 60 \pi^2 \times 10,700,000 \times (\frac{.05}{75})^2 = 258 \text{ psi}$$

The external pressure to produce this circumferential stress in the cylinder wall would be

$$F_{ext} = F_{cr} \frac{t}{r} = 258 \times .05/50 = .258 \text{ psi}$$

Fig. C8.22 is now used to find the increase in the torsional shear buckling stress due to
C8.22 Buckling Strength of Monocoque Cylinders

the internal pressure of 5 psi. The lower
scale parameter is,

\[ P_{nt} = P_{ext} = 5 - .563 = 4.44 \]

Using this value we read from C8.22 that

\[ \frac{P_{nt}}{P_{ext}} = 9280 \text{ psi}. \]

This buckling stress will
develop a torsional moment of

\[ T_e = \frac{P_{nt}}{2} (2m \cot^2) \]

\[ = 3840 (2m \times 50^2 \times .06) = 4,150,000 \text{ in. lb}. \]

Reference to the interaction curve in Fig.
C8.22 shows most of the best data falling
below the curve. A 90 or 93 percent probability
would give a curve considerably below the
curve shown. A correction factor of around .8
would seem to be in order or .8 x 3840 =
3072 psi.

The buckling stress can be calculated by
use of the interaction equation:

\[ R_{st}^2 - R_{pt}^2 = 1, \quad R_{st} = \frac{P_{nt}}{P_{ext}} \]

\[ R_{pt} = 5.0/.563 = 9.2. \quad \text{(use minus sign)} \]

\[ \frac{R_{pt}}{R_{st}} = 1 + 19.4, \quad \text{or} \quad R_{pt} = 3340 \text{ psi}. \]

Thus to cause torsional buckling with 5
psi internal pressure requires a torsional
stress of 3340 psi.

Then the torsional moment developed =

\[ 3340 (2m \times 50^2 \times .06) = 3,100,000 \text{ in. lb}. \]

Problem 14. Combined Compression, Bending,
Torsion and Internal Pressure.

The same cylinder is subjected to a
combined loading of \( P = 40^\circ 30, M = 1,770,000, \)
\( T = 700,000 \) and internal pressure of 5 psi.
Will cylinder buckle.

From Problem 10, \( P_a = 85,000 \)

From Problem 11, \( M_a = 3,770,000 \)

From Problem 12, \( T_a = 3,100,000 \)

The stress ratios then are:

\[ \frac{R_a}{R_{st}} = 40,750/35,000 = .51 \]

\[ \frac{R_a}{R_{pt}} = 1,770,000/3,770,000 = .46 \]

\[ \frac{R_a}{R_{nt}} = 700,000/3,100,000 = .22 \]

The interaction equation is \( R_a = R_{pt} + R_{nt} - R_{st}^2 \).

\[ .46 + .46 + .22 = .94 \]

This is less than 1.0, thus cylinder will
not buckle under the given combined loading.

C8.17 Buckling Strength of Thin-Walled (Monocoque)
Conical Shells.

In a multi-stage missile, the upper stages
normally have smaller diameters than the lower
stages, thus a conical shell provides a type of
structure to permit changing the missile
diameter between the various stages.

As for the case of the cylindrical
monocoque shell, the theoretical analysis for
the buckling strength of conical shells by
either the small or large deflection theories
would give results considerably above those given
by tests, thus design of conical shells at
present is based primarily on results obtained
by tests. The material presented in this
chapter will thus be limited to presenting a
few design buckling curves.

C8.18 Allowable Compressive Buckling Stress for
Thin-Walled Conical Shells.

Fig. C8.24 shows design curves for the
compressive buckling stress of a conical shell
as derived from a statistical study of test
data by Hausrath and Ditto (Ref. 16). The
expression for the buckling compressive stress
along a generator at mid-height of cone is,

\[ \frac{P_{cr}}{t} = 2 \frac{P}{P_a} \left( \frac{1}{\cos \theta} \right). \]

where, \( P_{cr} \) is the buckling axial compressive
load on cone.

\( R \) = radius of curvature of cone at mid-
height.

\( t \) = cone wall thickness.

\( \theta \) = semi-vertex angle of cone.

Data from 170 tests by various investiga-
tors were statistically evaluated for expected
men, 90 percent and 93 percent probability
strength levels. Dispersion of data was found
to be slightly less than that of monocoque
shells. Non-linear effects of radius to
thickness ratio or strength deterioration with
length to radius ratio were not discernible.

The "A" level curve in Fig. C8.24 is that
level which could be exceeded by at least 50
percent of the entire test population with 95
percent confidence; that is, the confidence is
95 percent that at least 50 percent of the
compressive strengths all cones will be
attained to exceed the "A" level curve. The
"B" level curve is for 50 percent probability
and 95 percent confidence.
The "A" and "B" curves are recommended for practical design. The "A" level curve is recommended for use for those structures where a single failure would result in catastrophic loss or injury to personnel. The stress level at the small end of the cone should be checked to preclude the possibility of an early failure precipitated by inelastic stresses.

**C8.19 Additional Design Buckling Curves for Thin-Walled Conical Shells.**

Figs. C8.25, 26, 27 gives curves for determining the allowable buckling stress for thin-walled conical shells in compression, bending and torsion respectively. These curves are reproduced by permission of the Boeing Company from their report 02-3617. Fig. C8.28 gives a design curve for the allowable external buckling pressure for thin-walled conical shells, also from same Boeing Report. These curves were used for minuteman interstage and skirt design and the conical shells had ring stiffeners designed by method given in (Ref. 17).

**DEFINITION OF TERMS:**

- $F_C$ = allowable compressive buckling stress (psi)
- $F_B$ = allowable bending buckling stress (psi)
- $F_T$ = allowable torsional buckling stress (psi)
- $F_{cr}$ = compressive stress
- $F_B$ = bending stress
- $F_T$ = transverse shear stress
- $F_{cr}$ = slant height of conical shell
- $t$ = wall thickness
- $a$ = one half cone apex angle
- $V_o$ = Poisson’s ratio
- $r$ = minimum radius of curvature
- $r_{avg}$ = average radius of curvature
- $q_{cr}$ = allowable external pressure (psi)

**Buckling Under Combined Loading**

From Boeing Report (02-3617) the effects of simultaneous compression, bending, shear and overpressure can be considered using the interaction equation:

$$(R_C + R_B + R_V)_{cr} + R_q_{cr} = 1$$

where, $R_C = F_C/F_{cr}$, $R_B = F_B/F_{cr}$, $R_V = \frac{F_V}{1.4 F_{cr}}$

$R_q = q/q_{cr}$

**C8.20 Example Problem.**

A conical shell with wall thickness $t = .05$ and other dimensions shown in Fig. (a) is fabricated from aluminum alloy ($E = 10,700,000$). Determine the following values:

1. Allowable compressive load $P_C$.
2. Allowable bending moment $M_B$.
3. Allowable torsional moment $T_{cr}$.
4. External buckling pressure $q_{cr}$.

**Solution:**

- $p = 10/\cos 10^\circ = 10.16$ in.
- $p/t = 10.16/0.05 = 203$
- $L = 20/\cos 10^\circ = 20.32$ in.
- $L/p = 20.32/10.16 = 2.0$

To find $P_C$:

From Fig. C8.28 with $p/t = 203$ and $L/p = 2.0$, we read $P_C = 10^4 E/\pi$ equals 1.20. Whence $P_C = 1.20 \times 10,000,000/1000 = 12,320$ psi. Then $P_C = P_C \times ( \pi r)^2$

= $12320 \times \pi \times 10 \times .05 = 40,500$ lbs.

To find $M_B$:

From Fig. C8.26 with $p/t = 203$ and $L/p = 2.0$, we read $P_B = 10^4 E/\pi = 1.48$. Whence $P_B = 1.48 \times 10,000,000/1000 = 15820$ psi. Then $M_B = P_B \times r t$

= $15820 \times \pi \times 10^3 \times .05 = 248,500$ in. lb.

To find $T_{cr}$:

From Fig. C8.27, with $p/t = 203$ and $L/p = 2.0$, we read $F_T = 10^4 E / 0.61$. Whence $F_T = .61 \times 10,000,000/1000 = 5560$ psi. Then $T_{cr} = F_T \times ( \pi r t)^2$

= $5560 \times \pi \times 10^3 \times .05 = 205,300$ in. lb.
Fig. C9.25
Allowable Compressive Buckling Stress for Thin-Walled Conical Shells.
(Ref. Boeing Report D2-8047)

NOTE: Curves are valid for elastic range only.
$\theta = \alpha < 119^\circ$

$P_c = \frac{F_c}{E} \times 10^3$

$L/\rho = 4$
$2$
$1$
$0.5$

$F_c = P_c \times 2 \pi r$

Fig. C9.26
Allowable Bending Buckling Stress for Thin-Walled Conical Shells.
(Ref. Boeing Report D2-8011)

NOTE: Curves are valid for elastic range only.
$\phi = \beta < 110^\circ$

$M_{cr} = \frac{F_B}{E} \times 10^8$

$L/\rho = 4$
$2$
$0.5$

$F_B = M_{cr} \pi r^2 t$
BUCKLING STRENGTH OF MONOCOQUE CYLINDERS

To find external buckling pressure \( q_{cr} \):

Use is made of Fig. C8.28.

\[
P_{avg} = \frac{10 + 13.32}{2 \cos 10^\circ} = 11.32 \text{ in.}
\]

\[
\frac{P_{avg}}{t} = 11.92/0.05 = 238
\]

The lower scale parameter of Fig. C8.28 is,

\[
Z = \frac{1 - \nu^2}{E_{avg}} \sqrt{1 - \nu^2}
\]

\[
Z = \frac{20.32}{11.92 \times 0.05} \sqrt{1 - 0.3^2} = 660
\]

From Fig. C8.28 for \( Z = 660 \), we read

\( K_y = 27.5 \). The equation for buckling pressure from Fig. C8.28 is,

\[
q_{cr} = \frac{P_{avg}}{L_{12}} \left( 1 - \frac{\nu^2}{E_{avg}} \right)
\]

\[
q_{cr} = \frac{27.5 \times 10,700,000 \times 0.05 \times 0.3^2}{11.32 \times 20.32 \times 0.05 \times 0.3} = 5.37 \text{ psi}
\]

PROBLEMS

(1) A monocoque circular cylinder is fabricated from aluminum alloy for which \( E \) is 10,500,000. Wall thickness is 0.064. Cylinder diameter is 30 inches. Length is 60 inches. Using a probability factor of 30 percent, determine the following values:

(a) Buckling compressive load for cylinder.
(b) Buckling bending moment for cylinder.
(c) Buckling torsional moment for cylinder.

(2) The cylinder in Problem (1) is subjected to the following design loads. Will the cylinder fail under these various design conditions:

<table>
<thead>
<tr>
<th>Condition 1</th>
<th>Condition 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P = 30,000 \text{ lbs. compression} )</td>
<td>( P = 25,000 \text{ lbs. compression} )</td>
</tr>
<tr>
<td>( M = 50,000 \text{ in. lbs.} )</td>
<td>( M = 45,000 \text{ in. lbs.} )</td>
</tr>
<tr>
<td>( T = 50,000 \text{ in. lbs.} )</td>
<td>( T = 50,000 \text{ in. lbs.} )</td>
</tr>
</tbody>
</table>

(3) If the cylinder in Problem 1 is pressurized to 6 psi, what ultimate compressive load will it carry. What ultimate bending moment will it carry. What ultimate torsional moment will it carry.

(4) A conical shell has a small end diameter of 30 inches, a length of 30 inches, a semi-vertex angle of 8 degrees and a wall thickness of 0.057. Material is aluminum alloy with \( E = 10,500,000 \). Determine the ultimate compressive load, the ultimate bending moment, and the ultimate torsional moment the cone will sustain when each load is acting separately.

Determine the external pressure that will buckle the cone.

REFERENCES


(14) Crant, Bando and Bied. The Effect of Internal Pressure on the Buckling Stress of Thin Walled Circular Cylinders Under Torsion. NACA WRL-67, 1940.


CHAPTER C9
BUCKLING STRENGTH OF CURVED SHEET PANELS. ULTIMATE STRENGTH OF STIFFENED CURVED SHEET STRUCTURES.

PART 1. BUCKLING STRENGTH OF CURVED SHEET PANELS

C9.1 Introduction.

Curved sheet panels represent a common external part of flight vehicle structures. Examples are the skin of the fuselage and the missile. If the curved sheet has no longitudinal stiffeners, failures will occur when buckling occurs. If the curved sheet has stiffening elements attached then the composite unit will not fail on the development of sheet buckling, that is, the composite unit has an ultimate strength much greater than the load which caused initial buckling of the curved sheet panels between the stiffening units commonly referred to as sheet stiffeners. In some flight vehicle designs it may be specified that under a certain percentage of the limit loads that no buckling of the curved sheet panels shall occur. Thus it is necessary to be able to determine what stresses will cause curved sheet panels to buckle and also to determine what external loads will cause a stiffened sheet panel to fail.

C9.2 State of the Theory.

In Chapter 18, the small deflection linear theory was used to determine the compressive buckling stress of flat sheet panels. The theoretical results compare favorably with experimental test results. However, when the same theory is applied to unstiffened curved sheet panels or thin-walled cylinders, the theoretical results are considerably above the test results on such units. A large deflection theory gives closer comparison to test results, but this theory involves a consideration of initial imperfections in the sheet and thus an unknown quantity. For the application of both small and large deflection theories to the buckling of curved sheet and cylinders, the reader is referred to the references listed at the end of Chapter C9, particularly those reports which give the results of such investigators as Donnell, Batdorf and Gerard.

C9.3 Compressive Buckling Stress of Curved Sheet Panels.

The expression for the buckling stress under axial compression is of the same form as for flat sheets, the value of the buckling coefficient $K_b$ having a higher value.

$$ F_{cr} = \frac{K_b \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{b} \right)^2 $$  C9.1

Fig. C9.1 gives curves for determining the buckling coefficient $K_b$. The theoretical derived curve is shown along with the recommended design curves which were dictated by test results. These curves apply for buckling stresses in the elastic range.

C9.4 Shear Buckling Stress of Curved Sheet Panels.

The equation for the buckling shear stress is,

$$ F_{scr} = \frac{K_b \pi^2 E}{12(1 - \nu^2)} \left( \frac{t}{b} \right) $$  C9.2

Figs. C9.2 to C9.5 give curves for finding the buckling stress coefficient $K_b$ to use in Equation C9.2. These curves apply when buckling stresses are in the elastic range.

C9.5 Buckling Strength of Curved Sheet Panels Under Combined Axial Compression and Shear.

The studies by Schildcrout and Stein (Ref. 1) gave the following interaction equation for combined longitudinal compression and shear on curved sheet panels:

$$ R_s^* + R_L = 1 $$  C9.3

where, $R_s = F_{scr}$, $R_L = F_{cr}$

If the longitudinal stress is tension instead of compression, then $R_s$ is considered as a negative compression using the compression allowable. The equation for margin of safety is:

$$ M.S. = \frac{2}{R_L + \nu R_L^* + 4 R_s^*} - 1 $$  C9.4

C9.6 Compressive Buckling Stress of Curved Panels with Internal Pressure.

As for the case of monocoque cylinders, internal outward pressure increases the axial buckling compressive stress of the curved sheet panel. Rafel and Sandlin (Ref. 3) and Rafeil (Ref. 4) performed tests on curved panels under axial compression and internal outward pressure. The results correlate with the interaction equation:

$$ R_c^* + R_p = 1 $$  C9.5
Fig. C9.1 Axial Compressive Buckling Coefficients for Long Curved Plates
Fig. C9.2 Shear Buckling Coefficients for Long Clamped Curved Plates
Fig. C9.3 Shear Buckling Coefficients for Wide Clamped Curved Plates.
Fig. C9.4 Shear Buckling Coefficient for Long Simply Supported Curved Plates.

\[ F_{scr} = \frac{K_a \pi^2 E}{12 (1 - \nu_e^2) \frac{b}{r}} \]

\[ Z_b = \frac{b^2}{rt} (1 - \nu_e^2)^{1/2} \]

(From Ref. 2)

Fig. C9.5 Shear Buckling Coefficient for Wide Simply Supported Curved Plates.

\[ F_{scr} = \frac{K_a \pi^2 E}{12 (1 - \nu_e^2) \frac{b}{r}} \]

\[ Z_b = \frac{b^2}{rt} (1 - \nu_e^2)^{1/2} \]

(From Ref. 2)
BUCKLING STRENGTH OF CURVED SHEET PANELS.
ULTIMATE STRENGTH OF STIFFENED CURVED SHEET STRUCTURES.

where \( R_C = P_C / P_{acr} \)

\[ R_P = \text{ratio of applied internal pressure to external inward pressure that would buckle the cylinder of which the curved panel is a section. Found by use of Fig. C9.16 in Chapter C8.} \]

In using Equation C9.5, the value of \( R_P \) is negative as inward pressure is opposite to the inward acting outward pressure.

C9.7 Shear Buckling Stress of Curved Sheet Panels with Internal Pressure.

As in the case of monocoque cylinders, internal pressure increases the shear buckling stress of the curved sheet panel. Brown and Hopkins (Ref. 5) solved the classical equilibrium equations to determine the effect of radially outward pressure upon the shear buckling stress of curved panels and obtained fair agreement with test data by Rafel and Sandlin (Ref. 3). The test data also correlates with the interaction curve used for the effect of internal pressure upon cylinders in torsion (see Chapter C8). The interaction equation is:

\[ R_S^2 + R_P = 1 \quad \text{---- C9.6} \]

where \( R_S = f_S / P_{acr} \)

\[ R_P = \text{ratio of applied internal pressure to external inward pressure that would buckle the cylinder of which the curved sheet panel is a section. Found by use of Figure C9.16 of Chapter C8.} \]

In using Equation C9.6, \( R_P \) is given a minus sign.

C9.8 Example Problems.

**PROBLEM 1.**

Fig. C9.6 illustrates a circular fuselage section with longitudinal stringers represented by the small circles. The area of each stringer is .18 sq. in. The skin thickness is .04 inches. All material is aluminum alloy with \( E_C = 10,700,000 \). The fuselage frame spacing (a) is 15.75 inches. The fuselage section is subjected to the following load system:

\[ M_y = 600,000 \text{ in.-lb. (causing compression on top half)} \]

\[ V_z = 5175 \text{ lbs. (acting up)} \]

\[ T^* = \text{Torsional moment} = 210,000 \text{ in.-lbs. (acting counterclockwise)} \]

The problem is to determine whether skin panels (A) and (B) will buckle under the given combined loading on the fuselage section.

**Solution.**

To find the bending stresses, the moment of inertia of the cross section about axis y-y is necessary, which axis is also the neutral axis since all material is effective. The moment of inertia will equal 4 times that due to material in one quadrant.

\[ I_y \text{ due to stringers is,} \]

\[ I_y = 4 \left[ .075 \times 20^2 + .15 \left( 19.30^2 + 17.34^2 + 14.14^2 + 10^2 + 5.18^2 \right) \right] = 720 \text{ in.}^4 \]

Due to skin:

\[ I_y = \pi r^2 t = \pi \times 20^2 \times .04 = 1005 \text{ in.}^4 \]

Total \( I_y = 720 + 1005 = 1725 \text{ in.}^4 \).

Consider Skin Panel (A):

\[ r/t = 20/0.04 = 500, \quad a/b = 15.75/6.25 = 3.0 \]

To determine the compressive buckling stress, use will be made of Fig. C9.1 to find the buckling coefficient \( K_C \).

\[ Z = \frac{b^4}{t^2} \left( 1 - \nu^2 \right)^{1/2} \]

\[ = \frac{5.25^2}{20 \times 0.04} \left( 1 - .3^2 \right)^{1/2} = 32.9 \]

From Fig. C9.1, \( K_C = 14 \)
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[ F_{cr} = \frac{K_a \pi^2 S}{12 \left(1 - \nu^2\right)} \left(\frac{t}{b}\right)^2 \]

\[ = \frac{14 \pi^2 \times 10,700,000}{12 \left(1 - 0.3^2\right)} \left(\frac{0.04}{5.25}\right)^2 = 7860 \text{ psi} \]

To find the shear buckling stress, we use Fig. C9.2. Z is the same as calculated above or 32.9. Thus from Fig. C9.2 we read for \( a/b = 3 \), that \( K_a = 20 \).

\[ F_{s,cr} = \frac{20 \pi^2 \times 10,700,000}{12 \left(1 - 0.3^2\right)} \left(\frac{0.04}{5.25}\right)^2 = 11,200 \text{ psi} \]

The bending stress at midpoint of Panel (A) will be calculated:

\[ f_c = \frac{MZ}{I_y} = \frac{600,000 \times 19.7}{1725} = 5850 \text{ psi} \]

Thus stress ratio \( R_c = \frac{f_c}{F_{s,cr}} = 5850/11200 = .5274 \)

Shear stress on Panel (A) due to torsion is:

\[ f_s = T/2A_t \text{, where } A \text{ is inclosed area of fuselage cell.} \]

\[ f_s = 210,000/2 \times \pi \times 20^2 \times \frac{0.04}{5} = 2090 \text{ psi} \]

The panel is also subjected to shear stress due to transverse shear of \( V_z = 5175 \text{ lbs.} \)

The shear flow equation is,

\[ q = \frac{V_z}{I_y} \text{ZZA} = \frac{5175}{1725} \text{ZZA} = 3 \text{ ZZ}A. \]

The shear flow will be zero on Z axis. The shear flow at top edge of Panel (A) will be due to effect of one-half the area of stringer (1).

\[ q_{a1} = 3 \times 0.075 \times 20 = 4.5 \text{ lb./in.} \]

Area of skin between stringers (1) and (2) is 5.25 \times 0.04 = 0.21 \text{ in.}^2.

Distance from centroid of Panel (A) from neutral axis is \( Z = r \sin \alpha / \alpha. \quad \alpha = 15^\circ \). which gives \( Z = 19.8 \text{ in.} \)

Thus \( d_{za} = 4.5 + 3 \times 21 \times 19.8 = 17 \text{ lb./in.} \)

Then average shear flow on panel is \( (17 + 4.5)/2 = 10.75. \)

Then shear stress = \( q/t = 10.75/0.04 = 269 \text{ psi}. \)

This shear stress has the same sense or direction as the torsional shear stress so we add the two to obtain the true shear or:

\[ f_s(\text{total}) = 259 + 2090 = 2359 \text{ psi}. \]

Then shear stress ratio \( R_s = f_s/F_{s,cr} = 2359/11200 = .21 \).

The interaction equation for combined compression and shear is,

\[ R_c + R_s^2 = 1 \]

\[ .874 + .21^2 = .913 \text{. This is less than 1.0 so panel will not buckle.} \]

**M.S. = \( \frac{2}{.874 + \sqrt{.874^2 + 4 \times .21}} - 1 = .09 \)**

Consider Skin Panel (B).

Arm \( Z \) to midpoint of panel = 15.88 in.

\[ f_c = Mz/I = 600,000 \times 15.88/1725 = 5520 \text{ psi} \]

\[ R_c = 5520/7850 = .704 \]

The torsional shear stress is the same on all panels or \( f_s = 2090 \text{ psi} \) as previously calculated.

Shear flow \( q \) due to transverse shear load:

\[ q = \frac{V_z}{I_y} \text{ZZA} = 3 \text{ ZZ}A. \]

**Calculation of ZZ A at upper edge of panel:**

For stringers = \( .075 \times 20 + .15 \times (19.3 + 17.34) = 70 \)

For skin: Area = \( 2 \times 5.25 \times .04 = .42 \)

Vertical distance \( Z \) to centroid of skin portion = \( r \sin \alpha / \alpha. \quad \alpha = 30^\circ \). The result is \( Z = 19.1 \text{ in.} \)

\[ q = \frac{0.075}{5.25} \times 15.03 = 0.03 \text{ ZZ}A. \]

A similar calculation for shear flow at lower edge of Panel (B) would give \( q \) equal 55.0. Thus average shear flow on panel is \( (55 + 45.09)/2 = 50.04 \).

Then \( f_s = q/t = 50.14/0.04 = 1251 \text{ psi.} \)

The total shear stress \( f_s \) on panel then equals 1251 + 2090 = 3341 psi.

\[ R_s = f_s/F_{s,cr} = 3341/11200 = .299 \]

**Substituting in interaction equation \( R_c + R_s^2 = .704 + .299^2 = .793 \). The result is less than 1.0, thus panel will not buckle.**

The student should check other panels for buckling and compare their margins of safety.
BUCKLING STRENGTH OF CURVED SHEET PANELS.

ULTIMATE STRENGTH OF STIFFENED CURVED SHEET STRUCTURES.

General Comment:-

In general the compressive stress is the dominant factor in causing the panel buckling. Thus to increase the buckling stress of the panels and also to give a more effective stringer arrangement to carry the bending moment, the stringers should be spaced closer in the top and bottom regions of the cross-section and with increased spacing as the neutral axis is approached.

PROBLEM 2.

The fuselage section in Problem 1 is subjected to an internal outward pressure of 5 psi. What would be the compressive buckling stress of a panel and also the shear buckling stress with this internal pressure existing.

Solution.

From Art. C9.6, the interaction equation is,

\[ R_0^2 + R_p = 1 \]

\[ R_0 = \frac{R_p}{F_{cr}} \]

From Problem 1, the compressive buckling stress \( F_{cr} = 7850 \) psi.

To evaluate \( R_p \), the external inward radial acting pressure that would cause buckling of a circular cylinder having a radius equal to that of the curved sheet panel must be determined. Use is made of Fig. C9.18 of Chapter C8 to determine the buckling stress under such a loading. The lower scale parameter for Fig. C9.18 is,

\[ \sqrt{\frac{1}{r_t}} = \sqrt{1 - \nu_s^2} \]

\[ = \frac{15.75}{20} x .04 \sqrt{1 - .3^2} = 296 \]

From Fig. C9.16, we read \( K_y = 19 \)

\[ F_{cr} = \frac{K_y \pi^2 E}{12 (1 - \nu_s^2)} \left( \frac{t}{L} \right)^4 \]

\[ = \frac{19 \pi^2 \times 10,700,000 \times .04}{12 (1 - .3^2)} \left( \frac{15.75}{20} \right)^4 = 1215 \text{ psi} \]

The external radial pressure to produce this buckling stress is,

\[ p = F_{cr} t/t = 1215 x .04/20 = 2.43 \text{ psi} \]

\[ R_p = 5.0 \]

Subt. in interaction equation with a minus sign for \( R_p \),

\[ \frac{f_s^2}{F_{cr}} - R_p = 1 \]

\[ \frac{f_s^2}{7850} - 2.06 = 1, \text{ or } f_s = 13750 \text{ psi} \]

Thus the internal outward pressure of 5 psi increases the axial compressive buckling stress from 7850 to 13750 psi.

Shear Buckling Stress Under 5 psi Internal Radial Pressure.

From Art. C9.7, the interaction equation is,

\[ R_s^2 + R_p = 1 \]

From Problem 1, \( F_{cr} \) or our panel was 11200 psi. The value of \( R_p \) is determined as above or \( R_p = 2.06 \). Subt. in interaction equation:

\[ \frac{f_s^2}{F_{cr}} = 2.06 = 1 \]

\[ \frac{f_s^2}{11200} = 2.06, \text{ or } f_s = 19500 \text{ psi} \]

The internal pressure of 5 psi thus increases the shear buckling stress from 11200 to 19500 psi.

PART 2. ULTIMATE STRENGTH OF STIFFENED CYLINDRICAL STRUCTURES.

C9.9 Introduction.

A cylindrical structure composed of a thin skin covering and stiffened by longitudinal stringers and transverse frames or rings is a common type of structure for airplane fuselages, missiles and various types of space vehicles, and such structures are often referred to as the semi-monocoque type of structure. The design of a semi-monocoque structure involves the solution of two major problems, namely, the stress distribution in the structure under various external loadings and the check of the structure to see if these stresses can be safely and efficiently carried by individual components of the structure as well as the structure acting as a whole.

C9.10 Types of Instability Failure of Semi-Monocoque Structures.

1. Skin Instability.
In general, thin curved sheet panels buckle under relatively low compressive stress and thus if design requirements specified no buckling of the sheet under limit or design loads, the sheet would have to be relatively thick or the stringers placed very close together and the fuselage or body structure would be unsatisfactory from a strength weight standpoint. In missile structures, internal pressurization increases the buckling stress greatly, thus the buckling weakness of thin sheet is improved, but keeping a structure pressurized under all operating conditions has its difficulties.

In a semi-monocoque body, the longitudinal stringers provide efficient resistance to compressive stresses and buckled sheet panels can transfer shear loads by diagonal tension field action, thus the buckling of the sheet panels is not an important factor in limiting the ultimate strength of the over-all structural unit. When buckling of the skin panels takes place, a stress redistribution over the entire structure takes place, thus it is important to know when skin buckling begins. Furthermore, design requirements may often specify that no skin buckling should take place under a certain percent of the limit or design loads. The equations and design curves in Part 1 of this chapter can be used to determine the buckling stress of curved sheet panels under various stress systems.

(2) Panel Instability.

The internal rings or frames in a semi-monocoque structure such as a fuselage divide the longitudinal stringers and their attached skin into lengths called panels. If these frames are sufficiently rigid, a semi-monocoque structure if subjected to bending will fail on the compression side as illustrated in Fig. C9.7a. The stringers act as columns with an effective length equal to the panel length which is the ring spacing. Initial failure thus occurs in a single panel and thus is referred to as a panel instability failure. In general, this type of failure occurs in most practical aircraft and aerospace semi-monocoque structures because the rings are sufficiently stiff to promote this type of failure. Since the inside of a fuselage carries various loads, such as passengers, cargo, etc., the rings must act as structural units to transfer such loads to the shell skin, thus requiring rings of considerable strength and stiffness. Even lightly loaded frames must be several inches deep to provide conduits required in various installations to pass through the web of the ring cross-section, thus providing a relatively stiff ring for supporting the stiffeners in their column action. When the skin buckles under shear and compressive stresses, the skin panels transfer further shear forces by semi-diagonal tension field action which produces additional axial loads in stringers and also bending which must be considered in arriving at the panel failing strength. This subject is treated in detail in Part 2 of Chapter C11.

(3) General Instability.

In general instability, failure is not confined to the region between two adjacent frames or rings but may extend over a distance of several frame spacings as illustrated in Fig. C9.7b for a stiffened cylindrical shell in bending. In panel instability, the transverse stiffeners provided by the frames on rings is sufficient to enforce nodes in the stringers at the frame support points as illustrated in Fig. C9.7a. Any additional stiffeners in excess of this amount does not contribute to additional buckling strength. General instability may thus occur when the stiffeners of the supporting frames is less than this minimum value.


The stresses in a stiffened cylindrical structure such as used in typical fuselage or missile design can be fairly accurately determined by the modified beam theory as presented in Chapter A20. A more rigorous approach is given in Chapter A8 involving matrix formulation but this approach requires the use of a large electronic computer to handle the required calculations. For details of applying the modified beam theory, the reader should refer to Articles A20.5 and A20.4 of Chapter A20. In the example problem solution as given in Article A20.4, the effective area to use for the curved sheet was based on the ratio of the buckling stress of the curved panel to the bending compressive stress on the panel due to bending of the entire effective cross-section of the fuselage under the design loads. In the example problem as given in Table A20.2, a conservative buckling compressive stress equal to .3 E/\(r^2\) was used for the curved panel and no consideration of the effect of shear stress on the compressive buckling stress was considered.

A more accurate procedure would be to calculate the effective area of the curved panels taking into account the influence of combined compression and shear on the buckling strength of the panel. Thus in Table A20.2 on page A20.5 of Chapter A20, the shear stress on each curved panel should also be calculated and then the buckling stress of the panel under the combined compression and shear calculated.

The buckling stress under pure compression
and shear should be calculated using Equations C9.1 and C9.2. The buckling stress under combined compression and shear is given by the interaction equation:

\[ R_C + R_S^2 = 1, \text{ where } R_C = \frac{f_c}{F_{cr}}, R_S = \frac{f_s}{F_{s,c}}. \]

The expression for margin of safety is,

\[ \text{M.S.} = \frac{2}{R_C + \sqrt{R_C^2 + 4R_S^2}} = 1 \]

Let \( f_c \) be the compressive stress that will buckle the curved sheet panel when subjected to combined compression and shear when the ratio of the applied compressive and shear stresses is a constant. Then,

\[ f_c = f_c \left( \frac{2}{R_C + \sqrt{R_C^2 + 4R_S^2}} \right) \]

These \( f_c \) values should then replace the values in column 5 of Table A20.2.

The author has noted that one aerospace company in their missile design uses only 90 percent of the theoretical buckling stress in computing the effective area of the buckled curved panels. This correction assumes that the curved sheet fails to hold the buckling stress as the fuselage section as a whole is further loaded and the curved sheet suffers more buckling distortion.

C9.12 Panel Instability Strength.

Panel instability means failure of the stringer and its effective skin between two adjacent frames. The bending of the stiffened shell as a whole produces a compressive load or stress on the stringer. The semi-tension field action of the skin after buckling produces an additional compressive load on the stringer and also a bending moment.

The compressive stress due to bending of the stiffened shell as a whole is found by the methods discussed in Article C9.11. The additional stringer loads due to semi-tension field action are determined by the theory and procedure given in Part 2 of Chapter C11.

These calculated stringer loads are then compared to the stringer strength to determine whether a positive margin of safety exists. The local crippling and column strength of a stringer plus its effective skin can be found by the theory and analysis methods given in Chapter C7. The bending strength of the stringer cross-section can be found by the theory and analysis method given in Chapter C3. The strength of the stringer in combined compression and bending is found by use of the proper interaction equation.


A great deal of theoretical and experimental work has been done on the subject of general instability of stiffened shells. The general goal in the design of such structures is to insure the frames have sufficient stiffness so as to prevent the type of failure illustrated in Fig. C9.7b or, in other words, to insure the type of failure illustrated in Fig. C9.7a which is panel instability.

Shauley (Ref. 6) has derived an expression for the required frame stiffness to prevent general instability failure of a stiffened shell in pure bending.

\[ (EI)f = C_f MD^2/L \]

In a study of available test data, \( C_f \) was found to be 1/16000. Thus,

\[ (EI)f = MD^2/16000L \]

where, \( E \) = modulus of elasticity

\( I \) = moment of inertia of frame section

\( D \) = diameter of stiffened shell

\( L \) = frame spacing

\( M \) = bending moment on shell

Becker (Ref. 7) in a comprehensive study of most published theoretical and experimental material relative to the general instability of stiffened shells, summarizes the results of his studies as given in Table C9.1.

Bending.

For the case of bending, the constant of 4.80 in the equation given in Table C9.1 is for the condition where the frames are attached to the skin between the stringers. For frames not attached to the skin between stringers, the constant should be 3.25.

The effective sheet width for use with the stringers may be found from the equation,

\[ \frac{b}{d} = 0.5 \left( \frac{F_{cr}}{F_c} \right)^{1/2} \]

where, \( w_s \) = effective width of skin per side of stringer (in.)

\( b \) = stringer spacing (in.)

\( F_{cr} \) = compressive buckling stress for curved skin panel

\( F_c \) = compressive stress at bending general instability (psi)
# Table C9.1

(Ref. 7)

## Theoretical General Instability Stresses of Orthotropic Circular Cylinders

(Results are based on the assumption that spacings of longitudinal stiffeners and circumferential frames are uniform and small enough to permit assumption that cylinder acts as orthotropic shell)

<table>
<thead>
<tr>
<th>Loading</th>
<th>Moderate-length cylinders</th>
<th>Long cylinders</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bending</strong></td>
<td>$F_C = gE (t^2 t)^{1/2} / R t$</td>
<td>$F_C = gE (t^2 t)^{1/2} / R t$</td>
</tr>
<tr>
<td></td>
<td>$g = 4.80 \left[ (b/d) (\omega / \omega) (t_b / t) (\rho_b / \rho)^2 \right]^{1/4}$</td>
<td></td>
</tr>
<tr>
<td><strong>External radial or hydrostatic pressure</strong></td>
<td>$F_Y = 5.51 E \left( \frac{t}{R} \right)^{1/2} \left( \frac{t}{R} \right)^{1/2} \left( \frac{R}{L} \right)^{1/2}$</td>
<td>$F_Y = 3E (\rho / R)^{2}$</td>
</tr>
<tr>
<td><strong>Torsion</strong></td>
<td>$F_{ST} = 3.46 \left( \frac{t}{R} \right)^{1/2} \left( \frac{t}{R} \right)^{1/2} \left( \frac{R}{L} \right)^{1/2}$</td>
<td>$F_{ST} = 1.754 \left( \frac{t}{R} \right)^{1/2} \left( \frac{t}{R} \right)^{1/2} \left( \frac{R}{L} \right)^{1/2}$</td>
</tr>
</tbody>
</table>

$F_C$ = Compressive stress at bending general instability (psi)

$F_Y$ = Circumferential normal stress under external pressure at general instability (psi)

$F_{ST}$ = Shearing stress at torsional general instability (psi)

$b$ = Stringer spacing (in.)

$d$ = Frame spacing (in.)

$R$ = Cylinder radius (in.)

$t$ = Skin thickness (in.)

$A_S$ = Stringer area (in. $^2$)

$A_f$ = Frame section area (in. $^2$)

$t_b$ = Distributed stringer area = $A_S / b$

$t_f$ = Distributed frame area = $A_f / d$

$I_b$ = Bending moment of inertia of frame section (in. $^4$)

$I_f$ = Distributed bending moment of inertia of frame = $I_b / d$

$\omega_S$ = Stringer section radius of gyration (in.)

$\omega_f$ = Frame section radius of gyration (in.)

$L$ = Length of cylinder (in.)

$E$ = Modulus of Elasticity.
For the frames the effective skin width should be taken as the total frame spacing (d). For inelastic stresses, the use of the secant modules appears to be applicable on the basis of limited test data.

**External Radial or Hydrostatic Pressure.**

The effective skin width to be used in computing the stringer and frame section properties may be determined from the following equation.

\[
\frac{W_{es}}{b} = \frac{W_{sf}}{d} = 0.5 \left( \frac{F_{cr}}{F_{st}} \right)^{1/2} \tag{C9.9}
\]

The subscript s refers to stringer and f refers to frame. The term d is the frame spacing.

**Torsion.**

The effective skin to be used can be determined from the following equation:

\[
\frac{W_{es}}{b} = \frac{W_{sf}}{d} = 0.5 \left( \frac{F_{sr}}{F_{st}} \right)^{1/2} \tag{C9.10}
\]

where \( F_{sr} \) is the shear buckling stress for the curved skin panel. \( F_{st} \) is the torsional shear stress at torsional general instability (psi).

The equations for torsion as given in Table C9.1 would not apply to shells in which there is a strong tension field that could introduce appreciable secondary stresses in the frame. The reader should refer to Part 2 of Chapter C11 for a treatment of this subject involving the effect of semi-tension field action in the skin panels.

**Transverse Shear General Instability.**

From Ref. 7, it is stated that a conservative shear general instability shearing stress may be made by utilizing the relation.

\[
F_{s} = 0.35 \frac{F_{st}}{d} \tag{C9.11}
\]

where \( F_{s} \) is the transverse shear stress under transverse shear general instability.

**General Instability in Combined Torsion and Bending.**

From (Ref. 7) the following interaction relation may be used to compute the permissible combinations of applied torsion and bending moments to a stiffened cylinder.

\[
\frac{M}{M_0} + \left( \frac{T}{T_0} \right)^2 = 1 \tag{C9.12}
\]

where \( M \) = applied moment
\( M_0 \) = moment causing bending general instability acting alone
\( T \) = applied torsional moment
\( T_0 \) = torsional moment causing torsional general instability acting alone

**General Instability in Combined Transverse Shear and Bending.**

(Ref. 7) concludes there is no interaction for this combination of transverse shear and bending loads. General instability occurs only for either type of loading acting alone and thus both loadings may be examined separately.

**C9.14 Buckling of Spherical Plates Under Uniform External Pressure.**

Classical Theory using 0.3 for Poisson’s ratio gives the following buckling stress for a perfect spherical shell subjected to a uniform external pressure:

\[
F_{p} = 0.606 \frac{E}{t} \tag{C9.13}
\]

Available test data on practical shells show this theoretical buckling stress to be much too high. Thus to satisfy experimental results, reduced values must be used. The buckling equation which is similar to that for curved plates, under external pressure (from Ref. 2) is,

\[
F_{p} = \frac{K_{p} \pi^2 E}{12 \left( 1 - \nu^2 \right)} \frac{t}{d^2} \tag{C9.14}
\]

Fig. C9.8 shows curves for determining the buckling coefficient \( K_{p} \) and also shows how test data falls considerably below the theoretical buckling curve. Equation C9.14 is for buckling stresses below proportional limit stress of material.

Report AS-D-569 of the Aeronautics Division of General Dynamics Corp. from a statistical study of test data gives the following equations for the buckling stress of spherical shells under uniform external pressure for use in preliminary design.

For Mean expected value:

\[
F_{p} = \frac{0.1561 \frac{E}{t}}{r \sin \alpha^{2/3}} \tag{C9.15}
\]

For probability = 90 percent and .95 confidence factor:

\[
F_{p} = \frac{0.1138 \frac{E}{t}}{r \sin \alpha^{2/3}} \tag{C9.16}
\]

For 99 percent probability and .95 confidence factor:
BUCKLING STRENGTH OF CURVED SHEET PANELS.

PROBLEMS

(1) The fuselage cross-section as given in Fig. C9.6 of example problem 1 is changed by increasing the skin thickness to .05 inches. The design loads are increased to the following values:

- \( M_y = 700,000 \text{ in. lb.} \), \( V_z = 6040 \text{ lbs.} \)
- \( T = 245,000 \text{ in. lb.} \)

Will any of the skin panels buckle under this combined loading.

(2) The fuselage section as given in problem 1 above is subjected to an internal outward pressure of 6 psi. What would be the compressive and shear buckling stress for the skin panels under this internal pressure.

REFERENCES

(1) Schildcrout & Stein. Critical Combinations of Shear and Direct Axial Stress for Curved Rectangular Panels. NACA T.N. 1926.


(3) Rafel & Sandin. Effect of Normal Pressure on the Critical Compression and Shear Stress of Curved Sheet. NACA WRL-37.

(4) Rafel. Effect of Normal Pressure on the Critical Compressive Stress of Curved Sheet. NACA WRL-266.


CHAPTER C10
DESIGN OF METAL BEAMS. WEB SHEAR RESISTANT (NON-BUCKLING) TYPE.

PART 1. FLAT SHEET WEB WITH VERTICAL STIFFENERS.

C10.1 Introduction.

The analysis and design of a metal beam composed of flange members riveted or spot-welded to web members is a frequent problem in airplane structural design. In this chapter, the general theory for beams with non-buckling webs is considered. In Chapter C11, the more common case where the beam web wrinkles and goes over into a semi-tension field condition is considered. The advantages and disadvantages of the non-buckling and the buckling or semi-tension field web are discussed in Chapter C11. The general beam theory as given in this chapter is basic to that given in Chapter C11, thus the student should study this chapter before C11.

Fig. C10.1 (From A&A Circular No. 522)

Fig. e Truss type of beam with channel section flange member.

C10.2 Flange Design.

For strength/weight efficiency, the beam flange should be designed to make the radius of gyration of the beam section as large as possible, and at the same time maintain a flange section which will have a high local crippling or crushing stress. Furthermore, the flange sections for large cantilever beams which are frequently used in wing design should be of such shape as to permit efficient tapering or reducing of the section as the beam extends outward. This tapering of section should also be considered from a fabrication or machining standpoint. The most efficient flange from a strength/weight standpoint might be very costly or entirely impractical from a fabrication and assembly standpoint.

Fig. 1 Wing beam of Boeing "Clip- per" flying boat. Basic flange section is square tube.

Fig. C10.1 shows a few typical metal beam sections for externally braced wings. Such beams must carry large axial compressive loads as well as bending loads. Fig. C10.2 shows typical beam flange sections for cantilever
metal covered wing construction. Sections (a), (b), (c) and (d) are typical beam flange sections for wide box beams where additional stringers or skin stiffeners are also used to provide bending resistance. These flange sections are generally of the extruded type although such sections as (b) and (c) are frequently made from sheet stock. These flange sections are almost always used with a beam web composed of flat sheet, which is stiffened by vertical stiffeners riveted or spot-welded to the web or stiffened by beads or flanged lightening holes.

Figs. (a) and (f) illustrate two types of flange sections used in truss beams which lend themselves readily to connection with truss web members. Beam flange sections (g) and (h) are typical sections for wing construction in which no additional spanwise stringers are used. In section (g) tapering of sectional area is provided by first omitting the reinforcing places, and then gradually decreasing the extruded shape by machining until only a small angle remains. In section (a) a gradual decrease in section area is produced by milling out the center portion to form an H section and then cutting this section down finally to a simple angle.

C10.3 Allowable Flange Design Stresses.

The calculating of the stresses in the beam flanges is in general not a difficult procedure if the usual assumptions are made in the flexure theory. The question as to what flange stresses will cause failure is the difficult one from a theoretical standpoint. The only sure way to determine the design allowable is to make sufficient static tests of specially designed test beams.

For beam sections as illustrated in Fig. C10.1, the following tests are usually necessary for forming the basis of design allowable stresses:

1. A test subjecting the beam to pure bending.
2. A test of a short length of beam in bending so that failure will occur in web instead of flange.
3. A test of a short length of beam in pure compression to obtain local crippling strength.
4. Several tests of beams in combined bending and compression, using different ratios of bending to compressive loads.

Enough data from the above tests can usually be obtained to give rather complete allowable stress curves for the design of the beam. For the approximate design of beam flanges, the method given in this chapter can be used.

The beam flange sections (a), (b), (c) and (d) of Fig. C10.2 are stabilized by the shear covering and also by the beam web; thus compressive tests of short lengths to obtain crippling stress, and a test of a length equal to the wing rib spacing should give sufficient information on which to base design allowances. Several lengths of the flange section for the truss type of beam should be tested in compression to obtain the column curve since the distance between panel points of the truss will vary.

Before designing any test beams, the structural designer would like to know approximately what his proposed beam flange sections will carry from a stress standpoint, since it is desirable to make test specimens closely approximate to the sections to be finally used in the completed structure. For most of the sections of Fig. C10.2, the ultimate stresses can be calculated approximately by the methods of Chapter C7. For heavy sections similar to (g) and (h) of Fig. C10.2, where the ultimate stresses fall far above the yield strength of the material, and where some parts of the section buckle before other parts, and also where two different kinds of material are used in the same flange section, a logical procedure in trying to calculate ultimate strength of the section would be to make use of the stress-strain diagrams of the materials.

C10.4 Use of Stress-Strain Diagrams in Computing Beam Flange Bending Allowable Design Stresses.

In the beam type of wing construction, where the flange material is concentrated over the web members instead of distributed over the surface in the form of stringers, the allowable ultimate compressive stress which can be developed is considerably above the yield strength of the material since the flange is composed of a section with thick elements which promotes a high crippling stress and since the beam flange is stabilized in both vertical and horizontal directions by the web and skin covering respectively, the influence of column action is negligible. Fig. (a) illustrates this type of beam. The general flexure formula assumes that stress is proportional to strain which is correct for stresses below the proportional limit of the beam material, but the ultimate resisting stresses for the flange of a beam such as in Fig. (a) is far above the proportional limit, thus the actual stress distribution is more like the dashed line in Fig. (b) instead of the triangular distribution as assumed in the classical beam theory. Thus to obtain beam fiber stresses above the proportional limit, it is necessary to consider strain and the stress which accompanies such a strain, which relation-
ship can be obtained from the stress-strain diagram of the material. A straight line distribution for strain, that is, plane sections remains plane after bending is a reasonable one, and verified by tests.

In a beam in bending, one side is in tension and the other in compression. The tensile and compressive stress-strain diagrams for materials like aluminum alloy are different above the proportional limit, and the same unit strain will cause different stresses on the two sides of the beam. In frequent cases of large beam design, the beam flange may be composed of two kinds of material and certain portions for attaching to skin or web may buckle before the ultimate strength of the section as a whole is obtained. The solution for the ultimate internal resisting moment for such a beam requires that consideration be given to the stress-strain diagram of the various materials and units making up the beam section. This general method of approach in studying the ultimate internal resisting moment of a beam section can best be explained by an example problem.

**Example Problem**

Fig. C10.3 shows the cross-section of a beam in a metal covered wing. The main flange members are composed of heavy 24ST extruded shapes. The extrusions are reinforced by 24ST sheet strips. The beam web is made from 24ST alclad material. The problem is to calculate the ultimate internal resisting moment for this beam section.

Fig. C10.4 shows the stress-strain diagram for these various materials. The 1/8 inch thick outstanding legs of the extrusions act as a plate stiffened on three sides and free on the fourth. These legs will buckle at a stress of 35,000 psi in compression as determined by the methods given in Chapter C7. The stress-strain diagram of the two outstanding legs will be horizontal at 35,000 psi as shown in Fig. C10.4. Although the legs buckle, they will tend to hold the buckling stress under further flange strain. In Fig. C10.5, each beam section has been broken down into narrow horizontal strips designated from (a) to (w). Only that portion of the web in way of flange has been considered in this example. Fig. C10.5a shows the strain distribution assumed for the trial solution. A heavy aluminum alloy section in compression will usually fail at a strain between .008 to .01 inch per inch if column failure is prevented. In Fig. C10.5a, the compressive unit strain at the upper beam fiber has been taken as .008"/". The neutral axis of the section has been assumed at 1.25" above center line of beam. Taking zero strain at this point the beam section strain is as shown in Fig. C10.5a.

For equilibrium the total compressive bending stresses above the neutral axis must equal the total tensile bending stresses below the neutral axis.

Tables C10.1 and C10.2 give the detail calculations for calculating the resisting moment. The bending unit stress in column (5) is obtained from Fig. C10.4 using the unit strain in column (4). If the summation of column (5) in each Table is the same, the assumed location of the neutral axis is correct. The total ultimate resisting moment for this section equals 10332000 + 1360110 = 2393000 in. lb. Using the ordinary beam formula with properties about the geometric neutral axis as given in Fig. C10.3 and taking extreme fiber stresses of 46000 and 62000 psi which correspond to stresses as per strain diagram of Fig. C10.5a, the internal resisting moment would equal, $M = \int f/\sigma = 46000 \times 436/969 = 2297000$ in. lb. as compared to 2392000 in. lb. by the more logical solution, which is a difference of 16 percent. The discrepancy would still be larger if the outstanding leg (a) did not buckle, since the more exact solution only allowed 35000 psi on this element whereas the general beam formula stresses it to 46000 psi.

**Trial and Error Approach**

The location of the neutral axis is unknown, thus the calculations in Tables C10.1 and 2 are for the final assumption which is the true location of the neutral axis. The general procedure would be as follows: Assume neutral axis as center line axis of beam, and find total axial load on each side of axis. The results will usually show excess load on one flange. For the next trial move neutral axis so as to give excess load on other flange. Plot the results as indicated in Fig. C10.5b to obtain true location of neutral axis and then make final calculations as illustrated in Tables C10.1 and 2.
C10.5 Flange Strength (Crippling).

In many cases of beam design, the flange is braced laterally because it is part of a cell construction and the sheet covering which is fastened to the beam flange provides a continuous lateral bracing to the flange or prevents lateral bending-column action for the beam flange. The beam web prevents column bending of the flange in the web direction, thus the flange fails by local crippling action and the crippling stress is determined by the methods of Chapter C7. In many cases the beam loads are relatively small and thus the area required for the flange may be relatively small, which means a flange shape with elements of small thickness, thus the failing local strength may be in the elastic stress range or relatively low.
In many cases, such as a frame in a fuselage, the inner flange of the frame cross-section is free from a lateral brace, thus provisions must be taken to brace the flange laterally or the flange must be designed for lateral column action. This subject is discussed further in Chapter D3.

C10.6 Web Strength. Stable Webs.

A stable web beam is one that carries its design load without buckling of the web, or in other words it remains in its initially flat condition. The design shear stress is not greater than the buckling shear stress for the individual web panels and the web stiffeners have sufficient stiffness to keep the web from buckling as a whole.

In general, a thin web beam with web stiffeners designed for non-buckling is not used widely in flight vehicle structures as its strength/weight ratio is relatively poor. In built-in or integral fuel tanks, it is often desirable to have the beam webs and the skin undergo no buckling or wrinkling under the design loads in order to give better insurance against leaking along riveted web panel boundaries.

The student should realize that the buckling web stress is not a failing stress as the web will take more before collapse of the web takes place, thus in general the web is not loaded to its full capacity for taking load and the web stiffeners are only designed for sufficient stiffness to prevent web buckling and not for the full failing strength of the web.

Equation for Web Buckling Shear Stress

Equation C5.7 of Chapter C5 gives the bending buckling stress. It is

$$ F_{cr} = \frac{\eta_b n_b \pi^2 E}{12(1 - \nu_e^2)} \left(\frac{b}{t}\right)^2 $$

In solving this equation, Figs. C4.15 and C5.8 of Chapter C5 are used.

Buckling of Web Panels Under Combined Shear and Bending

From Art. C5.11 of Chapter C5, the interaction equation for a flat sheet panel under combined bending and shear is,

$$ R_b^2 + R_s^2 = 1 $$

The expression for margin of safety is,

$$ M.S. = \frac{1}{\sqrt{R_b^2 + R_s^2}} - 1 $$

C10.7 Web Bending and Shear Stresses.

Since the web is designed not to buckle under the design loads, the web will be effective in taking bending stresses and the following well known flexural shear stress equation applies.

$$ f_b = My/I_x $$

where $I_x$ is the moment of inertia of the beam section and the web is included.

In Art. A14.3 of Chapter A14, the well known flexural shear stress equation was derived, namely,

$$ f_s = \frac{V}{1b} / ydA $$

Since the term $/ ydA$ is maximum for a section at the neutral axis, the shearing stress in a beam will be maximum at the neutral axis. In general, the webs of aircraft beams are relatively thin; thus the term $/ ydA$ for the web is quite small so that the shearing stress intensity over the web is approximately uniform. Thus a simple formula $f_s = V/hb$ has been widely used for calculating the maximum web shearing stress. In this equation $h$ is a distance which will make the shear stress $f_s$ check the maximum value of the shear stress as given by equation C10.5.

A simple consideration of the internal stresses on a small element cut from a beam in bending and sheal will indicate what value of $h$ to use in the simplified shear equation $f_s = V/hb$.

Fig. C10.6a shows a beam element of length $dx$ cut from a beam which is subjected to a
The bending moment which produces compression on the upper portion. The bending moment on section (AB) is M and on section (CD) M + dM, thus the vectors representing the stress on face (CD) are drawn longer than on face (AB). The beam element is also subjected to a total shear force V on each face. C and C', represent the resultant of the total compressive forces on each face and T and T', the resultant of the tensile stresses. Fig. (b) shows the same free body but with the tensile and compressive stresses on each face replaced by a simple force C and T which tends to move the block with the same results as the system of Fig. (a). In Fig. (c) the beam element of Fig. (b) has been cut along the neutral axis, and a force applied to the cut faces equal to f_s b dx, where f_s equals the horizontal shear stress intensity.

Writing the equilibrium equation, that the sum of the horizontal forces on the upper portion must equal zero, we obtain

\[ C - f_s \text{ b dx} = 0, \text{ hence } C = f_s \text{ b dx} \]

and likewise for the lower - T + f_s \text{ b dx} = 0, hence T = f_s \text{ b dx}.

Fig. (d) shows the free body of Fig. (b), but with C and T replaced by their above equivalent values.

Taking moments about point (O)

\[ M_0 = f_s \text{ b dx} \cdot h - V \text{ d x} = 0, \text{ hence } f_s = V / h, \quad \text{(C10.6)} \]

Thus to obtain a value f_s equal to the maximum value given by equation (a) use an effective arm (h) equal to the distance between the bending stress centroids. For a rectangular section the effective arm is obviously equal to 2/3 the beam height, but for the common beam sections as illustrated in Figs. C10.11 or C10.12, the distance between bending stress centroids is not so obvious particularly if the web is considered effective in bending. A close approximation of the effective arm (h) and a procedure which is common practice is to assume (h) equal to the distance between the centroids of the web-flange rivets. The student should take several example beam sections and check this assumption for (h) using Eq. (C10.6) against the exact values by Eq. (C10.5).

Some structural designers make assumptions as to the proportion of the total vertical beam shear which is carried by the beam web. For example, it is sometimes assumed that the web takes the entire beam shear, or it may take only 90 percent. The percentage of the shear load carried by the web depends of course upon the size of the flange sections and the form of the web section. For example, in Fig. C10.7, the same flange is connected to a web which is attached to the flanges in two ways as illustrated in Figs. (a) and (b). In Fig. (a), the shear flow on portions (AB) and (CD) help resist the external shear, whereas the shear flow on these portions in Fig. (b) act in the same direction as the external shear load; thus causing the shear load on the web to be greater than the external shear load. (See Chapter A14 and A15 for general discussion of shear flow in open and closed sections).

C10.8 Shear Resistance Provided by Sloping Flanges.

A large majority of the beams in airplane wing and tail surfaces have sloping flanges because of the taper of the structure in both planform and thickness. This sloping of the beam flanges relieves the beam web of considering shear load and should not be neglected. Fig. C10.8 shows a beam (abcd) carrying a load system P, P_e, etc. The top flange is sloping as shown. If both flanges were extended, they would intersect at point (O).

Let M = bending moment at section (ac)
Then \( C = \frac{T_h}{h} = \frac{M}{h} \) (\( h \) = distance between flange stress centroids)

The vertical component \( T_v \) of the load \( T \) in the upper flange equals \( T_h \frac{h}{L_0} \), but \( T_h = \frac{M}{h} \), hence

\[
T_v = \frac{M}{h} \frac{h}{L_0} = \frac{M}{L_0}
\]

Let \( V_p \) = shear load carried by beam flanges

Then \( V_p = \frac{M}{L_0} \)  \hspace{1cm} \text{(C10.7)}

Thus the shear component carried by the axial loads in the sloping flange members equals the bending moment at the section being considered divided by the distance from the section to the point of intersection (0) of the flanges.

The above derivation was based on the assumption that the entire resisting moment \( M \) was developed by the flanges. With the web effective in bending, the moment developed by the web should be subtracted from the total bending moment \( M \).

Let \( I = I_{fl} + I_{web} \)

The moment developed by web = \( M \frac{h_{web}}{I} \), where

\[ M = \text{total bending moment on section} \]

In airplane construction the cantilever beam with sloping flange members is the common case, and the shear resistance by the flange axial loads is an important factor which should not be neglected if an efficient structure is desired from a strength-weight standpoint.

C10.9 Effect of Variable Moment of Inertia on Flexural Shear Stress Distribution.

The fundamental shear stress equation C10.5 as derived in Chapter A14 applies strictly to beams of constant moment of inertia. For airplane beams the common case is one with variable moment of inertia; thus the stresses obtained by equation C10.5 are incorrect, although the discrepancy in most cases is not large. The student should realize this fact in studying the shear flow picture in tapered wing structures. See Art. A15.15 of Chapter A15.

C10.9a Flange Discontinuities.

From a weight saving standpoint, it is necessary to taper flange sections in order to approach a beam of constant strength relative to the applied loads.

Fig. a illustrates how such tapering of the flange section may produce local eccentric flange loads. Between sections (1) and (2) the upper flange tapers in side view as shown which

\[
\text{Fig. a}
\]

displaces the flange neutral axis as shown. Assuming there is no change in bending moment over the beam portion as shown, the force \( F \) must be greater than \( F \) since resisting arm \( d \) is less than \( d \). For equilibrium this moment due to \( F_1 \) and \( F \) not being collinear must be balanced in some manner. If the flanges are rigidly connected to web and stiffeners, this moment can be balanced by an additional shear stress on the web panel between points (1, 2, 3 and 4) as illustrated in Fig. b. Thus in cases of rather abrupt changes in flange sections which produce the eccentricity as illustrated the web and stiffeners should be checked for the additional shear flow load on the web. If such displacements in the flange neutral axis occur in the plan view, the skin covering should be investigated for the additional shearing stresses.

C10.10 Stiffener Size to Use with Non-Buckling Web.

A web stiffener is used to decrease the size of web panel; thus when buckling of the web starts, the stiffener tends to keep buckles from extending across the stiffener or causes the sheet to buckle in two panels instead of one. Mr. H. Wagner in a paper presented before a meeting of the A.S.M.E. in 1930 offered the following expression as the required moment of inertia of a stiffener to be used with a shear resistant web.

\[
I_v = \frac{2.29d}{t} \left( \frac{h_w}{s} \right)^\frac{3/2}{5} \quad \text{(C10.8)}
\]

where

\[
I_v = \text{moment of inertia of stiffener}
\]

\[
d = \text{center line distance between stiffeners}
\]

\[
h_w = \text{depth of web plate}
\]

\[
t = \text{thickness of web plate}
\]
V  = vertical shear at section

t  = web thickness

E  = modulus of elasticity

A more recent criteria for stiffener stiffness (Iv) for both flat and curved webs is given by the curve in Fig. C10.9. When the stiffener is used purely as such and not as a means to transfer a concentrated external load to the beam web, the question arises as to what is the minimum number of fasteners required in attaching the stiffener to the web. For non-buckling webs, two criteria are suggested:

1. The stiffener should be attached to the flange at each end.

2. The rivet pitch (spacing) should according to (Ref. 1) be at least equal to 1/4 times the stiffener spacing, or 1/4 the web height if this is smaller, in order to justify the assumption of stable support at the edges of the web panel. Normal practice uses more rivets.

C10.11 Notes on Beam Rivet Design.

Except for very small beam sections which may be extruded as one piece, the usual beam consists of separate web and flange members, fastened together by rivets, bolts, spot welding or continuous welding. In the design of such beams it is thus necessary to know what loads the rivets, bolts, etc., are subjected to in order to provide the proper connective strength. It is quite easy to substitute in simple formulas to find the loads on beam flange rivets, however, the student should be sure that he understands the fundamental beam action behind these formulas.

Fig. C10.10 illustrates a beam portion equal in length to the flange rivet pitch p. The beam section at (AA') is subjected to a bending moment M and M + ΔM at section (CC'). The bending stress distribution on the beam faces is indicated by the stress triangles. In this example section it will be assumed that the web takes no bending stresses.

Let Fp = the total pull on the flange angles due to bending stresses at section AB due to bending moment M. Then total pull on flange angles at section CD due to a moment M + ΔM on beam section CC' equals Fp + ΔFp.

Under the action of these two forces the flange angles would move to the left, but this movement is prevented by the rivet which ties...
Eq. (C10.11) is easier to use since the terms $I$ and $/ydA$ of Eq. (C10.12) are not required, and the distance $h$ can be estimated closely without calculation, and will be greater than the distance between the centroids of the flange areas.

Equation (C10.11) was derived on the assumption that the beam web took no bending stresses. In general this is not true or only partially true. With the web taking bending, equation (C10.9) would be wrong because the moment of the bending stresses on the web about our moment center is not included. Thus to make the simplified equation (C10.11), check the exact result as given by equation (C10.12), the distance ($h$) would have to be greater than the distance between the flange stress centroids. In fact, taking ($h$) equal to beam depth would not be far off from the results of equation (C10.12).

C10.12 Loads on Rivets Attaching Reinforcing Plates to Flange Member.

The beam flange is commonly composed of a main unit plus several reinforcing plates or parts which are held to the main unit by rivets or spot welds. Fig. C10.11 illustrates typical beam flanges. The basic section of the upper flange is reinforced by the plates (a) and (b), and the lower flange by the plates (c) and (d). The purpose of the rivets is to keep the reinforcing plates from sliding along the flange tee section due to the bending of the beam; thus making the plates effective in bending. This horizontal force tending to slide reinforcing plates and which is prevented by the rivets in shear is given by the fundamental shear flow equation (C10.12), namely

$$ q = \frac{V}{I} / ydA, $$

which equals the horizontal shear per inch along the beam. To find the load on rivet section 1-1 of the upper flange of Fig. C10.11, the term $/ydA$ equals the area of the plate (a) times the distance from its centroid to the neutral axis. For shear load on rivet at section (2-2), the term $/ydA$ equals the area of plate (b) times distance to neutral axis. On the lower flange rivet section (3-3) is critical since both reinforcing plates are on same side, and the entire shear flow produced by plates (c) and (d) must be resisted by rivet section (3-3). The term $/ydA$ would thus equal area of both plates times the distance to the neutral axis of the beam.

A simplified method which yields good results is based on the relative areas of the units of the total flange. To design connection of flange to beam web, the total horizontal shear $q$ produced by entire flange is always necessary and involves the use of the entire flange area in the shear flow calculation

$$ q = \frac{V}{I} / ydA. $$
The shear flow produced by a reinforcing plate is then taken as proportional to the area of the plate over the total flange area times the total flange shear flow. Using simplified equation (C10.11), we can write

\[ q = \frac{V}{h' A_p} \]  \hspace{1cm} (C10.13)

where

- \( a_p \) = area of plates under consideration
- \( A_p \) = total area of flange
- \( q \) = load per inch on rivet
- \( V \) and \( h' \) same as explained before

For rivet loads in beams with sloping flanges the shear \( V \) is the net shear as explained before in discussing web shear stresses.

### C10.13 Web Splices.

Usually in designing a sheet girder beam, it is necessary from a weight saving standpoint to use several web sheet thicknesses, which means web splices. Fig. C10.12 illustrates typical web splices. Fig. (a) is typical for a comparatively heavy web which prevents joggling or web as in the case of Fig. (b). In the case of Fig. (b), the lap is usually made under a web stiffener which provides a support for the web in driving the rivets through the thin web sheets.

#### Loads on Web Splice Rivets

The web is subjected to shear loads and for stable webs, the web undergoes bending stresses.

For rivet design it is usually assumed that the web shear stress is constant over the depth. Thus the vertical component of load on each web splice rivet is the same or

\[ R_y = \frac{V}{n} \]  \hspace{1cm} (C10.14)

where

- \( V \) = net web shear
- \( n \) = number of rivets in splice. If butt splice, \( n \), equals the number of rivets on one side of splice.

In bending the splice rivets must transfer the bending moment due to the bending moment \( M \) developed by the web. The largest rivet load on a rivet due to bending will be on the most remote rivet, e.g., rivet (a) at distance \( ra \) from center of rotation of the rivet group. Then load on this rivet due to web moment equals

\[ R_{ma} = \frac{M_w ra}{Z_f^2} \]  \hspace{1cm} (C10.15)

where

- \( Z_f^2 \) = moment of rivet group which equals the sum of the squares of the distances of the rivets from the center of rotation of bolt group.

The resultant load on the critical rivet will equal the vector sum of the values of equations (C10.14) and (15).

Since in most cases only two rows of rivets are used in a web splice, a close approximation for the moment load on critical rivet can be written by using the vertical distances (\( y \)) to the rivets instead of the radial distances (\( r \)), the resulting force acting in the horizontal direction. Hence

\[ R_{ma} = \frac{M_w y}{Z_f^2} \]  \hspace{1cm} (C10.16)

The resultant combined load on critical rivet is

\[ R = \sqrt{R_y^2 + R_{ma}^2} \]  \hspace{1cm} (C10.17)

The student should review Chapter D1 for more detailed information on rivet loads due to moment loads on riveted connections.

When the web of one beam is joined to that of another beam using shear "clips," a special problem may sometimes arise regarding the adequacy of the clips. This design problem is discussed in Chapter D3.

### C10.14 Example Rivet Problem.

Fig. C10.13 shows the cross-section of a riveted beam. If the design vertical shear on the section is 3000 lb., check the strength of the riveted connection.
Solution:

The loads on the rivets will be calculated by the "exact" and also by the simplified approximations.

The exact shear flow equation is

$$ q = \frac{V}{I} / ydA $$

The first step will be the calculation of the moment of inertia of the beam section about the neutral axis. The bending loads are such as to put the upper flange in compression.

The moment of inertia will be calculated about the centerline axis of the beam section and then transferred to the neutral axis. Table C10.3 gives the detailed calculations.

<table>
<thead>
<tr>
<th>Part</th>
<th>Area (A)</th>
<th>$A'$</th>
<th>$A'P'$</th>
<th>$I''$</th>
<th>$I'' = A'P'^2$</th>
<th>$I''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper angles</td>
<td>.1780</td>
<td>.201</td>
<td>.588</td>
<td>1.94</td>
<td>.008</td>
<td>1.948</td>
</tr>
<tr>
<td>Upper reinforcing plate</td>
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<td>.511</td>
<td>.386</td>
<td>1.26</td>
<td></td>
<td>1.360</td>
</tr>
<tr>
<td>Web</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>.008</td>
<td>.008</td>
</tr>
<tr>
<td>Lower angles</td>
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<td>.301</td>
<td>.588</td>
<td>1.94</td>
<td>.008</td>
<td>.008</td>
</tr>
<tr>
<td>Rivet hole</td>
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<td>.022</td>
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<td></td>
<td>.022</td>
</tr>
<tr>
<td>Rivet lower flange</td>
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<td>.453</td>
<td>.453</td>
<td>.453</td>
<td></td>
<td>.453</td>
</tr>
</tbody>
</table>

\[ \gamma = \frac{.453}{.453} = .57 \]
\[ I'' = 6.43 \times 7.943 = 6.17 \text{ in.}^4 \]

$I'' = \text{moment of inertia about its own centroidal axis.}$

Rivet load on upper flange rivets which attach angles to web:

Rivet pitch = 1-1/3 inch

Horizontal shear load per 1-1/3 inch distance equals

$$ q = 1.125 \frac{V}{I} / ydA $$

Substituting in equation (A)

$$ q = 1.125 \times \frac{3000}{6.17} \times .810 = 443 \text{ lb.} $$

Substituting in equation (A)

$$ q = 1.125 \times \frac{3000}{6.17} \times .810 = 443 \text{ lb.} $$

Total = .810

h' = beam depth = 7.062" (see Art. C10.10)

\[ h' = \text{beam depth} = 7.062" \]

hence

$$ q = 1.125 \times \frac{3000}{7.062} = 478 \text{ lb. which is conservative compared to 443 by exact expression.} $$

The web is attached to angles by 1/8 diameter 2117-T3 aluminum alloy rivets and are acting in double shear.

From Chapter D1:

Double shear strength of 1/8 - 2117-T3 rivet $= 2 \times 388 = 776 \text{ lb.}$

Bearing strength on 2024-T3 - .051 web plate = 630 lb., thus bearing is critical and

Margin of safety = $\frac{630}{443} - 1 = .42$ and with the approximate method the margin of safety would equal ($\frac{630}{478}$) - .1 = .31.

Check of Cover Plate Rivets:

Rivet spacing = 1.5" with two rows of rivets.

By exact equation, load on two rivets

$$ q = 1.125 \frac{V}{I} / ydA $$

$$ = 1.125 \times \frac{3000}{6.17} \times .1093 \times 2.961 = 226 \text{ lb.} $$

Load per unit = 226/2 = 118 lb.

By simplified formula:

$$ q = \left( \frac{V}{h'} \right)_{\text{plate}} = \left( \frac{3000}{7.062} \times .473 \right) \times 1.5 $$

$$ = 243 \text{ lb.} = 122 \text{ lb./rivet.} $$

The rivets are in single shear which is critical and equals 388 lb. as given above.

Hence margin of safety = $\frac{388}{118} - 1 = 2.28$
C10.12 DESIGN OF METAL BEAMS. WEB SHEAR RESISTANT (NON-BUCKLING) TYPE.

Fig. C10.14. Sketch of Beam (See Fig. C10.15 for Cross-section of Beam)

C10.15 Example Problem. Strength Check of Beam.

Fig. C10.14 shows a built up (I) section beam, simply supported and carrying its three concentrated loads as shown as the design load for the beam.

Check on Bending Strength of Beam

Since the beam cross-section is constant, the critical section is at the midpoint of the beam where the bending moment is:

\[ M_{\text{max}} = 3750 \times 50 - 2400 \times 25 = 127500 \text{ in.lb} \]

As indicated in Fig. C10.14, the beam is riveted to the sheet covering. On the upper flange which is in compression under the given loading, a certain effective sheet width will act with the beam flange. This effective sheet width depends upon the beam flange stress which is unknown at this stage. As a preliminary value, assume that a width of skin sheet equal to 30 ft as acting with each rivet line is a reasonable one. On the tension or bottom side, the entire skin sheet is effective or 2 inches to each side of the beam which is the distance midway to the first skin stiffener on each side.

Fig. C10.15 shows the details of the effective cross-section at midspan of the beam. Three rivet holes are assumed in the tension flange. Table C10.4 shows the calculations for the section properties, first about centerline reference axis, then transferred to the neutral axis.

Bending stress at midpoint of horizontal lag of upper angles: \( \text{N.A.} = 10.44 \).

\[ f_b = M_y/l = 127500 \times 3.514/10.44 = 44200 \text{ psi} \]

The assumption of 30 ft = 0.75 in as the skin effective width *w* will be checked. From Chapter 07,

\[ w = 1.95 \sqrt{E/Fl_d} \]

---

**Table C10.4**

| Part         | Area A | \( y \) | \( A_y \) | \( A_y \) | \( l_d \) | \( l_d + A_y \)
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
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<td>Upper angles</td>
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<td>1.000</td>
<td>3.786</td>
<td>0.012</td>
<td>3.798</td>
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<td>0.1504</td>
<td>0.604</td>
<td>-</td>
<td>0.604</td>
</tr>
<tr>
<td>Web</td>
<td>0.4990</td>
<td>0</td>
<td>0</td>
<td>1/12 x</td>
<td>0.072</td>
<td>7.875</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7/16</td>
<td></td>
<td>2.311</td>
</tr>
<tr>
<td>Lower angles</td>
<td>1.780</td>
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<td>-0.676</td>
<td>2.570</td>
<td>0.008</td>
<td>2.578</td>
</tr>
<tr>
<td>Lower skin</td>
<td>0.080</td>
<td>-4.01</td>
<td>-0.321</td>
<td>1.290</td>
<td>-</td>
<td>1.290</td>
</tr>
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<td>Skin rivet holes</td>
<td>0.0206</td>
<td>-3.959</td>
<td>0.082</td>
<td>-0.323</td>
<td>-</td>
<td>-0.323</td>
</tr>
<tr>
<td>Web flange rivet holes</td>
<td>-0.0227</td>
<td>-3.562</td>
<td>0.081</td>
<td>-0.288</td>
<td>-</td>
<td>-0.288</td>
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<td>0.316</td>
<td>9.980</td>
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<td></td>
</tr>
</tbody>
</table>

\[ y = 0.316/0.9249 = 0.34 \]

\[ \text{N.A.} = 9.98 - (0.925 \times 0.34) = 9.87 \text{ in.}^2 \]

Section with 0.072 web:

\[ \text{N.A.} = 10.44 \text{ (N.A. Location Same.)} \]
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

**Check of Web Buckling Stress**

The maximum shearing stress occurs at the support point and equals 3750 lbs. The web thickness at this point is .067 and the web stiffener spacing is 3.57 inches. The maximum shearing stress on the web by the simplified equation is,

\[ f_s = \frac{V}{ht} = \frac{3750}{7.125 \times 0.067} = 9230 \text{ psi} \]

By the more exact equation,

\[ f_s = \frac{V}{It} \int y dA = \frac{3750}{9.87 \times 0.067} \left( \frac{.264 \times 3.446 + .0375 \times 3.673 + .067 \times 3.6 \times 1.9}{9.87 \times 0.067} \right) = 9450 \text{ psi} \]

Since the bending moment in the first web panel adjacent to beam end support is practically zero, the web can be considered as subjected to shear stress only.

The buckling stress is given by equation C10.1.

\[ F_{cr} = \frac{\gamma_s \pi^2 E}{12 \left( 1 - \frac{a}{b} \right)^4} \left( \frac{b}{h} \right) \]

\[ b = 3.57 \text{ inches} \]

\[ a/b = 7.125/3.57 = 2.0 \]

Assuming simply supported edge conditions, we obtain \( k_s = 6.4 \) from Fig. C5.11 of Chapter C5.

\[ F_{cr} = \frac{\pi^2 \times 6.4 \times 10,700,000}{12 \left( 1 - \frac{3.57}{6.4} \right)^4} \left( \frac{0.067}{3.57} \right) = 15,800 \text{ psi} \]

This stress is below the proportional limit stress of the material so no plasticity correction is necessary or \( \gamma_s = 1.0 \).

The margin of safety against buckling is therefore (15,800/9450) - 1 = .67. This value is somewhat conservative as boundary condition for web panel is no doubt larger than for simply supported, which condition was assumed.

**Check of .067 Web at End of Bay 1**

The web will be more critical at this point because the beam is still subjected to an external shear load of 3750 lbs., plus a bending moment of 3750 x 23.22 = 87000 in.lb. at the midpoint of the end panel.

Since the web is clamped between the flange angles, buckling of the web will occur adjacent to the lower end of the upper flange angles at a distance of 2.91 inches from neutral axis.

\[ f_b = \frac{M_y/I}{87000 \times 2.91/3.97} = 26600 \text{ psi} \]

The bending buckling equation is,
DESIGN OF METAL BEAMS. WEB SHEAR RESISTANT (NON-BUCKLING) TYPE.

\[ F_{bc} = \frac{n^2 \cdot Kb \cdot \sigma}{12 \cdot (1 - \frac{a}{b})} \cdot \left(\frac{a}{b}\right)^{0.57} \]

\( b = 6.5 \text{ in., taken as distance between edges of flange angles since web is clamped in between flange angles.} \ a = \text{stiffener spacing} = 3.57. \ \text{The} \ a/b = .55. \ \text{From Fig. C5.15 of Chapter C5:}

\( Kb \text{ for simply supported edges} = 36 \)

\( Kb \text{ for clamped edges} = 50 \)

\( \text{We will assume an average value of} \ Kb = 43. \)

\[ F_{bc} = \frac{n^2 \cdot 43 \cdot 10,700,000}{12 \cdot (1 - .3^2)} \cdot \left(\frac{3.57}{6.5}\right)^{0.57} = 32200 \text{ psi} \]

The shear stress at this section will be same as at support since the external shear load is the same. Thus \( f_s = 9450 \text{ and } F_{bc} = 15800. \)

The interaction equation for combined bending and shear is,

\( R_b = R_s = 1 \)

\( R_b = f_s / F_{bc} = 9450 / 15800 = .598 \)

\( M.S. = \frac{1}{\sqrt{R_b^2 + R_s^2} - 1 = \sqrt{.598^2 + .598^2} - 1 = 0.01} \)

Check of .072 web at Centerline of Beam

The bending moment at the midpoint of the web panel adjacent to the beam centerline is 123,200 in. lb. The shear load on the panel is 1350 lb.

\( f_b = M_y / L = 123200 \times 2.91 / 10.44 = 34200 \text{ psi} \)

\[ F_{bc} = \frac{n^2 \cdot 43 \cdot 10,700,000}{12 \cdot (1 - .3^2)} \cdot \left(\frac{3.57}{6.5}\right)^{0.57} = 51500 \text{ psi} \]

This value is in the inelastic stress range so correction must be made for plasticity. The curves of Fig. C5.8 of Chapter C5 will be used. For 2024-T3 sheet, \( F_o = .95000 \) and \( n = 11.5. \)

The value of the lower scale parameter in Fig. C5.8 will be \( F_{bc} / F_o = .51500 / .95000 = 1.32. \) From Fig. C5.8 for \( n = 11.5 \) we read \( F_{bc} / F_o = .93. \) Therefore, \( F_{bc} = 39000 \times .93 = 36300 \text{ psi.} \)

The shear stress is,

\( f_s = \frac{V}{12} / yda = \frac{1350}{10.44 \times .072} (1.51) = 2710 \text{ psi} \)

Check of Stiffeners Required

\[ F_{cr} = \frac{n^2 \cdot 6.4 \cdot 10,700,000}{12 \cdot (1 - .3^2)} \cdot \left(\frac{3.57}{6.5}\right)^{0.57} = 25300 \text{ psi} \]

\( R_s = 2710 / 25300 = .107 \)

\( R_b = 34200 / 35300 = .94 \)

\[ M.S. = \frac{1}{\sqrt{1.07^2 + .94^2} - 1 = 0.01} \]

Check of Shear Stiffeners Required

The web is stiffened by 1/2 x 1/2 x .04 angles on one side of the web. From Equation (C10.8), the moment of inertia required of web stiffener to prevent web buckling is,

\[ I_s = \frac{2.29 \cdot V_{bw} \cdot V_{bw}^2}{t \cdot (33 \cdot E)} \]

\[ = \frac{2.29 \cdot 1.25 \cdot (3.750 \times 7.875)}{t \cdot (33 \cdot 10,700,000)} = .00060 \]

The moment of inertia of stiffener cross-section is .00094, thus stiffness is satisfactory.

The required moment of inertia will also be checked using curves in Fig. C10.9. The lower scale parameter in Fig. C10.9 is \( d/h \sqrt{k_s} \) where \( k_s \) as used in buckling equation must be multiplied by \( n^2 / 1 - .3^2 = .905. \) Thus, \( d/h \sqrt{k_s} = 3.57/7.125 \times 8.4 \times .905 = .208 \)

From Fig. C10.9 we read 1.1 for \( I_v/d^3. \) Whence \( I_v = 1.1 \times 3.57 \times .072 = .00073. \) (Stiffener G.K.).

Check of Rivets Attaching Web to Flange Angles.

End Bay. \( V = 3750 \text{ lb.}, \ t = .067 \)

The horizontal shear load q per inch by approximate formula C10.11 is

\[ q = V/h = 3750 / .067 = 56000 \text{ lb. per inch} \]

By the most exact C10.12 equation,

\[ q = \frac{V}{T} / yda = \frac{3750}{.957} \cdot \frac{.264 \times 3.464 + .0375 \times 3.673}{.072} \]

\[ q = 398 \]

The single shear strength of 1/8 diameter rivets made from 2117-T3 aluminum alloy is 398 lbs. Since rivets are in double shear and 1 inch spacing, rivet shear strength is \( 2 \times 398 = 796 \text{ lbs.} \) versus the design load of 938 lbs.

The bearing strength of a 1/8 diameter rivet on .067 web is 790 lbs. as against load.
of 398 lb. Thus rivets could be spaced further apart as margin of safety is rather large.

The rivets attaching the skin to the upper flange should be spaced to prevent inter-rivet buckling of the skin, since the skin was assumed effective in computing the beam moment of inertia. The rivets attaching the lower skin to the lower angles would be checked for the shear flow load and not for inter-rivet buckling since skin is in tension. The general subject of rivet design for structures is treated in Chapter D1.

The web stiffeners at the external load points and at the beam support points must be designed to transfer the concentrated load to the beam web. Refer to Chapter A21, which treats of loads in such stiffeners.

General Comments

The reader should understand that the margin of safety for the beam web in the preceding beam check was based on the design requirement of no initial buckling under the design load. The buckling stress as calculated is not the stress that would cause web to fail or collapse, as the web could take considerably more load in the buckled state. The subject of beam design with buckling webs is treated in Chapter C11.

In general, this type of beam web design is not widely used in flight vehicle structures because it is heavy construction since the web thickness must be relatively large to prevent buckling and the cost of fabrication and assembly is relatively high because of many parts and much riveting. The aerospace structures engineer decreases these disadvantages by using a web sheet with closely spaced beads or a series of flanged lightening holes which stabilize web against buckling and provide low fabrication and assembly cost. This type of web design is treated in Part 2 of this chapter.

C10.15a Use of Longitudinal Stiffener to Increase Bending Buckling Stress of Web Sheet.

The strength check of the beam in the example problem showed that the compressive stress on the web due to beam bending was the factor that had great effect in producing the web buckling. This web weakness can be improved by adding a single longitudinal stiffener on the compression side as illustrated in Fig. C10.15. Theoretical and experimental information on the effect of such a stiffener is quite limited (see Ref. 3). Such a stiffener can raise the buckling coefficient \( k_b \) to around 100 or more, thus the web can be made somewhat thinner if bending stresses are critical. Adding this longitudinal stiffener means another structural part and more assembly cost and therefore such construction is not widely used although it is a structural arrangement that will save structural weight under certain conditions of beam depth, span and external loading.

![Fig. C10.15](image-url)
PART 2. OTHER TYPES OF NON-BUCKLING BEAM WEBS.

(By W. F. McCombs)

C10.16 Other Types of Web Design.

At this point it should be noted that the web design discussed in Part I, required a large number of parts (the stiffeners) to achieve lightness. To keep manufacturing costs down, the number of parts must be kept to a minimum. A balance, or economic compromise must be found between manufacturing expense and weight penalty. This can be arrived at since in every airplane design some "dollor" penalty can be assigned to every extra pound of weight. The exact figure will depend upon the type of airplane being designed. Thus, eliminating parts may incur some weight penalty but if the savings in manufacturing cost is greater, then the reduction in parts is economical.

Three types of shear resistant, non-buckling webs are frequently used in aircraft design to save the expense of stiffeners. Actually, the web in most cases is as light, or lighter, than a web with separate stiffeners. There is a general limitation, however, in that a stiffener must be provided wherever a significant load is introduced into the beam. The web types are:

a) web with formed vertical beads at a minimum spacing,

b) web with round lightening holes having 45° formed flanges at various spacing,

c) web with round lightening holes having formed beaded flanges and vertical formed beads between holes.

The webs with holes, (b) and (c), also provide lots of built-in access space for the many hydraulic and electrical lines and control linkages in airplanes.

C10.17 Beaded Webs.

Figure C10.17 shows a web having "male" beads formed into it at the minimum spacing forming will allow. The cross-section of the web is described in Table C10.5.

REFERENCE: STRUCTURAL DESIGN MANUAL SECTION 4.230
when the beads collapse. An example will follow later in C10.19. A suitable stiffener must be used (replacing a bead) wherever a load is introduced to prevent earlier collapse of the bead. The allowables are for pure shear only, no normal loads (see Art. D3.8).

C10.18 Webs With Round Lightening Holes Having Formed 45° Flanges.

This is a simple easily formed web. Fig. C10.19 shows such a beam with a cross-section through the web at the holes. The geometry of the hole and its flanges is given in Table C10.6 for typical forming.

![Diagram of web with holes and stiffeners](image)

**Fig. C10.19**

The N.A.C.A. has developed, from an extensive test program, an empirical formula that gives the allowable shear flow (collapse) for webs having this type of hole. Referring to Fig. C10.19, the allowable shear flow is, from Ref. (2),

\[
g_{\text{all}} = k \left[ f_{sh} \left( 1 - \left( \frac{2}{h} \right)^2 \right) + f_{sc} \sqrt{\frac{b}{h}} \right] \frac{c'}{b}
\]

where

\[
k = 0.85 - 0.0005 \frac{h}{t}
\]

\[
f_{sh} = \text{Collapsing shear stress of a long plate of width } h \text{ and thickness } t; \text{ obtained from Fig. C10.20.}
\]

\[
f_{sc} = \text{Collapsing shear stress of a long plate of width } c \text{ and thickness } t; \text{ obtained from Fig. C10.20.}
\]

\[
b = \text{hole centerline spacing.}
\]

\[
h = \text{height of web, between flange to web rivets.}
\]

\[
c' = C - 2B \text{ where } B \text{ is given in Table C10.6 for a typical hole.}
\]

\[
C = b - D
\]

For this type of web design, in general, those webs designed to take ultimate load will probably not show any permanent set at yield.
Table C10.6

<table>
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<tr>
<th>D</th>
<th>B</th>
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<tr>
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<td>.19</td>
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<td>5.61</td>
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</tr>
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</table>

load if the shear stress at the yield load over the net sections, \( C' \times t \) and \( H - D \times t \), is about equal to the yield stress of the web material, \( F_{y} \).

In general, it will be found, from formula (18), that the larger holes (about \( D = .8h \)) with wider spacings, \( b \), will give the lightest web in designing for a given shear flow \( q \), but the stiffness will be less, of course.

Again, the allowables per formula (18) are for pure shear only, no normal loads on the web. Thus a stiffener must be provided wherever a load is introduced into the beam and also in areas where the beam may have significant curvature as in round and elliptical bulkheads, see Chapter D3.5. An example is given in Art. C10.19.

In addition to the above (collapse), the web should be checked for net shear stresses through the holes to be sure that \( f_{s,\text{net}} = F_{s,\text{u}} \) at ultimate load, \( q \times h \).

For a more complete discussion of this type of web design and the test data, the reader should review Refs. (2) and (1). Design charts can be prepared from formula (18), Fig. C10.18, and Table C10.6 for use in designing without having to resort to formula (18). Figure C10.21 shows such a chart taken from Ref. (2) for the cases of \( D/h = .80 \) and \( D/h = .50 \).


A third type of web has round holes with beaded flanges and vertical "male" beads between the holes. Such a beam is shown in Fig. C10.22. The vertical bead is as described in Table C10.5. The beaded flange is shown in Table C10.7. For the particular case where \( \frac{D}{h} = .6 \) and the hole spacing is equal to \( h \), the allowable shear flows shown in Fig. C10.23 apply.

---

**Graph:**

![Graph of Allowable Collapsing Shear Flows for 45° Flanges](Fig. C10.21)
The solid lines of Fig. C10.23 give the ultimate strength, \( q \), of the web as a function of web height, \( h \). This represents the total collapsing strength of the web. The dotted lines indicate the shear flow, \( q \), at which initial buckling begins. If the O.D. is greater than \( 0.5 \times h \), or if the spacing of holes is reduced, the allowances will be reduced but no data is included for other geometries. As in other cases, Arts. C10.15 and C10.17, the data is for pure shear only, no normal loads. Hence loading stiffeners must be used (replacing the vertical bead locally) and Article D3.8 should be reviewed.

Of course, in all three types of formed webs, a small amount of normal loading can be tolerated, but no limit is defined. Typical cases are wing ribs and fuselage frames where only relatively light normal airloads are distributed by the rib without loading stiffeners. In such cases some extra margin of safety above the shear load is usually used, depending upon the designer's judgment and/or substantiating element tests. Where large distributed loads are involved, as in a beam supporting a fuel cell, or where the beam curvature is significant (See D3.8), stiffeners should be used for the lightest design.

As a final note, whenever using webs with formed beads as in C10.16 and C10.18, it is important that the beads be formed with a length long enough to extend as close to the beam flanges as assembly will allow. Short beads, ending well away from the flanges, will not develop the strength indicated by the allowances given in the figures. Rivets attaching the web to the flange above a hole also need be more closely spaced to take the higher "net" shear locally.

C10.20 Example Problems.

For the beam shown in Fig. C10.24, determine the web gages required for

a) a beaded web, as in C10.16.

b) a 45° flanged hole web as in C10.17.

c) a beaded flange hole web with intermediate beads as in C10.18.

The beam shown in Fig. C10.24 has the same external dimensions and design load system as the beam shown in Fig. C10.14 and used for the example problem in Part I of this chapter. In these example problems, we are replacing the web design by the three variations (a), (b) and (c). Since the webs cannot be considered fully

2400#  2700#  2400#

7-1/8"  Bay "A" Bay "B"  Bay "B"  Bay "A"

3750#  25"  25"  25"  3750#

Fig. C10.24
effective in resisting beam bending stresses, the flanges would be stressed slightly higher than found in the example problem of Part 1. Using the same beam dimensions and beam design loading will provide a comparison of the web weights for the various web designs.

Using the simplified formula, the shear flows in bays A and B are

\[
q_A = \frac{V}{h} = \frac{3750}{7.125} = 526 \text{ lb./in.}
\]

\[
q_B = \frac{3750 - 2400}{7.125} = 189 \text{ lb./in.}
\]

Thus, for lightest weight, 2 web gages should be used, the lighter one for bays "B", and these could be utilized at the loading stiffeners introducing the 2400 lb. loads.

a) Beaded Web Gage Requirements.

Entering Fig. C10.18 with \( q = 526 \text{ lb./in.} \) and \( h = 7.125" \) we find that the minimum acceptable gage is .051". Thus the web of Bay A should be .051", giving \( q_{\text{allow}} = 770 \text{ lb./in., from Fig. C10.18.} \)

M.S. = \( \frac{770}{526} = 1.46 \)

Repeating for the web of Bay B, enter with \( q = 189 \text{ lb./in.} \) and \( h = 7.125" \) and find the minimum acceptable gage is .032", which has \( q_{\text{ALL}} = 310 \text{ lb./in.} \). Thus for Bay B,

M.S. = \( \frac{310}{189} = 1.64 \)

b) A 45° Flange Round Lightening Hole.

Since the larger diameter (and more widely spaced) holes are more efficient weight-wise, try a hole having \( D = .38" \) or \( D = 5.61" \) from Table C10.6. Spacing two holes in the 25" bays would indicate a spacing \( b = 12.5" \). Then, from C10.19

\[
C = b - D = 12.5 - .38 = 6.9
\]

\( S = .19, \text{ from Table C10.6} \)

\( C' = C - 2S = 6.9 - 2(.19) = 6.52 \)

Now determine the allowable value of \( q \) for, say, \( t = .081" \) for Bay A.

\[
\frac{h}{t} = \frac{7.125}{.081} = 88
\]

\[
k = .38 - .0005(38) = .30
\]

From Fig. C10.20, for \( \frac{h}{t} = 88 \) and \( \frac{C}{t} = 88 \)

\( f_{s_n} = 13,800 \)

Substituting values in equation (13)

\[
q_{\text{ALL}} = .79(.051) \left[ 13,800 \left[ 1 - (.51) \times \frac{7.125}{12.5} \right] + 13,800 \left( 7.125 \right) \frac{5.61}{12.5} \right] = .064 \left[ 13,800 (.38) + 13,800 (.887) \right] (.521)
\]

\( = 375 \text{ lb./in.} \)

Also, the net shear stresses on any section through the holes must be less than \( f_{s_n} \). Checking a vertical section through the hole we get

\[
f_{s_{\text{NET}}} = \frac{V}{(R-D)t} = \frac{526(7.125)}{(7.125-.51)(.081)} = 3750 = 30,500
\]

Thus, M.S. COLL. = \( \frac{3750}{526} = 1.46 \)

M.S. NET = \( \frac{f_{s_n}}{f_{s_{\text{NET}}}} = \frac{37,000}{30,500} = 1.21 \)

Repeating with the above geometry and using \( t = .051" \) for Bay "B" where \( q = 189 \text{ lb./in.} \), we get

\[
q_{\text{ALL}} = .77(.51) \left[ \frac{7500(.38) + 3500(.887)}{10.650} \right] (.521)
\]

\( = .0393 \left( 10.650 \right) (.521) = 217 \text{ lb./in.} \)

M.S. COLL. = \( \frac{217}{189} = 1.15 \)

M.S. NET = \( \frac{37,000}{17,400} = 1.12 \)

The reader can repeat the above for smaller sizes of holes and thickness to see if a lighter arrangement (NET web weight) can be obtained. It can be seen that the example gages are confirmed by the data of Fig. C10.21 extrapolating in the case of the .081 web.

c) A Beaded Flange Hole with Intermediate Vertical Beads.

The geometry for this is as previously discussed (O.D. = .68 and \( b = h \)).

First determine gage for Bay A. Entering Fig. C10.23 with \( h = 7.125" \) and \( q = 526 \text{ lb./in.} \), the closest acceptable gage is \( t = .064" \) (solid line), giving
\[ Q_{\text{ALL}} = 700 \text{ lb/in.} \]

\[ M.S. = \frac{700}{328} - 1 = 0.33 \]

Repeating for Bay B with \( q = 189 \) and \( h = 7.125 \)" the closest acceptable gage is \( t = 0.032 \).

\[ Q_{\text{ALL}} = 208 \text{ lb/in.} \]

\[ M.S. = \frac{208}{139} - 1 = 0.10 \]

Actually, a hole with \( O.D. = 4.43 \) (from Table C10.7) could be used even though
\[ \frac{O.D.}{h} = \frac{4.43}{7.125} = 0.62 \] is a little larger than .6, since the M.S. above is well above zero.

A comparison of the total weights of each of the 3 types of webs is given in Table C10.8, using a hole diameter of 5.61" for webs (b) and 3.31" (from Table C10.7) for webs (c) and a density of .10 lb/in. for aluminum.

<table>
<thead>
<tr>
<th>Type of Web</th>
<th>Weight in Pounds</th>
<th>Total Web Wt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bays &quot;A&quot; (2)</td>
<td>1.82</td>
<td>1.14</td>
</tr>
<tr>
<td>Bays &quot;B&quot; (2)</td>
<td>2.08</td>
<td>1.30</td>
</tr>
<tr>
<td>Bays &quot;C&quot; (2)</td>
<td>2.06</td>
<td>1.03</td>
</tr>
</tbody>
</table>

Web of Fig. C10.14 (with extra stiffeners) 5.82

Thus for a "non-distributed" load, web (a), a beaded web, is lightest for Bays "A" and (c) is lightest for Bays "B". If "access" for lines is required then a web of type (c) or (b) should be used for Bays "A".

The weight of the web and vertical stiffeners designed for no initial buckling under combined bending and shear gave a value of 5.62 lbs., or much heavier than the other designs.

**PROBLEMS**

(1) Fig. C10.25 shows the cross-section of a wing beam. Calculate the ultimate resisting moment for the beam section using the stress-strain curve for the extruded 24ST material given in Fig. C10.4. Use .008 unit strain at the extreme fiber of the upper flange. Compare the results with the resisting moment given by the general beam formula \( M = f_0 I / y \).

(2) Design a butt web splice for the beam section of C10.13 for a design shear load \( V = 3000 \text{ lb/ft} \) and a bending moment of 50,000 in. lbs.

(3) Fig. C10.26 shows a simply supported beam carrying a 2000 lb. design load located as shown. The cross-section of the beam is shown in Fig. C10.27. The design require-ment for the web is no initial buckling under the design external load. Check the given design for strength and modify if too weak or too strong, or in other words, improve the design. Assume beam flanges are braced against lateral column failures.

(4) Re-design the beam of Problem (3) to use the 3 types of stiffened web as presented in Part 2 of this chapter. Compare the web weights with the web weight required in Problem 3.

(5) Fig. C10.28 shows the external dimensions of a tapered cantilever beam carrying the distributed design load as shown. The web is not to buckle under the design load.
Make a complete structural design of beam showing size of all parts. For flange use 7075-T6 extrusion material and 7075-T6 clad material for web and web stiffeners. Show rivet design.

(5) Same as Problem (4) but use web with vertical beads.

REFERENCES

(1) Kuhn, Paul:- The Strength and Stiffness of Shear Webs With and Without Lightening Holes. NACA. ARR. June 1942.

(2) Kuhn, Paul:- The Strength and Stiffness of Shear Webs with Round Lightening Holes Having 45° Flanges. NACA. ARR. L-323 Dec. 1942.

CHAPTER C11
DIAGONAL SEMI-TENSION FIELD DESIGN

PART 1. BEAMS WITH FLAT WEBs. PART 2. CURVED WEB SYSTEMS.

PART 1

C11.1 Introduction.

The aerospace structures engineer is constantly searching for types of structures and methods of structural analysis and design which will save structural weight and still provide a structure which is satisfactory from a fabrication and economic standpoint. The development of a structure in which buckling of the webs is permitted with the shear loads being carried by diagonal tension stresses in the web is a striking example of the departure of the design of aerospace structures from the standard structural design methods in other fields of structures, such as beam design for bridges and buildings. The first study and research on this new type of structural design involving diagonal semi-tension field action in beam webs was by Wagner (Ref. 1), and because of this fact, this type of beam design is often referred to as a Wagner beam.

In Chapter C10, Part 1, the subject of beam design with shear resistant (non-buckling) flat webs was covered. This type of web design leads to a comparatively heavy weight, which fact prevents its wide use in aerospace structures. Part 2 of Chapter C10 dealt with webs stiffened by closely spaced beads, flanged lightening holes, etc., a design which shows an improvement relative to web weight over the flat sheet web with vertical stiffeners. However, a large proportion of sheet panels used in aerospace structures is part of the external surface and holes and deep beads in the surface skin cannot be permitted, thus continuous sheet is required and to save structural weight, semi-tension field action in the webs and surface sheet panels must be permitted, which means a wrinkling type of structure.

Since the original work by Wagner, much further study and testing of structures involving semi-tension field design has been carried out by both industry and government agencies, hence a fairly accurate procedure for the design of such structures has been developed and this chapter is concerned with a limited presentation of the principles involved and the design procedures that have been developed.

C11.2 Elementary Approximate Explanation of Tension-Field Beam Action.

Fig. C11.1 shows a single bay truss with double diagonal members (A) and (B) and carrying an external load P. The load P will cause a compressive load in member (A) and a tensile load in member (B). If member (A) is quite flexible it will buckle as a long column as shown in Fig. C11.1b, when load P is relatively small, however, panel will not collapse. As the load P is increased, the member (A) cannot take any more load but it will practically hold its column buckling load as the bending deflection or bowing gets larger. However, member (B) being in tension can take further load until it reaches its ultimate tensile strength. Thus any increase of the shear in the panel due to an increase of load P after diagonal (A) has buckled can be carried by a further increase of tension load in member (B).

Fig. C11.2 shows the same panel but with the two diagonals replaced by a flat sheet web. Under a small load (P), the web will not buckle and the stress picture on a small web element is shown in Fig. C11.2a, and $f_0 = f_t = f_s$ where $f_s$ is the shear stress in the web at this particular point on the web. Now thin flat sheet is relatively weak under compression, thus when panel load P is increased, the compressive stress $f_0$ reaches the buckling stress of the panel in the diagonal direction and the web buckles, however, the panel does not collapse as further increase in load P can be handled by
further increase in $f_T$ or diagonal tension in the web sheet. The web has an ability to hold the $f_T$ stress that caused buckling but cannot increase it. Fig. C11.2b shows the web stress picture after the load $P$ has been increased considerably, thus increasing the diagonal web tensile stress as shown by the length of the vector for $f_T$. Since the shear load on the panel is transferred by diagonal tension in the web and since flat sheet is efficient in tension, this method of carrying the shear load permits the use of relatively thin webs because of the high allowable design stresses in tension. Fig. C11.3 shows a photograph of a thin web beam under load. Since the diagonal wrinkling appears severe, the external load being carried is no doubt approaching the failing point of the web. The student should realize that such a degree of web wrinkling does not occur under normal flying accelerations since the loads carried in normal flying conditions are only a fraction of the design loads, and thus the buckling and wrinkling is barely noticeable under accelerations of 1/2 gravity, which may be encountered often in flying in gusty weather conditions.

C11.3 Elementary Derivation of Approximate Tension-Field Beam Formulas.

In order to give the student a general picture of the influence of web tension field action upon the beam component parts, an elementary approximation of the beam equations will be given.

Fig. C11.4 shows a cantilever beam with parallel chords and vertical stiffeners subjected to a single shear load $V$ at the free end. The dashed diagonal lines indicate the direction of the wrinkles as the thin sheet buckles under the load $V$.

Assuming that the flange angles develop all the bending resistance, the vertical and horizontal shearing stress is constant over the web and equals

$$f_s = \frac{V}{ht} \tag{1}$$

where

$t$ is web thickness
$h$ is taken as the distance between centroids of flange rivets.
$V$ = vertical shear load.

(See Fig. C11.5)

---

Fig. C11.4

Fig. C11.3 Loaded Cantilever Beam Showing Severe Diagonal Web Wrinkling (Ref. 3)
Fig. C11.5 shows the free body diagram of a small triangular segment of the web cut from the upper portion of the beam.

From elementary mechanics the horizontal and vertical shearing stresses produce compressive and tensile stresses on planes at 45° with the shearing planes.

The web, being very thin, can carry very little compression stress on the surface (AC) of Fig. C11.5 before buckling; thus this small stress which produces buckling on AC will be neglected or $f_c = 0$. The edge BC is subjected to tensile stresses which the sheet can carry effectively. The forces acting on the sheet element are shown in Fig. C11.5.

For equilibrium of the element:

$$Sx = 0 \text{ or } -f_s \frac{t}{2} dx + f_t \frac{t}{2} dx \cdot \frac{1}{\sqrt{2}} = 0$$

whence

$$f_t = 2 f_s \quad (2)$$

or the web tensile stress equals twice the web shearing stress:

From (1') $f_s = V/ht$

whence

$$f_t = 2\frac{V}{ht} \quad (3)$$

Likewise for equilibrium:

$$Sy = 0 \text{ or } f_v \frac{t}{2} dx - f_t \frac{t}{2} dx \cdot \frac{1}{\sqrt{2}} = 0$$

whence

$$f_t = 2 f_v \quad (4)$$

hence

$$f_v = f_s \quad (5)$$

Rivet Loads

The sheet element is held to the flange angles by the rivets along line (AB). The rivets are subjected to two loads, one parallel to AB and one normal to AB; and each force equals $f_s \frac{t}{2} dx$. The resultant rivet load therefore equals $\sqrt{2} f_s \frac{t}{2} dx$; and if $dx$ is taken as one inch, the resultant load will be $\sqrt{2} f_s t$.

But $f_s = V/ht$, hence

$$\text{Rivet load/inch} = 1.41 \frac{V}{ht} \quad (6)$$

Web Stiffener Load

The tendency of the web is Fig. C11.4 which has broken down into the tension bands, is to pull the flanges together; this action is prevented by the vertical stiffeners which keep the flanges apart. Thus if a pure tension field is assumed, the axial compressive load $P_s$ in the stiffeners from Fig. C11.5 equals the vertical component of the web tensile stresses over a distance $d$, the stiffener spacing.

whence

$$P_s = P \sin \theta$$

But

$$P = \int f_t \, dt \cdot \frac{1}{\sqrt{2}}$$

hence

$$P_s = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} f_t \, dt = \frac{f_t \, dt}{2}$$

but

$$f_t = 2 f_s \text{ and } f_s = V/ht$$

whence

$$\text{Stiffener load } P_s = Vd/h \text{ (compression)} \quad (7)$$

Flange Axial Loads

Fig. C11.7 shows a free body of the portions of the beam to the right of a section a distance $x$ from the end of the beam.

Let $M_x = \text{external bending moment at Section AB}$. For equilibrium the internal resisting moment on Section AB must equal external bending moment $M_x$.

Taking moments about point B:

$$M_B = M_x - f_t h' \cdot f_t \cos 45° h' \cdot t \cdot \cos 45° \frac{h}{2} = 0 \quad (8)$$
where
\[ h' = \text{distance between flange centroids.} \]

But
\[ f_t = 2 f_e = \frac{F}{h't}, \text{ hence } f_t = \frac{2V}{h't} \]

Substituting this value of \( f_t \) in Eq. (8)
\[ M_x - F_t h' - \frac{V h'}{2} = 0 \]

whence
\[ F_t = \frac{M_x h'}{h'} - \frac{V h'}{2} \]

Then to make \( 2h = 0 \) on Section AB:
\[ F_c = \frac{M_x}{h'} + \frac{V h'}{2} \]

Thus the compressive flange axial load due to bending is increased by a value equal to \( V/2 \) lbs. and the tension flange load due to bending is decreased by \( V/2 \) due to the horizontal components of the web tension field.

C11.4 General Wagner Equations for Tension Field Beams.

The approximate elementary derivation given in the previous article was for the purpose of giving the student a general idea of the influence of a complete tension field beam on the various beam stresses. The angle \( α \) is in general not 45° but depends on flange areas, beam height, stiffener spacing, etc.

The general equations derived by Wagner (Ref. 1) are as follows:

For beams with infinitely rigid and parallel flanges with vertical web stiffeners:

Diagonal tensile stress in web:
\[ f_t = \frac{2V}{ht} - \frac{1}{\sin 2α} \]

Axial load in tension flange:
\[ F_t = \frac{M}{h} - \frac{V}{2} \cot α \]

Axial load in the compression flange:
\[ F_c = \frac{M}{h} - \frac{V}{2} \cot α \]

Axial force in stiffeners:
\[ F_{\text{Stiff}} = -\frac{V d}{h} \tan α \]

In the above equations:
\[ V = \text{applied shear load} \]
\[ h = \text{distance between centroids of flange-web rivets} \]
\[ h' = \text{distance between centroids of flange sections} \]
\[ t = \text{web thickness} \]
\[ d = \text{vertical web stiffener spacing} \]
\[ α = \text{angle of web buckle (see Fig. C11.7)} \]

In deriving equations (10) to (13) Wagner assumed that the beam flanges were infinitely stiff in bending. Actually the flanges due to the lateral pull of the web tension field will act somewhat as a continuous beam over the web stiffeners as supports, as illustrated in Fig. C11.8a. The deflections of the beam flanges relieves the web stress in the midportion of the panels and concentrates the web stress near the stiffeners where deflection of the flange is prevented (see Fig. C11.8b).

Wagner (Ref. 2, Part III) has developed a correction factor \( R \) to take care of this web stress concentration due to flange deflection. This stress ratio factor is obtained from Fig. C11.9 and the following equations:

\[ w = 1.25 d \sin α \sqrt{\frac{5}{(I_u + I_L)}} \]

\[ w = \frac{d}{\text{stiffener spacing}} \]

\[ I_u \text{ and } I_L = \text{moment of inertia of upper and lower beam flanges about their own neutral axis.} \]

Wagner also has derived the following equations for determining the web buckle angle \( α \) as follows:

\[ \sin^2 α = \sqrt{a^2 - a - a} \]

\[ a = \frac{1}{\frac{ht}{ht} - \frac{ht}{ht}} \]

\[ \frac{ht}{ht} - \frac{ht}{ht} \]

\[ \frac{ht}{ht} - \frac{ht}{ht} \]
Modified Wagner Equations for Use in Design.

The Wagner equations are too conservative for design, particularly in the web stresses and the loads in the vertical stiffeners. The shear stress carried by the web before it buckles is in many cases an appreciable part of the total resistance, as the buckled sheet normal to the diagonal compressive stresses has the ability to hold this buckling shear load after wrinkling. The load in the vertical stiffener is too conservative, since the stiffener is riveted to the web and the web is riveted to the flange; thus the web acts as a large gusset plate instead of a pin end condition as assumed in the Wagner equation.

The following general method of analysis of Wagner beams has been used by many airplane companies. Instead of assuming that the entire beam shear is taken by the web in diagonal tension, the following assumptions are made relative to the resistance for carrying the beam vertical shear.

(1) The shear strength of the beam flanges is not neglected.

(2) The shear carried by the web before it buckles, that is, as a shear resistant member is considered as an effective resistance and not neglected.

(3) The remainder of the beam shear after subtracting that carried by (1) and (2) is considered carried by the web in a buckled state in the form of a diagonal tension field.

The above assumptions apply to beams with parallel flange members. If the beam has sloping flanges, part of the beam shear will be carried by the shear component of the flange axial loads, and thus the assumptions should be applied to the net beam shear.

C11.6 Shear Carried by the Beam Flanges.

The general flexural shear flow equation is

\[ q = \frac{V/ya}{I} = \frac{V}{I/\eta h} (\text{if} /ya \text{ is given symbol } Q) \]

For beam flange and web arrangements commonly used in aircraft structures the shear stresses are approximately constant over the web, that is, between the centroids of the flange-web rivets. Using this assumption, the value Q in the above equation equals the static moment of the flange area about the neutral axis of the beam, and I equals the moment of inertia of the flange area about neutral axis. (Web area is neglected).

The shear load resisted by the web alone therefore equals

\[ V_w = \frac{Qh}{I} = \frac{V}{I/\eta h} \]

where

\[ h = \text{effective web depth} = \text{distance between centroids of flange-web connection rivets.} \]

The total beam shear V which equals the resistance of both web and flange, equals

\[ V = \frac{V_w I}{Qh} \]

The difference between (17) and (18) gives the shear carried by flanges.

C11.7 Shear Load Carried by Web.

Up to the buckling stress of the web plate, the shear flow is assumed to be of constant intensity over the effective web depth. When the web buckles, it is assumed that the web maintains the diagonal critical compressive stress that produced the buckling of the plate. For further increase of shear load on the web, the entire resistance is provided by the increase in diagonal tensile stresses with no increase in the diagonal compressive stresses. In other words, for loads above the web buckling point, the web acts as a pure tension field beam. Fig. C11.10 illustrates these assumptions.

Shear Load Carried by Web at Web Buckling Stress

Fig. (C11.10a) shows the shear resistance
on the web face (a) when the web is shear resistant. The vertical shear stresses on (aa) have been replaced by the diagonal compressive and tensile stresses. The intensity of these diagonal stresses equals the intensity of the critical shearing stress $F_{scr}$.

Hence the shear load carried by the web at the web buckling stress equals

$$V_{cr} = F_{scr} h t - - - - - - - - - - (19)$$

The critical shearing buckling stress is given by the following equation from C5.4 of Chapter CS.

$$F_{scr} = \frac{n^3 k_s E}{12(1-\nu^2)} \frac{a^3}{b^3} - - - - - - - - (20)$$

where $k_s$ is a function of the aspect ratio $a/b$ of the shear panel and of the edge conditions. Fig. C5.11 of Chapter CS gives the value of $K_E$ for edges simply restrained. Taking a value of $K_E$ for this edge condition is no doubt slightly conservative.

Shear Load Carried by Web after Buckling

Fig. C11.10b shows the web stress distribution that is assumed to be subjected to the web when the web shear load is increased above that which caused the web to break down into a tension field. The diagonal tensile stress $f_t$ tends to pull the beam flanges together, and thus to bend the flanges. The diagonal tensile stress $f_t$ for the shear resistant web does not produce such action. To obtain the maximum combined tensile stress in the web, the stress $f_t$ must be multiplied by a concentration factor $1/R$ from Fig. C11.9.

Hence

$$f_t^{(max.)} = \left( \frac{f_t}{R} + F_{scr} \right) - - - - - - - (21)$$

Solving for $f_t$

$$f_t = (f_t^{(max.)} - F_{scr}) R - - - - - - (22)$$

If the web is riveted to the flanges and web stiffeners, part of the web material is cut away due to the rivet holes, thus the tensile stress of equation (21) must be multiplied by a rivet correction factor $1/K_R$ to obtain the true $f_t^{max}$.

Hence for riveted connections:

$$f_t^{(max.)} = \left( \frac{f_t}{R} + F_{scr} \right) \frac{1}{K_R} - - - - - - (23)$$

whence

$$f_t = \left( f_t^{(max.)} - \frac{F_{scr}}{K_R} \right) R - - - - - - (24)$$

where

$$K_R = \frac{rivet spacing - rivet diameter}{rivet spacing}$$

Equation (22) would apply for webs spot welded to flange members.

The vertical components of the total $f_t$ stresses on the web effective depth (h) would thus equal the shear load $V_t$ developed by the web after buckling.

For spot welded flange, web and stiffeners connections:

$$V_t = (f_t^{(max.)} - F_{scr}) R h \sin \alpha \cos \alpha - - - - (25)$$

For riveted connections:

$$V_t = (f_t^{(max.)} - \frac{F_{scr}}{K_R}) R h \sin \alpha \cos \alpha - - - - (26)$$

If $f_t^{max}$ is taken as the tensile yield point stress $F_{ty}$ or the ultimate tensile stress $F_{tu}$ equations (25) and (26) will give the shear load carried by the web above the buckling load when the web is stressed to the yield and ultimate tensile strengths respectively.

Thus for yield strength:

$$V_{ty} = (F_{ty} - \frac{F_{scr}}{K_R}) R h \sin \alpha \cos \alpha \text{ (riveted connect.)} - - - - (27)$$

$$V_{ty} = (F_{ty} - F_{scr}) R h \sin \alpha \cos \alpha \text{ (Spot welded connect.)} - - - - (28)$$

For ultimate strength:

$$V_{tu} = (F_{tu} - \frac{F_{scr}}{K_R}) R h \sin \alpha \cos \alpha \text{ (riveted connect.)} - - - - (29)$$

$$V_{tu} = (F_{tu} - F_{scr}) R h \sin \alpha \cos \alpha \text{ (spot welded connect.)} - - - - (30)$$

The total yield shear resistance of the web equals
V_{Wey} = (V_{Cr} + V_{Ty})

Total ultimate web shear resistance:
V_{Wtu} = (V_{Cr} + V_{Tu})

Total beam shear resistance equals equations (31) or (32) multiplied by \( \frac{1}{QR} \) (Ref. Equation 13)

Hence

\[ V_{yield} = \frac{1}{QR} (V_{Cr} + V_{Ty}) \]

\[ V_{ult.} = \frac{1}{QR} (V_{Cr} + V_{Tu}) \]

Substituting values from equations (27 to 30) for \( V_{Ty} \) and \( V_{Tu} \),
For spot welded connections:

\[ V_{yield} = \frac{It}{Q} \left[ F_{Scr} + \left( F_{Ty} - F_{Scr} \right) R \sin \alpha \cos \alpha \right] \]

\[ V_{ult.} = \frac{It}{Q} \left[ F_{Scr} + \left( F_{Tu} - F_{Scr} \right) R \sin \alpha \cos \alpha \right] \]

For riveted connections:

\[ V_{yield} = \frac{It}{Q} \left[ F_{Scr} + \left( F_{Ty} - \frac{F_{Scr}}{K_T} \right) K_T R \sin \alpha \cos \alpha \right] \]

\[ V_{ult.} = \frac{It}{Q} \left[ F_{Scr} + \left( F_{Tu} - \frac{F_{Scr}}{K_T} \right) K_T R \sin \alpha \cos \alpha \right] \]

C11.8 Beams with Parallel Flanges but with Oblique Web Stiffeners.

Wagner has developed equations for the condition where stiffeners are placed at an angle with the parallel flange members, (Fig. A). In this case equations (27) to (30) should be multiplied by a correction factor equal to \( (1 - \tan \alpha \cot \beta) \) where \( \beta \) is the upright angle. For oblique stiffeners the wrinkle flange angle \( \alpha \) is equal to \( \beta/2 \) and the concentration factor \( R \) can be taken as unity. Hence the equations for \( V_{ty} \) and \( V_{tu} \) become as follows:

\[ V_{ty} = \frac{1}{2} \left( F_{Ty} - \frac{F_{Scr}}{K_T} \right) K_T h \tan \frac{1}{2} \cos \beta \sin \beta \]

\[ V_{tu} = \frac{1}{2} \left( F_{Tu} - \frac{F_{Scr}}{K_T} \right) K_T h \tan \frac{1}{2} \cos \beta \sin \beta \]

(For spot welded joints, omit term \( K_T \))

C11.9 Rivet Loads.

The loads on the rivets connecting the flanges to the web consists of two parts, (1) due to the web acting as a shear resisting member subjected to stresses which cause web buckling, and (2) due to the web tension field for web stresses above the web tension stress.

Rivet Load at Web Buckling Stress

\[ P_{xcr} = \frac{V_{cr} I_{y}}{h \sin \alpha} \]

where

\( P_{xcr} \) = load on rivet parallel to flange in lb./in.

\( I_{y} \) = moment of inertia of flanges about beam neutral axis

\( h \) = distance between flange rivet centroids

\( V_{cr} \) = buckling shear strength of web, lbs.

Rivet Loads for Tension Field Action

In Fig. C11.11 the web element (abc) is attached to the flange along line (ab). A vertical depth (bc) of 1 inch has been taken. The shearing stress on this length due to \( V_{tu} \) equals \( V_{tu}/h \). This load represents the vertical component of the tension field, hence

\[ P_t = \frac{V_{tu}}{h \sin \alpha} \]

Resolving the tension load \( P_t \) into x and y components on rivet line (ab) and dividing by the length ab to get rivet loads per inch, we obtain

\[ P_y = \frac{V_{tu} \sin \alpha \sin \alpha}{h \sin \alpha \cos \alpha} = \frac{V_{tu}}{h} \tan \alpha \]

\[ P_x = \frac{V_{tu} \cos \alpha \sin \alpha}{h \sin \alpha \cos \alpha} = \frac{V_{tu}}{h} \]

Combining the three component rivet forces as given in equations 39, 42 and 41 to obtain resultant load \( R \) on rivet:—
C11.8 DIAGONAL SEMI-TENSION FIELD DESIGN

C11.10 Flange Loads.

The axial flange loads are due to two primary causes, namely

1. Stresses due to primary bending of the beam by the usual flexural theory.

2. Additional stresses produced by the web tension field.

In addition to these two primary effects, secondary bending stresses due to the bending of the flange because of the tension field are produced as illustrated in Fig. C11.8.

Stresses for Primary Bending:

\[ f_b = \frac{M_{cr} V_t}{I} \left( \frac{M - M_{cr}}{I} \right) \]  \hspace{1cm} (44)

where

- \( I \) = moment of inertia of total section including web about neutral axis
- \( I_F \) = moment of inertia of section without web about neutral axis
- \( M_{cr} \) = bending moment for load which causes web buckling
- \( M \) = total bending moment on section

The first term in the above equation gives the bending stresses at the point where the web breaks down into a tension field. The web is thus effective in computing the moment of inertia \( I \). The second term in the equation gives the bending stresses when the beam web acts as a tension field web, or in other words the buckled web is assumed ineffective in bending.

To be slightly conservative the bending stresses can be computed by the following equation which neglects the resistance of the web in bending before buckling.

\[ f_b = \frac{M_{cr} V_t}{I} \]  \hspace{1cm} (45)

The total flange loads can be calculated by equating the internal resisting couple to the external resisting couple to the external bending moment, or

\[ F_C = F_t + \frac{M}{h'} \]  \hspace{1cm} (46)

where

- \( F_C \) and \( F_t \) = total compressive and tensile flange load respectively
- \( h' \) = distance between flange centroids

Equation (46) neglects the resistance of the web in bending.

Flange Axial Stresses Due to Web Tension Field

Due to the horizontal components of the diagonal tension field each flange is subjected to a compressive load equal to

\[ F_C = F_t = \frac{V_t}{2} \cot a \]  \hspace{1cm} (47)

(Reference see Equation (12) and general derivation when \( a = 45^\circ \) see Equation (8) and (9)).

In Equation (47) \( V_t \) = shear load carried by tension field action.

Secondary Bending Stresses

For estimating the secondary bending moments on flanges due to lateral pull of web tension field, the flange can be treated as a continuous beam with spans equal to the stiffener spacing.

The component of the web diagonal tensile stresses normal to flange.

\[ w_v = \frac{V_t}{h} \tan a \text{ (pounds per inch)} \]  \hspace{1cm} (48)

where

- \( V_t \) = shear carried by web in diagonal tension

For a continuous beam of equal spans, the moment over the supports = \( 1/12 \ w_v \ d^2 \), where \( d \) equals the stiffener spacing. The deflection of the flanges tends to relieve the pull in the center portion between stiffeners which then decreases the continuity moment over the support. Wagner (Ref. 1) gives a relieving factor \( C \) (See
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Fig. C11.9) for use with the bending moment expression.

Therefore, the secondary bending moment on flanges is

\[ M_{sec} = \frac{1}{122} C \frac{V_t}{h} d^2 \tan \alpha \]  \hspace{1cm} (49)

C11.11 Loads in Web Vertical Stiffeners.

The following method of checking the strength of vertical web stiffeners was originally proposed by Wagner (Refs. 1, 2) and can be considered conservative.

The axial column load in the web stiffener equals

\[ F_{stiff} = -V_t \frac{d}{h} \tan \alpha \]  \hspace{1cm} (50)

This is the same as equation (13) except \( V \) is replaced by \( V_t \), the shear carried by the web in diagonal tension field action. Under this load Wagner considers the stiffeners as columns with elastic supports as the web tension restrains the struts from buckling out of the plane of the web. Wagner's calculations yield a reduction factor \( C_s \) which the actual length of the stiffener is multiplied by to obtain a reduced length \( L' \), or

\[ L' = C_s L \]  \hspace{1cm} (50a)

With this reduced length \( L' \), and the actual cross section of the strut, the column strength can be calculated by the methods of Chapters 81 and 85. Fig. C11.12 by Wagner shows a plot of the reduction factor \( C \) which is a function of the parameter as shown.

Since the work of Wagner, the NACA has conducted an extensive research program on the strength and design of semi-tension field beams and the strength design of the web stiffeners has been based on a more rational basis. The NACA method is given later in this chapter and it is recommended that for final check of web stiffener design, the NACA method be used. The method above, proposed originally by Wagner, can be used for quick preliminary design of the web stiffener.

C11.12 Beams with Non-Parallel Flanges.

In many aircraft beams the flanges are not parallel but have a slight taper. For this case equations (46) and (47) give only the horizontal components of the flange loads. The total flange forces and their vertical components can be computed from the horizontal components and the slope of the flanges. Thus from Fig. C11.12a subtracting the vertical components of the flange forces we obtain the net web shear load \( V_{wn} \).

\[ V_{wn} = V_w - (F_t \tan \theta_t + F_c \tan \theta_c) \]  \hspace{1cm} (51)

where

\[ V_{wn} = \text{net web shear for beam with non-parallel flanges} \]
\[ V_w = \text{web shear for parallel flanges} \]

![Fig. C11.12a](image)

C11.13 Example Problem Using Method 1.

Fig. C11.13 shows a cantilever beam of constant cross-section carrying a 13500 lb. load at the free end. The beam will be strength checked for the given load. The material properties are:

- Web: 2024-T3 alum. sheet, \( F_{tu} = 64000 \), \( F_{ty} = 42000 \), \( E = 10,500,000 \).
- Flanges: 7075-T6 alum. alloy extruded, \( F_{tu} = 78000 \), \( F_{ty} = 70000 \), \( E = 10,300,000 \).

Investigation of Web Strength

Web panel size between web stiffeners and centroid of flange rivets \( = 10 \) by 28.56.

Aspect ratio of panel \( = a/b = 28.56/10 = 2.856 \).

The critical buckling shear stress is given by Eq. C5.4 from Chapter C5:
The angle of the diagonal tension field with horizontal is
\[
\sin^2 \alpha = \sqrt{a^2 + a - a} \quad \text{(see Eq. 15)}
\]
where
\[
a = \frac{1}{\frac{ht}{A}} + \frac{\text{ht}}{a} \quad \text{(see Eq. 16)}
\]
Substituting
\[
a = \frac{1}{\frac{10\times 0.025}{0.675 + 0.375} + 0.025} = 4.05
\]
hence
\[
\sin^2 \alpha = \sqrt{4.05^2 + 4.05 - 4.05} = 0.47
\]
\[
\sin \alpha = 0.685, \quad \cos \alpha = 0.73
\]
To determine the web stress concentration factor, we solve for term \(w_d\) and use curve of Fig. C11.9.
\[
w_d = 1.25 \times \sin \alpha \sqrt{\frac{t}{(1 + I_d)h}} \quad \text{(Ref. Eq. 14)}
\]
\[
w_d = 1.25 \times 10 \times 0.685 \times \sqrt{\frac{0.025}{(1.075 + 0.0291) 23.56}} = 2.43
\]
From curve Fig. C11.9, \(R = 0.86\).
Since the web is riveted to the flange, correction must also be made for net area of web.
\[
K_r = \text{rivet spacing} - \text{rivet diameter} = 0.75 - 0.162
\]
\[
\text{rivet spacing} = 0.75
\]
\[
\text{For 2024-T3 material, } F_{ty} = 42000 \text{ and } F_{tu} = 64000 \text{ psi.}
\]
\[
V_{ty} = (42000 - \frac{34.2}{0.79}) \times 0.36 \times 28.56 \times 0.025 \\
x 0.685 \times 0.73
\]
\[
V_{tu} = 41565 \times 0.242 = 10100 \text{ lb.}
\]
\[
V_{tu} = (64000 - \frac{34.2}{0.79}) \times 0.242 = 15400 \text{ lb.}
\]
The total shear load including strength of flanges is given by equations (33) and (34), namely
\[
V_{yield} = \frac{1}{2h} (V_{cr} + V_{ty})
\]
V \text{ult.} = \frac{1}{Qh} (V_{Tr} + V_{Tu})

From Fig. C11.13

1. N.A. of section without web = 210 in.²
Q of flange about N.A. = 7.15

V_{yield} = \frac{210}{7.15 \times 28.56} (244 + 10,100) = 10,870 lb.

V \text{ult.} = \frac{210}{7.15 \times 28.56} (244 + 15,400) = 16,070 lb.

The design ultimate shear load = 13,500 lb.

hence

M.S. = (16,070/13,500) - 1 = .19

The applied shear load using a factor of safety of 1.5 is 13,500/1.5 = 9000 lb.

The yield strength is 10,870 lb. Thus for margin of safety of yield strength over applied load, we have (10,870/9000) - 1 = .18.

The results show that the beam shear resistance is proportioned as follows for a total ultimate strength of 16,070 lb.
(244/16,070 x 100 = 1.5% carried by web in pure shear, and 15,400/16,070) 100 = 96.0% carried by web as a diagonal tension field.

The remainder or 2.5% is carried by the shear strength of the flanges. This value is relatively low for this particular beam, as in general the flange may provide considerably more of the resistance. Due to the thin web thickness and the large web panels, the web buckles as a relatively low stress, thus the percent of the external shear load carried by the web at the buckling stress is quite small.

Check of Rivet Attachment - Web to Flange

The .025 web is attached to the flange members by 2 rows of 5/32 diameter 2117-T3 rivets at 3/4 inch spacing.

The resultant load on the rivets per inch of flange is given by equation (43), namely

\[ P_r = \left[ \frac{V_{Tr}}{I} \left( \begin{array}{c} 1 + V_{Tu}^a \tan a \end{array} \right) \right]^{1/2} \]

The web is not stressed to its ultimate tensile stress since we have 4% margin of safety. We will therefore solve equation (A) for V_{Tu} using the given external shear load \( V = 13,500 \) lb., and call this value \( V_r \), the shear load carried by the tensile field under the given shear load of 13,500 lb.

\[ 13500 = \frac{210}{7.15 \times 28.56} (244 + V_r) \]

whence

\[ V_r = 12,850 \text{ lb., which represents } V_{Tu} \text{ in equation (B) for } P_r \]

Substitute in Equation (B)

\[ P_r = \left[ \frac{244 - 210}{28.56} \cdot 12850 \cdot 12850 \cdot .325 \right]^{1/2} \]

\[ P_r = \left[ 6 + 450^a + 425^a \right]^{1/2} = 624 \text{ lb./per inch.} \]

Load per rivet pitch of .75 inch = .75 x 624 = 468 lb.

Single shear strength of 5/32 - 2117-T3 rivet = 686 x .36 = 248 lb. (See Chapter D1).

Bearing strength on 2024-T3 sheet = 1.24 x 392 = 486 lb. (See Chapter D1).

Bearing is critical and the rivet strength per rivet pitch is 2 x 486 = 972 lb. (2 rows of rivets).

Margin of Safety = (972/468) - 1 = 1.08

Due to large M.S., the rivet diameter could be made 1/8 inch, or 5/32 rivets could be spaced farther apart.

CHECK OF FLANGE STRENGTH

The end bay of the cantilever beam involves special design considerations such as the two beam fittings and the special end stiffener. Therefore, the basic flange areas will be checked at Section A-A, which is 50 inches from the load point, which is where the end fitting would possibly begin to take load out of the flange members.

Design bending moment at beam Section A-A is,

\[ 50 \times 13,500 = 675,000 \text{ in. lb.} \]

Since \( V_{Tr} = 244 \text{ lb., which is the shear load the web carries without buckling, the bending moment due to a shear of 244 lb. will be resisted by the entire cross-section including the web. Above this value the remaining moment is resisted by the section with the web neglected in computing the moment of inertia.} \]
Thus bending moment at web buckling load = 244 x 50 = 12200 in. lb.

Bending moment for tension field beam = 675,000 - 12200 = 662800 in. lb.

**Bending Stresses**

**Upper Flange Bending Stresses:**

\[
f_b = \frac{-12200 \times 12.58 - 662800 \times 10.94}{270.5} = \frac{567 - 34600 = 35167}{210} \text{ psi.}
\]

(See Eq. 44)

If the entire bending moment were assumed resisted by the flanges alone, then,

\[
f_b = \frac{675000 \times 10.94}{210} = 35200 \text{ psi.}
\]

**Lower Flange Bending Stresses**

\[
f_b = \frac{-12200 \times 17.42 - 662800 \times 19.06}{270.5} = \frac{783 + 60200 = 60980}{210} \text{ psi.}
\]

(Note: Section properties were computed without taking out one 5/32 rivet hole in vertical leg of flange tee, thus stresses are slightly unconservative. To be on the safe side, the net section properties should be used in figuring stresses).

**Average Axial Flange Loads Due to Bending**

Total Flange Load = \( M = \frac{675000}{29.45} = 22900 \text{ lb.} \)

Upper Flange \( f_c(\text{aver.}) = \frac{-22900 \times 575}{-33500} = \frac{33500}{-33500} \text{ psi.} \)

Lower Flange \( f_t(\text{aver.}) = \frac{22900 \times 378}{6050} = 3850 \text{ psi.} \)

**Flange Axial Loads Due to Tension Field Action**

The axial load in each flange due to diagonal tension in web equals

\[
f_t = \frac{F_c}{5 \times \theta \text{ cot } \alpha} \quad (\text{See Eq. 47})
\]

\[
= \frac{-3860}{5 \times 12860 \times 1.07 \times -6380}
\]

Average stress on upper flange

\( f_c = \frac{-5860 \times 375}{-10,170} \text{ psi.} \)

Lower flange, \( f_t = -5860 \times 378 = -18180 \text{ psi.} \)

**Combined Flange Axial Stresses**

**Upper Flange - Extreme Fiber**

\( f_c = -33167 - 10770 = -43337 \text{ psi.} \)

Average stress:

\( f_c(\text{av.}) = -33900 - 10170 = -44070 \text{ psi.} \)

Lower Flange - Extreme Fiber

\( f_c = 60980 - 18180 = 42800 \text{ psi.} \)

Average \( f_t = 60500 - 18180 = 42320 \text{ psi.} \)

Since the tension field action tends to decrease the tension load in the lower flange, it is good practice not to include the entire relieving effect as tension field action is not an exact theory.

**Flange Secondary Bending Stresses**

Since the web is in a diagonal tension condition, it pulls on the flange members, or in other words, each flange acts as a continuous beam with the web stiffeners as the support points and the transverse load on the flange is equal to the vertical component of the web diagonal tensile stresses. This secondary bending moment is approximated by equation (49).

\( M_{sec.} = \frac{1}{12} C \frac{V_Y}{h} d^2 \text{ tan } \alpha \)

From Fig. C11.9, \( C = 0.985 \) when \( wd = 2.43 \)

whence

\( M_{sec.} = \frac{1}{12} 0.925 \times 1.2860 \times 10^4 \times 0.933 = 3230 \text{ in. lb.} \)

**Secondary Bending Stresses on Upper Flange**

\( I_{M.A.}, \text{ upper flange} = 0.1075 \) (See Fig. C11.13)

\( I/y \text{ lower fiber} = 0.1075/1.16 = 0.0925 \)

\( I/y \text{ upper fiber} = 0.1075/338 = 0.319 \)

\( f_b(\text{lower fiber}) = 3230/0.925 = 34900 \text{ psi} \)

Compression as flange bends over the support.

\( f_b(\text{upper fiber}) = 3230/0.319 = 10140 \text{ psi.} \)

(tension)

The combined stress on the lower fiber then would be -34900 - 44070 = -78970 \text{ psi.} \)

The combined stress on the upper fiber would be 10140 - 45337 = -35197 \text{ psi.} \)

**Secondary Bending Stresses on Lower Flange**

\( I/y \text{ lower fiber} = 0.0291/0.217 = 0.134 \) (from Fig. C11.13)

\( I/y \text{ upper fiber} = 0.0291/0.939 = 0.311 \)

\( f_b(\text{upper fiber}) = 3230/0.311 = 10400 \text{ psi} \)

(Compression)

\( f_b(\text{lower fiber}) = 3230/0.134 = 24100 \text{ psi} \)

tension
The combined stresses on lower flange are:

\[ f_{\text{upper fiber}} = \frac{42320 - 104000}{61980} \text{ psi} \]
\[ f_{\text{lower fiber}} = \frac{42320 + 24100}{56900} \text{ psi} \]

### Calculation of Failing Compressive Stress for Upper Flange.

It will be assumed that lateral column action is prevented by lateral bracing from adjacent structure, thus failure of flange will be by local crippling.

The crippling stress for the flange as a whole will be calculated by the Gerard method as given in Chapter C7. The bulbs on the Tee Section (see Fig. C11.13) will be assumed as only partially effective in producing a simple support to the adjacent plate. The g factor for the Tee Section will be taken as 4 which is no doubt slightly conservative. The design curve of Fig. C7.7 of Chapter C7 will be used.

\[ \frac{A}{g} \left( \frac{F_{\text{cy}}}{R_c} \right)^{1/2} = \frac{0.675}{4 \times 1.562} \left( \frac{70,000}{10,300,000} \right)^{1/2} = 0.572 \]

From Fig. C7.7 we read \( F_{\text{cy}}/R_c = 0.90 \), hence \( R_c = 0.9 \times 70000 = 63000 \) psi: compression.

Without the effect of secondary bending of flanges, the average axial stress on upper flange was previously calculated to be -44070 psi. Thus margin of safety would be \( (63000/44070) - 1 = 0.43 \). However, the upper flange must also carry the stresses due to the flange acting as a continuous beam with the web vertical stiffeners as the support points. This bending moment over the support points was previously calculated to be 3230 inch lbs. The bending moment midway between the support points would be about 50 percent of this value since lateral deflection of flange produces a relieving effect. The previous calculations give the combined stress on upper fiber as -35197 psi, and on the lower fiber as -78970 psi. This compressive stress is above the \( F_{\text{cy}} \) value of 70000 for 7075-T6 material and thus the vertical leg of the Tee is obviously weak. The permissible average stress on the vertical leg can be calculated by considering it part of an equal angle section and computing the crippling stress by the Needham method of Chapter C7. The result would be -51500 psi, thus a stress of -78970 existing weakness is indicated and re-design is necessary.

The question now arises, what changes can be made without adding any weight to the present flange. Two obvious faults exist in the flange design. (1) For a beam 30 inches deep, increasing the depth of the flange by a small amount such as 0.5 inch would not effect appreciably the over-all beam bending strength, however, the 0.5 inch increase in depth of the Tee would raise the moment of inertia of the Tee considerably. (2) The strength of the vertical leg of the Tee can be improved by adding a lip. This addition will likewise increase the flange moment of inertia, which is needed to decrease the stresses due to the secondary bending action.

Fig. (A) shows a re-design of the flange section with the same area as before, thus no weight is added. The Tee has been made 0.5 inch more in height and an 0.5 inch wide lip has been added to bottom of Tee. The thickness and bulb size has been reduced to cancel the area added. The new section properties are given in Fig. A.

![Fig. A](image)

Area = 0.875
\( \frac{t}{t} \) upper fiber = 0.584
\( \frac{t}{t} \) lower fiber = 0.25

The secondary bending stresses will now be recalculated.

\[ f_{\text{b(upper)}} = \frac{3230}{0.65} = -12900 \]
\[ f_{\text{b(lower)}} = \frac{3230}{0.584} = 5530 \]

The combined stress on the lower fiber is -44070 - 12900 = -56970 psi as against the previous value of -78970 before Tee was modified. Adding the lip on the bottom of the Tee would raise the allowable stress above the -51500 previously calculated, thus stress of -56970 can now be carried.

This example problem has brought out the fact that the secondary bending stresses can be a major stress factor unless proper attention is given to designing the flange to reduce the secondary bending stresses.

Another approach to calculating or checking the strength of the flange would be to use the interaction equation,

\[ R_c + R_b = 1 \]

where \( R_c \) is equal to the flange load due to beam
bending and web diagonal tension effect, divided by the crippling stress \( F_{c}\) of the flange section.

\( R_0 \) is the ultimate bending moment that the flange can develop as per the method presented in Chapter C3, divided by the design secondary bending moment.

**Strength Check Lower Flange (Tension Flange)**

As previously calculated, the stress on bottom fiber was 66900 psi tension and -61800 psi compression on the upper fiber of the lower flange. The tension is O.K. since the \( F_{ty} \) of the material is 78000 psi which should give enough margin of safety to take care of rivet holes. However, the compressive stress of -61800 is too high and re-design is necessary. The difficulty is due to the large stress from secondary bending. Thus making similar changes for the lower flange Tee as was done for upper flange Tee would solve the problem without adding appreciable weight.

**Check of Web Vertical Stiffener Strength.**

The column load in web stiffener is given in Eq. (50).

\[
F_{\text{stiff}} = -V_t \frac{d}{2h} \tan \alpha = -12850 \frac{10}{22.56} \times .333 = -4200 \text{ lb.}
\]

The reduced column length factor \( C \) of the web stiffener is obtained from Fig. C11.12 for a parameter of

\[
\frac{d}{h (\cot \alpha \cot \beta)} = \frac{10}{22.56 (\cot 40^\circ)} = .304
\]

and gives \( C_s = .36 \)

hence

\[
L' = .36 \times 23.56 = 10.8"\]

Assuming an effective width of web sheet equal to .30\( t \) as acting with the \( L \times l \times 1/3 \) angle stiffener the radius of gyration equals .267 and the area equals .253,

hence

\[
L' / \rho = 10.8 / .267 = 40.5
\]

The crippling stress will be calculated by Needham method of Chapter C7,

\[
b' / t = .9375 / .125 = 7.5
\]

From Fig. C7.5 of Chapter C7, \( F_{cs} / \sqrt{F_{cy}} = .07 \) .

\[
F_{cs} = .07 \times 55000 \times 10,700,000 = 32500 \text{ psi.}
\]

Using Johnson-Euler equation of Chapter C7,

\[
F_c = F_{cs} \frac{F_{as}}{4\pi E} (L' / \rho)^2
\]

\[
F_c = 52500 - \frac{52,500}{4\pi^2} \times 10,700,000 \times (40.5)^2 = 41800 \text{ psi.}
\]

Stiffener strength = \( P = F_{PA} = 41800 \times .253 = 10600 \text{ lb.} \)

The load being only 4200 lb., the stiffener is far overstrength and should be re-designed to save structural weight.

**METHOD 2**

C11.14 NACA Method of Strength Analysis for Semi-Tension Field Beams with Flat Webs.

Method 1, or the modified Wagner equations and procedure used in example problem, are somewhat conservative relative to web and web stiffener design. To eliminate this conservatism and place the web and stiffener design on a more rational or truer basis, the NACA carried on a comprehensive study and testing program to develop a better understanding of semi-tension field beam action and to present a design procedure for use by the aeronautical structures engineer. The results of this program are summarized in (Refs. 3 and 4). The material which follows is taken from those reports.

**NACA SYMBOLS**

The list of symbols which follows is the same as used in Refs. 3 and 4 except \( \sigma, \tau \) and \( \phi \) have been replaced by \( f_r, f_s \) and \( f_t \) respectively, in order to be consistent with the symbols used in the first part of this chapter.

- \( A \): cross-sectional area, square inches.
- \( E \): Young's modulus, KSI.
- \( G \): shear modulus, KSI.
- \( P \): force in KIPS.
- \( Q \): static moment about neutral axis of parts of cross-section as specified by subscripts. in.
- \( R \): coefficient of edge restraint.
- \( S \): transverse shear force, kips.
- \( d \): spacing of uprights, inches.
- \( e \): distance from median plane of web to centroid of (single) upright, inches.
- \( h \): depth of beam, inches.
- \( k \): diagonal-tension factor.
t thickness, inches (used without subscript) signifies thickness of web.
\( \alpha \) angle between neutral axis of beam and direction of diagonal tension, degrees.
\( \rho \) centroidal radius of gyration of cross-section of upright about axis parallel to web, inches (no sheet should be included).
\( f_n \) normal stress ksi.
\( f_s \) shear stress ksi.

**Subscripts**

- DT diagonal tension
- F flange
- S shear
- U upright
- W web
- Cr critical
- ult ultimate
- e effective

**Special Combinations**

- \( P_u \) internal force in upright, kips.
- \( R'' \) shear force on rivets per inch run kips.
- \( R_{tot} \) total shear strength (in single shear) of all rivets in one upright, kips.
- \( d_u \) upright spacing measured as shown in Fig. C11.16a.
- \( h_w \) depth of web as measured as shown in Fig. C11.16a.
- \( h_e \) depth of beam measured between centroids of flanges, inches.
- \( h_r \) depth of beam measured between centroids of web to flange rivet patterns, inches.
- \( h_d \) length of upright measured between centroids of upright to flange rivet patterns, inches.
- \( K_{ss} \) theoretical buckling coefficient for plates with simply supported edges.
- \( F_u \) "basic" allowable stress for forced crippling of uprights.
- \( W_d \) flange flexibility factor \((.7d \sqrt{1/(I_e+I_T)}) \) where \( I_e \) and \( I_T \) are moments of inertia of compression and tension flanges respectively.

**C11.15 Engineering Theory of Incomplete Diagonal Tension.**

In a beam with a thin flat sheet as a web, if the external shear load is less than the buckling load for the web, then the web is in a state of pure shear at the neutral axis as indicated in Fig. C11.14 (Fig. a). If we neglect the normal stresses due to bending over the depth of the web, this shear stress arrangement can be assumed constant over the full depth of the web.

If a web is thin, it will buckle under a certain critical load and if the load is increasing beyond this critical buckling value, the buckle pattern will approach a pure tension field as indicated in (b) of Fig. C11.14b.

![Fig. C11.14 - State of Stress in a Beam Web.](image)

In the usual practical thin web beam in aircraft construction, the state of stress in the web is intermediate between pure shear and pure diagonal tension. The engineering theory as developed by the NACA considers that this intermediate state of incomplete diagonal tension may be based on the assumption that the total shear force in the web can be divided into two parts, a part \( S_g \) carried by pure shear and a part \( S_{PT} \) carried by pure diagonal tension.

Thus under this assumption one can write,

\[ S = S_g + S_{PT} \]

which can be written in the form

\[ S_{PT} = kS \]

\[ S_g = (1-k)S \]

where \( k \) is the "diagonal-tension factor" which expresses the degree to which the diagonal tension is developed by a given load. Thus the state of pure shear is measured by \( k = 0 \) and the state of pure diagonal tension by \( k = 1 \). Fig. C11.15 illustrates the stress condition for the limiting cases of \( k = 0 \) and \( k = 1 \) and for the intermediate case. The letters \( P_k \), \( DT \) and \( PDT \) as labeled on Fig. C11.15 mean pure shear, diagonal tension and pure diagonal tension respectively relative to web stress conditions.

**C11.16 Formulas for Stress Analysis.**

**Limitations of Formulas:**

The NACA believes the formulas which follow will give reasonable strength predictions if...
C11.15 Resolution of Web Stresses at Different Stages of Diagonal Tension.

Common design practices are used. The following limitations should be observed.

1. Uprights on web stiffeners should not be too thin. \( \frac{t}{b} = 0.5 \)

2. The upright or web stiffener spacing should not be too much outside the range \( 0.2 < \frac{d}{h} < 1.0 \)

3. The tests by NACA did not cover beams with very thin or very thick webs, hence some possibility of inconsiderate predictions may exist if \( \frac{t}{b} > 1500 \) or less than 200.

C11.17 Critical Shear Stress.

In the elastic range, the critical shear stress between two web uprights is calculated by the formula:

\[
F_{SR} = k_{SS} \frac{E}{d_c} \left( \frac{t}{d_c} \right)^4 \left[ R_h + \frac{1}{2} (R_d - R_h) \left( \frac{d_c}{h_c} \right) \right] \tag{53}
\]

where,

- \( k_{SS} \) = theoretical buckling coefficient (given in Fig. C11.16a for panel length of \( h_c \) and width \( d_c \) with simply supported edges.
- \( d_c \) = width of sheet between uprights measured as shown in Fig. C11.16a.
- \( h_c \) = depth of web measured as shown in Fig. C11.16a.
- \( R_h \) = restraint coefficient for edges of sheet along upright (See Fig. C11.16b).
- \( R_d \) = restraint coefficient for edges of sheet along flanges. (See Fig. C11.16b).

Curves of the critical shear stresses for plates of 2024 aluminum alloy with simply sup-

ported edges are given in Fig. C11.17. To the right of the dashed line the curves in Fig. C11.17 are plots of the theoretical equation,

\[
F_{SR} = k_{SS} E \left( \frac{t}{d_c} \right)^4 \tag{54}
\]

and may be used for most aluminum alloys. To the left of the dashed line, the curves represent straight line tangents to the theoretical curves in a nonlogarithmic plot and are valid only for 2024 aluminum alloy.

If the critical shear buckling stress is above the proportional limit stress for the material, a plasticity correction must be made. Fig. C11.18 presents curves for correcting the calculated elastic values for this plasticity effect. The plasticity correction for other materials can be obtained as explained in Art. C5.8 and Fig. C5.13 of Chapter C5.

C11.18 Loading Ratio.

The loading ratio is the ratio \( f_S/F_{SR} \) where \( f_S \) is the depth-wise average or nominal shear stress.

When the depth of the flanges is small compared with the depth of the beam and the flanges are angle sections, the stress \( f_S \) may be computed by the formula

\[
f_S = \frac{S_W}{h_c} \left( \frac{t}{d_c} \right) \tag{55}
\]

In beams with other cross-sections, the average nominal shear stress should be computed by the formula

\[
f_S = \frac{S_W Q_f}{1 + \frac{2}{3} Q_f} \tag{56}
\]

Where \( Q_f \) is the static moment about the neutral axis of the flange material and \( Q_d \) is the static moment about the neutral axis of the effective web material above the neutral axis. For the computation of \( I \) and \( Q \), the effectiveness of the web must be estimated in the first approximation. As second and final approximation, the effectiveness of the web may be taken as equal to \( 1 - k \), where \( k \) is the diagonal-tension factor determined in the next step. Thus in computing \( I \) and \( Q \) the effective web thickness is \( 1 - k \) t.

C11.19 Diagonal-Tension Factor k.

Having determined the loading ratio \( f_S/F_{SR} \), the diagonal tension factor \( k \) can be read from Fig. C11.19.
C11.20 Average and Maximum Stress in Upright or Web Stiffener.

The average stress over the length of the upright for a double upright (stiffener on each side of web) can be calculated by the formula,

\[ f_u = \frac{k f_s \tan \alpha}{A_u + .05 (1 - k)} \quad (57) \]

This equation can be evaluated with the use of Fig. C11.20. The stress \( f_u \) is uniformly distributed over the cross-section of the upright until buckling of the upright begins.

Eq. 57 assumes the values \( f_s \), \( k \), and \( \alpha \) to be the same in the panels on each side of the stiffener. If they are not, then average values should be used, or a conservative check should be made using the largest shear value.

Single Upright (or Web Stiffener on one side of Web)

The stress \( f_u \) for a single upright is obtained in the same manner, except that the ratio \( A_u/d_t \) is replaced by \( A_{ug}/d_t \), where

\[ A_{ug} = \frac{A_u}{1 + (\theta/p)^2} \quad (58) \]

For the single upright \( f_u \) is still an average over the length of the upright, but it applies only to the median plane of the web along the line of rivets connecting the upright to the web. In any given cross-section of the upright, the compressive stress decreases with increasing distance from the web, because the upright is a column loaded eccentrically by the web tension. Thus formulas for local crippling based on uniform distribution of stress over the cross-sections do not apply.

Maximum Stress in Upright

The stress \( f_u \) in the upright varies from a maximum at (or near) the neutral axis of the beam to a minimum at the ends of the upright ("gusset effect"). The maximum value is given by the following equation:

\[ f_{umax} = f_u \left( \frac{f_{umax}}{f_u} \right) \quad (59) \]

where, \( (f_{umax}/f_u) \) is the value of the ratio when the web has just buckled, Fig. C11.21 gives the value of this ratio.

C11.21 Angle of Diagonal Tension.

Having determined \( k \) and \( f_u/f_s \), the angle \( \alpha \) between the direction of the diagonal tension and the axis of the beam can be determined by the use of Fig. C11.22.

C11.22 Allowable Stresses in Uprights.

Four types of failure are conceivable.

1. Column Failure.
2. Forced Crippling Failure.
3. Natural Crippling Failure.
4. General elastic instability failure of web and stiffeners.

Column Failure

Column failures in the usual meaning of the word (failure due to instability, without previous bowing) are possible only in double uprights. When column bowing begins, the uprights will force the web out of its original plane. The web tensile forces will then develop components normal to the plane of the web which tend to force the uprights back. This bracing action is taken into account by using a reduced "effective" column length \( L_e \) of the upright, which is given by the following empirical formula,

\[ L_e = \frac{h_u}{\sqrt{1 + k^2 (3 - 2 \frac{d}{h})}} \quad (60) \]

The stress \( f_u \) at which column failure takes place can be found using the standard column curve with the slenderness ratio \( L_e/p \) as shown.

The problem of "column" failures in a single upright has not been investigated to any extent and test results are greatly at variance with theoretical results. Two criteria are suggested for strength design, namely:

1. The stress \( f_u \) should be no greater than the column yield stress for the upright material. This accounts for the upright acting as an eccentrically loaded compression member.

2. The stress at the centroid of the upright (which is the average stress over the cross-section) should be no greater than the allowable column stress for the slenderness ratio \( h_u/2p \). This is an attempt to take in account a two-wave type of buckling failure that has been observed in very slender uprights.

Forced Crippling Failure

The shear buckles in the web will force buckling of the upright in a leg attached to the web, particularly if the upright leg is thinner than the web. These buckles give a lever arm to the compressive force acting in the leg and therefore produce a severe stress condition. The buckles in the attached leg will in turn induce buckling of the outstanding legs.
In single uprights the outstanding legs are relieved to a considerable extent by virtue of the fact that the compressive stress decreases with distance from the web; the allowable stresses for single uprights are therefore somewhat higher than those for double uprights. Because the forced crippling is of local nature, it is assumed to depend on the peak value $F_{\text{max}}$ of the upright stress rather than on the average value.

The upright stress at which final collapse occurs is obtained by the following empirical method:

1. Compute the allowable value of $F_{\text{max}}$ for a perfectly, elastic upright material by the formula:

   For 2024-T3 Aluminum Alloy:
   
   $$F_u = 21300k^{1/2}(tu/t)^{1/2} \text{ (for double uprights)} \quad (61a)$$
   $$F_u = 26000k^{1/2}(tu/t)^{1/2} \text{ (for single uprights)} \quad (61b)$$

   For 7075-T6 Aluminum Alloy:
   
   $$F_u = 26000k^{1/2}(tu/t)^{1/2} \text{ (for double uprights)} \quad (61c)$$
   $$F_u = 32000k^{1/2}(tu/t)^{1/2} \text{ (for single uprights)} \quad (61d)$$

2. If $F_u$ exceeds the proportional limit for the upright material, use as allowable value the stress corresponding to the compressive strain $F_u/2$.

$F_u$ can also be obtained for various materials from Fig. C11.38.

Natural Crippling Failure

The term "natural crippling failure" is used to denote a crippling failure resulting from a compressive stress uniformly distributed over the cross-section of the upright. By this definition it can occur only in double uprights. To avoid natural crippling failure, the peak stress in the upright $F_{\text{max}}$. In the upright should be less than the crippling stress of the section for L/φ = 0. It appears that crippling failure does not appear to be a controlling factor in actual designs.

General Elastic Instability of Web and Uprights

Test experience so far has not indicated that general elastic instability need be considered in design. Apparently the web system is safe against general elastic instability if the uprights are designed to fail by column action or by forced crippling at a shear load much less than the shear strength of the web.

C11.23 Web Design

For design purposes, the peak value of the nominal web shear stress within a bay is taken as,

$$f_{\text{s, max}} = f_{\text{s}} \left(1 + \frac{t_u}{t}\right) \quad (62)$$

Refer to Figs. C11.23 and C11.24 to determine values of factors $C_t$ and $C_s$. The $C_s$ term constitutes a correction factor to allow for the angle $\alpha$ of the diagonal tension field differing from 45 degrees. $C_s$ makes allowance for the stress concentration due to the flexibility of the beam flanges.

Allowable Shear Stress $f_{\text{s}}$

The allowable shear stress $f_{\text{s}}$ is determined by tests and depends on the value of the diagonal-tension factor $k$ as well as on the details of the web to flange and web to upright fastenings. Fig. C11.25 gives empirical allowable curves for two aluminum alloys. It should be noted that these curves contain an allowance for the rivet factor; inclusion of this factor in these curves is possible because tests have shown that the ultimate shear stress based on the gross section (that is, without reduction of rivet holes) is almost constant within the normal range of rivet factor ($C_R = 0.4$).

For the allowable stress for other materials refer to Fig. C11.42.

Permanent Buckling of Web

A check for the development of permanent shear buckles can be made using Fig. C11.46. In this figure $F_{\text{sp}, \theta}$ is the allowable web gross shear stress for no permanent buckles. The Air Force usually specifies no permanent buckles at limit load.

C11.24 Rivet Design

Web to Flange Rivets

The load per inch run acting on web to flange rivets is taken as,

$$R_i = \frac{F_{\text{u}}}{k} \left(1 + 0.443k\right) \quad (63)$$

With double uprights the web to upright rivets must provide sufficient longitudinal shear strength to make the two uprights act as an integral unit until column failure occurs. The total shear strength (single shear strength of all rivets) required in an upright is

$$R_{\text{total}} = \frac{2F_{\text{u}} \theta \Sigma b}{b L_e} \quad (64)$$
where

\[ F_{cy} = \text{column yield strength of upright material. (If } F_{cy} \text{ is expressed in ksi, } R_{total} \text{ will be in kips.)} \]

\[ Q = \text{static moment of cross-section of one upright about an axis in the median plane of the web in}^3 \]

\[ \delta = \text{width of outstanding leg of upright.} \]

\[ \eta_0/Le = \text{ratio obtainable from formula (60).} \]

The rivets must also have sufficient tensile strength to prevent the buckled sheet from lifting off the stiffener. The necessary strength is given by the criterion.

\[ \text{Tensile strength per inch of rivets} = 0.15 \times t \times F_{tu} \quad \text{(65)} \]

where \( F_{tu} \) is the tensile strength of the web.

**For Web to Upright Rivets on Single Uprights.**

The required tensile strength is given by the tentative criterion,

\[ \text{Tensile strength per inch of rivets} = 0.22 \times t \times F_{tu} \quad \text{(66)} \]

(The tensile strength of a rivet is defined as the tensile load that causes any failure; if the sheet is thin failure will consist in the pulling of the rivet through the sheet.)

No criterion for shear strength of the rivets on single uprights has been established; the criterion for tensile strength is probably adequate to insure a satisfactory design.

The pitch of the rivets on single uprights should be small enough to prevent inter-rivet buckling of the web (or the upright leg if thinner than the web), at a compressive stress equal to \( f_{max} \). The pitch should also be less than \( d/4 \) in order to justify the assumption on edge support used in the determination of \( F_{cy} \).

**Upright to Flange Rivets**

These rivets must carry the load existing between upright and beam flange. These loads are,

\[ P_u = f_u A_u \text{ (for double uprights)} \quad \text{(67)} \]

\[ P_u = f_u A_{ue} \text{ (for single uprights)} \quad \text{(68)} \]

These formulas neglect the gusset effect (decrease of \( f_u \) towards the ends of the upright) in order to be conservative. Two fasteners should be used to attach the upright to the flange, especially when the upright is jogged. (See Chapter D2.)

**C11.25 Secondary Bending Moments in Flanges.**

The secondary moment in a flange, caused by the vertical component of the diagonal tension may be taken as,

\[ M = \frac{1}{12} k f_d \times d \times C_3 \quad \text{(69)} \]

where \( C_3 \) is a factor given in Fig. C11.24. The moment given by this formula is the maximum moment and exists at the ends of the bay over the uprights. If \( C_3 \) and \( k \) are near unity, the moment in the middle of the bay is half as large as that given by formula (69) and of opposite sign.

**C11.26 Shear Stiffness of Web.**

The theoretical effective shear modulus of a web \( G_e \) in partial diagonal tension is given in Figs. C11.26a and C11.26b. In Fig. C11.26a, \( G_{eq} \) is valid only in the elastic range. The correction factor for plasticity is given for 2024-T3 aluminum alloy in Fig. C11.26b.

The effective modulus should be used in deflection calculations.

\[ G_e = \frac{G_{IDT} \times C_s \times G}{G_{IDT}} \quad \text{(70)} \]

\( G_e \), for example, should be used in place of \( G \) in the flexibility coefficients for shear panels in Art. A7.10 of Chapter A7, if buckling skins or webs are present.

**C11.27 Example Problem. Using NACA Method.**

(Method 3). The beam used in example problem of Art. C11.13 will be checked by the NACA Method.

First check to see if given beam falls within the limitations of the NACA formulas.

From Art. C11.16:

\[ \frac{t_u}{t} \text{ should be greater than 0.6} \]

For our beam, \( t_u \), the leg thickness of the upright is .125" and the web thickness is .025", hence \( .125/0.025 = 5 \), which is greater than .6.

Also from Art. C11.16, the \( d/h \) value for the beam should fall between .8 and 1.0.

The \( d/h \) value for our beam is \( 10/30 = \frac{1}{3} \), which falls within the given range.
Calculation of the Critical Shear Stress ($F_{scr}$)

Use will be made of Fig. C11.17,

$$d_c = \frac{10}{0.025} = 400 = \text{Stiffener spacing (See Fig. C11.16)}$$

$$h_c = \frac{27.94}{10} = 2.79, \text{ See Fig. C11.16 for } h_c.$$  

Using the above values, we find from Fig. C11.17 that

$$F_{scr} = 370 \text{ psi}.$$  

Calculation of the Loading Ratio $f_s/F_{scr}$

From equation 55

$$f_s = \frac{S_W}{h_t} = \frac{13500}{29.45 \times 0.025} = 18350 \text{ psi}.$$  

where, $S_W = \text{external shear load on web}$

$$h_t = \text{distance between flange centroids}.$$  

Equation 55 was used because the flanges will take very little of the shear load.

Therefore the loading ratio,

$$\frac{f_s}{F_{scr}} = \frac{18350}{370} = 49.6$$

Calculation of Diagonal-Tension Factor $k$

With $f_s/F_{scr} = 49.6$, we use Fig. C11.19 to find value of $k = 0.69$, for zero curve since sheet is flat or R = 0.

Calculation of Average Stress in Upright

Upright consists of one $1 \times 1 \times 1/8$, 2024 extruded angle section.

To obtain effective area, we use eq. 58

$$A_{ue} = \frac{A_u}{1 + \left(\frac{e}{\rho}\right)^2} = \frac{0.254}{1 + \left(\frac{0.025}{0.025}\right)^2} = 0.112$$

where,

$$A_u = \text{area of web upright or stiffener}$$

$e = \text{distance from median plane of web to centroid of single upright.}$

$\rho = \text{centroidal radius of gyration of cross section of upright about axis parallel to web.}$

Using Fig. C11.20, with $A_{ue}/d_t = \frac{0.112}{10 \times 0.025} = 0.45$ and $k = 0.69$, we obtain value of $f_u/f_s = 0.92$.

Hence, $f_u = 18750 \times 0.92 = 17090 \text{ psi}$, which represents the average stress over the length of the upright but applies only along the median plane of the web or along the line of rivets connecting upright to web.

Calculation of Maximum Stress Upright

Value of $\frac{d}{h_u} = \frac{10}{28.50} = 0.352$

where, $d = \text{upright spacing}$

$$h_u = \text{distance between centroids of upright-flange rivet connections}.$$

Using 0.352 as value of $d/h_u$ and $k = 0.69$, we find from Fig. C11.21 that $f_u_{max}/f_u = 1.17$. Hence,

$$f_u_{max} = -16900 \times 1.17 = -19800 \text{ psi}.$$  

Calculation of Diagonal Tension Angle $\alpha$

From Fig. C11.22 for values of $k = 0.69$ and $f_u/f_s = 16900/19350 = 0.92$, we obtain $\tan \alpha = 0.815$.

Calculation of Allowable Stresses for Upright

Check allowable stress for failure as a column:

Since the design is of the single upright type, we check the following two criteria:

1. Stress $f_u$ should be no greater than the column yield stress for the upright material.

In the previous solution of this beam by Method 1, the crippling stress of the web upright was calculated to be $-52600 \text{ psi}$, which corresponds to the column yield stress $F_{oc}$. Since $f_u$ (average) was $-16900 \text{ psi}$, the upright is far overstrength for this particular strength check.

2. The stress at the centroid of the upright should be no greater than the allowable column stress for a slenderness ratio of $h_u = 28.50/2 \times 0.29 = 49$

The Johnson-Dulest column equation is,

$$F_C = \frac{F_{oc}}{4 \pi^2 \frac{L}{\rho}}$$

$$F_C = \frac{52600}{4 \pi^2 \times 10,700,000} (49)^2 = -35000 \text{ psi}$$

$$M.S. = (38000/16900) = 1 = 1.14.$$  

Allowable Stress for Upright for Forced Crippling

For a single member upright equation (61b) applied.
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

\[ F_u = 26000 \frac{k^2}{t} \left( \frac{t_u}{t} \right)^{1/2} \]
\[ F_u = 26000 \times 0.69^{1/3} \left( \frac{1.25}{0.025} \right)^{1/3} = 34600 \text{ psi} \]

The \( F_{uy} \) for 2014-T6 Ext. = 53000

Correcting from \( F_{uy} = 42000 \) for 2024 material by directly as the \( \sqrt{t} \) of the yield stresses, which is approximately true, we obtain,
\[ F_u = \left( \frac{57000}{42000} \right)^{1/2} \times 34600 = 38600 \text{ psi} \]
M.S. = (38600/19800) - 1 = 0.96

The stress is below the proportional limit stress of the material so no correction is necessary.

The stress \( F_u \) could also be found by use of curves in Fig. C11.33.

The web stiffener or upright is too much overstrength and should be re-designed.

**Check Web Design**

The peak value of the nominal web shear stress within a beam bay is taken as,
\[ f_{s\text{max}} = f_s (1 + k C_e) (1 + k C_a) \]

From Fig. C11.23 using tan \( \alpha = 0.815 \), we find \( C_e = 0.922 \).

Using Fig. C11.24, we must find value of term
\[ \frac{4}{wd} = 0.7 \frac{d}{\sqrt{I_C + I_T}} \rho_a \]

\( C_a \) can be found.
\[ wd = 0.7 \times 10 \sqrt{\frac{0.025}{(1.075 + 0.0291) 29.45}} = 1.96 \]

hence from Fig. C11.24, \( C_a = 0.075 \)

Substituting:-
\[ f_{s\text{max}} = 18850 (1 + 0.69 \times 0.022)(1 + 0.69 \times 0.075) \]
\[ = 19600 \text{ psi} \]

**Allowable Web Shear Stress**

From Fig. C11.25a for \( k = 0.69 \), the allowable \( f_{s\text{all}} \) for 24ST (\( f_u = 62000 \text{ psi} \)) equals
\[ 21000 \text{ psi for web without rivet washer and} \]
\[ 23700 \text{ psi for web with rivet washer. The margin of safety for web with rivets without} \]
\[ \text{washer is (21000/19600) - 1 = .07. For rivets with washer the margin of safety would be} \]
\[ (23700/19600) - 1 = .21. \]

**Check of Rivets**

Loads per inch of run web to flange rivets is given by the following equation.
\[ R^* = \frac{S_w}{R_T} (1 + 0.414 k) \]
\[ = \frac{13500}{28.50} (1 + 0.414 \times 0.69) = 610 \text{ lb.} \]

Load per rivet pitch of 3/4 inch = .75 x 610 = 456 lb.

These values are practically the same as by the first method of solution in Art. C11.13.


Strength per rivet pitch = 2 x 486 = 972 lb.
M.S. = (972/456) - 1 = 1.12

**Web to Upright Rivets**

The required tensile strength is given by the criterion, that the tensile strength of rivets per inch of stiffener should be greater than 0.22 to \( F_{uy} \) which equals 0.22 x .025 x 62000 = 340 lb./inch.

In our rivet problems we have specified no web-stiffener rivets. Thus rivets should be specified that will develop 340 lb. tension strength per inch of stiffener. See Fig. C11.37a for tension strength for some fastener-skin combinations. Refer to further discussion in latter part of Art. C11.32.

**Upright to Flange Rivets**

The web uprights are fastened to the flange both upper and lower by 1/4 dia. AN steel bolt.

The load on the upright at its ends is,
\[ F_u = f_u A_{ue} = 16900 \times 0.112 = -1885 \text{ lb.} \]

Shear strength of 1/4 bolt = 3661.

Bearing on vertical 3/32 lag of lower flange
\[ = 1.25 \times 2340 = 2925 \text{ lb. (critical).} \]

M.S. = (2925/1895) - 1 = .54

Since a 1/4" dia. rivet A177ST, \( F_{ru} = 38 \text{ ksi} \) has a shear strength of 1970 lb. it could be used instead of the 1/4" bolt and the M.S. would be .04.

**Check of Flange Strength**

Section 50° from end. Design external
bending moment = 50 x 13500 = 675000° lb.
Diagonal tension factor = .69
Thus bending moment developed as a shear resistant beam = (1 - .69) 675000 = 209000° lb.

The remainder of the bending moment is developed as a pure tension field beam, or a moment of 465000° lb.

### Bending Stresses

**Upper flange bending stresses, (extreme fiber)**

\[ f_b = \frac{209000 \times 12.58 - 465000 \times 10.94}{270.5} = \frac{-9700 - 24300}{270.5} = -34000 \text{ psi.} \]

**Lower Flange Bending Stresses (extreme fiber)**

\[ f_b = \frac{-209000 \times -17.42 + 465000 \times -10.96}{270.5} = \frac{13450 + 42400}{55850} = 55650 \text{ psi.} \]

**Average Axial Flange Loads Due to Bending**

As a shear resistant beam average axial flange load equals

\[ \frac{I - I_w}{l} = \frac{M (1 - k)}{t} \]

where \( M (1 - k) = 675000 (1 - .69) = 209000° lb. \)

at time of web buckling.

\[ h_w = \text{distance between flange centroids} \]
\[ I = \text{moment of inertia of entire beam section about neutral axis} \]
\[ I_w = \text{moment of inertia of web about beam N.A.} \]

### Substituting

\[ \text{Flange load} = 209000 \left( \frac{270.5 - 29.45}{270.5} \right) = 3310 \text{ lb.} \]

### Average Axial Flange load due to bending with beam in diagonal tension state

\[ 465000 \times 29.45 = 15840 \text{ lb.} \]

### Total Flange Average Axial Loads Due to Bending:

Upper flange \( F_C = 15840 - 5510 = 10330 \text{ lb.} \)

Lower flange \( F_t = F_C = 12350 \text{ lb.} \)

### Flange Axial Loads Due to Tension Field Action

\[ F_t = \frac{S_{D_{F}}}{2} \cot \alpha \]

\[ S_{D_{F}} = \text{shear load carried by diagonal tension field action} = \frac{\lambda s}{k}, \text{where} \]
\[ k = .69 \]

Hence \( F_t = \frac{.69 \times 13500}{2} \times 1.228 = 5710 \text{ lb.} \)

### Total Flange Average Axial Loads

Upper flange \( F_C = 21350 - 5710 = 15640 \text{ lb.} \)

Lower flange \( F_t = 21350 - 5710 = 15640 \text{ lb.} \)

Average stress upper flange = \( \frac{-27060}{-40000} = .675 \text{ psi.} \)

Average stress lower flange = \( \frac{15640}{40000} = .391 \text{ psi.} \)

Comparing these values with the values obtained by the first method of solution we find -44000 and 42450 respectively. Thus the NACA method decreases the load on the flanges.

### Check of Flange Stresses Due to Secondary Flange Bending Moments

From Formula 65

\[ M = \frac{1}{12} k f_s t d^2 c_s \]

From Fig. C11.24, \( c_s = .97 \) when \( w_d = 1.96 \)

Hence,

\[ M = \frac{1}{12} x .69 \times 18350 \times .025 \times 10^9 \times .97 = \]

\[ 2560 \text{ in. lb.} \]

This compares with 3210 in. lb. by the first method of solution. The flange stresses could be found as in solution method I.

### C11.28 General Conclusion.

In general the NACA method gives higher margins of safety. The NACA method is recommended for actual design of semi-tension field beams. The NACA method has been extended to cover curved webs and this subject is presented in Part 2 of this chapter.

In the example problem the web was relatively thin and the diagonal tension factor of .69 means that wrinkling is quite severe as 69 percent of the shear load is carried by diagonal tension. In heavily loaded and rather shallow depth beams, such as wing spars or wing bulkheads subjected to large external loads, the webs are much thicker and the \( k \) factor much less.

The great saving in web weight over that required for a non-buckling web design easily exceeds the weight increase in the flanges and the web uprights that diagonal tension field action produces. The rivet design in semi-tension field design presents more detailed design problems since the rivet loads are larger.
and more complex than for a beam with non-buckling web.

C11.29 End Bay Effects.

The previous discussion has been concerned with the "interior" bays of a beam. The vertical stiffeners in these areas are subject, primarily, to only axial compression loads, as discussed. The outer or "end bay" is a special case. Since the diagonal tension effect results in an inward pull on the end stiffener, it produces bending in it, as well as the usual compression axial load. This action can be clearly seen in Fig. C11.7. Obviously, the end stiffener must be considerably heavier than the others, or at least supported by additional members to reduce the stresses due to bending.

Actually an end bay effect exists wherever a buckled panel ends and structural members along the edge of the panel must carry the bending due to the diagonal tension loads.

Typical examples, in addition to the end stiffener discussed, would be the edge members bordering a cut-out or a non-structural door in a flat beam, or a curved fuselage or wing panel. Illustrations are shown in Fig. C11.27.

![Diagram showing end bay effects](attachment:image)

**Fig. C11.27**

The component of the running load per inch that produces bending in such edge members is given by the formulas

\[ w = k \cdot q \cdot \tan \alpha \]

for edge members parallel to the neutral axis (stringers) and

\[ w = k \cdot q \cdot \cot \alpha \]

for members normal to the neutral axis (stiffeners or rings). The longer the unsupported length of the edge member subjected to \( w \), the greater will be the bending moment it must carry.

For example, consider the end stiffener of the beam of Fig. C11.13 in Art. C11.13. Using the data from Art. C11.27,

\[ w = k \cdot q \cdot \cot \alpha \]

\[ = 0.69 \times (18,350 \times 0.025) \times 1.223 = 385 \text{ lb./in.} \]

This is a severe loading for the end stiffener to carry in bending in addition to its other loads. A heavy member would be required.

There are in general 3 ways of dealing with the edge member subjected to bending, the object being to keep the weight down.

1. Simply "beef-up" or strengthen the edge member so it can carry all of its loads (this is inefficient for long unsupported lengths).

2. Increase the thickness of the end bay panel to either make it non-buckling or to reduce \( k \) and thereby the running load producing bending in the edge member (this is usually inefficient for large panels).

3. Provide additional member (stiffeners) to support the edge member and thereby reduce its bending moment due to \( w \) (this requires additional parts).

Actually a combination of these methods might be best.

Consider method (3) above. This will sometimes increase the local shear in the bay and should be considered. Assume that 3 additional stiffeners are added (equally spaced in this case) to support the end stiffener against bending and analyze the end bay internal loads resulting for the beam of Fig. C11.13 and the analysis used for it in Art. C11.27 (NACA Method).

Fig. C11.28a shows the end bay and the loads applied to it.
Note that in Fig. (a) there are two sets of applied loads. One is the basic applied shear load of 13,500 lb. and the other is the component of the tension field load, \( w \), calculated as 388 lb./in. earlier in this article.

Fig. (b) shows the shear flows in the end bay panels, taken as constant in all the panels, due to the 13,500 lb. load applied. The shear flows are shown as they act on the edge members.

Fig. (c) shows the shear flows in the panels due to the applied load \( w \). Average shear flows (in the center of each panel) are shown. Actually the shear flow in the end bay will vary from a maximum of \( q = 388 \times \frac{29.45}{2 \times 10} = 571 \) lb./in. to 0 lb./in. at the center. The values of this variation at the center of each bay are shown for analysis purposes.

In Fig. (d) the load systems of (b) and (c) are added to obtain the final (preliminary) loads. It can be seen here that the shear flow in the upper two panels of the end bay is significantly increased over the nominal value of \( q = \frac{13,500}{29.45} = 467 \text{ lb./in.} \) existing in the other bays of the beam. This means that the diagonal tension effects in this area will be greater (for the same web thickness) and must be considered locally here in checking the upper flange, the end stiffener, the added support stiffeners, the rivets, the web, etc.

Having determined the basic internal loads, the members involving the end bay can be checked for strength using the methods and data of Art. C11.14 to C11.26. When, as in the case of the upper added support stiffener, the shear flows are different in the adjacent bays, average values of \( q \) and \( k \) should be used in checking the stiffener for strength. Formula 57 assumes equal shears in adjacent bays.

In general, there is no simple analytical way of calculating exact tension field load variations when shear flows vary from panel to panel in a structural network. The procedure outlined above is but an elementary "approximation" that can be used for design purposes. If all margins are near zero, substantiating element tests are in order.
Fig. C11.16 Graphs for Calculating Buckling Stress of Webs.

(a) Theoretical coefficients for plates, (b) Empirical restraint coefficients, with simply supported edges.

Fig. C11.17 Buckling Stresses $F_{cr}$ for Plates with Simply Supported Edges. $E = 10,600$ ksi. (To left of dashed line, curves apply only to 24ST aluminum alloy.)

Fig. C11.18
Fig. C11.19 Diagonal-tension factor $k$. (If $h > d$, replace $\frac{ld}{Rh}$ by $\frac{th}{Rd}$; if $\frac{d}{h}$ (or $\frac{h}{d}$) $> 2$, use 2.)
For Flat Sheet Use Zero Curve.

Fig. C11.20 Diagonal-tension analysis chart.
Fig. C11.21 Ratio of Maximum Stress to Average Stress in Web Stiffener.

NOTE for use on curved webs:
For rings, read abscissa as $\frac{d}{h}$; for stringers, read abscissa as $\frac{h}{d}$.

Fig. C11.22 Incomplete Diagonal Tension.

Fig. C11.23 Angle Factor $C_1$.

Fig. C11.24 Stress-Concentration Factors $C_2$ and $C_3$.

$$wd = 0.7d \sqrt{\frac{1}{(C_2 \cdot wd)^2}}$$
(a) 2024 Aluminum Alloy.  
(b) Alclad 7075 Aluminum Alloy.

Fig. C11.25 Allowable Values of Nominal Web Shear Stress.

(a) Modulus Ratio for Elastic Web.

(b) Plasticity Correction for 24S-T3 Aluminum Alloy.

Fig. C11.26 Effective Shear Modulus of Diagonal-Tension Webs.
PART 2. CURVED WEB SYSTEMS.

(By W. F. McCOMBS)

CII. 30 Diagonal Tension in Curved Web Systems - Introduction.

This type of structure has an important place in the design of light metal structures. The structural designer should have as good an understanding of it as he must have for the somewhat simpler plane web system. Actually, most airframe shear web systems are curved rather than flat, the fuselage of a modern aircraft being the outstanding example. To make a fuselage skin entirely non-buckling would require a very thick skin and/or a closely spaced substructure supporting it. This would involve a considerable weight penalty compared to the buckling skin arrangement. The typical metal skin in a modern fighter or transport airplane thus carries its limit and ultimate loads with a considerable degree of skin buckling. In view of this, the need for an understanding of diagonal tension effects in curved web systems is obvious.

CII. 31 General Discussion.

Before getting into the details of design, a general discussion of what happens in a curved web system as the web buckles is helpful. Consider a semi-monocoque structure with a circular (or elliptical) cross-sectional shape as shown in Fig. CII.29.

The structure consists of a number of axial members (called "stringers" if they are numerous and "longerons" if they are few in number, say 3 to 9) which are supported by frames or rings and covered with a skin. The rings may be attached to the skin and "notched" to let the stringers pass through, as in Fig. (b), or they may be located entirely under the stringers and not, therefore, attached to the skin. In this latter case they are called "floating" rings, as in Fig. (c). Sometimes both types of rings are present. Comparing this structure to a plane web beam, the stringers correspond to the flanges, the rings correspond to the uprights and the skin corresponds to the web. Thus, the stringers carry (or resist) axial loads. The rings support the stringers and, if not of the "floating" type, also divide the skin panels into shorter lengths. The skin carries (or resists) shear loads.

Now, assume the structure to be subjected to a pure torsion, T, as shown in Fig. CII.29. Before the skin buckles, this torsion produces a shear in the skin panels given by the well-known formula

\[ \tau = \frac{T}{2A} \]

Only the skin is loaded. As in the case of the uprights of a plane web beam, the rings are not loaded. There is no load in the stringers.

As the torsion is increased, however, the skin shear stress eventually becomes larger than the critical buckling stress and the panels buckle. Any further increase in torsion must now be carried as diagonal tension. Five main things then occur as the torsion is increased above the buckling value, as illustrated in Fig. CII.30.

1. The skin panels buckle and flatten out between rings fastened to the skin, from their original curved shape. This gives a polygonal cross-section (away from a ring). The angle of diagonal tension is less than that for a plane web beam, however, in the range of 20° - 30°.

2. The stringers now feel an axial load, due to the pulling on the ends, (CII.30b), or the structure by the buckled skin, just as in the case of the plane web beam.

3. The stringers also feel a normal loading that tends to bend, or "bow", them inward between supporting rings, as shown in Fig. CII.30c.

4. The supporting rings feel an inward loading which puts them in "hoop" compression. For rings attached to the skin this loading is applied by the stringers and the skin, and is thus "spread out". For floating rings this loading is applied only by the
stringers, coming entirely from (3) above, and is thus concentrated at points. This concentration produces not only compression but also internal bending moments in the floating rings. This is shown in Fig. C11.30e.

5. Any fasteners splicing skins together or fastening the skins to the end rings feel not only a shear panel type of loading but also a normal loading, as in the case of a plane web beam. Also, the "folds" in the skin due to the diagonal buckles try on the rivets as they "attempt" to extend across the rivet lines at the stringers and rings.

The important thing to realize here is that, although only a pure torsion has been applied, considerable axial loads have been generated in the stringers and rings. And even some bending moments have been induced in the stringers and in the rings of the "floating" type.

Now, assume that at the same time the torsion load is being applied an increasing compressive axial load, P, is being applied, simultaneously, as shown in Fig. C11.31.

The following will now happen due to the presence of P.

1. The stringers will, of course, have to carry the compressive load, P, which will be divided among them. There will be some "effective" skin to help.

2. Less obvious, but very important, is the fact that the diagonal tension loads due to the torsion, T, will be considerably affected by the presence of the axial load, P. The larger P is, with respect to T, the greater will be its effect upon the diagonal tension effects. This is as follows.

   a) The skin panels will now buckle at a lower amount of applied torsion since they are now also strained axially in compression. Actually there is a "combined" buckling consisting of compression and shear buckling. This can be obtained from an interaction equation, discussed later.

   b) Since the critical shear buckling stress is now lower the diagonal tension factor, k, is larger.

   c) All of the diagonal tension effects depend upon k, which are increased. These include the axial loads induced in the stringers, the normal loads bending the stringers inward, the loads induced in the rings and the loads felt by the fasteners.

   d) The angle of diagonal tension will be larger, closer to 45°.

Thus we see that the effect of compression is to increase the loads due to diagonal tension.

Now assume that instead of being compression, the axial load, P, is tension. In this case, the effects of diagonal tension, due to T, are reduced.

1. The stringers and skin feel tension due to P, which opposes compression due to diagonal tension.
2. The skin panels can carry a larger shear stress before buckling in shear. In fact, a relatively small amount of axial tension can prevent them from buckling at all, as will be discussed later.

3. The diagonal tension factor, \( k \), will be smaller (or zero if no buckling occurs).

4. All diagonal tension effects dependent upon \( k \) are reduced and the diagonal tension angle is smaller.

Finally, suppose that instead of an axial load, \( P \), a bending moment is being applied simultaneously with the torsion, \( T \), as in Fig. C11.32.

![Fig. C11.32](image)

In this case, also, as the torsion is being increased from a small amount, \( M \) is also being increased, (the ratio of \( M \) to \( T \) being constant, as was the case for axial load, \( P \)). Then, from the usual bending theory, \( f = \frac{Mx}{I} \), the following will occur.

1. The stringers (and skin) above the neutral axis will feel compression loads, the further away the greater the load. The upper skin panels will thus buckle earlier in combined compression and shear and produce the largest diagonal tension loads on the stringers and rings.

2. The skins below the neutral axis will buckle later (or not at all) due to the tension strain produced by \( M \). Thus the diagonal tension effects will be smaller (or non-existent) in the rings and stringers in this region.

3. The skins near the neutral axis will feel little or no strains due to bending and will buckle about as in the case for the pure torsion, \( T \), producing equivalent effects.

In an actual airplane structure, a fuselage for example, the applied loading is more complex. Instead of simply an applied torsion, there is also, usually, a vertical and perhaps a sideward set of loads which produce shears that vary from panel to panel. And there may be not only a bending moment, which changes, along the fuselage, but also axial loads due to landings, catapulting requirements, etc. Obviously all of this complicates the calculations and experience and judgment are of great help; but the method of going about it is fundamental and will now be discussed.

The N.A.C.A. has conducted an extensive program over the years with the object of determining a system for the design of structures having curved webs in diagonal tension. The theory for this system, as in the case for the plane web beam, was given by Wagner and others and modified as necessary from the results of many tests. The design method is fully discussed in Ref. (3) and the substantiating test program in Ref. (4). The reader is encouraged to consult these references for a fuller presentation of the theoretical development and test results.

As mentioned earlier, curved web systems are of two general types. One of these has an arrangement which results in the web, or skin panels being longer in the axial direction, \( d \), than in the circumferential direction, \( h \). This is typical of the stringer system in a fuselage, as shown in Fig. C11.33.

![Fig. C11.33](image)

That is, the geometry of the stringer spacing, \( h \), and the ring spacing, \( d \), is such that

\[
\frac{d}{h} = 1 \quad d > 1
\]

"Floating" rings, not being attached to the skin, do not determine the spacing, \( d \).

A second type of curved web structure may be referred to as the longeron system. Its main characteristic is that the skin panels are long in the circumferential direction. This type of structure would, typically, in a fuselage, consist of a few axial members (a minimum of 3 but more usually 4 to 8 for a "fail safe" design) and a large number of closely spaced frames. The frames would, in this system, be at about a 4" to 6" spacing as compared to a 15"-20" spacing for a stringer system. This gives

\[
\frac{d}{h} < 1
\]

as indicated in Fig. C11.34.
In the longeron system the frames are attached to the skin and longeron, there are no "floating" rings.

Most of the NACA data and design method is for the stringer system. The design method for the longeron system, also presented herein, has evolved from Ref. (3) and also from the results of an investigation and test program at Chance-Vought Aircraft Corp. These latter checks are in some placed different from the ones in the stringer system. The main problem is the determination of the angle of diagonal tension, for either system. Once this is known all of the diagonal tension stresses are known.

The stringer system will be discussed first and the longeron system will be discussed secondly. The basic approach is similar to that for plane web beams.

C11.32 Analysis of Stringer Systems in Diagonal Tension.

Before the diagonal tension effects can be calculated, the primary internal loads in the structure, due to the applied loads, must be determined. This can be done as discussed in Chapter A20. The engineers theory of bending can usually be used to determine the axial loads in the stringers, as in Art. A20.4-420.5. The shear flows in the various skin panels can be determined as in A20.6-420.9. In the case of stringer construction a more accurate determination of the skin shear flows may, if desired, be obtained if the skin panels are considered flat, rather than curved, between stringers. This is because the panels, after buckling, actually flatten out, giving a polygonal cross-sectional shape, except immediately adjacent to non-floating rings, as shown in Fig. C11.30a.

The diagonal tension effects can now be evaluated.

1. Determination of Critical Buckling Stress of Skin Panels.

The buckling strength of curved panels under pure shear and compressive stresses is covered in Chapter C9. The equation for buckling shear stress $F_{Scr}$ is,

$$F_{Scr} = \frac{\pi^2 \pi}{12 \left(1 - \nu_e^2\right)} \frac{E}{b}$$

where simple support is usually conservatively assumed in determining $E$.

When axial loads (and strains) are present, as in practical structures subjected to bending as well as shear, the situation is more complicated. As discussed in Chapter C9, the presence of compressive stresses together with shear stresses causes the panel to buckle at a lower value of shear than if no compression were present. The presence of tension stresses, along with shear stresses, enables the panel to stand larger shear stresses before buckling occurs. In fact, a relatively small amount of tension stress may prevent the panel from ever buckling in shear.

It is important to determine the actual shear buckling stress when axial stresses, particularly compression, are present. The reason is that this affects the diagonal tension factor, $k$, and all of the ensuing stresses affected by $k$.

First consider a shear panel subjected to a shear stress $f_s$ and a compression stress $f_c$. It has been demonstrated experimentally at the NACA, Ref. (5) that a curved panel, thusly loaded, buckles according to the interaction formula

$$\frac{f_c}{F_{r,cr}} + \left(\frac{f_s}{F_{s,cr}}\right)^2 = 1.0 \quad (71)$$

where $F_{r,cr}$ and $F_{s,cr}$ are the critical panel buckling stresses for pure compression and pure shear respectively. From Chapter C9, the buckling stress for a curved shear panel, in end compression is given by the equation,

$$F_{cr} = \frac{\pi^2 \pi}{12 \left(1 - \nu_e^2\right)} \frac{E}{b}$$

Now for any particular panel,

$$\frac{F_{r,cr}}{F_{s,cr}} = A, \ (a \ constant) \quad (72)$$

and

$$F_{cr} = A F_{s,cr}$$

Now, also for any given applied loading condition being checked we can calculate $f_c (-\nu_e)$ at the center of the panel (before buckling) and also $f_s$ (previously done). These stresses will bear a constant ratio to each other until buckling occurs, after which the
compression stress no longer increases. Thus we can write

\[ \frac{f_c}{f_s} = B \quad \text{(73)} \]

or \[ f_c = B f_s \]

Substituting these formulas for \( F_{oc} \) and \( f_c \) back into the interaction formula (71) we obtain,

\[ \frac{B f_s}{A f_{oc}} + \left( \frac{f_s}{f_{oc}} \right)^2 = 1.0 \]

Solving this quadratic for \( f_s \) we get

\[ f_s = f_{oc} \left[ \frac{-3 + \sqrt{3A^2 + 4}}{2A} \right] \quad \text{(74)} \]

where \( f_s \) is the actual shear stress at which the panel buckles due to the presence of compression stresses. Calling this stress \( f_{oc} \) and calling the expression in the brackets \( R_c \)

\[ f_{oc} = f_{oc} R_c \quad \text{(75)} \]

where \( R_c \) is always less than 1.0 when compression stresses are present.

Next, consider a shear panel subjected to a shear stress, \( f_s \), and a tension stress, \( f_t \). For this case it has been experimentally demonstrated, Ref. (6), that the interaction relationship defining buckling is

\[ \frac{f_s}{f_{oc}} - \frac{1}{2} \frac{f_t}{f_{oc}} = 1.0 \quad \text{(76)} \]

where \( f_{oc} \) and \( f_{oc} \) are as previously discussed.

The shear stress, \( f_s \), at which the panel will buckle when the tension stress present is \( f_t \), can be solved directly from equation (76),

\[ f_{oc} = 1.0 + \frac{f_t}{2f_{oc}} \quad \text{(77)} \]

or,

\[ f_{oc} = (1.0 + \frac{f_t}{2f_{oc}}) f_{oc} \quad \text{(78)} \]

Calling the expression in parenthesis in (78) \( R_t \) we get,

\[ f_{oc} = f_{oc} R_t \quad \text{(79)} \]

Since \( R_t \) is always greater than 1.0 when tension stresses are present, the actual shear buckling stress will always be greater than \( f_{oc} \), the buckling stress for pure shear loads only. Inspection of the term in parenthesis in (78) making up \( R_t \) shows that as the tension stress becomes several times larger than \( f_{oc} \) the value of \( f_{oc} \) will become several times larger than \( f_{oc} \). Thus, if \( f_t \) is large enough no buckling will occur even when large shear stresses are present.

2. DIAGONAL TENSION FACTOR, \( k \).

The next step is the determination of the diagonal tension factor, \( k \). This is a function of \( f_{oc} \) where, as discussed above,

\[ f_{oc} = f_{oc} R_c \quad \text{or} \quad f_{oc} = f_{oc} R_t \]

depending upon whether compression or tension stresses are present.

For a curved panel the formula for \( k \) as determined from much test data, Ref. (3) is the empirical relationship,

\[ k = \tanh \left( \frac{0.5 + 300 \frac{t_d}{R}}{\log_e \frac{f_s}{f_{oc}}} \right) \quad \text{(80)} \]

with the auxiliary rules that

a) If \( \frac{d}{h} > 2 \) use only 2

b) If \( h > d \) replace \( \frac{d}{h} \) by \( \frac{h}{d} \) ("longeron" system) and, in this case, if \( \frac{h}{d} > 2 \) use only 2.

Rather than calculate \( k \) from the formula, it can more easily be obtained from Fig. C11.19.

3. STRINGER LOADS, STRESSES, AND STRAINS

As in the case of the plane web system, the total stringer load will consist of the primary axial loads, \( P_p \), due to applied bending moments and/or axial loads, plus the diagonal tension induced loads, \( P_{bt} \).

\[ P_{STR} = P_p + P_{bt} \quad \text{(81)} \]

\( P_p \) is determined as in Chapter A20.2-A20.5. \( P_{bt} \) is determined, similarly to the case for the flanges of a plane web beam, from the "pulling" of the buckled skin on the end frames.
As shown in Fig. CII.35, the diagonal tension load in a stringer bounded by panel "a" on one side and panel "b" on the other. Let the width of panel (a) be \( h_a \) and the width of panel (b) be \( h_b \). (For equally spaced stringers, of course, \( h_a = h_b \).) Let the shear flow in panel (a) be \( q_a \) and that in panel (b) be \( q_b \). Then we can write

\[
P_{D.T.} = \frac{k_a q_a h_a \cot a}{2} + \frac{k_b q_b h_b \cot b}{2}
\]

To keep the calculations simpler (as will be appreciated later) we can accept some inaccuracy and use average values for the respective terms \( \frac{h_a + h_b}{2} \) and get

\[
P_{D.T.} = k q h \cot a
\]

remembering that this is an "average" load, which, for closely spaced stringers, is sufficient, especially for preliminary design.

The total stringer load is then, from 80 and 82

\[
P_{STR} = P_T + k q h \cot a
\]

where all terms are known except \( a \).

The stringer stress is obtained by dividing the terms on the right side by their respective effective areas

\[
\sigma_{STR} = \frac{P_{STR}}{A_{EFF}} = \frac{P_T}{A_{EFF}} + \frac{k q h \cot a}{A_{EFF}}
\]

\( a \): \( A_{EFF} \) is the total effective area, stringer and skin, used in determining the primary loads as in Chapter 20. If \( P_T \) is tension the area is equal to \( hT \), panel is fully effective.

\( b \): \( A_{EFF} = A_{STR} + \frac{.5}{2} hT (1-k) R_{D,T} \) see Ref. (3)

Ref. (3) suggests a more accurate calculation.

Ref. (2) depends upon whether axial compression of tension is present. Again, \( k \) and \( R_{D,T} \) are also average values for the panels on each side of the stringer. Thus we can write

\[
f_{STR} = \frac{P_T}{A_{STR} + A_{SKIN}} + \frac{k q h \cot a}{A_{STR} + .5 hT (1-k) R_{D,T}}
\]

or

\[
f_{STR} = \frac{P_T}{A_{STR} + A_{SKIN}} + \frac{k f_s \cot a}{A_{STR} + .5 hT (1-k) R_{D,T}}
\]

where all terms are known on the right hand side of the equation except \( a \).

The total stringer strain is then

\[
e_{STR} = \frac{f_{STR}}{E_T}
\]

If \( f_{STR} \) is larger than the proportional limit stress, or in the neighborhood of the yield stress, \( E_T \) is not a constant and a stress strain diagram should be used to read \( e \) directly, using \( f_{STR} \). Again, \( f_{STR} \) and hence \( e_{STR} \) cannot be determined until \( a \) is known (later).

There is also a secondary loading on the stringer which tends to bend or "bow" it inward. This is caused by the fact that the taut skins are pulled flat on each side of the stringer. Thus as in Fig. CII.30 there is an inward component of the skin diagonal tension loading that pulls the stringer inward. This loading is not a simply distributed one; it is largest in the middle of the stringer and becomes smaller at the supports. Ref. (13) recommends that the effect of this loading be considered as producing secondary bending moments in the stringer, taken as

\[
M_{STR} = \frac{f_s h d^2 k \tan a}{24R}
\]

This represents a "peak" moment at the middle and at the ring supports. It will produce tension on the inside of the stringer at the middle and compression on the inside of the stringer at the supports. The recommended value of \( M_{STR} \) is the result of many test measurements, therefore it is of a semi-empirical nature.

4. STRESSES AND STRAINS IN RINGS

There are two types of rings, those attached to the skin and those not attached to the skin, called "floating rings," which support, and are therefore loaded only by the stringers.

Rings attached to the skin are usually "notched" to let the stringers pass through.
The stringers are attached to these rings locally by some shear clip arrangement. The rings feel an inward acting loading which puts them in "hoop compression". These loads come from the stringers and from the skin. The stringer being pulled inward by the skin, as described above and in Fig. CII.30, in turn pushes inward on the supporting rings. The skin, not being flat at the ring, also pulls inward on the ring. The result of all of this is, essentially, according to Ref. (3) a hoop compression stress in the ring for the case of a cylinder under pure torsion. This is because the loading is approximately equivalent to an evenly distributed inward angular loading. For this case the radial loading can be taken as, per inch along the ring,

\[ f_{R0} = \frac{f_s \, dt \cdot k \cdot \tan \alpha}{R_{RING}} \]

The axial (hoop) compression load in the ring will then be

\[ P_{RO} = P_K = f_s \, dt \cdot k \cdot \tan \alpha \]

(88)

The axial compression stress in the ring will be

\[ \varepsilon_{R0} = \frac{f_{R0}}{A_{RO} + \text{SKIN}} = \frac{f_s \, \tan \alpha}{A_{RO} + \frac{A_{SKIN}}{dt} + 0.5 (1-k)} \]

(89)

where all terms on the right are known except \( \alpha \).

The axial strain in the ring will be

\[ \varepsilon_{R0} = \frac{f_{R0}}{E} \]

(90)

where \( f_{R0} \) and hence \( \varepsilon_{R0} \) are unknown until \( \alpha \) is determined.

When loads other than pure torsion are applied \( f_s \) and \( k \) will vary from panel to panel and the hoop compression stress will not be constant around the ring. There will also be some varying secondary shear stresses in the panel due to unequal "nulls" on each side of the panel, at the stringer, by the buckled skin.

When "floating" rings are used concentrated inward acting radial loads are applied by the stringers. This produces hoop compression and, since all the loads are concentrated, also some bending moments. There is, of course, no effective skin acting with these rings. The axial compression load is

\[ P_{DT} = f_s \, dt \cdot k \cdot \tan \alpha \]

The axial compression stress is then,

\[ f_{R0} = \frac{k \, f_s \, \tan \alpha}{A_{RO} \, \frac{dt}{d}} \]

(91)

where all terms on the right hand side are known except \( \alpha \).

The axial strain is given by

\[ \varepsilon_{R0} = \frac{f_{R0}}{E} \]

(92)

or can, of course, be gotten from a stress strain diagram for the material.

The maximum bending moment present in a floating ring is given by

\[ M_{R0} = \frac{f_s \, t \cdot h^2 \cdot d \cdot \tan \alpha}{12R} \]

(93)

where \( R \) = radius of cvr. curv. This occurs at the junction with the stringer. There is a secondary moment, half as large, midway between stringers in the ring.

5. STRAINS IN THE SKIN PANELS

The strain in the skin panels is given in Ref. (3) as

\[ \varepsilon = \frac{f_s}{E} \left( \frac{2k}{\sin 2\alpha} + \frac{\sin 2\alpha}{\sin 2\alpha} (1-k) \right) \]

(94)

where \( u \) = Poisson's Ratio = .32 for aluminum, every term on the right hand side of the equation is known except \( \alpha \). Fig. CII.36 is of help in calculating \( \varepsilon \), giving the value of the bracketed term.

6. DETERMINATION OF \( \alpha \)

For the stringer system \( (d = h) \) Ref. (3) shows that \( \alpha \) is related to the stringer strain, the ring strain and the web, or skin panel, strain by the formula

\[ \tan^2 \alpha = \frac{\varepsilon - \varepsilon_{ST}}{\varepsilon - \varepsilon_{R0} + \frac{1}{24} \frac{R_h}{R}} \]

(95)

where \( R \) = radius of curv. curv. \( \varepsilon \) is a tension strain (+) and \( \varepsilon_{R0} \) and \( \varepsilon_{ST} \) are entered as negative members for compression, thus adding to \( \varepsilon \).

\( \alpha \) is determined by successive approximation using the three prior formulas for \( \varepsilon \), \( \varepsilon_{ST} \) and \( \varepsilon_{R0} \) and then checking with the formula (95) above. That is,

* For a faster estimate or for preliminary design, step 6 can be skipped for panels in compression and shear and \( \alpha \) simply assumed to be 45°.
11.36  DIAGONAL SEMI-TENSION FIELD DESIGN

1) Assume a value for \( a \).

2) Determine \( \epsilon \), \( e_{\text{yy}} \) and \( e_{\text{yxx}} \) using this assumed value of \( a \).

3) Substitute \( \epsilon \), \( e_{\text{yy}} \) and \( e_{\text{yxx}} \) into the formula for \( \tan^2 a \) and get a "new" value for \( a \).

4) Repeat steps (1) to (3) as many times (say 3) as necessary to get \( a \) to a "converged" value.

If rings attached to the skin are being used then \( e_{\text{yy}} \) in step (2) above is obtained from (90), using (89) for \( f_{\text{rg}} \). If "floating" rings are used then \( e_{\text{yy}} \) is obtained from (92), using (91) for \( f_{\text{rg}} \).

Once \( a \) is determined, the stringer stresses and ring stresses are known, having been used in getting the strains \( e_{\text{yy}} \) and \( e_{\text{yxx}} \) for the final check of \( a \) in (95). The bending moments in the stringers can then be calculated from (97) and the bending moments in floating rings, if used, from (93).

7. LOADS ON THE RIVETS

The only remaining internal loads to be calculated are those acting on the rivets. These are of two types.

a) There are the primary loads, in the plane of the skins which try to cause shear or bearing failures at the riveted joints, in any spliced skin.

b) There are also, "prying" forces on the rivets which try to "pop off" the rivet heads or to pull the skin up and around the rivet heads. This latter may occur, particularly, when counter-sunk or dimpled skins and flush head rivets are used and the rivet diameter is too small or the rivet spacing is too large. These are "secondary" loads.

The primary rivet loads occur whenever the skin is spliced, which is usually, but not necessarily, along a stringer or along a ring. These loads would also be present if the skin panel ended at stringer or ring, as at an opening or "cut-out" and the panel had not been re-enforced by a doubler to prevent buckling.

At a splice parallel to a stringer the load per inch along the rivet line is due to the same effects as discussed for the plane web beam. It is

\[
\text{Load/inch} = f_{\text{st}} \left[ 1 + k \left( \frac{1}{\sin a} - 1 \right) \right] - - \quad (96)
\]

At a splice (or opening) along a ring (a vertical splice) the loading is

\[
\text{Load/inch} = f_{\text{s}} t \left[ 1 + k \left( \frac{1}{\sin a} - 1 \right) \right] - - \quad (97)
\]

Note that in either case, if the panels involved are made non-buckling (on each side of the splice) \( k = 0 \) and the load per inch is the same as for a non-buckled web. It is only around a "cut-out" or opening that the panels are made either non-buckling or made to buckle to a lesser extent, and this is done to "relieve" the loading on the edge member rather than on the rivets (see Art. 11.29).

The second type of rivet loads, the prying loads, are not determinable by any formula. Ref. (3) recommends that an arbitrary criteria be used as follows. The tensile strength of the rivet-skin combination, \( P_{\text{r}} \), should be such that it is as large as the number given by

\[
P_{\text{r}} \text{ per inch} = 0.22 t P_{\text{tu}} \quad - - - - - - \quad (98)
\]

where \( P_{\text{tu}} \) is the ultimate tensile strength of the skin or web material being used.

\( P_{\text{r}} \) is usually most critical for flush attachments. As an aid in getting this, Figs. C11.37a and C11.37b give information on the tensile strength of various rivet types and sizes.

C11.33 Allowable Stresses (and Interactions)

1. STRINGERS

Just as there are two types of basic loads (and stresses) in the stringers and rings (the primary ones and the ones due to diagonal tension effects) there are also two types of allowable stresses for local failure. An interaction formula is thus used to predict adequate strength. This is, for the stringer

\[
\frac{f_p}{F_{\text{cc}}} + \frac{f_{\text{STM}}}{F_{\text{ST}}} = 1.0 \quad - - - - - - - - \quad (99)
\]

where

\[
f_p = \text{stringer stress due to the applied loads, this is the first term on the right hand side of equation (85)}.
\]

\[
F_{\text{cc}} = \text{the allowable crippling stress for the stringer, obtained as in Chapter C7}.
\]

\[
f_{\text{STM}} = f_{\text{ST}} x \frac{f_{\text{STM}}}{F_{\text{ST}}}
\]

\( f_{\text{STM}} \) is the second term on the right hand side of equation (85).
\( \frac{f_{\text{ST-MAX}}}{f_{\text{ST}}} \) is a ratio obtained from Fig. C11.21. It is explained in the plane web beam discussion.

\( f_{\text{fழ}} = \text{the "forced crippling" (diagonal tension effect) allowable stress for the stringer is obtained from Fig. C11.38, as in the case for the uprights in the plane web beam, or calculated from the formulas (also for } f_{\text{fழ}} \text{ and } f_{\text{fழ}}). \)

\( f_{\text{ST}} = 26,000 \text{ k}\varepsilon/\text{m}^2 \left( \frac{f_{\text{ST}}}{f_{\text{WEB}}} \right)^{1/3} \text{ for 24ST material.} \)

\( f_{\text{ST}} = 32,500 \text{ k}\varepsilon/\text{m}^2 \left( \frac{f_{\text{ST}}}{f_{\text{WEB}}} \right)^{1/3} \text{ for 75ST material with the same restrictions noted in C11.32. (The total axial stringer stress is } f_0 + f_{\text{f谮}} \text{ at the area near the supporting rings, and } f_0 + f_{\text{f谮-MAX}} \text{ in the middle between supports.)} \)

The stress in the stringer due to the bending moment, \( M_{\text{STR}} \), of equation (97) can be calculated and added to \( f_0 \) in the proper manner.

Ref. (3) also suggests the stringer be checked as a column, fully fixed at the supporting rings and carrying the stress \( f_{\text{f зло}} \). Some allowance should be made for beam column action, due to the bending moment present in the middle of the stringer. This can be done using some effective loading producing the moments discussed or by carrying some extra margin of safety over and above the column buckling stress.

2. RINGS

The rings have allowable stresses similar in nature to the stringers. The interaction equation is,

\[ \frac{f_D}{F_{\text{CO}}} + \frac{f_{\text{F-MAX}}}{F_{\text{F-LG}}} = 1.0 \ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \ (100) \]

for the case of rings attached to the skin.

\( f_D \) is the stress due to the ring carrying loads other than the tension field ones (as a bulkhead analysis would show). \( F_{\text{F-LG}} \) is the allowable forced crippling stress for the ring, obtained in the same manner as that for the stiffener, (Fig. C11.39).

\[ f_{\text{F-MAX}} = f_{\text{F-LG}} \times \frac{f_{\text{F-MAX}}}{f_{\text{F-LG}}} \], obtained in the same manner as discussed for the stiffener. It occurs in the ring midway between stiffeners. \( F_{\text{CO}} \) is the "normal" allowable crippling stress for the ring. If \( F_D > F_{\text{CO}} \) then use \( F_D \) for \( F_{\text{F-LG}} \).

Floating rings, not being subject to forced crippling by the skin, are checked in the usual manner for the stresses due to hoop compression loads and the accompanying bending moment, equations (17) and (19). The interaction equation is,

\[ \frac{f_D}{F_{\text{CO}}} - 1 \geq 0 \ - \ - \ - \ - \ - \ - \ - \ - \ - \ (101) \]

\( f_D \) is a constant stress between stringer junctions; it does not have a maximum peak as do the rings attached to skin.

3. GENERAL INSTABILITY

A general instability check for the stringers and skin can be made from the empirical criteria presented in Fig. C11.39. This is obtained from test data and recommendations of Ref. (7) and Ref. (3). The allowables are based upon pure torsion tests. The adequacy of the structure is checked by

\[ \frac{f_S}{f_{\text{S INST}}} - 1 \geq 0 \ - \ - \ - \ - \ - \ - \ - \ - \ - \ (102) \]

It is suggested that a respectable margin of safety be held \( (\text{M.S.} \geq .15) \). The radii of gyration in Fig. C11.39 should be made assuming the full width of sheet to act with the stringer or ring respectively, and that the sheet is flat because the criterion was obtained under these assumptions.

4. ALLOWABLE STRESSES IN THE SKIN (OR WEB)

Ref. (3) recommends that the allowable stress in a web or skin be taken as, for non-flush attachments,

\[ f_{\text{S ALL}} = f_{\text{S ALL}} \left( .65 + \Delta \right) \ - \ - \ - \ - \ - \ - \ - \ - \ - \ (103) \]

where \( \Delta = .3 \tan h \frac{\text{AR}}{\text{d}} + .1 \tan h \frac{\text{AST}}{\text{t}} \); \( \Delta \) can, more easily, be read from Fig. C11.40. \( f_{\text{S ALL}} \) is obtained from Fig. C11.41b or C11.41c after obtaining all stresses from Fig. C11.41a. Data from tests by the Chance-Vought Aircraft Corp. indicate that the allowable web stress can simply be read from Fig. C11.42. In either case, the allowables apply to \( f_S \), the gross stress in the panel. The net shear stress between rivet holes can be carried up to the ultimate shear stress \( F_{\text{SU}} \) of the material.
C11.34 Example Problem.

The foregoing explanation can be better explained or clarified through the presentation of an example problem.

Assume that we have a fuselage with a structural arrangement as in the example problem of A20.5 and Fig. A20.3 and A20.4 of Chapter A20. Also, assume the moment of inertia and neutral axis for a linear bending stress distribution to apply. These values are given on page A20.8. Also let it be assumed that the supporting rings are attached to the skin and spaced at 15". The rings are 1" x 3" x 1" 24ST # sections, .040" in thickness (see Fig.C11.43).

An analysis will be made of stringer #3, skin panels 2-3 and 3-4, and the ring. It is assumed also that an axial skin splice occurs along stringer #3 and a vertical splice along the ring to show a check of the fasteners.

![Diagram of fuselage structure with stringers and skin panels](image)

Fig. C11.43

First the internal loads in the stringer and skin panels are determined. The "average" loads at the middle of the bay will be used in this example to get the diagonal tension effects.

At the center of the bay,

\[ M = 1,475,000 \text{ in. lb.} \]

\[ T = 634,000 \text{ in. lb.} \]

\[ V = 11,700 \text{ lb.} \]

Max. stringer stress occurs at Ring B:-

\[ f_{PMAx} = \frac{M}{I} = \frac{1,475,000}{2382} = 624,000 \text{ psi.} \]

\[ f_{PAV} = \frac{1,475,000}{2382} = 20,400 \text{ psi.} \]

Shear flows in skin panels:–

\[ q_{2-3} = \frac{VQ}{T} \]

\[ = 11,700 \left[ \frac{.167(36.3) + .169(36.4)}{2382} \right] \]

\[ = 61.6 + 113 = 175 \text{ lb./in.} \]

\[ q_{3-4} = \frac{11,700 \left[ .167(36.3) + .169(36.4) + .216(36.9) \right]}{2382} \]

\[ + 113 = 96 + 113 = 209 \text{ lb./in.} \]

The diagonal tension effects will now be calculated assuming average q's, bending stresses, etc., for the panels 2-3 and 3-4.

\[ q_{AV_{2-3}} = \frac{175 + 209}{2} = 192 \text{ lb./in.} \]

\[ f_{SAV_{2-3}} = \frac{192}{.040} = \frac{6000}{0.040} = 150 \text{ psi.} \]

The average critical shear buckling stress will now be calculated using equation (75).

First, if no skin buckling occurred the average compression stress in the two panels would be, approximately, the stress in the stringer between them, thus

\[ f_{cp} = f_{PAV} = 20,400 \text{ psi.} \]

then the constant, B, would be

\[ B = \frac{f_c}{f_s} = \frac{20,400}{6,000} = 3.40 \]

The critical pure shear buckling stress equation from Chapter C9 is,

\[ F_{Scr} = \frac{n^2 k_s E}{12 \left( 1 - \nu_c^2 \right) b} \]

From Chapter C9, \( k_s = 15.9 \)

\[ F_{Scr} = \frac{n^2 k_s E}{12 \left( 1 - \nu_c^2 \right) b} \]

\[ = \frac{15.3 \times 10,600,000}{12 \left( 1 - .3^2 \right) b} \]

\[ = 3140 \text{ psi.} \]

For pure compression the critical buckling stress equation from Chapter C9 is,

\[ F_{Cr} = \frac{n^2 k_c E}{12 \left( 1 - \nu_c^2 \right) b} \]

From Chapter C9, \( k_c = 15.9 \)

\[ F_{Cr} = \frac{n^2 k_c E}{12 \left( 1 - \nu_c^2 \right) b} \]

\[ = \frac{15.2 \times 10,600,000}{12 \left( 1 - .3^2 \right) b} \]

\[ = 2980 \text{ psi.} \]

Thus, \( A = F_{Cr}/F_{Scr} = 2980/3140 = .950. \)

Since buckling stresses are below the proportional limit stress of the material, no plasticity correction was necessary as is usually the case in thin walled structures. Thus stress ratio (A) could be obtained directly as \( k_c/k_s. \)

\* A non-linear bending stress distribution can be used, but it also is affected by the diagonal tension compressive stresses and involves considerable iteration. Linear ones are often used.

\[ \begin{align*}
2 & 26 \times 29.8 \times 7.25 \\
= 61.6 + 113 = 175 \text{ lb./in.} \\
q_{3-4} &= \frac{11,700 \left[ .167(36.3) + .169(36.4) + .216(36.9) \right]}{2382} \\
+ 113 = 96 + 113 = 209 \text{ lb./in.} \\
q_{AV_{2-4}} &= \frac{175 + 209}{2} = 192 \text{ lb./in.} \\
f_{SAV_{2-4}} &= \frac{192}{.040} = \frac{6000}{0.040} = 150 \text{ psi.} \]
\]
Next: 
\[ R_c = \frac{-3 + \sqrt{3^2 + 4}}{2} = \frac{-3.40}{.950} + \sqrt{\frac{3.40^2}{.950^2} + 4} = .28 \]
and, from equation (75)
\[ f_{scr-2-4} = \frac{f_{cr}}{R_c} = 3140 \times .28 = 815 \text{ psi} \]
which is quite small, due to the presence of compression.

Next,
\[ \frac{f_{SAV}}{f_{scr-2-4}} = \frac{6000}{815} = 7.36 \]
now \( k \) can be determined:
\[ 300 \frac{t_d}{RR} = \frac{300 \times (.032)}{30} = .64 \]
From Fig. C11.19, \( k = .75 \).

The expressions for stringer and ring stresses can now be written in terms of \( \alpha \):
Substituting into equation (65) and using \( f_{PAV} \) for the first term on the right side,
\[ f_{STR} = 20,400 + \frac{.75 (6000) \cot \alpha}{7.25 (.032)} + .5 (1 - .75) (.26) = 20,400 + 5570 \cot \alpha \]
this is the "average" stress in the stringer.

Substituting into equation (89) for rings, and using \( ARG = (1^2 + 3^2 + 1^2)(.040^2) = .20 \text{ in.}^2 \),
\[ f_{RG} = \frac{.75 (6000) \tan \alpha}{(15)(.032)} + .5 (1 - .75) \]
\[ = 8,300 \tan \alpha \]
\( \alpha \) will now be determined, by successive approximation using equations (84), (86), (92) and the above expressions for stresses in them, and also Fig. C11.36 for equation (94). Since there is so much compression involved, assume \( \alpha = 45^\circ \), to start with. From Fig. C11.36
\[ \frac{E}{f_s} = 1.83 \quad \varepsilon = \frac{1.83 (6000)}{10.6 \times 10^6} = 1036 \times 10^{-6} \]
\[ \varepsilon_{STR} = \frac{20,400 + 5570 (1.0)}{10.6 \times 10^6} = 2450 \times 10^{-6} \]
* Since \( \frac{d}{h} = 2.0 \), use 2.0 (See page C11.33)
and, along the circumferential splice, from (97)

\[
\text{Load/in.} = 6000(0.032)\left[1 + 0.75(0.675) - 1\right] = 262 \text{ lb./in.}
\]

From the above stresses, \( f_{\text{MAX}} / f \) ratios, and the allowances, checks for adequate strength can be made as follows:

**STRINGERS:**

At the junction with ring (b),

\[
f_{\text{STR}} = (21,600 + f_b) + 6090 \text{ psi.}
\]

For adequate local strength,

\[
\frac{f_c + f_b}{f_c} = \frac{f_{\text{STR}}}{f_{\text{STR}}} = 1.0
\]

\[
F_{\text{ST}} = 26,000 \frac{k^a}{a}\frac{(1.05)}{t} = 28,000(0.75)\frac{a}{a}(0.05) = 24,900 \text{ psi}
\]

(Also determinable from Fig. C11.38).

Thus,

\[
\frac{21,600 + 5200 + 6090}{39,500} = 24,900 = .92 \text{ (sufficient)}
\]

At the center of the stiffener,

\[
f_{\text{STMAX}} = f_{\text{ST}} \left(\frac{f_{\text{STMAX}}}{f_{\text{ST}}}\right)
\]

from Fig. C4.21, \( f_{\text{STMAX}} = 1.02 \).

Therefore,

\[
f_{\text{STMAX}} = 6090 (1.02) = 6210 \text{ psi.}
\]

Then we get, using the interaction formula,

\[
\frac{20,400 + 5200 + 6210}{39,500} = .90 \text{ (sufficient)}
\]

Checking as a fixed end column,

\[
\rho = \frac{L}{\Delta} = \frac{15}{412} = 0.038
\]

\[
F_{\text{COL}} = 39,500 \text{ (very short column range)}
\]

**RINGS**

The rings attached to the skin are subject only to the forced crippling stresses, \( f_{\text{RG}} \).

Midway between stringer junctions this action is a maximum and for adequate local strength,

\[
\frac{f_{\text{RGMAX}}}{f_{\text{RG}}} = 1.0
\]

Using the data in Fig. C11.21,

\[
\frac{f_{\text{RGMAX}}}{f_{\text{RG}}} = f_{\text{RG}} \times \frac{f_{\text{RGMAX}}}{f_{\text{RG}}}
\]

\[
= 7,610 \times 1.0 = 7,610 \text{ psi.}
\]

and thus

\[
\frac{f_{\text{RGMAX}}}{f_{\text{RG}}} = \frac{7,610}{28,100} = .33 \text{ (sufficient)}
\]

The same compression stress (and load, \( P = f_{\text{RG}} \times A_{\text{RG}} \)) exists at the stringer junction. If the ring is notched to let the stringer pass through, as is usually done, the net section at the ring must be made capable of carrying this load, which is located at the centroid of the un-notched ring cross-section. Usually this means some "beef-up", locally, around the notched section; sometimes incorporated into the clip attaching the stringer to the ring.

Actually the rings, like the stringers, are subject to the average of the shear stresses in the panels fore and aft of them. These have been assumed equal in this example.

**SKIN**

The allowable shear stress taken from Fig. C11.42 is

\[
F_s = 21,500
\]

Using the method of Ref. (3)

\[
F_{s\text{ALL}} = 20,800 \text{ (from Fig. C11.4la and b)}
\]

\[
\Delta = .38 \text{ (from Fig. C11.40)}
\]

Thus

\[
F_s = 20,800 (.65 + .38) = 21,400 \text{ psi.}
\]
EITHER CASE SHOWS THERE TO BE A LARGE
MARGIN OF SAFETY FOR THE ACTUAL SHEAR STRESS
OF $f_s = 6000$ psi, (AVERAGE VALUE SHOWN).

(PANELS NEARER THE NEUTRAL AXIS WILL, OF
COURSE, FEEL A LARGER SHEAR STRESS.)

**FASTENERS**

Any fasteners used in this area must, of
course, be able to transfer the load/inch at
the splices as previously calculated. This
criterion might not design the spacing however.
They must also have a tension allowable, when
installed in .032 skin, or, equation (98),

$$
Tens.\text{All. Load/Inch} = .22 \times (.032)(62,000)
= 435 \text{ lb.}
$$

From Fig. C11.37 it can be seen that 1/8" flush
head aluminum rivets in a dimpled .032 skin at
.50" spacing should be adequate, or 1/8"
brazier head rivets at a spacing of .5" if a
"flush" joint is not required. The tension
requirement is most severe in this case.

**GENERAL INSTABILITY**

A check against overall instability of the
"network" of rings and stringers can be made
using Fig. C11.39. This is analogous to the
column stability check for the uprights of a
plane web beam. In the case of curved webs,
however, no "help" is given by the taut skin
in preventing instability.

For this example problem the values used in
calculating the shear stress in the skin at
which "collapse" due to instability would occur
are:

$$
\rho_{ST} = .414 \text{ (Stringer } S_e \text{ of Fig. A20.4 in Chapter A20)}
$$

$$
\rho_{RG} = .819
$$

(Both of these radii of gyration are gotten by
including full width of skin acting with the
stringer, $w_e = h$, and with the ring, $w_e = d$,
per Ref. (3)).

Then

$$
\frac{(\rho_{ST} \cdot \rho_{RG})^{1/2} \times 10^4}{(dn)^{1/4} \cdot R_{1/4}^{1/4}} = \frac{(\rho_{ST} \cdot \rho_{RG})^{1/2} \times 10^4}{(\rho_{ST} \cdot \rho_{RG})^{1/2} \times 10^4}
$$

$$
= \frac{.346 \times 10^4}{133} = 25
$$

From Fig. C11.39

$$
\frac{F_{S\text{INST}}}{F_e} = 3.0; \text{ hence } F_{S\text{INST}} = 3 \times 10.3 \times 10^5 = 30,900 \text{ psi}
$$

and the margin of safety against collapse is,

$$
M.S. = \frac{F_{S\text{INST}}}{f_s} - 1 = \frac{30,900}{6,000} = \text{large}
$$

Other panels, of course, might have a larger
shear stress and a smaller allowable $F_{S\text{INST}}$
and thus be more critical, but there is too
much stiffness available, as seen, to expect
any instability troubles.

**LONGERON TYPE SYSTEM**

**C11.35**

The longeron type of structural system is
somewhat simpler from a total analysis stand-
point. This is, of course, primarily because
there are fewer members carrying the axial
loads and not as many shear panels with varying
shear loads. This type of structure may, or
may not, be the most optimum arrangement from
a weight and manufacturing cost consideration
for a particular airplane. But this is not the
subject of this discussion since it depends
upon an optimization study. The methods of
analysis presented here would, however, have a
place in the calculations behind such a study.

Some typical types of longeron structural
systems cross-sections for a fuselage are shown
in Fig. C11.44.

**Fig. C11.44**

Fig. (a) shows the minimum arrangement, as
to number of longerons, since at least 3 axial
load carrying members are necessary for
equilibrium when bending moments in more than
one plane are involved. This arrangement, how-
ever, has a disadvantage in that it is not a
"fail safe" design. This means that the failure
of any one member will not leave a structure
capable of still carrying some arbitrary
percentage (usually 50% to 67%) of the design
ultimate loads.

The system shown in Fig. C11.44b is capable
of doing this and is, therefore, the minimum
type acceptable from the "fail safe" stand-
point (4 longerons). More longerons may be used, as
in Fig. (c), occasionally, depending upon other
factors of design and manufacturing.

The longeron system, however, requires
more closely spaced rings than does the stringer
DIAGONAL SEMI-TENSION FIELD DESIGN

The engineering procedure for calculating stresses and allowances for the longeron system is somewhat similar to that used for the stringer system. The reader will note the differences.

1. First, at any bay being checked, determine the primary internal load distributions in the longerons and shear panels due to the applied loads. This can be done as in Art. C11.34 using the engineers theory of bending in most cases. In other cases where judgment and experience and the nature of the structure indicate it, the method of Chapter A8 may be used in determining the primary load distribution due to the applied loads.

2. Next determine the critical shear buckling stresses in the skin panels. Since compression stresses are nearly always also present in practical situations, pure shear buckling does not occur. Thus, as discussed in the case for stringer design, some rational interaction must be used to obtain a "reduced" shear buckling stress. This can be done, for example, by using some "average" compression stress in the panel, weighted toward the high side for conservatism. Thereby the interaction method of Article C11.32 can be used where

\[ R_c = \frac{-B + \sqrt{B^2 + 4}}{2} \text{ as in (74)} \]

and \((A)\) is determined for a curved panel of length "d" between rings and height (h) between longerons measured along the circumference as in Fig. C11.42. \((B)\) is the ratio of the compression stress to the shear stress, \(\left(f_c/f_s\right)\), for the particular loading condition being investigated. The compression stress should be calculated as if the panel being calculated had not yet buckled. Then, as in equation (76)

\[ f_{scr} = R_c f_{sc} \]


gives the reduced shear buckling stress, \(f_{scr}\).

When tension strains, rather than compressive ones, are present with the shear we have, as in equation (78)

\[ R_T = 1.0 + \frac{f_t}{2f_{scr}} \]

and then

\[ f_{scr} = R_T f_{sc} \text{ as in equation (79).} \]

3. Next, the loading ratio, \(f_s/f_{scr}\) can be calculated using \(f_{scr}\) as determined in (2) above.

4. Following this, the diagonal tension factor, \(k\), can be obtained from Fig. C11.19.

5. The total axial stress in the longeron can now be written as

\[ f_L = f_p + f_{D.T.} \]

\[ = f_p - \frac{ZA_L}{h_1h_2} \left[ \frac{.25 (1-k)}{R_c} \right] k_s f_s \cot \alpha \]

\[ - \frac{ZA_L}{h_2h_t} \left[ \frac{.25 (1-k)}{R_c} \right] k_s f_s \cot \alpha \]

\[ = \frac{ZA_L}{h_1h_2} \left[ .25 (1-k) \right] R_c \]

\[ = \frac{ZA_L}{h_2h_t} \left[ .25 (1-k) \right] R_c \]

where

\[ f_p = \text{primary longeron stress from Step (1)} \]

\[ f_s = \text{primary shear stress from Step (1)} \]

\[ A_L = \text{Longeron Area} \]

\[ f_{D.T.} = \text{Diagonal Tension Stress} \]

\[ k, f_s, \cot \alpha, R_c \text{ (or } R_T, \text{ h, and } t \text{ are as previously defined. One set, script (1), is for the panels above the longeron and the other, (2) for the panel below the longeron.} \]

6. The average stress in the supporting ring (or frame) due to diagonal tension effects is given by the following formula (and note that this is different from the stringer case)

\[ \ldots \]
The angle, \( \alpha \), must satisfy the equation, from Ref. (3), and this is different from the case for stringers,

\[
\tan^2 \alpha = \frac{\epsilon - \epsilon_L}{\epsilon - \epsilon_{\text{pl}}} \left( 1 - \frac{b^2}{R^2} \right) \tan^2 \alpha
\]

where, as before, from equation (94)

\[
\epsilon = \frac{f_s}{E} \left( \frac{2k}{\sin 2\alpha + \sin 2\alpha (1-k)(1-k)} \right)
\]

or use Fig. C11.26 for bracketed expression.

\[
f_r = \frac{f_{\text{pl}}}{\alpha} \quad (f_r \text{ from equation (104)})
\]

\[
f_{\text{RG}} = \frac{f_{\text{RG}}}{E} \quad (f_{\text{RG}} \text{ from equation (105)})
\]

Equation (105) must be solved by successive approximation as was done in the stringer system discussion.

8. Strength checks can then be made as follows, using stress ratios.

a) Longerons:

\[
\frac{f_r}{f_{\text{cc}}} + \frac{f_{0.1}}{f_{\text{L}}} \leq 1.0
\]

where \( f_{\text{cc}} \) is the "natural" crippling strength and \( f_{\text{L}} \) is obtained from Fig. C11.23.

Frequently, in the case of thicker longerons, the following is used as a strength check:

\[
\frac{f_r + f_{0.1}}{f_{\text{cc}}} \leq 1.0
\]

That is, the primary and diagonal tension effects are simply added and checked against \( f_{\text{cc}} \).

b) Rings:

\[
\frac{f_{\text{RG}}}{f_{\text{cc}}} \leq 1.0
\]

If the rings have stresses in them due to loads other than diagonal tension (i.e. bulkhead type loads, see Chapter A21), then an interaction equation is used.

\[
\frac{f_r + f_{\text{RG}}}{f_{\text{cc}}} \leq 1.0
\]
where $f_p$ is the stress due to loads other than diagonal tension effects.

c) Skins:

Strength check:

$$\frac{f_S}{F_S} \leq 1.0$$

where $F_S$ is obtained from Fig. C11.42.

Permanent Buckling:

Usually no perm. buckling is allowed at limit load.

$$\frac{f_{S,\text{perm}}}{F_S} \leq 1.0$$

where $F_{S,\text{perm}}$ is obtained from Fig. C11.46.

d) General Instability:

$$\frac{f_S}{F_{S,\text{INST}}} \leq 1.0$$

where $F_{S,\text{INST}}$ is obtained from Fig. C11.39. It is recommended that a margin of safety of .15 to .20 be maintained here. Note in Fig. C11.39 that no effective skin is used and that $P_{LONG}$ is at most .34.

e) The loads on rivets at splices, etc. are the same as for the stringer system, eq. (96) and (97).

The procedure outlined in (1) to (5) above can best be illustrated through the use of an example problem.

Fig. C11.37 Example Problem.

Consider a longeron type fuselage structure having a cross-section as shown in Fig. C11.47. The details and section properties of the longerons and rings are included in the figure.

The properties of the structural cross-section are given in Fig. C11.48. The properties shown are for the case where the bending moment causes the upper skin to be in compression and the lower in tension. The upper skin thus buckles out early, as indicated.

For this example problem assume that the applied loads are,

$$M = 2,135,000 \text{ in. lb. (compression in upper skin)}$$

Shear clip

Ring

Long. Cross-Section

I = .0252 in.3

P = .565 in.

Clad 7075-T6 Extruded Mtl.

$F_{p} = 70,000$ psi

$E_C = 10.5 \times 10^6$

Cross-Section and Details of Members

Fig. C11.47

$T = 600,000$ in. lb. (reversible)

$V = 12,000$ lb. (produces comp. in upper skin)

These are the loads at the middle of the bay being checked.

Geometry

Cross-Section Area = 2830 in.2

Effective Section in Bending - Skin 'Dashed' is not Effective

In. $A = 1895$ in.2

Fig. C11.48

1. Using the geometry and properties of Fig. C11.48 and the engineering theory of bending, we find the primary internal stresses to be,

$$f_{\text{upper long.}} = \frac{Mz}{I} = \frac{-2,135,000(33)}{1895} = -37,200 \text{ psi (comp.)}$$

$$f_{\text{stop skin}} = \frac{T}{2At} = \frac{600,000}{2(2830)(.025)} = 4240 \text{ psi}$$

$$f_{\text{side skins}} = \frac{T}{2At} + \frac{VQ}{2It} = 4240 + \frac{12,000(2 \times 60 \times 33)}{2 \times (1895)(.025)} = 11,780 \text{ psi}$$

$$f_{\text{lower long.}} = \frac{Mz}{I} = \frac{-2,135,000(-13.2)}{1895} = +14,800 \text{ (tens.)}$$
2a. Check top skin panel for $f_{scr}$ and $k$.

Use properties of Fig. C11.47 and the applied loads to get $R_o$, eq. (74). The skin compression stresses will be "fictitious" since the skin buckles out early, but will give the proper constant, $B$, for interaction. Assume \( f_{skin} \) at a point 2/3 up from upper longeron to top surface as governing compressive buckling.

\[
f_{skin} = \frac{M_z}{I} = \frac{-2,135,000(27.7)}{1395} = -31,200 \text{ psi (comp.)}
\]

\[
f_s = \frac{T}{2\pi t} = \frac{500,000}{2(2350)(0.025)} = 4240 \text{ psi}
\]

Then, from eq. (73),

\[
B = \frac{f_s}{f_s} = \frac{31,200}{T} = 7.35
\]

For this upper panel, 8' long (a), 41.8" wide (b) and $R = 30^\circ$, the critical pure shear and compression buckling stresses can be found from buckling equations in Chapter 9.

From Chapter 9, for our panel dimensions, $K_p = 10$ and $K_a = 12.85$, thus

\[
F_{scr} = \frac{n^2 x 10 x 10,300,000}{12 (1 - 0.3^2)} = \frac{(0.25)^2}{5} = 2310 \text{ psi}
\]

\[
F_{scr} = \frac{n^2 x 12.85 x 10,300,000}{12 (1 - 0.3^2)} = \frac{(0.25)^2}{5} = 2980 \text{ psi}
\]

hence, from eq. (72)

\[
A = \frac{F_{scr}}{F_{scr}} = 2310 = 0.774
\]

Then, from eq. (74)

\[
R_o = \frac{B}{A} + \sqrt{\frac{B}{A}} = 4 + \sqrt{\frac{7.35}{0.774}} = 7.35 + \sqrt{7.774} = 10.5
\]

and

\[
f_{scr} = R_o F_{scr} = 0.105 (2980) = 313 \text{ psi}
\]

3a. The loading ratio, \( \frac{f_s}{f_{scr}} \), is

\[
\frac{f_s}{f_{scr}} = \frac{4240}{313} = 13.54
\]

3b. The loading ratio is then

\[
\frac{f_s}{f_{scr}} = \frac{11,780}{313} = 11.8
\]

4a. $k$ is then, from (3) and Fig. C11.19, for

\[
\frac{300 R_{th}}{d} = \frac{300 (0.025)}{30} = 0.5
\]

\[
k = 0.51
\]

Next, repeat steps 2a-4a for the side panels. This involves an arbitrary interaction to serve as a criterion for buckling under-combined shear and compression.

2b. The stresses at the top (upper longeron) and bottom (lower longeron) of the side panels are $-37,200$ (comp.) and $14,800$ (tension) respectively. This gives an average "fictitious" stress of $-11,200$ axial and $\pm 26,000$ psi bending stress (equivalent). The actual reduced buckling stress for the side panels could be obtained from an interaction for these axial and bending stresses.

Another way is to use an axial stress only, "weighted" on the high side of the average, arbitrarily, to account for the effect of the bending stress. For this problem a compression stress value half-way between the average, $-11,200$, and the max, $-37,200$, is used.

Thus\[
\frac{f_o}{-11,200 + -37,200}{-24,200} = 24,200 \text{ psi}
\]

then, since $f_s = 11,780$,

\[
B = \frac{f_s}{f_s} = \frac{24,200}{11,780} = 2.05
\]

and, as before,

\[
A = 0.774 (\frac{F_{scr}}{F_{scr}})
\]

Hence

\[
R_o = \frac{2.05}{0.774 + \sqrt{0.774}} = 0.336
\]

and

\[
f_{scr} = R_o F_{scr} = 0.336 (2980) = 998 \text{ psi}
\]

3b. The loading ratio is then

\[
\frac{f_s}{f_{scr}} = \frac{11,780}{998} = 11.8
\]

4b. From Fig. C11.19, using 11.8 from (3b)

\[
k = 0.79
\]

5. The upper longeron stress can then be gotten from eq. (104);
**C11.46**

**DIAGONAL SEMI-TENSION FIELD DESIGN**

\[
\frac{f_{L}}{f_{p}} = \frac{k_{x} f_{s} \cot a_{x}}{k_{x} f_{s} \cot a_{x}} - \frac{2A}{R} + \frac{.81 (4240) \cot a_{x}}{(41.8)(.025)} + .5 (1-.81)(.108)
\]

\[
= -37,200 - \frac{.73 (11,780) \cot a_{x}}{52.4 (.025)} + .5 (1-.79)(.335)
\]

\[
= -37,200 - 2970 \cot a_{x} - 9510 \cot a_{x}
\]

**6a.** The stress in the rings supporting the upper skin is obtained from eq. (105).

First,

\[
A_{RG} \frac{f_{RG}}{f_{s}} = \frac{A_{RG}}{1 + \left(\frac{d_{a}}{R}\right)^{2}} = \frac{.0787}{1 + (.75)^{2}} = .0285
\]

Then,

\[
\frac{f_{RG}}{f_{s}} = \frac{.81 (4240) \tan a_{x}}{.0285} = 10,500 \tan a_{x}
\]

As discussed earlier, this assumes the upper panels on each side of the frame to have the same shear stress.

**6b.** The stress in the rings supporting the side panels is, similarly,

\[
\frac{f_{RG}}{f_{s}} = \frac{.79 (11,780) \tan a_{x}}{.0285} = 27,900 \tan a_{x}
\]

\[\frac{A_{RG}}{A_{s}} = \frac{.0285}{.0255} \approx 1.125
\]

\[\frac{A_{RG}}{A_{s}} = \frac{.228}{.125} \approx 1.82
\]

\[\frac{A_{RG}}{A_{s}} = \frac{.6}{.0255} \approx 23.5
\]

\[A_{RG} \frac{f_{RG}}{f_{s}} = \frac{.0285}{.0255} = .6
\]

\[A_{RG} \frac{f_{RG}}{f_{s}} = \frac{23.5}{.0255} = .95
\]

Then from Fig. C11.45, using the above parameters,

\[\frac{f_{RG}}{f_{s}} = 23.5
\]

\[A_{RG} \frac{f_{RG}}{f_{s}} = .95
\]

As discussed earlier, this assumes the upper panels on each side of the frame to have the same shear stress.

**7.** Angle of diagonal tension.

The compression in the upper panel is so large that \(a_{x}\) can be reasonably taken as 45°.

For the side panel the same is probably true, in the upper portion, but this will be checked using the method of successive approximation and equations (106), (105), (104) and (94).

Assume \(a = 44°\) (and \(u = 45°\)). From Fig. C11.38 and equations above,

\[\epsilon = 1.86 \left(\frac{f_{s}}{E}\right) = 1.86 \left(\frac{11,780}{10.3 \times 10^{6}}\right) = .00215
\]

\[\epsilon_{L} = \frac{f_{L}}{E} = -.00061 - .00281(1.19) = .000995
\]

\[\frac{f_{RG}}{E} = \frac{-27,900 (\epsilon_{L})}{10.3 \times 10^{6}} = .00022
\]

Then,

\[\epsilon_{c} = \epsilon_{L} = .00022
\]

\[\frac{E}{R} = \frac{10.3 \times 10^{6}}{11,780} = 23.5
\]

\[\frac{E}{R} = \frac{.0285}{.0255} = .6
\]

\[\frac{E}{R} = \frac{.6}{.0255} = .95
\]

\[\frac{E}{R} = \frac{23.5}{.0255} = 1.125
\]

Thus, \(a_{x} = 45°\) which is close to the 44° assumed. Let \(a_{x} = 45°\). This bears out the statement that any significant compression stress (or strain) forces \(u\) towards 45°.

Had there been no significant average axial strain (or stress) present a could be gotten from Fig. C11.45 which is based on pure shear (no axial loads present). Using this for the above side panels, for example, we would proceed as follows:

\[\frac{E}{R} = \frac{10.3 \times 10^{6}}{11,780} = 23.5
\]

\[A_{RG} \frac{f_{RG}}{f_{s}} = \frac{.0285}{.0255} = .6
\]

Then from Fig. C11.45, using the above parameters,

\[\frac{f_{RG}}{f_{s}} = 23.5
\]

\[A_{RG} \frac{f_{RG}}{f_{s}} = .6
\]

Thus the effect of compression, as previously calculated, is significant in forcing \(u\) towards 45°, as frequently assumed for simplicity.

8. Now that \(a\) has been established as 45° for the upper skin panels and for the upper portion of the side skin panels, the stresses are available (from \(5\) and \(6\) ) and strength checks can be made.

a. **Longerons:**

\[f_{p} = -37,200; \ f_{0.7} = -2970(1.0) - 9810(1.0)
\]

\[= -12,780
\]

\[F_{cc} = 55,000
\]

Using Fig. C11.38, and the formula for "c" in the figure,

\[F_{RG} = N x C = 43,500 (1.28) = 55,600
\]

then using the interaction equation suggested,

\[37,200 + 12,780 = 55,600 = .801 < 1.0 \text{ (adequate)}
\]
b. Rings:
The ring under the side panel skin will be most critical \( (f_g \) is larger):

\[
f_{RG} = 27,900 \ (1.0) = 27,900
\]

\[
f_{RG\text{MAX}} = 27,900 \ (1.15) = 32,050 \quad \text{(Using Fig. C11.21)}
\]

\[
k' = \frac{10}{5} = 2
\]

Then, using Fig. C11.38,

For \( R = 30'' \)

\[
N = 19,200
\]

\[
F_{RG} = N \times c = 19,200 \times 1.197 = 22,800
\]

\[
M.S. = \frac{F_{RG}}{R} - 1 = \frac{22,800}{30,000} - 1 = -0.25
\]

Thus the ring is not adequate. Either more area (thickness) is required or a closer ring spacing (or both) are needed to lower \( f_{RG} \).

c. Skins:
The side panel has the highest stress and will be, therefore, more critical from an ultimate shear strength than the top panel.

\[
f_g = 11,750 \ \text{psi}
\]

From Fig. C11.42, for \( k = 0.79 \) and \( F_{pq} = 72,000, F_{all} = 25,000 \ \text{psi} \). M.S. = \( (25,000/11,750) - 1 = 1.13 \).

d. Check for Permanent Buckling

Usually there is the requirement that no permanent skin buckles shall occur at limit load. This is checked as follows:

\[
f_{g\text{limit}} = \frac{f_g}{1.5} = \frac{11,750}{1.5} = 7,840 \ \text{psi}
\]

Using Fig. C11.46

\[
4 \ F_{st} \times 10^9
\]

\[
\frac{E \times 10^9}{F_{pq} = 4 \ (2.090) \times 10^9 = 1.37}
\]

\[
F_{sp,B} = 2.8, F_{sp,R} = 2.8(2960) = 8,340 \ \text{psi}
\]

\[
M.S. = \frac{F_{sp,B}}{f_{g\text{limit}}} - 1 = \frac{8,340}{7,840} - 1 = 0.065
\]

e. General Instability

Using Fig. C11.39, with

\[
\rho_{LONG} = 0.554 \quad \rho_{RG} = 0.565
\]

(Using no effective skin, per the figure)

\[
0.39 (\rho_{RG})^{1/4} \frac{(dn)^{1/4} R^{1/3}}{K_{R}^{1/4}} \times 10^6 = \frac{0.39 \times 0.565}{(5 \times 52.3)^{1/4}} \times 10^6 = 11.35
\]

From Fig. C11.39

\[
F_{s\text{INST}} = 3.0 \times 10^{-3} = 30,000 \ \text{psi}
\]

Thus

\[
M.S. = \frac{F_{s\text{INST}}}{f_{g}} - 1 = \frac{30,000}{11,750} - 1 = 1.62
\]

(large)

f. Rivets

The rivet requirements can be checked in the same manner as was done for the stringer type structure along any skin splices and for the tension field "prying" forces.

The load transferred between the longeron and the ring can be calculated as

\[
F_{RG} = F_{rg} \times A_{rg} \quad \text{from (8b) and (6a)}
\]

\[
= 27,900 \times 0.0285
\]

\[
= 795 \ \text{lb.}
\]

This load is actually carried by the rivets attaching the outer ring flange to the longeron flange (next to the skin) and also by the gusset action of the skin at the ring-longeron junction. Two fasteners, staggered if necessary, should be used to attach the outer ring flange to the longeron flange. (See discussion of joggles in Chapter 03).

c11.38 Summary

From these examples involving both stringer and longeron type construction it can be seen that the effect of axial compression stresses (or strains) along with the shear stresses in the panels is two-fold:

a) It brings about, through interaction, earlier buckling of the panels than would result from shear stresses alone. The result is, of course, a higher value of \( k \) and, hence, of all the ensuing loads and stresses that are a function of \( k \).
b) The angle of diagonal tension is forced to approach 45°, more than would result from shear stresses only.

The effect of tension stresses is just the opposite and can thus be conservatively ignored, or evaluated if desired.

From a time-saving standpoint, in preliminary design, an arbitrarily large value of \( k \) and an angle of diagonal tension of 45° can be assumed where significant axial compression strains are present.

The exact magnitude of the various diagonal tension effects throughout a network of skin panels defies simple evaluation from an analytical standpoint. This is particularly true when both shear stresses and axial stresses change from panel to panel as in most practical structures and loadings. No simple analytic expressions are available. But some rational approach is necessary to complete the design, or specimens for test programs, and the approaches given in this chapter represent one such procedure. If margins are extremely small, element tests for substantiation are in order. The reader is encouraged to consult the references for a more thorough understanding of the basic theory and its limitations, particularly with regards to areas where substantiating test data is relatively meager.

The stringer system is usually found, for example, in fuselage strumera where there are relatively few large "cut-outs" to disrupt the stringer continuity. This is more typical of transport, bomber and other cargo carrying aircraft. The longeron type structure is more efficient and suitable where a large number of "quick-access" panels and doors and other "cut-outs" are necessary to service various systems rapidly. These would "chop-up" a stringer system rather severely, making it quite expensive from both a weight and manufacturing standpoint. Therefore, longeron systems are more usually found in fighter and attack type aircraft, and in others with unusual features. There are also, of course, other factors influencing the choice of structural arrangement.

Some further notes concerning this general subject are included in Chapter C6. This includes "beef-up" of panels and axial members bordering cut-outs or non-structural doors, which is also related to "end-bay" effects discussed in C11.24.


1. In the example problem of Art. C11.34, what M.S. would exist if the ring were made of .032 2024 (instead of .040). (Assume \( a = 45° \)).

2. In the example problem of Art. C11.34, how wide could the ring spacing, \( d \), be made and still show a positive M.S. for the ring. (Assume \( a = 45° \)).

3. In the example problem of Art. C11.34, how much additional torsion, \( T \), could be applied before the ring would show a M.S. = 0. (Assume \( a = 45° \)).

4. Repeat example problem Art. C11.34 using applied loads

\[ M = 2,500,000 \text{ in./lb.} \]
\[ T = 0 \]
\[ V = 40,000 \text{ lb.} \]

Assume the general section properties (I and neutral axis location) remain the same as in C11.34.

5. In the example problem of Art. C11.37

a) What (standard) gauge of 7075-T6 sheet aluminum would the ring have to be to show a minimum positive M.S.

b) What ring spacing, \( d \), would be required for the .062" ring to show a minimum positive M.S. (M.S. = 0)

6. Repeat the example problem of Art. C11.37 using as applied loads.

\[ M = 2,400,000 \text{ in./lb.} \]
\[ T = 0 \]
\[ V = 28,000 \text{ lb.} \]

(Assume section properties are not changed.)

7. What is required to eliminate any negative margins of safety in Problem (6).


(1) The beam as shown in Fig. (A) is subjected to a shear load of 8000 lb. as shown. Determine the margin of safety for the given loading for the following units: (1) web, (2) web flange rivets, (3) web stiffeners. Use NACA method.

(2) Same as problem (1) but with web upright spacing = 4".

(3) Design a semi-tension field beam for the beams and loading of Fig. (B). Take web as 2024-T5 with minimum thickness being .020.
Web uprights and flange members to be 2014-T6 extrusions. Rivets to be 2117-T3. Show calculations for at least four sections along beam length. Assume the beam braced laterally by .025 skin on top and bottom.

(4) Fig. (C) shows the cross-section of a single spar wing beam. Strength check the following units of the beam section. (1) web, (2) web uprights. The upright spacing is 8 inches and the design shear load on section is 35000 lb. web material is 7075 alclad.

REFERENCES


(2) H. Wagner, "Flat Sheet Girder With Thin Metal Web."
   Part I - N.A.C.A. Technical Memo. 604
   Part II - N.A.C.A. Technical Memo. 606
   Part III - N.A.C.A. Technical Memo. 606


(5) N.A.C.A. T.N. 1347.


(7) N.A.C.A. T.N. 1197.

(8) N.A.C.A. T.N. 774.
Fig. C11.38 Graph for Calculating Web Strain (Ref. 3)

\[
\frac{SE}{Ts} = \frac{2k}{\sin 2\alpha} + \sin 2\alpha (1 - k)(1 - \mu)
\]

\[\mu = 0.32\]

Fig. C11.37a Tensile Strength of Four Types of A17S-T3 Aluminum-Alloy Rivet in 24S-T3 Aluminum-Alloy Sheet. (Ref. 3)

Fig. C11.37b Tensile Strength of NACA Machine-Countersunk Flush Rivets of A17S-T3 Aluminum-Alloy in 24S-T3 Aluminum Alloy Sheet. (Ref. 3).

Approximate \[\frac{S}{2} = \frac{\pi}{2}\]
GENERAL INSTABILITY CRITERIA FOR CURVED SHEAR-PANELS

(N.A.C.A. T.P. 1197)

NOTES

1) Compute Csp and Cpg Assuming Flat Width of Shear-Acting with Stringers or Ring Respectively and that Sheet is Flat.

2) Do not include Effective Skin in Computing Cpg and Rpg.

3) If Csp is Larger than 0.34, use Cpg = 0.34

4) If R = 150 Use Column Analysis for FIsG (Plane Web)-Tension Field Beam-Stiffeners in Place of this Curve.
Fig. C11.40 Correction for Allowable Ultimate Shear Stress in Curved Webs.

\[ R_R = \frac{ds}{d_R} \quad R_s = \frac{ht}{A_{ST}} \]

Fig. C11.41a

Fig. C11.41b 2024-T3 Aluminum Alloy. \( \sigma_{ult} = 62 \) ksi. Dashed Line is Allowable Yield Stress.

Fig. C11.41c Alclad 7075-T6 Aluminum Alloy. \( \sigma_{ult} = 72 \) ksi

Fig. C11.41d Angle of Pure Diagonal Tension.

\[ R_R \cdot \frac{dL}{\sigma_{RG}} = R_s \cdot \frac{dL}{\sigma_{ST}} \]
ALLOWABLE GROSS AREA WEB-SHEAR STRESS

NOTES:

1) These curves apply to all common structural materials used at room or elevated temperatures. Simply determine the ultimate tensile strength of the material under the critical condition and select the corresponding curve below. Where extrapolation is necessary use a straight line variation.

2) For curves not shown ratio the new tensile ultimate to that of the base curve \( F_{tu} = 52,000 \).

---

**Fig. CII.42**
Approximate Values of the Angle of Diagonal Tension $\alpha$

For panels with $h = d$ (Longeron Systems)

$$\frac{A_{RCd}}{dt}$$

NOTES:

1. This curve is for panels loaded in shear only, with no significant axial strains.

2. The curve is an approximation of the following equation:

$$\tan^2 \alpha = \frac{K + \sin^2 \alpha \left[ \frac{2}{(1-k)(1-k)} \cos^2 \alpha + \frac{E}{A_{FC}} \right]}{K + \cos^2 \alpha \left[ \frac{2}{(1-k)(1-k)} \sin^2 \alpha + \frac{E}{A_{FC}} \right]}$$

3. The range of $\frac{E}{A_{FC}}$ values represented on this curve is from 0.2 to 1.5.

Values of this parameter outside this range (for values of $\frac{E}{A_{FC}} < 0.2$)

must be applied to the equation above (Note 2) to determine $\alpha$.

Chance Vought Aircraft, Inc.

Fig. C11.45

NOTES:

1. This curve is good for flat or curved panels. Conventional panel stiffening members must be attached to edge members.

2. Curve does not apply to panels with stringers.

3. Curve may be used for any material at any temperature provided proper values of $F_{SCR}$, $E_c$, and $F_{CY}$ are used in the parameters.

$F_{SCR}$ = Panel Shear Buckling Stress

$E_c$ = Modulus of Elasticity in Compression

$F_{CY}$ = Compression Yield Stress

$$\frac{4 \times F_{SCR} \times 10^8}{E_c \times F_{CY}}$$

Fig. C11.46
CHAPTER DI
FITTINGS AND CONNECTIONS. BOLTED AND RIVETED.

Dl.1 Introduction.

The ideal flight vehicle structure would be the single complete unit of the same material involving one manufacturing operation. Unfortunately the present day types of materials and their method of working dictates a composite structure. Furthermore general requirements of repair, maintenance and stowage dictate a structure of several main units held to other units by main or primary fittings or connections, with each unit incorporating many primary and secondary connections involving fittings, bolts, rivets, welding, etc. No doubt main or primary fittings involve more weight and cost per unit volume than any other part of the aerospace structure, and therefore fitting and joint design plays an important part in aerospace structural design.

Dl.2 Economy in Fitting Design.

For structural economy the structural designer in the initial layout of the flight vehicle should strive to use a minimum number of fittings particularly those fittings connecting units which carry large loads. Thus in a wing structure splicing the main beam flanges or introducing fittings near the centerline of the airplane are far more costly than splices or fittings placed farther outboard where member sizes and loads are considerably smaller. Avoid changes in direction of heavy members such as wing beams and fuselage longerons as these involve heavy fittings. If joints are necessary in continuous beams place them near points of inflection in order that the bending moments to be transferred through the joint be kept of small magnitude. In column design with and fittings avoid introducing eccentricities on the beam and on the other hand make use of the fitting to increase column and fixity thus compensating some of the weight increase due to fitting weight by saving in the weight of the beam.

For economy of fabrication, the structural designer should have a good knowledge of shop processes and operations. The cost of fitting fabrication and assembly varies greatly with the type of fitting, shape and the required tolerances. Poor layout of major fitting arrangement may require very expensive tools and jigs for shop fabrication and assembly.

Fittings likewise add considerably to the cost of inspection and rejections of costly fittings because of faulty workmanship or materials are quite frequent, thus adding greatly to the unit cost.

Dl.3 Fitting Design Loads. Minimum Margins of Safety.

As discussed in other chapters, limit loads are the maximum loads which a flight vehicle may be subjected to during its lifetime, when carrying out the required ground and flight operations. The limit loads must be carried by the structure without exceeding the yield stress of the material used in the structure. The ultimate or design loads are the limit loads multiplied by a factor of safety, usually 1.5 for aircraft and less for missiles. The structure must have sufficient strength to carry the ultimate or design loads without failure.

The normal flight vehicle structure involves many parts which are joined together by various types of connections. In general, an additional blanket factor of safety is required for design of these connections. This blanket factor of safety is normally 1.15 to 1.20. The stress analysis of most connections or fittings is more complicated than for the primary structural members due to such factors as combined stresses, stress concentration, bolt-hole tightness, etc., thus an additional factor of safety is necessary to give a similar degree of strength reliability for connections as provided in the strength design of the members being connected.

Dl.4 Special or Higher Factors of Safety.

A blanket factor of safety for all types of fittings or load conditions is not logical. The manner in which a load is applied to a joint often involves a dynamic or shock load, as for example joints or fittings in landing gear. Single pin connections often undergo rotation or movement between adjacent parts, thus producing faster wearing away of material in operation. Repeated loads often present a fatigue problem. In an airplane or missile there are certain main fittings which, if they failed, would definitely cause the loss of the vehicle. Thus the design fitting requirements of the military and civil aviation agencies involve many special or higher factors of safety. This is so particularly in design involving castings.
DL. 5 Aircraft Bolts.

The aircraft bolt is used primarily to transfer relatively large shear or tension loads from one structural member to another. Fig. DL.1 shows three standard aircraft bolts in common use. There are other types but they will not be presented in this limited chapter on connections.

Hexagon Head

Clevis

Internal Wrenching

Fig. DL.1

The hexagon head bolt is an Army-Navy standard bolt made from SAE 2330-3.5 percent nickel steel, heat treated to an ultimate tensile strength of 125,000 psi. The bolt head is of sufficient size as to develop the full tensile strength of the bolt.

The internal-wrenching bolt is a high strength steel bolt usually heat treated from 160,000 to 180,000 psi. It is especially suitable for main splice fittings because of its high strength and the relatively small space required for the bolt head.

The clevis bolt is referred to as a shear bolt because its head is not designed to develop the full tensile strength of the bolt. The clevis bolt is usually used when a group or cluster of bolts is required to transfer a load by shear loads on the bolts. The smaller bolt heads thus save weight and also provide greater bolt head clearances. The clevis bolt will develop about one half the tensile strength of the standard AN hexagon head bolt.

The three bolts are also made from aluminum alloy for diameters over 1/4 inch. In many fitting designs weight can be saved by using aluminum alloy bolts.

General Rules in Using Bolts.

Bolt threads should not be placed in bearing or shear. The length of the bolt shank should be such that not more than one thread extends below fitting surface, which can be done by the use of washers.

Bolts less than 3/8 diameter should not be used in major fittings. For steel bolts, 3/16 inch diameter should be the smallest size to be used in any fitting.

Bolts connecting parts having relative motion or stress reversal should have close tolerances to decrease shock loads.

For bolts connecting members having relative motion a lubricator should be incorporated in the surrounding parts of the fitting; the fitting should not be drilled to provide lubrication.

Bolts should be used in double or multiple shear if possible in order to increase strength efficiency in bolt shear and to decrease bending tendency on bolt.

DL. 6 Aircraft Nuts.

Fig. DL.2 illustrates four standard steel aircraft nuts. Nut material is more ductile than bolt material, thus when the nut is tightened the threads will deflect to seat on the bolt threads.

The Castle nut is probably the most common aircraft nut. It develops the full rated strength of the bolts. The nut has slots milled in it so that nuts can be prevented from turning.

The Shear nut is only one half as thick as the Castle nut, and thus has only threads enough to develop one half of bolt tensile strength. It is used with a clevis bolt which has a screw driver slot to limit the torsion in tightening the nut. The nut is also castellated for cotter pin lock.

Fig. DL.2

CASTLE NUT AN - 310  SHEAR NUT AN - 320  PLAIN NUT AN - 315  SELF LOCKING AC - 303

The Plain nut which is very seldom used in present design is used for permanent locations and is locked by "peening" or riveting the end of the bolt over the nut, an operation which destroys the bolt protective coating or finish.

Aluminum Alloy Nuts. Aluminum alloy nuts are not used on bolts designed for tension. Sometimes aluminum alloy nuts are used on steel bolts on land planes to save weight provided the bolts are cadmium plated. If the bolt is used in places where the nut is repeatedly removed, neither bolt or nut should be aluminum alloy because of the danger of injuring the relatively soft threads.
Self-Locking Nuts. Self-locking nuts are widely used in aircraft industry and there are a number of types on the market. Fig. D1.2 shows one type of self-locking nut involving a fiber insert. When the bolt reaches the fiber collar it tends to push the fiber up because the hole in the collar is smaller than the bolt, and is not threaded. The fiber ring thus sets up a heavy downward pressure on the bolt automatically throwing the load carrying sides of the nut and bolt threads into positive contact. Thus all play in the threads is eliminated and a friction is set up between every bolt and nut thread in constant. This constant pressure which is mainly is maintained on threads, provides the friction which prevents nut from moving under vibration. The use of the self-locking nut reduces the assembly costs as it eliminates the bothesome cotter pin which takes extra operations of the mechanic and is very difficult to install on the nut in the many joints and corners of an airplane. In another type of self-locking nut, the locking force is provided by the spring action of the upper part of a specially designed nut.

D1.7 Bolt Shear, Tension & Bending Stresses.

Table D1.1 gives the section properties and the ultimate shear, tension and bending strengths for AN Standard Steel bolts at room temperature. Fig. D1.3 shows the correction to be applied to the strength values in Table D1.1 when bolts are subjected to elevated temperatures. Table D1.2 gives the tensile and double shear strengths of Steel Internal Wrenching bolts. Table D1.2a gives ultimate shear, tension and bending strengths for aluminum alloy bolts. The value of \( F_b \) the modulus of rupture, was determined by the method given in Chapter C3.

**Table D1.2**

<table>
<thead>
<tr>
<th>Size Dia.</th>
<th>Ultimate Tensile Strength lbs.</th>
<th>Double Shear Strength lbs.</th>
<th>Size Dia.</th>
<th>Ultimate Tensile Strength lbs.</th>
<th>Double Shear Strength lbs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>5,190</td>
<td>9,300</td>
<td>5/8</td>
<td>43,600</td>
<td>55,300</td>
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<tr>
<td>5/16</td>
<td>9,820</td>
<td>14,600</td>
<td>3/4</td>
<td>63,200</td>
<td>83,900</td>
</tr>
<tr>
<td>3/8</td>
<td>15,200</td>
<td>21,000</td>
<td>7/8</td>
<td>86,100</td>
<td>114,200</td>
</tr>
<tr>
<td>7/16</td>
<td>20,500</td>
<td>28,600</td>
<td>1.0</td>
<td>114,000</td>
<td>149,200</td>
</tr>
<tr>
<td>1/2</td>
<td>27,400</td>
<td>37,300</td>
<td>1-1/8</td>
<td>144,000</td>
<td>188,900</td>
</tr>
<tr>
<td>9/16</td>
<td>34,800</td>
<td>47,200</td>
<td>1-1/4</td>
<td>180,000</td>
<td>233,200</td>
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**Table D1.2a**

<table>
<thead>
<tr>
<th>SHEAR Fcu-35,000</th>
<th>TENSION Fu-52,000</th>
<th>BENDING Fb-72,000</th>
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<td>922</td>
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<td>10,738</td>
<td>14,719</td>
<td>1,730</td>
</tr>
<tr>
<td>15,463</td>
<td>21,972</td>
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</tr>
<tr>
<td>21,046</td>
<td>29,520</td>
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<tr>
<td>27,489</td>
<td>39,759</td>
<td>7,070</td>
</tr>
</tbody>
</table>

D1.8 Bolts in Combined Shear and Tension.

When bolts are subjected to both shear and tension loads, the resulting strength is given by the following interaction equation:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(D1.1)}
\]

where,

\[ x = \text{shear load} \]
\[ y = \text{tension load} \]
\[ a = \text{shear allowable load from Table D1.1} \]
\[ b = \text{tension allowable load from Table D1.1} \]

Figs. D1.4 and D1.5 is a plot of equation for the various AN Steel bolt sizes. The curves are not applicable where shear nuts are used. The curves are based on the results of combined load tests with nuts finger tight.

D1.8a Bushings.

It is customary to provide bushings in the lugs of single bolt or pin fittings subjected to reversal of stress or to slight rotation. Thus if wear and tear takes place a new bushing can be inserted in the lug fitting. Steel bushings are commonly used in aluminum alloy.
single bolt fitting lugs to increase the allowable bearing stress on the lug since the bushing increases the bearing diameter 1/8 inch since bushings are usually 1/16 inch in wall thickness. If bushings are not used on single bolt connections sufficient edge distance should be provided to ream hole for next size bolt in case of excessive wear of the unbushed hole. If considerable rotation occurs a lubricator should be provided for a plain bushing or an oil-impregnated bushing should be used.

D1.9 Single Bolt Fitting.

Possibly the simplest method of joining two members together is the use of a single bolt or pin connection. Such a joint can transmit relatively large loads and yet the joint is easily and quickly disconnected.

Fig. D1.6 illustrates the four general methods of connecting two members by a single bolt. First the connection is made symmetrical about the centerline of the load on the joint. Thus is Fig. D1.6a the load P on the male part of the fitting divides equally and symmetrically to the two female plates or units which make up part of the fitting unit. If the male and female parts of the connection are to be tied together by a single bolt it is evident that the connecting plates will be weakened due to the bolt hole unless extra material is added at the bolt hole section.

Fig. D1.6a shows a fitting consisting of three rectangular plates of uniform section throughout fastened together by a single bolt. Obviously the weak section for the plates in tension would be a section through the centerline of the hole. If this section is strong enough to carry the given loads, then the remaining part of the fitting members are considerably over strength. To avoid this over-strength which means extra weight of fitting units, a single bolt fitting unit is often made like one of the examples indicated in Fig. b, c and d of Fig. D1.6.

In Fig. b, the plates are made constant thickness but increased in width in the vicinity of the hole section. In Fig. c, the width of the plates is kept constant but the thickness of the plates are increased in the vicinity of the hole section, and in Fig. d, both width and thickness of plates are changed.

D1.10 Methods of Failure of Single Bolt Fitting and the Allowable Failing Loads.

As the load on a fitting is transferred from one side of the fitting to the other, internal stresses are produced which tend to cause the fitting to fail in several different ways.
Failure by Bolt Shear.

In Fig. D1.7 the bushing is not continuous between the plates, but each of the three plates have separate bushings. As the pull P is placed on the fitting it tends to shear the bolt at sections (1-1) and (2-2), (Fig. D1.7a). Fig. D1.7b illustrates the forces or pressures on the bolt and the failure which can take place if the stresses are sufficient.

Let $P_u$ represent the maximum or ultimate load on the fitting. This force $P_u$ must be resisted by the shear strength of the bolt at the two sections (1-1) and (2-2). Hence,

$$P_u(\text{bolt shear}) = F_{su} A/2$$

where $F_{su}$ = ultimate shearing stress for bolt material.

$A$ = cross-sectional area of bolt.

Failure by Bolt Bending.

The main concern regarding bolt bending is that under the limit loads any bending deflection of the bolt be not permanent as such deformation would make removal of the bolt difficult. The subject of bolt bending stresses is discussed in a later article.

Failure of Lug Portion of Fitting.

The lug portion of the fitting refers to that portion of the fitting that involves the hole for the single bolt that connects the male and female parts of the fitting unit. The simplified assumptions regarding failing action and the resulting equations which follow have been widely used for quick approximate check of the lug strength. The procedure which follows will be referred to as Method 1.

METHOD 1 OF BOLT & LUG STRENGTH ANALYSIS

Failure in Tension.

Fig. D1.8 indicates how a fitting plate can pull apart due to tension stresses on a section through the centerline of the bolt hole. Both the male and female parts of the fitting must transfer the load past the centerline of the hole, thus both parts must be considered in the design of a fitting. Equating the allowable load $P_u$ to the ultimate resisting tensile stresses at points (a) and (b) Fig. D1.8, we obtain,

$$P_u(\text{tension}) = F_{tu} (2R - D) t$$

where $F_{tu}$ = ultimate tensile strength of plate material.

Equation (2) assumes that the tensile stress on the cross-section is uniform. This is not true as the flow of stress around the hole causes a stress concentration. To take care of this stress concentration requires a margin of safety of 25 percent.

Failure by Shear Tear Out.

Fig. D1.4 illustrates the manner in which failure can occur by the shearing tear out of a plate sector in front of the bolt. In Fig. a,
the load \( P \) causes the bolt (not shown) to press on the plate around the bolt hole edge. Stresses are produced which tend to cause the portion (a) in Fig. (b) to tear out as shown. Equating the load \( P_u \) to the ultimate shearing resistance of the material times the area of the two shear out areas, we can write,

\[ P_u (\text{shear out}) = F_{su} A_s \]  \hspace{1cm} (3)

where \( F_{su} = \text{ultimate shearing strength of the plate material} \)
\[ A_s = \text{shear out area} \]

It is very common practice to take the shear out area \( A_s \) equal to the edge distance at the centerline of the hole times the thickness \( t \) of the plate times two since there are two shear areas. This is slightly conservative because the actual shear area is larger. Ref. 1 permits one to use the area along line 1-1 which is limited by the 40 degree line as shown in Fig. Dl.5c.

**Failure by Bearing of Bushing on Plate.**

In Fig. Dl.10a, the pull \( P \) causes the bolt (not shown) to press against bushing wall which in turn presses against the plate wall. If the pressure is high enough the plate material adjacent to the hole will start to crush and flow thus allowing the bolt and bushing to move which results in the elongated hole as illustrated in Fig. b. Equating the load \( P_u \) to the ultimate bearing strength on the bearing surfaces we can write,

\[ P_u = F_{br} D t \]  \hspace{1cm} (4)

where \( F_{br} = \text{allowable bearing stress} \)
\[ D = \text{diameter of bushing} \]
\[ t = \text{plate thickness} \]

It is good practice to require a margin of safety of 50 percent.

**Failure by Bearing of Bolt Bushing.**

A bushing is pressed into the plate hole and thus it is considered as a tight fit. A fitting bolt is usually considered as removable therefore a certain tolerance between the bolt and bushing inside diameters is necessary in order to insert and remove bolt. If fitting is subjected to reversible loads the small slip in the fitting tends to cause shock on the fitting. Also the fitting may be such that slight rotation takes place on the bolt, which tends to cause wear between bolt and bushing. It is therefore customary to check the bearing pressure between the bolt and the bushing since failure of the bushing could take place in a manner explained in the previous article dealing with bearing of bushing on plate. Then as before,

\[ P_u = F_{br} D t \]  \hspace{1cm} (5)

where \( D = \text{bolt diameter} \)

A margin of safety of 50 percent should be maintained. If the fitting is subjected to infrequent rotation under load but with load involving no shock or vibration, require a margin of safety of 100 percent. If shock or vibration with infrequent rotation is present, require a margin of safety of 150 percent. Shock is considered to occur in such structures as landing gears, gun mounts, hoisting, mooring and towing connections.

**General Comment on Margins of Safety for Lugs.**

In general it is good design practice to design lugs conservatively as the weight of lugs is small relative to their importance in insuring the safety of the flight vehicle. Inaccuracies in manufacture are difficult to control. It is good design practice to provide sufficient material to permit drilling for a bushing if bushing is not used in original design. If castings are used as fittings, much higher factors of safety on the limit loads are specified because of the low ductility of the material in castings.

**Dl.11 Method 2. Lug Strength Analysis Under Axial Loading.**

Due to a comprehensive study and test program by Cozzone, Melcon and Hobbit (Refs. 3 and 4), this procedure as given in Method 1 is somewhat modified. The important difference is that curves derived from test results give the stress concentration factor to use for tension on the net section and the shear-out failure as assumed in Method 1 has been replaced by a combined shear-out bearing failure. Fig. Dl.11 shows the lug-pin combinations and types of failure as taken from Ref. 3.
The methods of failure and the methods of lug strength analysis are as follows:

**Tension Across the Net Section.**

Because of stress concentration, the stress on the net cross-section cannot be taken as uniform. The ultimate allowable tension load $P_u$ for lug equals:

$$P_u = K_t F_{tu} A_t$$  

(6)

where $K_t$ is the stress concentration factor as found from Fig. D1.12 and Table D1.3. $F_{tu} =$ ultimate tensile strength of the material and $A_t =$ net tension area.

---

**Fig. D1.12 Lug Design Data**

Tension Efficiency Factors for Axially Loaded Lugs (Ref. 3, 4)

---

**Table D1.3**

<table>
<thead>
<tr>
<th>曲线代号</th>
<th>材料和应用</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curve 1</td>
<td>2014-T6 and 7075-T6 Die Forging (L)</td>
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<tr>
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<td>4130 and 6830 Steel</td>
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<tr>
<td></td>
<td>2014-T6 and 7075-T6 Plate &lt; 0.5 (L, LT)</td>
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<tr>
<td></td>
<td>7075-T6 Bar and Extrusion (L)</td>
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<td>2014-T6 Hand Forged Billet = 144 in.² (L)</td>
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<tr>
<td>Curve 2</td>
<td>2014-T6 and 7075-T6 Plate = 0.5 &lt; 0.8 (L, LT)</td>
</tr>
<tr>
<td></td>
<td>7075-T6 Extrusion (L, ST)</td>
</tr>
<tr>
<td></td>
<td>2014-T6 Hand Forged Billet = 144 in.² (L)</td>
</tr>
<tr>
<td></td>
<td>2014-T6 Hand Forged Billet = 36 in.² (L, LT)</td>
</tr>
<tr>
<td></td>
<td>2014-T6 and 7075-T6 Die Forgings (LT)</td>
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<tr>
<td>Curve 3</td>
<td>2024-T4, 2024-T2 Extrusion (L, LT, ST)</td>
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<tr>
<td>Curve 4</td>
<td>2014-T6 and 7075-T5 Plate = 1 in. (L, LT)</td>
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<td></td>
<td>2024-T4 Bar (L, LT)</td>
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<tr>
<td></td>
<td>2024-T3, 2024-T4 Plate (L, LT)</td>
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<tr>
<td>Curve 5</td>
<td>2014-T6 Hand Forged Billet = 36 in.² (LT)</td>
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<tr>
<td>Curve 6</td>
<td>Aluminum Alloy Plate, Bar, Hand Forged Billet and Die Forging (ST)</td>
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<tr>
<td>Curve 7</td>
<td>AZ31C-T8 Magn. Alloy Sand Casting</td>
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<tr>
<td></td>
<td>6061-T6 Aluminum Alloy Casting</td>
</tr>
</tbody>
</table>

---

**Fig. D1.13 Lug Design Data**

Shear-Bearing Efficiency Factors for Axially Loaded Lugs (Ref. 3, 4)

---

**Fig. D1.14** gives curves for finding $K_{by}$.

**Bushing Yield.**

Take $A_{by}$ as the smaller of the bearing areas of bushing on pin or bushing on lug. (The latter may be smaller as a result of external chamfer of the bushing, oil grooves, etc.) The allowable yield bearing load on bushing is then:

$$P_{by} = 1.25 F_{oy} A_{by}$$  

(3)

where $F_{oy}$ is compressive yield strength of bushing material.

**Bolt or Pin Shear Strength.**

The bolt shear strength is calculated in the same manner as given in Method 1.
Bolt or Pin Bending.

The subject of bolt bending strength is treated in Art. D1.14.

D1.12 Lug Strength Analysis Under Transverse Loading.

Cases arise where the lug of a fitting unit is subjected to only a transverse load. Malcon and Hobert in (Ref. 4) express the ultimate transverse or failing load by a single equation:

\[ P_{tu} = K_{ty} A_{br} F_{tu} \]  \hspace{1cm} (10)

Similarly the yield strength of lug is,

\[ P_{ty} = K_{ty} A_{br} F_{ty} \]  \hspace{1cm} (11)

The efficiency failing and yield coefficients \( K_{ty} \) and \( K_{ty} \) are given by the curves in Fig. D1.15. The curve nomenclature for the curves in Fig. D1.15 is given in Table D1.4. In using Fig. D1.15, a value called \( A_{uy} \) is needed, the value of which is shown in the equation shown on Fig. D1.15

D1.13 Lug Strength Analysis Under Oblique Loads.

Fitting lugs are often subjected to oblique loads. Ref. 4 gives the following approach to this loading case.

Resolve the applied load into axial and transverse components. Then use the following interaction equation:

\[ P_a \times P_{tr} = 1 \]  \hspace{1cm} (12)

\[ M.S. = \frac{1}{R_a + R_{tr}} \leq 0.3 \]  \hspace{1cm} (13)

where, \( R_a \) = axial component of applied ultimate load divided by the smaller of the

Fig. D1.15 Lug Design Data

Tension Efficiency Factors for Transversely Loaded Lugs (Ref. 3, 4)

(See Table D1.4 for Curve Nomenclature)

Table D1.4

<table>
<thead>
<tr>
<th>Curve</th>
<th>Load Condition</th>
<th>Notes</th>
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<tbody>
<tr>
<td>1</td>
<td>4130 and 8630 Steel thru 125 KSI H.T.</td>
<td>Reference to be used for all conditions.</td>
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<tr>
<td>2</td>
<td>4130 and 8630 Steel 150 KSI H.T.</td>
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<td>3</td>
<td>Key for All Aluminum and Steel Alloys</td>
<td>Reference to be used for all conditions.</td>
</tr>
<tr>
<td>4</td>
<td>4130 and 8630 Steel 180 KSI H.T.</td>
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<td>5</td>
<td>356-T6 and AZ31C-T8 Sand Castings</td>
<td>Reference to be used for all conditions.</td>
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<td>2024-T3 and 2024-T4 Plate ( \geq 0.5 ) in.</td>
<td>Reference to be used for all conditions.</td>
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<tr>
<td>7</td>
<td>220-T6 and 2024-T4 Sand Castings</td>
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<tr>
<td>8</td>
<td>2024-T6 and 7075-T6 Plate ( \leq 0.5 ) in.</td>
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<tr>
<td>9</td>
<td>2024-T3 and 2024-T4 Plate ( \geq 0.5 ) in. also 2024-T4 Bar</td>
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<tr>
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<td>Approximate Cantilever Strength for All Aluminum and Steel Alloys</td>
<td>Reference to be used for all conditions.</td>
</tr>
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<td>11</td>
<td>2014-T6 and 7075-T6 Plate ( \geq 0.5 ) in. ( \leq 1.0 ) in. 2014-T6 Extrusions</td>
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<td>Reference to be used for all conditions.</td>
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<td>14</td>
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</table>

or margin of safety is,
values obtained for equations (6) or (7).

\[ R_{tu} = \text{transverse component of applied ultimate load divided by the values of } P_{tu} \text{ in equation (10).} \]

**DL.14 Bolt Bending Strength.**

In general static tests of single bolt fittings will not show a failure due to bolt bending failure. However, it is important that sufficient bending strength be provided to prevent permanent bending deformation of the fitting bolt under the limit loads so that bolts can be readily removed in maintenance operations. Furthermore, bolt bending weakness can cause peaking up of a non-uniform bearing pressure on the fitting lugs thus influencing the lug tension and shear strength. The unknown factor in bolt bending is the true value of the bending moment on the bolt because the moment arm to the resultant bearing forces is unknown. An approximate method (Ref. 4) for determining the arm \( b \), to use in calculating the bending moment on bolt is given in Fig. DL.16, which gives \( b = \frac{5}{4} t_{l} + g \) where \( g \) is clearance or gap between lugs. The resulting bending moment is considered to be conservative. (See Ref. 4 for other refinements relative to determining moment arm \( b \).)

Shown. On the vertical scale we locate point \( y \) at tension load of 5800 lbs., and draw horizontal dashed line to give point \( a \). A straight line through points \( (o) \) and \( (a) \) is drawn and extended to intersect the curve for a 5/8 AN bolt at point \( b \). Projecting downward from point \( (b) \) to lower scale, we obtain \( P_{t}(allow) = 22400 \text{ lb.} \) and projecting horizontally we obtain \( P_{t}(allow) = 7700 \text{ lb.} \)

Then \( M.S. = \frac{(P_{t}(allow))}{x} = 1 = \frac{(22400)}{16700} = 1 = .34 \) or \( M.S. = \frac{(P_{t}(allow))}{/y} = 1 = \frac{(7700/5800)}{1} = .34. \)

The specified required M.S. was .25, thus bolt strength is satisfactory.

**PROBLEM 2.**

Fig. DL.17 shows a single pin fitting. The lug material is AISI Steel, heat treated to \( P_{tu} = 125000 \text{ psi.} \) The bolt is AN steel, \( P_{tu} = 125000. \) The bushing is steel with \( P_{tu} = 125000. \) The fitting is subjected to an ultimate tension load of 15650 lb. The fitting will be strength checked for the design load. The check will be made by both Methods 1 and 2.

**Fig. DL.17**

![Diagram of single pin fitting](image)

**SOLUTION BY METHOD 1.**

A fitting factor of safety of 1.15 will be used which is standard practice for military airplanes.

Design Fitting Load = 1.15 x 15650 = 18000 lb.

Check of Bolt Shear Strength.

Bolt is in double shear. From Table DL.1, single shear strength of 1/2 inch diameter AN steel bolt is 14700 lb.

Where \( P_{u} = 2 \times 14700 = 29400 \text{ lb.} \)

\[ M.S. = \frac{(29400/18000)}{1} = .62 \]
Fittings and Connections. Bolted and Riveted.

Check Bending of Bolt

Referring to Fig. D1.15, the moment arm for calculating bending moment on bolt is,

\[ t = 0.35t + 0.2s + g \] (take g as 1/4 inch)
\[ b = 0.5 \times 0.213 + 0.35 \times (0.33) + 0.156 = 0.313 \text{ inches} \]

Bending moment on bolt = \( 0.5b \times 12000 \times 0.213 = 1265 \text{ in} \cdot \text{lb} \)

\( F_0 = \frac{Mr}{I} = \frac{1265 \times 0.23}{0.003069} = 150000 \text{ psi} \)

From Table D1.1, \( F_0 \) for AN Steel bolts is 150,000.

Therefore M.S. = \( \frac{180000}{150000} = 1.2 \)

Check of Lug A

This lug is more critical than lug B since thickness of lug B is more than one-half of lug A.

Check of Tension Through Bolt Hole

\[ P_u = F_{tu} A_0 \]
\[ P_u = 125000 \times (1.1875 - .625) \times .375 = 26400 \text{ lb} \]

M.S. = \( \frac{26400}{18000} = 1.46 \)

To take care of stress concentration, Method 1 says maintain a M.S. of .25, thus lug tensile strength is OK.

Check of Shear Out Strength of Lug

\[ P_u = F_{su} A_0, \quad F_{su} = 92000 \]
\[ P_u = 35000 \times (0.337 - 0.3125) \times 0.375 = 17300 \text{ lb} \]

This value is under the design fitting load of 18000 lb. It is permissible to take a slightly greater shear out distance than the edge distance as used (see Fig. D1.9c). The additional distance to the 45 degree line would aid enough shear out area to give a positive margin of safety. This method of calculating shear out strength is conservative as will be brought out by the result in Method 2 solution.

Bearing Strength

The bushing and lug have the same ultimate strength \( (F_{tu}) \); thus bearing will be critical for bolt on bushing since bearing area is less.

\[ P_u = F_{tu} A_0 \]

An extra 50 percent margin of safety is required or the allowable bearing stress for should be divided by 1.5.

\[ P_u = \frac{134000}{1.50} \times 0.375 = 24500 \text{ lb} \]

M.S. = \( \frac{24500}{18000} = 1.35 \)

Solution by Method 2

Bolt Shear Strength and Bolt Bending Strength are calculated in the same manner as in Method 1 and thus the calculations will not be repeated.

Tension Net Section

\[ P_u = K_e F_{tu} A_t \]

\( K_e \) is the tension efficiency factor to take care of stress concentration due to the hole and is determined from Fig. D1.12. Table D1.3 says to use curve number 1 for all steels. To use Fig. D1.12 requires the ratio \( W/D = (1.1875/0.625) = 1.9 \). Then from Fig. D1.12 we read \( K_e = .95 \), whence,

\[ P_u = .95 \times 125000 \times (1.1875 - .625) \times .375 = 22800 \text{ lb} \]

M.S. = \( \frac{22800}{18000} = 1.26 \)

Fig. D1.12 says that a M.S. of .18 is appropriate over that of all required fitting factors of safety, thus our M.S. of .50 provides more than this additional M.S. of .15.

Shear Bearing Strength

\[ P_{br} = K_{br} F_{tu} A_{br} \]

\( K_{br} \) is the shear-bearing efficiency factor and is obtained from Fig. D1.13.

\[ D/t = 0.625/0.375 = 1.65 \]

\[ e/d = 0.3125/0.625 = 0.50 \]

From Fig. D1.13, we read \( K_{br} = .80 \)

\[ P_{br} = .80 \times 125000 \times .625 \times .375 = 23400 \text{ lb} \]

M.S. = \( \frac{23400}{18000} = 1.30 \)

The reader should note shear out strength by Method 2 is considerably larger than by Method 1. Fig. D1.12 says a .15 M.S. of appropriate, thus our 0.30 is satisfactory.

Bushing Yield

\[ P_{by} = 1.65 F_{by} A_{br}, \quad F_{by} = 113000 \text{ for steel with } F_{tu} = 165000 \]

\[ P_{by} = 1.65 \times 113000 \times .500 \times .375 = 58000 \text{ lb} \]

M.S. = \( \frac{58000}{18000} = 1.17 \)

General Conclusion

Since all margins of safety are positive, the strength of fitting unit is satisfactory. It could be redesigned to save weight. Moving
the hole slightly back of the center of the lug radius would help shear out strength. This change would permit decreasing the thickness of the lug slightly. Decreasing the lug thickness would decrease the bolt bending moment and possibly permit use of 7/16 diameter bolt. The student should redesign the lug.

**PROBLEM 3.**

A lug identical to lug A in Problem 2 is subjected to a transverse load, \( P_{tu} \), which is the failing load. Find the load \( P_{tu} \).

**SOLUTION:** From equation (10) the failing load is,

\[
P_{tu} = K_{tu} A_{pu} P_{tu}
\]

The failing coefficient \( K_{tu} \) is determined by use of curves in Fig. D1.15. The lower scale parameter for Fig. D1.15 is \( A_{pu} A_{br} \), where

\[
A_{pu} = \frac{3}{A_{1}} + \frac{1}{A_{2}} + \frac{1}{A_{3}} + \frac{1}{A_{4}}
\]

For the meaning of these \( (A) \) areas, refer to sketch of lug in Fig. D1.15. Simple calculations give the following values for these areas.

\[
A_{1} = A_{2} = .14 \quad A_{3} = A_{4} = .105
\]

whence, \( A_{pu} = \frac{3}{.14} + \frac{1}{.105} + \frac{1}{.105} + \frac{1}{.14} = .126 \)

\[
A_{br} = .375 \times .625 = .234, \quad A_{pu}/A_{br} = .126/234 = .54
\]

Table D1.4 indicates curve 1 is used for steel material with \( P_{tu} \) up to 125000 psi. Then from Fig. D1.15, we read \( K_{tu} = .74 \).

Then, \( P_{tu} = .74 \times .234 \times 125000 = 21500 \) lb.

**PROBLEM 4.**

Fig. D1.18 shows a steel forked fitting bolted to a double channel section made from 7075-T6 aluminum alloy. The hinge pin is 5/8 diameter AISI Steel, heat treated to \( P_{tu} = 150000 \). Steel fitting is also 150000 Steel. A strength check of the fitting will be carried out.
\[ P_b = 150000 + 146000 \times (1.7 - 1) = 258000 \text{ psf} \]

\[ M.S. = (253000/252000) - 1 = 0.14 \]

**Check Shear-Bearing Tear Out of Lug.**

\[ F_{bru} = K_{bru} F_{tu} A_{bru} \]

\[ D/t = 0.75/0.625 = 1.2, \quad e/D = 0.625/0.75 = 0.833 \]

From Fig. D1.13, \( K_{bru} = 0.72 \)

\[ F_{bru} = 0.72 \times 150000 \times 0.75 \times 0.625 = 51500 \text{ lb.} \]

\[ M.S. = (51500/33000) - 1 = 0.56 \text{ (margin desirable at least 0.15).} \]

**Check Tensile Strength of Lug Section Through Pin Holes.**

\[ F_u = K_f F_{tu} A_t \]

\( K_f \) obtained from Fig. D1.12 using curve 1.

\[ W/D = 1.125/0.75 = 1.5 \]

From Fig. D1.12, \( K_f = 0.99 \)

\[ F_u = 0.99 \times 150000 \times (1.125 - 0.75) \times 0.625 = 34800 \]

\[ M.S. = (34800/33000) - 1 = 0.06. \]

It would be desirable to have a M.S. of .15.

**Check Tensile Strength of Section 2-2 of Steel Fitting.**

From Fig. D1.18, four 5/16 diameter AN Steel bolts are used to attach fitting to channel members. Since bolts are the same size, it will be assumed that each of the 4 bolts transfers 1/4 of the total fitting load. Fig. D1.19 shows the load in the steel fitting and the channel members. The total load passes by section 2-2.

---

**A stress concentration factor \( K_f \) from Fig. D1.12 will be used to be conservative.**

\[ \frac{W}{D} = (1.125/0.3125) = 3.6. \]

Fig. D1.12 gives \( K_f = 0.91. \)

\[ F_u = 0.91 \times 150000 \times 0.305 = 41700 \text{ lb.} \]

\[ M.S. = (41700/33000) - 1 = 0.26. \]

**Check Section of Steel Fitting at Section 3-3.**

Load from Fig. D1.19 = 16500 lb. Net Section = (1.125 - .3125).25 = .204 sq. in. Use same \( K_f \) as before.

\[ F_u = 0.91 \times 150000 \times 0.204 = 27900 \text{ lb.} \]

\[ M.S. = (27900/16500) - 1 = 0.69 \]

**Check Shear Strength of 4 - 5/16 Dia. Bolts.**

Bolts in double shear.

\[ F_u = 4 \times 2 \times 5750 = 46000 \text{ lb.} \]

\[ M.S. = (46000/33000) - 1 = 0.39. \]

**Check Bearing of Bolts on Steel Fitting.**

Section 3-3 or 4-4 is critical as bearing area is less.

Allowable bearing stress for 150000 steel is 218000.

\[ F_u = F_{br} A_{br} = 218000 \times 0.3125 \times 0.25 = 17000 \text{ lb.} \]

\[ M.S. = (17000/21800) - 1 = 0.06. \]

**Bearing of Bolts on 7075-T6 Channels.**

\[ F_{br} \text{ for 7075 aluminum alloy} = 106000. \]

\[ F_u = 106000 \times 0.3125 \times 0.25 = 8300. \]

\[ M.S. = (8300/6850) - 1 = 0.01. \]

**Check Tensile Strength of Channels at Section 4-4.**

Load on Section = 33000.

Net area = 0.70 - 0.25 x 0.3125 = 0.522.

\[ F_{tu} \text{ for 7075 material} = 78000 \]

\[ F_u = 78000 \times 0.522 = 40650. \]

\[ M.S. = (40650/33000) - 1 = 0.47 \text{ (which will take care of any stress concentration).} \]

**Check Shear Out of Channels Behind Bolt B.**

Shear out distance = 0.625 - 0.156 = 0.469.

Shear out area = 2 x 0.469 x 0.25 = 0.234.
Pu = Fsu Ag = 43000 x .234 = 10100 lb.

M.S. = (10100/8220) - 1 = .23 (figured conservatively by Method 1)

Shear Out of Steel Fitting Behind Bolt B,

Fsu = 95000

Shear out area Ag = (4375 - .1655)2 x .25 = .141

Pu = .141 x 95000 = 13400.

M.S. = (12400/8220) - 1 = .62.

Bushing Yield Strength.

Fory = 1.85 Fcy Abr. Fcy = 113000 for 125000 steel.

Fory = 1.85 x 113000 x .625 x .625 = 81700 lb.

Load = 33000, not critical.

**DL.16 Bolt Loads for Multiple Bolt Fitting.**

Bolt Sizes Different. Concentric Loading.

In designing or strength checking a multiple bolt fitting, the question arises as to what proportion of the total fitting load does each bolt transfer. This distribution could be affected by many things such as bolt fit or bolt tightness in the hole; bearing deformation or elongation of the bolt hole; shear deformation of the bolt or pin; tension or compressive axial deformation of the fitting members and the member being connected, and a number of other minor influences.

Since aircraft materials have a considerable degree of ductility, if the fitting is properly designed, the loads on the bolts when the loads on the fitting approach their maximum value, will tend to be in proportion to the shear strength of the bolt. That is, if the combined shear strength of the bolts is the critical strength, the yielding of the fitting material in bearing, shear and tension will tend to equalize the load on the bolts in proportion to their shear strengths. For stresses below the elastic limit of the fitting plates the bolt load distribution no doubt is more closely proportional to the bearing area of each bolt. Since the primary interest is failing strength, the bolt load distribution in proportion to the bolt shear strengths is usually assumed.

Let:

\[ P = \text{design load on fitting} \]

\[ P_{sa} = \text{allowable shear strength of any bolt} \]

\[ P_{n} = \text{load on bolt n} \]

\[ P_{sn} = \text{allowable shear strength of bolt n} \]

Therefore we can write,

\[ P = \frac{P_{sa}}{P_{sn}} \]

**Example Problem of Bolt Load Distribution.**

Fig. DL.20 shows a multiple bolt fitting unit subjected to a concentric load of 100000 lbs. Determine the load transferred by each bolt.

![Fig. DL.20](image)

**Distribution of loads to each bolt:**

\[ P_{a} = \frac{46020}{35200} = .23 \]

\[ P_{b} = 100,000 x 37274/117340 = 31500 \text{ lb.} \]

\[ P_{c} = 100,000 x 22544/117340 = 19200 \text{ lb.} \]

\[ P_{d} = 100,000 x 11502/117340 = 9800 \text{ lb.} \]

Total load on the four bolts adds up to 100,000 lb.

**DL.17 Multiple Riveted or Bolted Joints Subjected to Eccentric Loads.**

Fig. DL.21 shows a plate fitting attached to another member by means of four bolts or rivets. (The circles represent the bolts). The fitting plate is subjected to the loads Pa and Pb acting as shown. Let it be required to find the resultant loads on the bolt group due to the given loads.

The centroid of resistance for the bolt group will then correspond to the centroid of the bolt areas. Fig. DL.22 shows the fitting unit with the force Pb and Py replaced by an equivalent force system at the bolt group centroid point (O). This equivalent force system will be:

\[ Py = -200 \text{ lb.}, \quad Pb = 1000 \text{ lb.} \]
an: $M_0 = -200 \times 1.75 - 1000 \times .25 = -600$ in.lbf.

Since $P_Y$ and $P_Z$ act through the bolt centroid they will be reacted equally by each bolt, hence $V$ load on each bolt due to $P_Y = -200/4 = -50$ lb.; $H$ load on each bolt due to $P_H = 1000/4 = 250$ lb.

The load produced on each bolt due to the moment load $M_0 = -600$ in.lbf. will vary directly as the distance of the bolt from the center of resistance which coincides with the bolt group centroid.

Let $r_a$ equal the distance from the bolt group centroid to bolt (a). Then the resisting moment developed by bolt (a) equals $F_a r_a$, where $F_a$ equals the load on bolt (a). Since the bolt loads are proportional to their distance from $O$, the resisting moment $M_0$ developed by bolt (b) will equal

\[(F_a r_a)P_Y + F_a r_a^2 = M_0 = F_a r_a^2\]

and similarly for bolt (c)

\[
M_0 = F_a r_a^2 + F_b r_b^2 + F_c r_c^2 + F_d r_d^2
\]

The total moment resistance of the bolt group therefore equals,

\[
M = \frac{F_a r_a^2 + F_b r_b^2 + F_c r_c^2 + F_d r_d^2}{r_a}
\]

hence,

\[
F_a = \frac{Mr_a}{r_a + r_b + r_c + r_d}
\]

where $I = r^2$ of bolt group.

Therefore for the loads on the other bolts,

\[
F_b = \frac{Mr_b}{1}, \quad F_c = \frac{Mr_c}{1}, \quad F_d = \frac{Mr_d}{1}
\]

For the bolt group of Fig. D1.22 $r_a = r_b = r_c = r_d = 1.25$.

Hence, $I = r^2 = 4 \times 1.25^2 = 6.25$ in.

Therefore the moment load $F_m$ on each bolt will equal

\[
F_m = \frac{Mr}{I} = \frac{-600 \times 1.25}{6.25} = 120 \text{ lb.}
\]

Fig. C4.31 shows the resulting $X$, $V$ and moment loads applied to each bolt. The resultant load can be found graphically by drawing the force polygons as shown in Fig. C1.23. The resultant bolt loads can likewise be determined analytically. For example consider bolt (c)

\[
Z_H = 250 \times 120 \times \frac{1}{1.25} + 0 = 346 \text{ lb.}
\]

\[
Z_V = -50 \times 120 \times \frac{75}{1.25} + 0 = -122 \text{ lb.}
\]

Hence $R = \sqrt{Z_H^2 + Z_V^2} = \sqrt{346^2 + 122^2} = 367$ lb.

Case where Bolts or Rivets are of Different Diameters.

When the joint bolts or rivets are not all the same size, the moment load on each bolt is proportional to the bolt area times its distance to the bolt group centroid. Thus the bolt areas must enter equation (15).

\[
F_a = \frac{M r_a}{r_a + r_b + r_c + r_d + r_e + r_f + r_g + r_h + r_i + r_j}
\]

\[
F_a = \frac{M r_a}{r_a + r_b + r_c + r_d + r_e + r_f + r_g + r_h + r_i + r_j}
\]

\[
n = \text{number of bolts of each size.}
\]

Since theory of loads on a multiple bolt group is only approximate, reasonable margins of safety should be maintained.

RIVETED CONNECTIONS

D1.18 Types of Rivets.

From an aerospace structural standpoint rivets may be placed into two general classifications, namely:

(1) The Protruding Head type of rivet.

(2) The Flush type rivet.

Fig. D1.24 illustrates the protruding head type of rivet. Fig. D1.25 illustrates a number of modifications of the protruding head type of rivet that have been used in the past.
For many years the round head rivet was used for all interior work and before the era of high speeds it was used as a surface rivet as well. When wind tunnel experiments showed that such rivets gave appreciable drag, designers turned to rivets with less head protrusion, thus the development of the Brazier and modified Brazier type of rivet head. Then as the age of relatively high airplane speeds arrived a flush surface was needed, particularly on certain sensitive portions of the airplane surface, thus various modifications of the countersunk head involving press and machine countersinking of the sheets were developed.

Fig. D1.26 illustrates the flush type of rivet. As illustrated in Fig. D1.26, this flush type rivet can be used in several different ways, thus the method shown in Fig. D1.26a is referred to as the machine countersunk type; that in Fig. D1.26b as the press countersunk double dimpled type; and that in Fig. D1.26c as the combined press and machine countersunk type or the dimpled machine countersunk type.

**Approx. Sheet Limitations For Machine Countersunk Rivets (AN-425)**

- 1/8 Dia. Rivet.
- 5/32 Dia. Rivet
- 3/16 Dia. Rivet

**Approx. Limitations For Press Countersunk or Double Dimpled Rivets (AN-425)**

- 1/8 Dia. Rivet
- 5/32 Dia. Rivet
- 3/16 Dia. Rivet
DL.19 Rivet Material.

Since aluminum alloy is by far the most widely used material in the aircraft industry, it follows that aluminum alloy is the material most widely used for rivets. Table D1.5 (column 1) lists the 5 aluminum alloys used for rivets and the ultimate shearing stress $F_{su}$ for each material. Rivets made from 2017-T3 ($F_{su} = 34000$) and 2024-T31 ($F_{su} = 41000$) are rivets that must be driven soon after heat treatment or before age hardening takes place. The aging or hardening is slowed by keeping rivets in refrigeration after heat treatment. The other rivet material is less hard or less brittle in the aged state and thus can be stored in air and driven anytime. These so-called softer rivets have less shear strength, but since a great deal of aircraft construction involves thin sheet, bearing is often critical and thus rivet shear is not critical. Most surface or skin riveting involves the softer rivet, usually 2117-T3 ($F_{su} = 30000$).

DL.20 Strength of Rivets. Protruding Head Type.

Rivets are widely used in airplane structures to fasten or tie together two or more structural units. Standard methods of stress analyses of riveted joints consider two primary types of failure, namely, the shear of the shank of the rivet and the bearing or compressive failure of the metal at the point where the rivet bears against the connecting sheet or plate.

Fig. D1.28 illustrates the main forces on a rivet in transferring a load from one plate to another. The load is transferred to the rivet from the plate by bearing of the plate on the rivet. The load is then transferred along the rivet and resisted by bearing action on the other plate. Since the plate bearing forces on the rivet are not in the same line, the forces tend to shear and bend the rivet. Bending of the rivet is usually neglected if there are no intermediate filler plates. In Fig. (a) of D1.28, the rivet is in single shear, whereas in Fig. (b) the rivet is in double shear.

The ultimate shear strength of a rivet is given by the following equation:

$$ F_a = F_{su} A n $$  \hspace{1cm} (17)

where, $F_a = $ ultimate shear strength of rivets (lbs.)
$F_{su} = $ ultimate allowable shear stress for rivet (psi)
$A = $ area of rivet cross-section = $\pi D^2/4$
where $D$ equals the nominal rivet hole diameter.
$n = $ number of shear areas per rivet.

Reference (17) shows that the shear strength of protruding head aluminum alloy rivets is affected by increasing $D/t$ (Diameter of rivet over sheet thickness) ratios. The conclusions in Reference (17) are as follows:-

Rivets in Single Shear:

For values of $D/t$ up to 3: -
Single shear strength = basic allowable single shear strength.

For values of $D/t$ greater than 3: -
Single shear strength = basic allowable single shear strength times \[1 - 0.04 (D/t - 3)]

For Rivets in Double Shear:

For values of $D/t$ up to 1.5: -
Double shear strength = basic allowable double shear strength.

For values of $D/t$ greater than 1.5: -
Double shear strength = basic allowable double shear strength times \[1 - 0.13 (D/t - 1.5)]

Table D1.5 (from Ref. 2) gives the shear strength of protruding and flush head aluminum alloy rivets and the corrections to take care of the $D/t$ influence on the rivet shear strength. Table D1.6 gives the allowable bearing strength between the protruding head rivet and the various aluminum alloy sheet and plate material. The bearing values are given for two a/D ratios, namely 1.5 and 2.0, where $a$ is the edge distance measured from the center of the hole to the edge of the plate. Any reduction in edge distance may cause bulging of the edge of the sheet due to driving energy. Edge distance should not be less than $a/D = 1.5$.

DL.21 Strength of Rivets. Flush Type.

Since flush rivets have no protruding head on the flush end of the rivet and also since flush riveting involves machine countersinking or press countersinking or both, the strength of the flush type rivet is different than the common protruding head type.
Fig. D1.29 illustrates a machine countersunk rivet. Due to the pull P on the two sheets which are held together by the rivet and induced force F sub i on the sloping side of the head of the rivet. This induced force tends to shear and bend the portion 1-1 of the rivet head. The sharp edge of the countersunk sheet at point (a) tends to cut into the rivet. These combined influences tend to cause excessive deflections and finally failure as roughly illustrated in Fig. D1.30.

![Fig. D1.29](image)

![Fig. D1.30](image)

In the press countersunk or dimpled type of flush rivet connection, see Fig. D1.26b and c, because of the interlocking of the sheets due to the dimple, the joint could transmit a load without a rivet if the sheets were held together. Since there is no clearly defined bearing or shear surface in this type of joint, the manner in which the loads are transferred is quite complex. As a result resort must be made to tests to establish design allowables. Tables D1.7, D1.11 and D1.12 give the ultimate and yield strength of flush type rivets (Sec. 2).

**D1.22 Blind Rivets.**

The name “Blind” rivet is given to that type of rivet which can be completely installed from one side of the joint, and is therefore almost exclusively used where it is impossible or impractical to drive the normal rivet, which requires access to both sides of the joint. There are two general types of blind rivets, namely where the inside or blind head is formed mechanically or where it is formed by an explosive force.

Fig. D1.31 illustrates the Du Pont explosive type and Figs. 32 to 34 inclusive illustrate the mechanically formed head type.

![Inserted Installed](image)

**D1.3 Riveted Sheet Splice Information.**

In splicing or connecting two sheets together by means of rivets or bolts, the joint or connection may fail in the various ways as explained in detail for single and multiple bolt fitting units. Thus one must check the shear strength of the rivets; bearing of rivets on the sheets; tear out of the sheet edges and tension on sections through the rivet holes.

**Types of Sheet Splices or Connections.**

Fig. D1.35 illustrates the various types of sheet splices. In the offset lap splice between two sheets of different gauges, the offset should be in the heavier material. For a single shear butt splice, the butt splice plate should be equal to the thinnest of the two sheets being spliced and likewise in the
double shear butt splice the splice plates should be of the same thickness and should be equal to the thinnest plate being spliced.

Lap Splice

Offset Lap Splice

Fig. D1.35

Butt-Double Strap Splice

Butt-Single Strap Splice

Proper Rivet Size to Use.

No exact rules can be given relative to the optimum rivet for a given splice, because a number of practical considerations usually enter into the design of a sheet splice. Small rivets, namely 1/16 and 3/32 diameters are hard to handle and are seldom used as structural rivets. The common size of rivets are 1/8, 5/32, 3/16 and 1/4 inch diameters. The larger sizes should not be used in sheet splices unless there is a backing up structure as the sheet may buckle under the driving of the large rivets. From a structural design standpoint, the optimum joint is one in which the allowable rivet shear strength and bearing strength of the given sheets are practically the same for the largest practical size rivet.

Rivet Spacing, Sheet Edge Distance.

The allowable rivet-sheet bearing loads as given in (Ref. 1) are based on an edge distance of two diameters. Therefore in general no edge distance in a joint should be less than 2 rivet diameters for protruding head rivets, and 2–1/2 diameters for press and machine countersunk flush rivets.

The minimum pitch (distance between centerline of rivets in the same row) for a given size rivet should not be less than that given in Tables A and B.

<table>
<thead>
<tr>
<th>Rivet Diameter</th>
<th>1/8</th>
<th>3/32</th>
<th>3/16</th>
<th>1/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal Minimum Spacing</td>
<td>1/2</td>
<td>9/16</td>
<td>11/16</td>
<td>7/8</td>
</tr>
</tbody>
</table>

Table A

<table>
<thead>
<tr>
<th>Rivet Diameter</th>
<th>1/8</th>
<th>3/32</th>
<th>3/16</th>
<th>1/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal Minimum Spacing</td>
<td>11/16</td>
<td>27/32</td>
<td>1-1/2</td>
<td>1-1/2</td>
</tr>
</tbody>
</table>

Table B

In general the minimum rivet row spacing should be such as to make the distance between any two rivets in the two rows not less than the minimum rivet spacing for the rivet size being used.

Splice Sheet Tension Efficiency.

When a sheet is spliced by means of rivets or bolts it means the sheet is weakened since the rivet holes cut away a part of the sheet material. The ratio of the tension strength of the spliced sheet to the unspliced sheet is called the sheet tension efficiency of the joint. If the minimum rivet spacing is used and only one row of rivets the sheet efficiency will be around 70 to 75 percent. The designer should strive for a higher efficiency.

D1.24 Illustrative Problems Involving Use of Rivets.

PROBLEM 1. Horn Connection to Torque Tube.
Fig. D1.38 illustrates a rudder horn attachment to the lower end of a rudder torque tube for a small airplane. The horn is fastened to the tube by two collars riveted to the horn and likewise to the tube. The design load table pull is 400 lb. which includes a fitting factor of safety of 1.15. The rivet material is 2024-T3 aluminum alloy. The horn and tube is 2024-T3 aluminum alloy. The margin of safety for the riveted connection will be calculated.

**Solution:**

The horizontal cable pull of 400 lb. can be replaced by an equivalent force system at the centerline of the tube, consisting of a torsional moment of 400 x 5 = 2000 in.-lb., and a horizontal 400 lb. force.

Consider attachment of collars to horn:

- 6 - 1/8 diameter rivets are used in double shear.

Load per rivet due to torque = 2000 x 3 x 1,000 = 334 lb. (Rivet arm is 1 inch).

Load per rivet due to horizontal force = 400 x 6 = 67 lb.

Resultant load on most critical rivet is 334 + 67 = 401 lb.

From Table D1.9 the single shear strength of a 1/8 diameter rivet of 2017-T3 material is 589 lb. Since the rivets are in double shear the shear strength of one rivet would be 2 x 589 = 1178 lb. Referring to the table at the bottom of Table D1.5, we find a rivet factor of 335 to apply for a 1/8 rivet on 0.063 sheet thickness. Therefore rivet strength is 0.335 x 766 = 713 lb.

M.S. in rivet shear = (713/401) - 1 = 0.79

Each rivet bears on two collars each of which is 0.063 in thickness. The bearing strength of a 1/8 rivet on 0.063 2024-T3 aluminum alloy material is obtained from Tables D1.9 and D1.3.

From Table D1.9, bearing strength of 1/8 diameter rivet on 0.063 base on an allowable bearing stress of 600,000 is 810 lb. Then referring to Table D1.6 for 2024-T3 material and a/D of 1.5, we find a correction factor of 0.8. Thus bearing strength is 0.8 x 810 = 648 lb. Since each rivet bears on two collars, the bearing strength for one rivet = 2 x 648 = 1296 lb.

M.S. = 1293/401 = 2.36

Consider attachment of collars to horn:

10 - 3/32 diameter rivets are used.

Load on each rivet due to torque = 2000/10 x 6 = 334 lb. The horizontal force of 400 lb. can be taken by direct bearing between collars and tube.

The rivets are in single shear. The single shear strength of a 5/32 rivet from Table D1.5 = 596 x 0.25 = 354 lb. (The value of 0.25 is the correction from middle table of Table D1.5.

M.S. = (354/334) - 1 = 0.73

Bearing strength on 0.063 tube wall from Table D1.9 for 0.063 thickness and Fb = 100,000 is 798 lb. Correcting to .45 = (0.45/0.50) 798 = 780. For 2024-T3 tube material and a/D = 2.0 we obtain a factor of 1.2. Therefore bearing strength of one rivet on tube wall is 1.24 x 780 = 957 lb.

M.S. = (957/334) - 1 = 1.9

**Problem 2:**

Fig. D1.37 shows plate fitting attached to a double channel section by 6 - 1/4 diameter rivets. The design fitting loads are shown in the figure. The riveted connection will be checked for strength under the given design fitting loads.

**Solution:**

The given force system will be replaced by an equivalent force system acting at the center of gravity of the rivet group. This force system will consist of:

W = 9000 lb., Vc.g. = 3000 lb.

W = 9000 x 0 = 3000 x 3 = 9000 in. lb.

Fig. D1.37

Fig. D1.38
### Table D.5 Shear Strengths of Protruding and Flush-Head Aluminum-Alloy Rivets

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear strength, lb:</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5056, $F_\tau = 28$ ksi</td>
<td>99</td>
<td>203</td>
<td>363</td>
<td>356</td>
<td>302</td>
<td>1,450</td>
<td>2,290</td>
<td>3,280</td>
</tr>
<tr>
<td>2117-T3, $F_\tau = 30$ ksi</td>
<td>106</td>
<td>217</td>
<td>388</td>
<td>356</td>
<td>302</td>
<td>1,550</td>
<td>2,460</td>
<td>3,510</td>
</tr>
<tr>
<td>2017-T3, $F_\tau = 34$ ksi</td>
<td>120</td>
<td>247</td>
<td>442</td>
<td>375</td>
<td>318</td>
<td>1,760</td>
<td>2,790</td>
<td>3,970</td>
</tr>
<tr>
<td>2024-T3, $F_\tau = 38$ ksi</td>
<td>133</td>
<td>273</td>
<td>494</td>
<td>375</td>
<td>318</td>
<td>1,870</td>
<td>3,110</td>
<td>4,450</td>
</tr>
</tbody>
</table>

#### Single-shear rivet strength factors

| Sheet thickness, in. | 0.016 | 0.018 | 0.020 | 0.025 | 0.032 | 0.036 | 0.040 | 0.045 | 0.050 | 0.063 | 0.071 | 0.080 | 0.090 | 0.100 | 0.125 | 0.160 | 0.200 | 0.250 |
|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|                      | 0.964 | 0.984 | 0.996 | 1.000 | 0.972 | 1.000 | 0.964 | 0.980 | 0.996 | 1.000 | 0.972 | 0.980 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

#### Double-shear rivet strength factors

| Sheet thickness, in. | 0.016 | 0.018 | 0.020 | 0.025 | 0.032 | 0.036 | 0.040 | 0.045 | 0.050 | 0.063 | 0.071 | 0.080 | 0.090 | 0.100 | 0.125 | 0.160 | 0.200 | 0.250 |
|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|                      | 0.688 | 0.714 | 0.825 | 0.856 | 0.868 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 | 0.880 |

**Note:** Values of shear strength should be multiplied by the factors given herein whenever the D/t ratio is large enough to require such a correction.

**Shear stresses in Table 5.1.1.1.1(e) corresponding to 50 percent probability data are used wherever available.**

**Shear values are based on areas corresponding to the nominal hole diameters specified in Table 5.1.1.1(d), note 1.**

*The *T3* designation refers to rivets that have been heat-treatd and then maintained in the heat-treated condition until driving.*
<table>
<thead>
<tr>
<th>Rivet size, in.</th>
<th>⅛</th>
<th>⅜</th>
<th>½</th>
<th>⅞</th>
<th>¾</th>
<th>¾</th>
<th>¾</th>
<th>¾</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drill No.</td>
<td>51</td>
<td>41</td>
<td>30</td>
<td>21</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal hole diameter, (in.)</td>
<td>0.067</td>
<td>0.066</td>
<td>0.1283</td>
<td>0.159</td>
<td>0.191</td>
<td>0.257</td>
<td>0.323</td>
<td>0.386</td>
</tr>
</tbody>
</table>

Table D1.7 Ultimate and Yield Strengths of Solid 1092 Machine-Countersunk Rivets

<table>
<thead>
<tr>
<th>Rivet material</th>
<th>2117-T3</th>
<th>2117-T3</th>
<th>2047-T3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clad sheet material</td>
<td>2024-T3, 2024-T4, 2024-T6, 2024-T81, 2024-T86, and 7075-T6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rivet diameter, in.</td>
<td>⅛</td>
<td>⅜</td>
<td>½</td>
</tr>
<tr>
<td>Sheet thickness, in.</td>
<td>0.020</td>
<td>1.32</td>
<td>163</td>
</tr>
<tr>
<td>0.025</td>
<td>1.56</td>
<td>221</td>
<td>250</td>
</tr>
<tr>
<td>0.032</td>
<td>1.78</td>
<td>272</td>
<td>348</td>
</tr>
<tr>
<td>0.040</td>
<td>1.92</td>
<td>309</td>
<td>416</td>
</tr>
<tr>
<td>0.050</td>
<td>2.20</td>
<td>340</td>
<td>479</td>
</tr>
<tr>
<td>0.063</td>
<td>2.16</td>
<td>363</td>
<td>523</td>
</tr>
<tr>
<td>0.071</td>
<td>2.72</td>
<td>542</td>
<td>739</td>
</tr>
<tr>
<td>0.080</td>
<td>2.56</td>
<td>560</td>
<td>769</td>
</tr>
<tr>
<td>0.090</td>
<td>2.57</td>
<td>575</td>
<td>795</td>
</tr>
<tr>
<td>0.100</td>
<td>2.81</td>
<td>618</td>
<td>1,054</td>
</tr>
<tr>
<td>0.125</td>
<td>3.83</td>
<td>1,090</td>
<td>1,773</td>
</tr>
<tr>
<td>0.160</td>
<td>1,591</td>
<td>2,000</td>
<td></td>
</tr>
<tr>
<td>0.190</td>
<td>1,970</td>
<td>2,084</td>
<td></td>
</tr>
<tr>
<td>Shear</td>
<td>2.17</td>
<td>388</td>
<td>596</td>
</tr>
</tbody>
</table>

Ultimate strength

| Yield strength |
|----------------|---------|---------|---------|
| 0.020 | 91 | 98 | 110 | 204 |
| 0.025 | 113 | 150 | 200 | 362 |
| 0.032 | 132 | 198 | 265 | 370 |
| 0.040 | 133 | 231 | 273 | 370 |
| 0.050 | 188 | 261 | 321 | 419 | 538 | 594 |
| 0.063 | 213 | 321 | 402 | 471 | 501 | 515 | 610 | 614 | 811 |
| 0.071 | 348 | 433 | 538 | 481 | 557 | 706 | 669 | 902 |
| 0.080 | 498 | 616 | 562 | 623 | 788 | 781 | 982 |
| 0.090 | 537 | 685 | 633 | 746 | 851 | 842 | 1,053 |
| 0.100 | 554 | 745 | 854 | 1,017 | 913 | 1,115 |
| 0.125 | 536 | 836 | 1,018 | 1,313 | 1,021 | 1,357 |
| 0.160 | 1,574 | 1,694 |
| 0.190 | 1,753 | 1,925 |

Note: The values in this table are based on "good" manufacturing practice, and any deviation from this will produce significantly reduced values.

1. Sheet gage is that of the countersunk sheet. In cases where the lower sheet is thinner than the upper, the shear-bearing allowable for the lower sheet-rivet combination should be computed.
2. Increased attention should be paid to detail design in cases where D/t > 4.9 because of possibly greater incidence of difficulty in service.
3. Yield values of the sheet-rivet combinations are less than 2/3 of the indicated ultimate values.
## Fittings and Connections: Bolted and Riveted

Table D1.8 Aluminum-Alloy Sheet and Plate Bearing Factors

(K = ratio of actual bearing strength to 100 ksi)

<table>
<thead>
<tr>
<th>Material</th>
<th>Thickness, in.</th>
<th>A values</th>
<th>B values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>K (Ultimate)</td>
<td>K (Yield)</td>
<td>K (Ultimate)</td>
</tr>
<tr>
<td></td>
<td>e/D=2.0</td>
<td>e/D=1.5</td>
<td>e/D=2.0</td>
</tr>
<tr>
<td></td>
<td>e/D=2.0</td>
<td>e/D=1.5</td>
<td>e/D=2.0</td>
</tr>
</tbody>
</table>

| 2024-T42 (heat treated by user) | <0.250 | 1.18 | 0.92 | 0.64 | 0.56 | 0.79 | 0.69 | 1.29 | 1.02 | 0.82 | 0.71 |
| 0.250-0.500   | 1.22 | 0.96 | 0.64 | 0.53 | 1.27 | 1.01 | 0.78 | 0.69 | 1.29 | 1.02 | 0.82 | 0.71 |
| 501-1.000     | 1.18 | 0.93 | 0.61 | 0.53 | 1.37 | 1.08 | 0.80 | 0.78 | 1.25 | 1.06 | 0.87 | 0.75 |
| 2024-T3       | <0.250 | 1.24 | 0.98 | 0.79 | 0.69 | 1.27 | 1.01 | 0.78 | 0.69 | 1.29 | 1.02 | 0.82 | 0.71 |
| 0.250-0.500   | 1.24 | 0.98 | 0.74 | 0.64 | 1.27 | 1.01 | 0.78 | 0.69 | 1.29 | 1.02 | 0.82 | 0.71 |
| 501-1.000     | 1.20 | 0.95 | 0.70 | 0.62 | 1.37 | 1.08 | 0.80 | 0.78 | 1.25 | 1.06 | 0.87 | 0.75 |
| 2024-T36      | <0.500    | 1.33 | 1.05 | 0.96 | 0.84 | 1.37 | 1.08 | 0.80 | 0.78 | 1.25 | 1.06 | 0.87 | 0.75 |
| 0.500-1.000   | 1.06 | 0.94 | 0.75 | 0.58 | 1.10 | 0.97 | 0.69 | 0.64 | 1.16 | 0.92 | 0.75 | 0.70 |
| Clad 2024-T42  | ≥0.065    | 1.10 | 0.88 | 0.85 | 0.64 | 1.16 | 0.92 | 0.75 | 0.70 | 1.11 | 0.92 | 0.75 | 0.70 |
| (heat treated by user) | <0.065 | 1.10 | 0.88 | 0.85 | 0.64 | 1.16 | 0.92 | 0.75 | 0.70 | 1.11 | 0.92 | 0.75 | 0.70 |
| Clad 2024-T3   | 0.63-2.499 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| 250-4.999     | 1.18 | 0.93 | 0.61 | 0.53 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| 500-1.000     | 1.14 | 0.90 | 0.58 | 0.50 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| Clad 2024-T4   | 0.63-2.499 | 1.10 | 0.87 | 0.75 | 0.68 | 1.16 | 0.92 | 0.75 | 0.70 | 1.16 | 0.92 | 0.75 | 0.70 |
| (coiled)      | <0.065    | 1.10 | 0.87 | 0.75 | 0.68 | 1.16 | 0.92 | 0.75 | 0.70 | 1.16 | 0.92 | 0.75 | 0.70 |
| Clad 2024-T5   | ≥0.065    | 1.14 | 0.90 | 0.75 | 0.66 | 1.14 | 0.90 | 0.75 | 0.66 | 1.14 | 0.90 | 0.75 | 0.66 |
| 0.63-2.499    | 1.18 | 0.93 | 0.61 | 0.53 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| Clad 2024-T8   | ≥0.63     | 1.14 | 0.90 | 0.75 | 0.66 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| 0.63-2.499    | 1.18 | 0.93 | 0.61 | 0.53 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| Clad 2024-T81  | ≥0.63     | 1.14 | 0.90 | 0.75 | 0.66 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| 0.63-2.499    | 1.18 | 0.93 | 0.61 | 0.53 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| Clad 2024-T86  | ≥0.63     | 1.14 | 0.90 | 0.75 | 0.66 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |
| 0.63-2.499    | 1.18 | 0.93 | 0.61 | 0.53 | 1.14 | 0.90 | 0.73 | 0.64 | 1.14 | 0.90 | 0.73 | 0.64 |

* For e/D values between 1.5 and 2.0 bearing factors may be obtained by linear interpolation. (r = edge distance, D = hole diameter.)
<table>
<thead>
<tr>
<th>Sheet thickness in.</th>
<th>1/8 in.</th>
<th>1/4 in.</th>
<th>1/2 in.</th>
<th>1/4 in.</th>
<th>1/2 in.</th>
<th>3/8 in.</th>
<th>1/2 in.</th>
<th>3/4 in.</th>
<th>1/2 in.</th>
<th>3/4 in.</th>
<th>7/8 in.</th>
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<td>1519</td>
<td>1850</td>
<td>2035</td>
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<td>2384</td>
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<td>2888</td>
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* Bearing values are based on areas computed using the nominal hole diameters specified in Table S.1.1.1.1(d).

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* Bearing values are based on areas computed using the nominal pin diameters indicated.
Since rivets are same size, all rivets are assumed to share equally in resisting H and V loads.

Load on each rivet due to H, \( M_{c.g.} = \frac{3000}{6} = 500 \) lb acting in H direction and to the right. Load on each rivet due to \( V_{c.g.} = \frac{3000}{6} = 500 \) lb acting down.

From equation (15), the load on a rivet due to \( M_{c.g.} \) on rivet group equals \( F = Mr/I \).

\[
I = 2r^2 = 1.625^2 \times 4 + 0.625^2 \times 2 = 11.4
\]

Consider rivet marked c;

\[
r = 1.625 \text{ arm to c.g. of bolt group.}
\]

\[
P_c = Mr/I = (3000 \times 1.625) \times 11.4 = 1200 \text{ lb.}
\]

Since rivets b, d and e are the same distance as rivet c from the c.g., the moment load on these bolts will also equal 1200. Fig. D1.39 shows the H, V, and M loads on the rivets b, c, d and e. Since the arm r to the rivets f and g is only 0.625, the load due to moment will be considerably smaller and thus these rivets will not be critical. Observation of Fig. D1.39 shows rivet c is the rivet with the largest residual load.

\[
R_c = \sqrt{2F_H^2 + 2F_V^2}
\]

\[
2F_H = 1.333 \times 1200 \times 1.5/1.428 = 2513 \text{ lb.}
\]

\[
2F_V = -500 \times 1200 \times 0.625/1.428 = -992
\]

Hence, \( R = \sqrt{2513^2 + 982^2} = 2700 \) lb.

The rivets are in double shear. Rivet material is 2117-T31.

From Table D1.5, single shear value = 1760 lb. or double shear strength = 3520 lb.

Bearing strength of 1/4 rivet on the .071 2024-T6 clad channel section from Tables D1.9 and D1.9 is 1325 x 1.20 = 2100. Since rivet bears on two channels, bearing strength of one rivet = 2 x 2100 = 4200 lb. Rivet shear is critical.

M.S. = (3520/2700) - 1 = .30

As a problem for the reader, change rivets f and g to 3/16 diameter and determine whether rivet attachment still shows a positive margin of safety (use equation 16).

**Problem 3.**

Fig. D1.39 shows a lap joint involving two rows of rivets as shown. Sheet material is 2024-T3 clad, and rivets are 5/32 diameter and 2117-T3 material and of the protruding head type.

The ultimate design tension load in the sheet including a 1.13 fitting factor of safety is 1000 lb/in. The limit fitting load is 2/3 x 1000 = 667 lb/in.

The margin of safety of the sheet splice will be determined.

\[
\text{Fig. D1.39}
\]

**Solution:**

As an analysis unit, a width of sheet equal to the rivet pitch of 1 inch will be used. Thus load on 1 inch unit = 1000 lb.

**Check Tension in Sheet at Section Through Holes.**

\[
Pt(allow) = A Ftu
\]

\[
A = \text{net area} = (1 - .159) .04 = .0336
\]

\[
(.159 = \text{drill diameter for } 3/32 \text{ rivet, Table D1.6})
\]

\[
Ftu \text{ for } 2024-T6 \text{ clad} = 60,000 \text{ psi.}
\]

\[
Pt(allow) = .0336 \times 60000 = 2016 \text{ lb.}
\]

M.S. = (2016/1000) - 1 = 1.01

**Check Shear of Rivets.**

Rivets are in single shear and two rivets act in the 1 inch unit which was assumed. From Table D1.5, single shear strength for 3/32, 2117-T3 rivet is 586 lb. The strength factor middle table of Table D1.5 for .04 sheet thickness is .964. Thus for two rivets the shear strength is 2 x .964 x 586 = 1150 lb.

M.S. = (1150/1000) - 1 = .15

**Check Bearing of Rivets on .04 Sheet.**

From Table D1.9, the ultimate bearing strength based on Por of 100,000 psi for 3/32 rivet on .04 sheet = 556 lb. Then referring to
Table D1.8 for 2024-T3 clad material and an e/o ratio of 2.0, we find correction factor $K = 1.14$. Therefore rivet bearing strength is 
$1.14 \times 635 \times 2 = 1450 \text{ lb.}$

$$M.S. = (1450/1000) - 1 = .45$$

Check Rivet Shear Out.

Since edge distance is 5/16 in. or e/o = 2.0, shear out strength is satisfactory.

**PROBLEM 4.**

Assume rivets are changed to the solid 1009 dimpled type. What would be the M.S. for the rivets. Referring to Fig. D1.27, we find the sheet thicknesses are such as to prevent double dimpling. From Table D1.11 and D1.12, we obtain the ultimate and yield strength of a 5/32 rivet on .04 sheet as 655 and 506 lbs., respectively.

Whence, Ultimate M.S. = $(2 \times 655/1000) - 1 = .27$

Yield M.S. = $(2 \times 506/657) - 1 = .49$

**NOTE:** In checking tensile strength of sheet through hole section, the drill size for dimpled rivets is slightly larger than for protruding head type.

**PROBLEM 5.**

This is a typical problem involving the rivet loads in a sheet-stringer type of construction as illustrated in Fig. D1.40.

Before the rivet size and spacing at the points (1) to (10) can be determined, the rivet loads at these points must be known. The shear flow in direction and magnitude on the web and skin are shown on the figure and are in lbs. per inch. These values represent the results in one of the flight conditions. The structural designer must look at all the shear flows in the various flight and landing conditions in order to obtain the critical rivet loads. It is assumed the shear flows as shown include any diagonal tension effect in the various sheet panels.

The rivet loads in lbs./in. at rivet lines 1 to 10 will be as follows:

**Rivet Line (1).** Since .06 vertical web ends at point (1), the shear flow of 1075 lbs./in. in the vertical web must obviously be reacted by the rivets in rivet line (1), thus load on rivet line (1) is 1075 lbs./in.

**Rivet Line (2).** By the same reasoning since skin ends at point (2), the load on rivet line (2) equals shear flow in panel 2-3 or 575 lbs./in.

**Fig. D1.40**

Rivet Line (3). The skin is continuous over stringer at point (3). Sketch (a) shows a free body of the skin and stringer at point (3).

Since the summation of the forces parallel to the stringer must equal zero, it is observed that the load transferred to the stringer is 150 lbs./in.

**Rivet Lines (4) and (5).** Since the sheets and over the stringer, the load in rivet lines (4) and (5) are 425 and 275 lbs./in. respectively.

**Rivet Line (6).** Rivet load = 275 - 125 = 150 lbs./in.

**Rivet Line (7).** The skin is lap spliced over the stringer at point 7. Sketch (b) shows a free body. The load produced on the stringer is 150 from equilibrium. Thus the worst shear load on the rivet is 150 lbs./in. which is greater than the shear on another cross-section of the rivet which equals 125 lbs./in. as the shear flow in panel 8-7.

**Rivet Load at (3) = 175 - 25 = 150 lbs./in.**

**Rivet Load at (9) = 175 lbs./in.**

**Rivet Load at (10) = 575 lbs./in.**

**DI.25 Rivets in Tension.**

Great judgment should be used in using rivets in tension. There is a general saying, "Never use a rivet in tension." If this requirement is strictly followed, it would be difficult to design a conventional airplane. For example, the skin on the upper surface of the wing, due to the upward suction air forces places the rivets that hold the skin to the stringers and ribs in tension, however these tension loads in most cases are relatively small.

The following general criteria apply relative to rivets in tension.
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<td>2024 T6, 2024 T86, and 2024 T11</td>
<td>2024 T6, 2024 T86, and 2024 T11</td>
<td>2024 T6, 2024 T86, and 2024 T11</td>
<td>2024 T6, 2024 T86, and 2024 T11</td>
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<td>0.255</td>
<td>0.280</td>
<td>0.305</td>
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<td>0.375</td>
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Note: The values in this table are based on "good" manufacturing practice and any deviation from this will produce a significantly reduced value.

*These allowables apply to double dimpled sheets and to the upper sheet dimpled into a machine-countersunk lower sheet. Sheet gage is that of the thinnest sheet for double dimpled joints and of the upper dimple sheet for dimpled, machine-countersunk joints. The thickness of the machine-countersunk sheet must be at least 1 tabulated gage thicker than the upper sheet. In no case shall allowables be obtained by extrapolation for skin gages other than those shown.
Table Dl. 12  Yield Strength of Solid 100° Dimpled Rivets

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<th>2021-T3 and 2021-T6 and 2024 T86 and 7075-T6</th>
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Note: The values in this table are based on "good" manufacturing practice and any deviation from this will produce significantly reduced values.

a These allowables apply to double dimpled sheets and to the upper sheet dimpled into a machine-countersunk lower sheet. Sheet gage is that of the thickest sheet for double dimpled joints and of the upper dimple sheet for dimpled, machine-countersunk joints. The thickness of the machine-countersunk sheet must be at least 1 tabulated gage thicker than the upper sheet. In no case shall allowables be obtained by extrapolation for skin gages other than those shown.
(1) Tension on rivets shall be restricted to conditions in which tension load is incidental to the major shear carrying purpose of the rivet. When it is difficult to determine if the tension component is incidental or major, a bolt shall be used.

(2) The following are examples of joints where rivets are considered to be satisfactory tension carrying mediums.

(a) Skin attachment to ribs and frames

(b) Attachment of sheet panels to beam flanges and stringers, where interrivet buckling or diagonal sheet wrinkling produce tension loads on rivets.

(c) Skin attachment on a pressurized nacelle or body.

(3) Do not use rivets to fasten control brackets to a supporting structure.

(4) If there is no load reversal on the assembly, the tension allowables given in the following tables can be used.

(5) If there is load reversal on the assembly, the tension load on the rivet should not exceed 25 percent of the values in the table.

(6) Rivets loaded in shear and tension should be checked for combined stresses, using the interaction equation,

\[ R_t^* + R_g^* = 1 \]

(7) A sufficient number of rivets shall be used to insure that failure of any one rivet due to improper installation, cracked head, etc., shall not result in the failure of the structure that is being held together by the rivets.

D1.26 Rivet Tension Strengths.

Reference to the structures design manuals of various aircraft companies shows that rivet tensile strengths are not the same or, in other words, not standardized as in the case of shear and bearing. Tables A, B & C have been taken from (Ref. 8). The values given are conservative relative to values found in other company manuals.

### Table A
**PROTRUDING HEAD RIVETS (AN470, AN442) ULTIMATE TENSILE STRENGTH**

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### Table B
**100° FLUSH HEAD RIVET (AN426) MACHINE COUNTERSUNK JOINT ULTIMATE TENSILE STRENGTH**

<table>
<thead>
<tr>
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### Table C
**100° FLUSH HEAD RIVET (AN426) DOUBLE DIMPLE ULTIMATE TENSILE STRENGTH**

<table>
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<tr>
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<td>1952</td>
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\[ t_{min} \]
(1) The single bolt fitting unit as shown in Fig. 1 is subjected to a design fitting load of 12,000 lbs. in axial tension. The pin is an AN Steel bolt 3/8 inch diameter. Bushing is 1/16 wall and steel $F_{tu} = 120,000$. Lug material is 2014-T6, $F_{tu} = 58,000$. The fitting is not subjected to shock or vibration. Strength check the bolt and lug (A) and give all margins of safety.

(2) Same fitting as in Problem 1 but design fitting load is a transverse load of 10,000 lb. Strength check and give all margins of safety.

(3) Same fitting as in Problem 1 but lug (A) is subjected to a design fitting load acting at 45° with a value of 10,000 lb. Strength check for this loading.

(4) A 1/2 inch diameter AN steel bolt is subjected to a combined shear and tension load. The shear load on the bolt is 10,000 lbs. and the tension load is 12,000 lbs. Find margin of safety under this combined loading.

(5) Design a hinge pin using a standard AN steel bolt and a male lug to carrying an axial tensile load of 25,000 lbs. Use fitting factor of 1.15. Use steel bushing. No shock or vibration. Assume lugs of female part of fitting 1/2 as thick as male lug. Design the male lug from two materials. (1) 2024-T4 aluminum alloy and (2) AISI steel, $F_{tu} = 180,000$.

(6) Fig. 2 illustrates an end fitting for a streamline strut. The tube is flattened slightly at the end to fit a simple block fitting. Loads shown are design strut loads. Using a fitting factor of 1.20, check the strength of the entire fitting unit. Assume no shock or vibration.

(7) Fig. 3 illustrates a fitting unit on the end of an extruded (I) section. The web on the (I) section extends out to form part of the fitting lug, which is reinforced by steel fitting plates. Strength check the fitting for a design load of 45,000 lb. Use fitting factor of 1.15.
(6) Fig. 6 shows a flap hinge fitting attached to a supporting structure by 4 - 1/4" dia. AN steel bolts. The bolts are in double shear. Check the most critical bolt in shear and bearing for load of 3500 lbs. acting as shown. Fitting material is 2014-T3.

(10) In the eccentrically loaded multiple bolt fitting of Fig. 6, determine the resultant load on each of the five AN steel bolts in resisting the 24000 lbs. load acting as shown.

(11) In Fig. 7 the fitting is attached to a fuselage frame by four 1-1/4 dia. AN steel bolts. The design fitting loads are as shown. Determine the margin of safety on the most critical bolt. Use Fig. B1.4 to obtain equation for combined tension and shear stresses.

(12) In Fig. 8 find the size of 2017-T3 rivets necessary to carry the ultimate fitting design load of 3000 lbs. as shown. The fitting plate is steel 1/8 thick, Flt = 36000, and the 2024-T3 channel frame is .081 inch thick.

(13) Design the lightest overlap sheet splice for .061 clad 2024-T3 and protruding head type rivet. Design tension load on sheet = 2350 lb./in. Give all details of joint.

(14) Rework design in Problem 13 to use double dimpled rivets.

REFERENCES


(6) Various Structural Manuals of Various Aircraft Companies.
CHAPTER D2
WELDED CONNECTIONS

D2.1 Introduction.

Since the overall structure of an airplane, missile or space vehicle cannot be fabricated as a single continuous unit, such structures involve many structural parts which must be fastened together. For certain materials and types of structural units, welding plays an important role in joining or connecting structural units. Research is constantly going on to develop better welding machines and welding techniques and also to develop new materials that can be welded without producing a detrimental strength influence on the base or unwelded material. A fair size book could be written on the subject of welding and design for welding. This brief chapter can only be a brief introduction to the subject.

D2.2 Gas Welding.

There are two types of gas welding, namely, oxyacetylene and oxyhydrogen. Practically all gas welding in aircraft work is oxyacetylene. Some welders prefer the oxyhydrogen flame in welding aluminum alloys because the flame is not so hot. The major aircraft structural units in which gas welding plays an important part are welded steel tubular fuselages, engine mounts, and landing gears and the attachment of plate and machined fittings to such structure.

D2.3 General Notes on the Practical Design of Welded Joints.

The designer of welded structures in steel can greatly help the welder obtain good joints or connections by adhering to the following general rules.

(1) It is much easier to obtain a good weld when the parts being welded together are of equal thickness. It is general design practice to try and keep thickness ratio between the two welded parts less than 3 to 1. Some designers try to keep within a 2 to 1 ratio in order to eliminate possibilities of welders burning the thinner sheet.

(2) Designers usually consider .025 as the minimum thickness to be welded in general practical structural joints as there is considerable danger that the average welder might burn a thinner gauge.

(3) In general avoid welds in tension since they produce a weakening effect. In some connections it is difficult to avoid all tension loads on welds, thus weld stresses should be kept low and if possible incorporate a fishmouth joint or finger patch to put part of weld in shear.

(4) A weld should not encircle a tube in a plane perpendicular to the tube length. Standard splices or joints for overlapping tubes, and end socket fittings in tubes have been developed, which require no strength check. These are the diagonal weld and fishmouth welds as illustrated in Fig. D2.1.

(5) Tapered gusset plates should be incorporated in all important welded joints to insure gradual change in stress intensity in members. These gussets lessen the danger of fatigue failure by reducing stress intensity.

(6) A weld over a weld should not be made.

(7) To prevent burning of sheet welds should not be made on both sides of a thin sheet.

(8) If two welds are placed close together lack of shrinkage space may cause cracking.

(9) Cracks usually develop if welding is done on bends.

(10) When tubes are spliced by welding locate splice near one end of the tube, to avoid affecting column properties. In general it is not good practice to weld brackets to the middle of column members. Clamps are preferable.

(11) In welding members together local internal stresses are set up. On most weld assemblies it is therefore customary to "normalize" the assemblies after welding. This heating permits the equalization of the internal localized stresses thus preventing cracking in service.

Fig. D2.1
D2.4 General Types of Welded Steel Fitting Units.

Fig. D2.2 taken from aircraft tubing handbook of the Summerville Tubing Company summarizes the common types of tube terminals and discusses their structural merit. Fig. D2.3 illustrates the conventional concentric butt welded fuselage joint which tests show is satisfactory where vibration is not present. Tests have shown that the fatigue strength of a welded joint as illustrated in Fig. D2.3 when the members are subjected to reverse bending is reduced considerably, thus it is common practice to add additional joint reinforcement such as finger plates or insert gussets as illustrated in Figs. D2.4 and D2.5 to joints subjected to vibration.

Figs. D2.6 to D2.8 illustrate methods of splicing a longeron at a truss joint. The vertical and diagonal members strengthen the butt weld on the spliced member.

Figs. D2.9 and D2.10 illustrate fitting plate attachments to tubes. Except for light fitting loads, the fitting plate should extend through to both sides of tube or to the adjoining members. The fitting type illustrated in Fig. D2.11 is only used for secondary conditions where loads or plate are relatively light.

Since eccentricity of member forces on a joint produce bending in the connecting members which may lower the fatigue strength of the joint, such cases of joint eccentricity as illustrated in Figs. D2.12 to D2.14 should be eliminated in joint design.

D2.5 Electric Arc Welding.

This method of welding is based upon the heat generated in an electric arc. Arc welding to a limited extent has been used for many years in aircraft fabrication. No doubt the flexibility and general all around good results obtained with gas welding retarded its extensive use, however in recent years its use is increasing rapidly as its economies and advantages become apparent to the designer. In arc welding the applied heat is more concentrated and quicker welding results with less expansion and warping as compared to gas welding. In the design of tubular joints, care should be taken to make all welds as accessible as possible. To secure proper stress distribution in arc welded joints the designer should follow the recommendations as illustrated in Figs. D2.15 to D2.18.

The fact that less expansion and warping takes place since the heat is concentrated makes it possible to hold to closer tolerances on parts requiring machining after welding an allowance of 1/16 inch is generally sufficient on most assemblies. Electric welding permits welding of thin sheets as low as .016 inch thickness.

D2.6 Effect of Welding on Base Metal.

Tests show that plain carbon and chromium molybdenum steels suffer very little in loss of tensile strength due to welding. For cold rolled sheet or tubing the refinement in grain.
### TYPICAL WELDED STEEL TUBE TERMINALS

**CORRECT**
- Satisfactory where a wide flange is required
- Drawing area of flange may vary with respect to tube
- Welded plug and spigot is generally satisfactory
- Not strong in compression in flange service if struck under proper loading conditions.

**INCORRECT**
- Requires excessive amount of cold working and too many welding operations.
- Not strong in compression in flange service if struck under proper loading conditions.

---

**Fig. D2.3**
- Tube Cluster

**Fig. D2.4**
- Fig. D2.5

**Fig. D2.6**
- Fig. D2.7

**Fig. D2.8**
- Fig. D2.9

**Fig. D2.10**
- Fig. D2.11

**Fig. D2.12**
- Bending on tube tends to lower weld in tension

---

Reproduced by permission from Summerill "Aircraft Tubing Data"
due to cold working is lost in the material adjacent to the weld which lowers strength to a small degree. Welding, however, does produce a more brittle material which has lower resistance to shock, vibration and reversal of stress, thus it is customary to assume an efficiency of weld joints less than 100 percent.

Table D2.1 gives the allowable ultimate tensile stress for alloy steels for material adjacent to weld when structure is welded after heat treatment.

Table D2.1 (Ref. 1)
Allowable Ultimate Tensile Stresses Near Fusion Welds in 4130, 4140, 4340, or 8630 Steels

<table>
<thead>
<tr>
<th>Type of joint</th>
<th>Ultimate tensile stress, ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tapered joints of 30° or less</td>
<td>90</td>
</tr>
<tr>
<td>All others</td>
<td>90</td>
</tr>
</tbody>
</table>

* Welded after heat treatment or normalized after weld.
* Gussets or plate inserts considered 90° taper with center line.

For welding members subjected to bending, the allowable modulus of rupture for alloy steels when welded after heat-treatment should not exceed the following as specified in (Ref. 1).

For tapered joints of 30° or less, use modulus of rupture $F_0$ equivalent to that for steel having $F_{tu} = 90000$ psi.

For all other types of welds, use $F_0$ equal to .9 of that for steel having $F_{tu} = 90000$ psi. Chapter C4 gives chart for determining the modulus of rupture $F_0$ for alloy steel tubes and Chapter C6 gives a procedure of determining $F_0$ for other shapes subjected to bending.

Strength of Base Material When Structure is Heat-Treated After Welding.

Reference (1) says that for materials heat-treated after welding, the allowable stresses in the parent material near a welded joint may equal the allowable stress for the heat-treated material, in other words, no reduction for welding. However, it is good design practice to be conservative on welded joints, thus a reduction of 10 to 20 percent of the heat-treated properties is often used in calculating the tensile or bending strength in the member adjacent to the weld.

D2.6 Weld-Metal Allowable Stress.

Table D2.2 (from Ref. 1) gives the allowable weld-metal strengths for the various steels. These design allowable stresses for the weld material are based on 85 percent of the respective minimum tensile ultimate test values.

Table D2.2 (Ref. 1) Strength of Welded Joints

<table>
<thead>
<tr>
<th>Material</th>
<th>Heat treatment subsequent to welding</th>
<th>$F_{tu}$, ksi</th>
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</thead>
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<tr>
<td>Carbon and alloy steels</td>
<td>None</td>
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<tr>
<td></td>
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<td>32</td>
</tr>
<tr>
<td>Alloy steels</td>
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<td>43</td>
</tr>
<tr>
<td></td>
<td>Stress relieved</td>
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</tr>
<tr>
<td></td>
<td>Stress relieved</td>
<td>60</td>
</tr>
<tr>
<td>Steel</td>
<td>Quench and temper</td>
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</tr>
</tbody>
</table>

D2.7 Allowable Load for Welded Seams.

The allowable load on a welded seam can be calculated by the following equation:

$$P_a = F_{tu} \times (L_t) = \ldots = D2.1$$

where,

- $P_a = \text{allowable load in lbs.}$
- $F_{tu}$ and $F_{tm}$ from Table D2.2
- $L_t = \text{length of welded seams in inches}$
- $t = \text{thickness of thinnest material joined by the weld in the case of lap welds between two steel plates or between plates and tubes. (inches)}$
- $t = \text{average thickness in inches of the weld metal in the case of tube assemblies, but not to be greater than 1.25 times the thickness of the welded stock.}$

D2.8 Brazing.

Brazing as applied to aircraft work is a process of uniting steel parts by means of a copper-zinc mixture, which is applied by melting with an air-gas flame or by dipping into the molten mixture. The strength of the joint depends on the surface areas of contact and the clearance between the parts to be joined.

Although the brazing mixture may develop a shearing strength of 40000 psi, a general allowable value of 10000 psi is used in aircraft work because many factors, particularly the skill of the workman, affects the strength of a brazed joint.

The requirements of the procuring or government agencies should be noted before using brazing in aircraft work.
D2.9 Welding of Aluminum Alloys.

The heat-treatable alloys, commonly referred to as the strength alloys, such as 2015, 2024, 2219, cannot be welded with the oxyacetylene torch without destroying their mechanical properties, which are not restored even if heat-treated after welding. These alloys are generally classed as unweldable. Constant research is going on to develop aluminum alloys that have relatively high strength which can be welded without appreciable decrease of the strength properties. A recent development by the Aluminum Company of America is a new alloy designated X7008, which develops high strength after welding.

The strain-hardened alloys, namely, 1100 and 3003, are readily joined by gas welding. Either an oxyacetylene, or an oxyhydrogen flame is used and sheet thicknesses as low as .020 are successfully welded. It is common practice to use these materials in welded fuel or oil tank for aircraft.

D2.10 Illustrative Problems Involving Welding.

PROBLEM 1. Fig. D2.19 shows two plates welded together to form a lap joint. The material is alloy steel F\text{su} = 90000 psf. Find the margin of safety of the welded seams under the load of 5200 lbs. acting as shown.

![Fig. D2.19](image)

From equation D2.1,

\[
\begin{align*}
P_a &= F_{su} Lt \\
F_{su} &= \text{from Table D2.2} = 43000 \\
L &= \text{total weld seam length} = 2 \times 1 = 2 \text{ in.} \\
t &= .065 \\
P_a &= 43000 \times 2 \times .065 = 5880 \\
M.S. &= (5880/43000) - 1 = .11
\end{align*}
\]

Tensile strength of .065 plate using a reduced allowable stress due to welding of 20000 as per Table D2.1, gives

\[
P_a = 1 \times .065 \times 20000 = 5200 \text{ lb.}
\]

This is conservative since weld does not extend across the plate, thus any decrease of tensile strength properties should be less than that assumed above.

PROBLEM 2.

Fig. D2.20 shows a gusset plate inserted between the ends of two tubes of a truss. The gusset is used as a fitting to take the pull from a 3/16 diameter steel tie rod. Determine the margin of safety of connection of gusset to tubes. All material steel, F\text{su} = 35000.

Resolving the wire pull into components parallel to the tubes, we obtain \( P = 2100 \times \sin 45^\circ = 1480 \times 1.2 \) (fitting factor) = 1770 lb. The allowable weld load is governed by thinnest material or .049 of the vertical tube.

\[
\begin{align*}
P_a &= F_{su} Lt \\
&= 43000 \times 1.125 \times .049 = 2370 \text{ lb.} \\
M.S. &= (2370/1780) - 1 = .33
\end{align*}
\]

PROBLEM 3.

In general welded fittings involving plates and tubes present conditions for which it is difficult to determine the actual stress flow through the joint, thus the general procedure is to make conservative assumptions regarding the stress flow distribution and check the fitting units for these conservative assumptions. The following example illustrates this approximate procedure of strength checking a welded fitting joint.

In Fig. D2.21 the fitting plate which is welded to the three steel tubes is subjected to a tension load of 14000 lb. as shown. The fitting will be investigated for possible weakness.
**WELDED CONNECTIONS.**

![Diagram](image)

**Solution**

**Shear Strength of Clevis Pin:**

Load on pin = 14000 lb.

Double shear strength of 1/2 diameter AN clevis bolt = 2 x 14722 = 29444. M.S.=(29444/14000) = 2.1

**Bearing of Clevis Bolt on Bushing:**

Bearing stress  

\[ f_b = \frac{14000}{1.5 \times 0.375} = 59500 \text{ psi} \]

Ultimate bearing stress = 175000 psi. Thus a large margin of safety is available to take care of wear due to slight rotation or shock.

**Bearing Stress Bushing on Lug:**

\[ f_o = \frac{14000}{0.525 \times 0.375} = 59500 \text{ psi} \]

Allowable bearing stress  

\[ F_{br} = 140,000 \text{ psi} \]  
(See Chapter B2).

The result shows that bearing on lug is not at all critical.

**Shear Out Strength of Fitting Plate:**

Shear area main plate = (0.75 - 0.3125) x 1875 x 2 = 0.184

Washers = (0.525 - 0.3125) x 1875 x 2 = 0.113

Total shear out area = 0.297

\[ f_s = \frac{14000}{0.297} = 49700 \text{ psi} \]

**Friction for steel when Fau = 55000 is 55000. See Chapter B2.**

This value will be decreased to 50000 because of welding effect on material properties.

\[ M.S. = \frac{50000}{49700} - 1 = 0.1 \]

This margin of safety is conservative since shear out area is conservative.

**Tension Stress on Section Through Bolt Hole:**

Area of Section Through Hole:

- Main plate = (1.5 - 0.25) x 1875 = 164
- Washers = (1.25 - 0.75) x 1875 = 113
- Total net area = 272 sq.in.

\[ f_t = \frac{14000}{272} = 51700 \text{ psi} \]

Fr from Table B2.1, allowing full correction for welding effect, equals 80000 psi.

\[ M.S. = \frac{80000}{49700} - 1 = 0.61 \]

Tension stress on fitting plate at Section 1-1 (See Fig. D2.21).

Net area = (2.5 - 1.25) x 1875 = 235 sq.in.

The entire load of 14000 will be assumed to pass this section, which is no doubt conservative.

\[ f_t = \frac{14000}{235} = 59500, F_t = 30000 \text{ psi. o.k.} \]

**Check of Connection Between Fitting Plate and Tubes:**

It will be assumed that the horizontal component of the wire pull will be transferred to tube (A) by the weld between the tube and the fitting plate. This is a conservative assumption.

Horizontal load component = 14000 x 2/\sqrt{5} = 12500 lb.

The weld length between tube (A) and the fitting plate is 1.5 inches on the upper tube surface and 2 inches on the lower surface. To be conservative, a total weld length of 1.5 = 3 inches will be assumed acting.

The fitting plate is welded to the tube on both sides and since twice 0.065, the tube thickness is less than the plate thickness, a total weld length based on tube strength is 2 x 3 = 6 inches.

\[ F_a = F_{su} \cdot L_t = 50000 \times 5 \times 0.065 = 19500 \text{ lb.} \]

\[ M.S. = \frac{19500}{12500} - 1 = 0.56 \]

It will be assumed that the vertical component of the wire pull will be taken into tube (B) by the weld along each side of the tube.

\[ \text{Load} = 14000 \times 1/\sqrt{5} = 6250 \text{ lb.} \]
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The weld length on one side of tube is 0.625 inches long and 1 inch on the other. A total weld length of only 2 x 0.625 = 1.25 inches will be assumed which is conservative.

\[ P_{w} = 50000 \times 1.25 \times 2 \times 0.058 = 7250. \]

Thus even under the assumed conservative assumption, the weld attachment for transferring vertical component to tube (g) is more than adequate.

SPOT WELDING

D2.11 Spot Welding.

After many years of research and testing, spot welding of aluminum alloys, magnesium alloys and corrosion resistant steels has become a reliable established practice of joining many parts or units of flight vehicle structures.

The spot welding process is accomplished by clamping two or more sheets of metal between copper or copper alloy electrodes, under comparatively high pressure and causing an electric current of low voltage to flow between the electrodes for a predetermined interval. The current creates an intense heat at point "A" (See Fig. a) which melts the metal locally due to the resistance set up by the sheets. As soon as the metal is molten to the extent shown at "B", the predetermined time of current flow is completed and the sheets are forged together by the pressure on the electrodes "P". This pressure depends on the thickness of the sheet.

![Fig. a](image)

D2.12 Spot Welding of Aluminum Alloys.

In general, aluminum alloy spot welded joints should not be used in primary or critical structures without the specific approval of the military or civil aeronautical authorities. The following are a few types of structural connections in aerospace structures where spot welding should not be used.

1. Attachment of flanges to shear webs in stiffened cellular construction in wings.
2. Attachment of shear web flanges to wing skin covering.
3. Attachment of wing ribs to beam shear webs.
4. Attachment of hinges, brackets and fittings to supporting structure.
5. At joints in trussed structures.
6. At juncture points of stringers with ribs unless a stop rivet is used.
7. At ends of stiffeners or stringers unless a stop rivet is used.
8. On each side of a joggle, or wherever there is a possibility of tension load component, unless stop rivets are used.
9. In general most aluminum and aluminum alloy material combinations can be spot welded. Table D2.3 gives information on this subject.

<table>
<thead>
<tr>
<th>Table D2.3 Acceptable Material Combinations for Spot Welding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
</tr>
<tr>
<td>--------------------</td>
</tr>
<tr>
<td>5052</td>
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<tr>
<td>5052</td>
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<tr>
<td>6061</td>
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<tr>
<td>6061</td>
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<tr>
<td>3003</td>
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<td>3003</td>
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<td>Clad 2024</td>
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</tbody>
</table>

D2.13 Spot Strengths.

Design shear strength allowable for spot welds in aluminum alloys are given in Table
D2.4, for magnesium alloys in Table D2.6, and for steels in Table D2.6. The minimum edge distances from spot welds is also given in the tables.

Fig. D2.26 gives the maximum static strength of spot welded joints having the same pitch in all rows in aluminum alloys together with the maximum pitch with which these values can be obtained. For joints having larger pitches, use Table D2.4.

Fig. D2.23 gives the tensile strength single spot welds in 7075-T6 clad material.

D2.14 Reduction of Tensile Strength of Parent Metal Due to Spot Welding.

Spot welding decreases the ultimate tensile strength of the sheet material being spot welded. Fig. D2.24 gives the efficiency in tension for spot welding of aluminum alloy sheets.

Table D2.4 (Ref. 1)
Shear Strengths and Minimum Edge Distances for Bare and Clad Aluminum Alloys

<table>
<thead>
<tr>
<th>Nominal Thickness of Thinner Sheet, Inches</th>
<th>Materials &amp; Tensile Strength, ksi</th>
<th>Minimum Edge Distance, Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Above 56</td>
<td>28 to 56</td>
<td>20 to 27</td>
</tr>
<tr>
<td>Shear Strength of Sheet, Pounds</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 0.012 | 60 | 52 | 24 | 16 | 3/16 |
| 0.018 | 88 | 78 | 56 | 40 | 3/16 |
| 0.020 | 112 | 106 | 80 | 62 | 3/16 |
| 0.025 | 148 | 140 | 116 | 88 | 7/32 |
| 0.032 | 206 | 188 | 166 | 132 | 1/4 |
| 0.040 | 276 | 248 | 240 | 180 | 9/32 |
| 0.051 | 384 | 354 | 328 | 240 | 9/16 |
| 0.064 | 552 | 500 | 451 | 320 | 11/32 |
| 0.072 | 678 | 589 | 524 | 394 | 3/8 |
| 0.081 | 842 | 691 | 620 | 424 | 13/32 |
| 0.091 | 1020 | 810 | 703 | 484 | 7/16 |
| 0.102 | 1230 | 960 | 760 | 548 | 7/16 |
| 0.114 | 1465 | 1085 | 803 | 591 | 7/16 |
| 0.125 | 1698 | 1300 | 840 | 629 | 9/16 |
| 0.156 | 2400 | | | | 5/8 |

Table D2.5 (Ref. 1) Shear Strengths for Magnesium Alloys

<table>
<thead>
<tr>
<th>Nominal Thickness of Thinner Sheet, Inches</th>
<th>Magnesium Alloy QQ-M-54: QQ-M-44</th>
<th>Minimum Edge Distance, Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear Strength of Sheet, Lbs.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 0.020 | 51 | 69 | 3/16 |
| 0.025 | 71 | 97 | 3/16 |
| 0.032 | 102 | 139 | 1/4 |
| 0.040 | 137 | 185 | 9/32 |
| 0.051 | 186 | 251 | 9/16 |
| 0.064 | 242 | 328 | 11/32 |
| 0.072 | 279 | 378 | 3/8 |
| 0.081 | 320 | 434 | 13/32 |
| 0.091 | 368 | 498 | 7/16 |
| 0.102 | 432 | 586 | 7/16 |
| 0.114 | 488 | 658 | 7/16 |
| 0.125 | 544 | 735 | 9/16 |

Table D2.5 Spot-Weld Maximum Design Shear Strengths for Uncoated Steels and Nickel Alloys (Ref. 1)

<table>
<thead>
<tr>
<th>Nominal Thickness of Thinner Sheet, in.</th>
<th>Material Ultimate Tensile Strength, lb.</th>
</tr>
</thead>
<tbody>
<tr>
<td>150 ksi and above</td>
<td>90 ksi to 150 ksi</td>
</tr>
</tbody>
</table>

| 0.006 | 70 | 57 | ..... |
| 0.008 | 120 | 85 | 70 |
| 0.010 | 165 | 127 | 92 |
| 0.012 | 220 | 155 | 120 |
| 0.014 | 270 | 198 | 142 |
| 0.016 | 320 | 235 | 170 |
| 0.018 | 390 | 270 | 198 |
| 0.020 | 425 | 310 | 225 |
| 0.025 | 580 | 435 | 320 |
| 0.030 | 750 | 565 | 403 |
| 0.032 | 835 | 623 | 453 |
| 0.040 | 1168 | 850 | 850 |
| 0.042 | 1275 | 920 | 712 |
| 0.050 | 1700 | 1205 | 955 |
| 0.056 | 2039 | 1358 | 1186 |
| 0.060 | 2265 | 1558 | 1310 |
| 0.063 | 2479 | 1685 | 1405 |
| 0.071 | 3012 | 2024 | 1856 |
| 0.080 | 3540 | 2405 | 1960 |
| 0.090 | 4100 | 2810 | 2290 |
| 0.095 | 4336 | 3012 | 2476 |
| 0.100 | 4575 | 3300 | 2545 |
| 0.112 | 5088 | 3633 | 3026 |
| 0.125 | 5665 | 4052 | 3440 |

a Refers to plain carbon steels containing not more than 0.20 percent carbon and to austenitic steels. The reduction in strength of spotwelds due to the cumulative effects of temperature-stress factors is not greater than the reduction in strength of the parent metal.

Fig. D2.22 (Ref. 1) Maximum Static Strength of Spot Welded Joints in Aluminum Alloys and Corresponding Maximum Spot Weld Pitch.
**ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES**

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**Fig. D2.23 (Ref. 1) Static Strength of Typical Single Spot Welds In Tension Using Star Coupons**

**Fig. D2.24 (Ref. 1) Efficiency in Tension for Spot Welding Aluminum Alloys**

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**PROBLEMS**

(1) Fig. (a) illustrates a welded plate fitting unit fastened to a round steel tube. Both fitting plate and tube are steel $F_{tu} = 450$,000. What is the maximum design load $P$ which the fitting can be subjected to if a fitting factor of 1.2 is used. Fitting is not subjected to vibration or rotation on hinge pin.

(2) Same as Problem 1 but tube and fitting is heat-treated after welding to $F_{tu} = 150$,000.

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**REFERENCES**


(2) ANI-8, March, 1955.
CHAPTER D3
SOME IMPORTANT DETAILS IN STRUCTURAL DESIGN

BY
WILLIAM F. MCCOMBS
(DESIGN SPECIALIST - CHANCE VOUGHT CORP.)

D3.1 Introduction.

In the design and fabrication of an airplane the major components receive a thorough review and evaluation. Many of the smaller parts, however, are designed at the last minute and, not receiving so much attention, sometimes have faulty details. It is these which frequently lead to trouble in service and in tests. This chapter represents an attempt to point out some of the more common details that seem, somehow, to be overlooked from time to time. This should be of help to those involved in designing or dealing in other ways with the structural components of airplanes or of similar types of structures. With regard to specific details, many aircraft companies have standard methods of design. The reader should always consult his company's data on these, if available. In the event such are not available, the following suggested practices should be of practical help.

D3.2 Shear Clips.

There are hundreds of these in a typical military airplane. They are used in joining together both primary structural components and secondary structural parts such as equipment mounting brackets, etc. The function of the shear clip is to transfer a shear load from one part to another. It is not intended to transfer axial load or bending moment or twist, only shear.

A typical example is shown in Fig. D3.1. Here bracket, or beam, (a) is supported by beams (b) and (c). The load P is thus "beamed out" to (b) and (c), passing as a shear load through the clips into the webs of (b) and (c) as illustrated.

Shear Clips

When a significant axial load or bending moment must also be transferred, additional members must be provided or the shear clip must be replaced by a heavier fitting. This is illustrated in Fig. D3.2. Here a beam, (a), is cantilevered off of a heavy piece of structure, (b). The load, P, passes through the shear clip as a shear load from web (a) to (b). There is also the bending moment, P x l, to be transferred. Additional splice plates, "S", are provided for this purpose. They transfer the moment in the form of axial loads from the flanges of beam (a) to member (b).

Shear clips are usually seen in two forms:

(a) bent up sheet metal or extruded angles (the angle being anywhere from 0° to 180° between the legs).

(b) extruded "tees".

These shear clips are shown in their "minimum acceptable" form in Fig. D3.3. The minimum requirement is that each leg of an angle type clip must have at least 2 fasteners through it. The "top" of the tee type clip and its leg must also have at least 2 fasteners each, as shown in (e) and (f).

The load "balance" of an angle type shear clip is illustrated in Fig. D3.4. The corner edge of the clip should be assumed to carry...
only the shear being transferred, taken as 1000 lb. in this figure. The net loads on the fasteners are then as illustrated. Once the loads are known, the clip and fasteners can be checked for strength using standard methods.

Example:

Let \( a = 40^\circ, b = 1.0^\circ \)

Then \( Q = 1000 \times \frac{4}{1.6} = 400 \# \)

Resultant Rivet Load:

\( R = \sqrt{500^2 + 400^2} = 640 \# \)

Use 2-5/32" Alum. Rivets

Then clip thickness required is \( t = .022", 7500 \text{ Ale.} \)

Fig. D3.4

Fig. D3.5 illustrates what will happen if only one fastener is provided in a leg of an angle clip. The single fastener cannot (ignoring friction) balance any shear at the corner. In other words, it can receive only a shear from the web to which it is fastened. This, in turn, puts a twist, \( P \times a \), into the other leg of the clip and, hence, into the other web. This is unacceptable, of course, since a much thicker leg would be needed to carry the tension, and an undue twist would be present in the other web being joined. Of course, several fasteners, rather than just two, may be used when space allows.

Clips of type (a), (c) and (e) in Fig. D3.3 are more efficient than are types (b), (d) and (f). The latter are used when this is all that space limitations will allow. In all cases the dimension "a" should be kept as small as practical installation will allow.

An "Unacceptable" Type Shear Clip

For loads on longer leg in figure, let \( a = 4^\circ, b = 1.0^\circ \).

Twist = 1000 x .4" = 400#

This is required to balance the 1000# which is out of the plane of the longer leg. This is unacceptable.

Fig. D3.5

Another type of deficiency sometimes arises when a minimum type shear clip is being used. This is illustrated in Fig. D3.6 where it has been necessary to "joggle" one leg of the angle clip, say to fit over some locally thicker part of the member being attached. If the joggle is a significant one, say to the order of the clip's thickness or greater, it can considerably reduce the clip's rigidity and cause it to function as 2 "one rivet clips" with the adverse twist effects mentioned previously. In this case at least 2 fasteners should be provided on one side of the joggle in the joggled leg, as illustrated in Fig. D3.6(b), to maintain rigidity and proper functioning. The load should be assumed to be carried by the 2 fasteners above the joggle, similar to case (b) or (d) in Fig. D3.3. Joggles are discussed further in Art. D3.4.

D3.3 Tension Clips

These are also quite numerous in military airplanes, being used to splice relatively light tension loads from one member to another. The tension clip is a very inefficient type of splice. It has a relatively poor fatigue life, particularly, and should be used only when the load is small and other design factors prevent the use of the more efficient lap shear splice.

It is usually resorted to when some structural member such as a bulkhead web or flange or fitting cannot be efficiently "opened up" to let an axially loaded member pass through. It is also frequently used to attach cantilevered brackets to bulkheads or ribs or other structure.

Consider Fig. D3.7. Member (a) is, say, on one side of a bulkhead and is to be spliced to member (b) on the other side. There is an axial tension load to be transferred and since the bulkhead cannot be cut, a tension clip arrangement must be used as shown. Angle clips in this case are illustrated.

Fig. D3.6

Fig. D3.7
Simple tension clips are used for relatively light loads, large loads requiring a machined housing of a "boltded" type. These clips are usually seen in 3 forms.

(a) single angle - either bent up sheet metal or extrusions.

(b) double angles (back to back)

(c) clips cut from extruded tee sections

Type (c) is the strongest and stiffest for a given thickness.

To obtain maximum strength and stiffness, bolts should be used for attachment purposes. Allowable load data is given in Fig. D3.6 for the single angle clip arrangement illustrated. The method of obtaining the allowable load is also illustrated by the dashed lines in the figure.

**YIELD LOAD FOR SINGLE ANGLES**

<table>
<thead>
<tr>
<th>BOLT SPACING</th>
<th>BOLT HEAD CLEARANCE - INCHES</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8 IN. AND OVER</td>
<td>0.15 IN.</td>
</tr>
<tr>
<td>1/4 IN.</td>
<td>0.18 IN.</td>
</tr>
<tr>
<td>1/2 IN.</td>
<td>0.25 IN.</td>
</tr>
</tbody>
</table>

**THICKNESS OF ANGLE - INCHES**

<table>
<thead>
<tr>
<th>LOAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
</tr>
</tbody>
</table>

**YIELD LOAD PER BOLT - LIVE.**

**NOTES:**
1. In these tests the angles protruded at outer end beyond the r of the bolt a distance of 1/2 the bolt spacing.
2. For thick angles the bolt may be critical.
3. Values are for room temperature use only.

**Fig. D3.8**

(Ref. Vought Structures Manual)

It is noted that for larger clip thicknesses the bolt may become critical and begin to yield (the same would apply to smaller bolts than those specified for the thicknesses shown). This is because of the prying action on the bolt. Because of the prying action, the load in the bolt is always greater than the applied load. Consider Fig. D3.9(a) illustrating an angle type clip arrangement. As the applied loads, P, begin to "open-up" the clips the bolt feels an increasing tension load, Q, and the "toes" of the clip bear down.

![Fig. D3.9](image)

More on each other. From statics, taking moments about the center of pressure on the toes, Q = P x (3 + S). Thus Q is greater than P. Obviously a small enough bolt will yield or fail in tension before a thick clip will yield or fail in bending near the washer (Mclip = P x e). There is also a prying action in the tee type clip, as illustrated.

This prying action is the reason why the designer should be cautious in using rivets even for light tension loads, as is sometimes done. When rivets are used, as in mounting equipment brackets, it is best to use steel types and carefully check the prying load maintaining an ample margin of safety. In any event, riveted clips are inferior and no design data for them is given here.

Another point in using tension clips is frequently overlooked. The structure to which the clip is attached must be capable of taking the loads applied to it. These loads consist of the tension load from the bolt and the load from the toe action. Several examples are shown in Fig. D3.10. In these examples the term "unacceptable" means that the allowable loads of Fig. D3.8 are not applicable.

![Fig. D3.10](image)

Cases (b), (c), and (e) require a rearrangement of, or additional structure in, the back-up
Some aircraft companies have specific strength data and practices for the design of joggled members. This should, of course, be consulted by the designer, if available. Some companies use a 9:1 joggle length to depth ratio, others use a 3:1 ratio, or both may be used. Strength or stiffness data for one ratio should not be used blindly for another. Some of this data indicates that when, in the case of angle members, the depth of the joggle is to the order of the thickness of the joggled leg or more, the loss in strength is about equivalent to the loss of the leg outboard of the bend radius. A "rule-of-thumb" design practice is therefore suggested as follows. Assume that the net effect of the joggle, from a strength and stiffness standpoint, is equivalent to a slot cut into the joggled leg that extends inward to the bend radius tangent point. This is illustrated in Fig. D3.12.

With this assumption, the flat portion of the joggled leg will carry no axial load across the joggled area but will provide support for the curved element. The effective net section, Fig. D3.12(c), can then be checked using standard methods of analysis for whatever forces are acting on it. It is obvious that the net section shown will have little strength for carrying bending moment normal to the remaining leg. Thus, care should be taken to insure that any axial loads are introduced as near the corner as possible - which in turn means that at least two fasteners should be used on each side of the joggle.

The above approach, considering the joggle equivalent to a slot, will give the designer a much better "feel" of what he is really doing when he specifies a joggled member. The basic reason for the loss of strength and stiffness can be seen in Fig. D3.13.
The axial loads in the jogged leg being inclined to each other require a balancing load. Such a balancing load is not available except as shear in the thin leg and this results in the loss of stiffness and strength. If the symmetrical leg of a tee member were jogged there would be more stiffness than in the case of an angle, but the same approach, though more conservative, is recommended in the absence of test data.

As an example of the foregoing discussion assume that an angle member is supporting another member locally which is loaded by the forces Q as shown in Fig. D3.14. If a skin is present, as shown, part of the load can be carried across the joggle by the gusset effect of the skin. This can be approximated by using the methods of calculating inter-rivet buckling of skins discussed in another chapter. The rest of the load must be carried across by the net effective section of the angle in the jogged area.

![Fig. D3.14](image)

Thus if the total load at the joint were 2Q and the load carrying ability of the skin were R, then the net section of the angle would be subjected to a load \( P = 2Q - R \) and a bending moment \( M = 2Q \times a - R \times b \). The stress at the lower curved edge would be the sum of the compressive stresses,

\[
f_c = \frac{MP}{I \text{ Net Section}} + \frac{P}{A \text{ Net Section}}
\]

For ultimate strength, \( f_c \) could be carried up to \( f_{cu} \), conservatively, in the typical case.

In order to realize the maximum strength and stiffness, the load in the net section must be applied in the "corner". This is to prevent stresses due to bending out of the plane of the remaining leg. This requires that a minimum of 2 fasteners be provided to receive the load at the joint. The reasoning here is the same as discussed in Art. D3.2 concerning minimum type shear clips, and the fastener loads can be calculated in the same manner as discussed there.

If the load is too large for the net section to carry, then an additional member should be provided locally. Two ways of doing this are shown in Fig. D3.15.

Sometimes local requirements are such as to necessitate both legs of an angle member being jogged. In such cases it should be assumed that the angle has no significant load carrying ability at the joggle. Thus, the existence of any significant load at the joint would require an additional member and the angle should be ended just short of the joint rather than jogged up onto it.

![Fig. D3.15](image)

The suggested effective net sections of members having other types of cross-sections are shown in Fig. D3.16 where the legs indicated by dotted lines are jogged. In general if the joggle is slight, considerably less than the thickness of the joggled leg, its effect can be ignored, but proper fasteners should still be provided as discussed. The smaller the length to depth ratio used for jogging, the greater the effect of the joggle. Joggled members lose stiffness and strength when subjected to tension loads as well as when under compression (but any skin present is, of course, much more effective as a gusset than when in compression).

![Fig. D3.16](image)

D3.5 Fillers.

As the name implies, fillers are used to fill up a void. It is when they become a part of the structural load path that they need particular attention. Fillers also represent an item that is quite common in typical large or complicated metal airplane structures.

As an example consider Fig. D3.17. Here two tees, "a" and "b" carrying an axial load, \( P \), are seen to be spliced together by a pair of angles "c". Since the lower leg of "b" is thicker than that of "a", a filler is needed. This filler is part of the structural load.
D3.6 SOME IMPORTANT DETAILS IN STRUCTURAL DESIGN

Fig. D3.17

path, from "c" to "a". In this case, to realize full strength of the fasteners, the filler must be "extended" and additional fasteners provided to tie the extended portion into member "a". If this is not done, it is said to be a "floating" filler. As explained later, a floating filler, if thick enough, will cause a loss in fastener strength.

In the above example let the total load P be 8000 lbs. and assume that 2000 lbs. of this must be transferred from "c" to "a" by the two fasteners in the filler area. The part of the load put into the filler by bearing pressure can be taken as

\[ P_{\text{Filler}} = P_{\text{Fasteners}} \times \frac{t_{\text{Filler}}}{t_{\text{Filler}} + t_a} \]

\[ = 2000 \times \frac{.04}{.04 + .06} = 900 \text{ lb.} \]

Sufficient fasteners should be put in the extended part of the filler to transfer this 900 lbs. into member "a".

Thus, whenever a thick filler is inserted between two members being spliced together in shear, the filler should be assumed to be a part of one of them. The part of the total splice load it will carry can be calculated as illustrated above. The fasteners can then be considered as being in two sets. One set must have the strength to splice the total shear load from the single member to the combination "filler plus member". The other set of fasteners must have the strength (meaning in shear and in bearing) to transfer the calculated filler load to the member it is assumed to be a part of. Another example uses Fig. D3.18.

Fig. D3.18

Assume filler to be integral with the lower member, "b".

Case I.

P = 3000 lbs.

1. Rivets required to splice 3000 lbs. from "a" to combination of "b" plus filler:
   \[ \text{No. Rivets} = \frac{3000}{1500} = 2 \text{, or } 2 \]

2. Load carried by filler (to be spliced to "b"):
   \[ P_{\text{Filler}} = 3000 \times \frac{t_{\text{Filler}}}{t_{\text{Filler}} + t_b} = 3000(0.072) = 1500 \text{ lbs.} \]

3. Fasteners required to transfer P to "b"
   \[ \text{No. Fasteners} = \frac{1500}{1500} = .92 \text{ or } 1 \]

   Total Fasteners required = 2 + 1 = 3.

   Splice is adequate since 3 rivets are present. Had no filler been present, 2 fasteners would have sufficed.

Case II.

P = 5000 lbs.

Repeating the same steps as in Case I:

1. No. Rivets required = \[ \frac{5000}{1500} = 3.33 \text{, or } 3 \]

2. Load in Filler = \[ 5000 \times \frac{.072}{.072 + .072} = 2500 \text{ lbs.} \]

3. No. Rivets required = \[ \frac{2500}{1500} = 1.67 \text{ or } 2 \]

Thus the 3 rivets are required to transfer load, P, from "a" and 2 additional rivets are needed to unload the filler into member "b". The filler should be extended over "b" and 2 additional fasteners added as shown by the dotted lines in Fig. D3.18.

Admittedly, the above procedure is approximate, but it provides a quick way of evaluating the effect of the filler. When the filler thickness is less than about 15% of the fastener diameter, its presence can be ignored.

The effect of the filler is to reduce the allowable strength of the fastener. The reason for this can be seen from Fig. D3.19 where the presence of the filler causes greater prying loads and hence more tension in the fastener, along with the shear load.

Any filler in the structural load path should, of course, be made from a material compatible in stiffness with that of the
structure around it. That is, one should not use a soft aluminum filler between high-heat-treated steel parts or a phenolic or fiberglass filler between aluminum parts. The need for fillers arises not only from design considerations but frequently from manufacturing problems. In the latter cases "mis-match" between parts sometimes occurs in assembly. To prevent expensive re-work, structural fillers must be used to make the spliced area adequate. In these cases detailed attention is necessary. In the occasional instances when floating fillers cannot be avoided, the fasteners should have quoted allowable wells in excess of the shear being transferred locally, if the filler is of significant thickness. It is common practice also to use a bonding agent (glue) in addition to the fasteners in installing fillers.

Fig. D3.19

D3.6 Cut-outs in Webs or Skin Panels.

The aircraft structure is continually faced with requirements for opening up webs and panels to provide access or to let other members such as control rods, hydraulic lines, electrical wire bundles, etc., pass through. The designer or liaison engineer should be familiar with some of the various methods of providing structurally sound cut-outs.

There are several ways of providing cut-outs. Three will be mentioned here. These are:

(a) Providing suitable framing members around the cut-out.

(b) Providing a doubler or "bent" where framing as in (a) cannot be done.

(c) Providing standard round flanged holes which have published allowables as discussed in chapter on beam design.

(a) Framing Cut-Outs in Webs

As an example assume that a beam web requires a cut-out as shown in Fig. C3.20.

There are 2 ways to determine the loads in the area framed around the cut-out. The first is to assume a shear flow equal and opposite to that present with no cut-out (q₀) in the figure above) and determine the corresponding balancing loads in the framed area. Adding this load system to the original one will give the final loads and, of course, q = 0 in the cut-out panel. The other method is to use standard procedures assuming the shear to be carried in reasonable proportions on each side of the cut-out. The first method will be illustrated here.

All shear flows are shown as they act on the edge members (on the flanges and stiffeners) in this discussion.

If there were no cut-out there would be a constant shear flow, q₀, in all of the panels, as shown in Fig. D3.21a. Next a shear flow equal and opposite to that in the center panel of Fig. (a) is applied to the center panel of Fig. (b).

Fig. D3.20

Before the cut-out was made the members shown by solid lines (flanges, stiffeners, webs) are present. The members "a" and "b" are added to frame the cut-out, as shown by the broken lines.

Fig. D3.21

Self-Balancing Internal Loads (Due to Application of Equal and Opposite q₀ Assumed in the Cut-Out Panel)
Fig. (b). Since this represents a self-balancing load system, no external reactions outside of the framing areas are required. This is an important concept and the reader should think about it. The loads in the framed areas due to $q_0$ in (b) are next determined. To eliminate redundancies, it is usually assumed that the same shear flow exists in the panels above and below the cut-out. It is also assumed that shear flows are the same in the panels to the left and right of the cut-out.

a) the shear flow in the panels above and below the center panel must statically balance the force due to $q_0$, or

$$E = 0, \quad 7/3 \times q_0 = 7/3 \times q_0$$

b) the shear flows in the panels to the left and right of the center panel must also statically balance the force due to $q_0$.

$$E = 0, \quad 12 \times q_0 = 12 \times q_0$$

c) the shear flows in the corner panels must also balance the force due to the shear flow in the (any) panel between them. Considering the panels in the right hand bay

$$E = 0; \quad 1/2 \times 7 = 7/2 = 7/3 \times q_0$$

$$q_0 = \frac{7}{3}$$

d) the final shear flows are gotten by adding the values in (a) and (b) together, algebraically. Note that:

1. the shear flow in the center panel (the cut-out) is $q = q_0 - q_0 = 0$, as it should be.

2. the shear flows above and below and to the left and right of the cut-out add, giving a number greater than the original $q_0$.

3. the shear flows in the corner panels are smaller than the original value of $q_0$.

This is the way the changes always occur in the area framed about a cut-out.

e) Finally, and importantly, there are axial loads developed in all of the framing members due to the cut-out.

These will add or subtract, depending upon their directions, to any loads present before the cut-out was made (as in the case of the beam flanges). The axial loads due to the cut-out can be gotten from Fig. (b). Or the total axial loads in all of the members can be gotten from (c). These are illustrated in Fig. D3.22.

Axial Load Distribution in Upper Flange from Fig. D3.21c (a)

Axial Load Distribution in Framing Member Above Cut-Out Obtained from D3.21c (Same Result Could Be Gotten from Fig. D3.21b).

Axial Load Distribution in Stiffener Bordering Cut-Out on Left Side, from Fig. D3.21c (Same Result Obtained from D3.21b).

Once the internal loads are known, the members can be checked for strength using standard methods of stress analysis.

The cut-out could have been framed without extending the framing members into the bay on the right of the cut-out. This case is illustrated in Fig. D3.23.

Had the 7" deep cut-out been required at the bottom of bay, the framing could have been done with only one member (as could the preceding cases also) as illustrated in Fig. D3.24. This represents the minimum or adequate framing for any cut-out. That is, there must be a minimum of one redistribution bay on one side of the cut-out and at least two redistribution bays on the other side, and there must be the framing members defining the bays. These framing members will always be loaded axially.

Note that in the previous examples in Case (b) the sum of the loads on all edge members (framing members) is zero. No external loads are needed for equilibrium. This is
Always the case when a set of self-balancing shear flows are applied to a flat panel structure or to a 3 dimensional box structure with a cut-out on any side. The reader should study the examples closely. Although the method is shown only for a flat beam it is also applicable to any structure with a cut-out, such as the box beam of Article A21.3. This has actually been illustrated in Solution No. 2 of that article and the reader should review it at this time.

Sometimes framing members for a cut-out are not conveniently available as were the stiffeners and flanges of the beam used in the previous examples. In such cases they must, of course, be provided.

(b) Framing Cut-Outs With Doublers or Bents

Frequently a cut-out in the web of a beam must be so deep that it removes nearly all of the web. In this case the method previously described cannot be used. Instead the "brute-force" approach is necessary and a heavy doubler, or bent, is provided around the cut-out to carry the shear. This is illustrated in Fig. D3.25.

As shown in (a) the doubler whose thickness is yet to be determined is made to fit around the cut-out as shown. Reasonable internal radii are in the cut web and doubler at the corners to keep stresses due to curved beam bending reasonable (see Chapter C11). Attachments are provided as shown to pick up the basic shear flow in the web.

The loading imposed on the doubler is shown in (b), namely the shear flow qo. Strictly speaking, the doubler should be analyzed as a frame. With reasonable symmetry the loading in (c) can be assumed at the center of the frame. That is, one half of the total shear, qo x h/2 is resisted in the top of the frame, one half in the bottom and a pin joint (no bending moment) exists at the cut. The bending moment axial loads and shears at any section of the frame follow as a matter of statics. For example,

At A-A,

\[ M = V \times \frac{w}{2} = q_o \times \frac{h \times w}{2} \]

(there may also be a little relieving moment due to qo)

\[ F_z = V = \frac{4q_o h}{2} \]

\[ F_y = q_o \times p_o \]

The thickness of doubler required to take the loads can thus be determined using standard methods of stress analysis. The doubler should have sufficient out of plane stiffness, also, to provide simple support for the beam web, as discussed in Chapter C10. This will normally be provided by the thickness required for strength purposes.

Sometimes the nature of the cut-out is such that the frame (doubler) can be deeper at the top (or bottom). In such a case, the
Deeper parts can be assumed to carry a greater portion of the total shear, \( V \), and the lower part a smaller portion. This is illustrated in Fig. D3.25 where the cut-out does not extend as high as in Fig. D3.25 and the upper part is assumed to carry 4/6 of the total shear and the lower part only 1/6.

The cases illustrated are for shear-resistant webs. If a tension field is present the doubler must also be designed for the end-bay effects discussed in Chapter C11.

![Fig. D3.25](image)

(c) **Providing Access With Standard Round Flanged Holes or "Donut Doublers"**

Frequently a cut-out size requirement is such that a standard round flanged hole will provide the needed open space and strength. In such cases either of the following can be done when the beam is of the shear-resistant type:

1. replace the web, locally, with a panel having a standard round hole that has a 45° flange, as discussed in Chapter C10.

2. Cut the required diameter hole into the web and attach a "donut" doubler (ring) that has the same (or greater) stiffness in a direction normal to the web as does the flange of (1) above.

In either case the allowable shear can be calculated as discussed in Chapter C10. These allowables apply only to round holes, not to elliptical or rectangular holes with flanged edges, which are weaker.

Item (2) above is illustrated in Fig. D3.27. Note that the hole spacing "b" of Chapter C10 will be quite large if only one hole is present and is not near the end of the beam. The hole spacing "b" is, of course, used in determining the allowable shear if no stiffeners are present.

If a beaded web is cut, a panel can be inserted locally, containing a hole with a beaded flange, as discussed in Chapter C10 and these allowables used conservatively.

If the beam is of the tension field type, method (c) does not apply. In such cases, a heavier frame, as in method (b), that can take the tension field effects should be used.

D3.7 Special Cases of Beam Design.

There are several cases involving beam design not discussed in Chapters C10 and C11, since these chapters are intended primarily to present fundamental design principles. The designer will encounter in practice, however, the following situations which include ordinary straight beams and beams in the form of bulkheads or frames:

a) Curved beams
b) Flanges with local changes in direction ("bent" flanges)
c) Flanges subject to "normal" loads which tend to bend them.
d) Flanges requiring "stabilization" against buckling out of the plane of the web.

(a) Curved Beams

Consider a portion of a curved beam as shown in Fig. D3.28. The curvature is taken as constant, as is the bending moment, for purposes of illustration.

When the outer (upper flange in (a)) is in tension and the inner flange in compression there will be a collapsing (compression) loading on the web as shown in (b). This is because a "hoop" loading is required to keep the flanges from moving inward, towards each other. For a constant load, \( P \), in the flange and from simple statics:

\[ 2P = \sigma D \]

\[ w = \frac{P}{D} \]
ANALYSIS AND DESIGN OF FLIGHT VEHICLE STRUCTURES

Fig. D3.28

w is, of course, larger at the inner flange, since R is smaller, than at the outer flange.

This compressive loading tends to collapse the web. If the web is also carrying shear, as is the usual practice, an interaction formula which includes compression and shear stresses should be used to predict buckling. This is the reason that the allowable stresses quoted for flanged panels and panels with flanged holes in Chapter C10 do not apply. These allowable stresses were for straight webs with no normal loads, as are present in curved beams.

If the curvature is enough to cause significant compression in the web, stiffeners should be provided to take this load. This can be done as shown in Fig. D3.29.

Fig. D3.29

internal load analysis. Then the stiffener loads will be,

\[ P_{st1} = 2P_0 \tan \theta \]

\[ P_{st0} = 2P_1 \tan \theta \]

Any difference in stiffener and loads will be shared into the web through the attaching fasteners.

The flanges should be designed for the axial loads, \( P \), in them. Some allowance for bending moment, \( M \), (Fig. D3.30) can be made by taking \( M \) as \( P \times \epsilon \times 1/2 \) at the center and \( -P \times \epsilon \times 1/2 \) at the stiffener junctions. These secondary effects are sometimes ignored when obviously small, in practice.

Fig. D3.30

"\( \epsilon \)" is the eccentricity between stiffener junctions due to the flanges being curved rather than straight. The same applies to the inner flange, except that it is in compression in this discussion.

When the outer flange is in compression and the inner one in tension, all of the loadings are reversed, as in Fig. D3.28(c) and (d). The web is then subjected to tension loads and there is no collapsing problem. No stiffeners are required, other than for the "normal" reasons. However, in practice, most loadings are reversible to some degree, and some compression on the webs will then, of course, be present.

(b) Flanges With Local Changes in Direction (Bent Flanges).

This is a special case of (a) above. Instead of a general curvature the flange has a local change in direction as in Fig. D3.31.

Fig. D3.31

The outstanding legs of the flanges will move upwards unless a stiffener provides the load required to hold them down. This can be calculated from statics at the joint. The
The loading shown will produce an upward load on the bolts, putting tension into the leg of the tee and the stiffener angle. A reversed loading would push down on the tee or column, producing compression. The above tee and angle combination could, of course, be replaced by a single machined fitting. The important thing is that an adequate stiffener, attached to the outstanding legs, be present if all of the outstanding leg is to function effectively.

(c) Flanges Subjected to Normal Loads Tending to Bend Them.

Frequently, the flanges of a beam are subjected to loads which pull outward or push inward. It is important in each case that the flange be "backed up" by a stiffener. The effect is the same as in the case just discussed, (b), involving kick loads due to bent flanges. A similar stiffener arrangement can be used.

(d) Flanges Requiring Stabilization Against Out-Of-Plane Buckling.

In the cases of flange design discussed in Chapters C10 and C11, it was assumed that any flange in compression was stabilized, or prevented from buckling as a column, by some supporting member. This member was usually shown there as a flange or a skin to which the flange was attached. Thus the flange could not buckle since it was restrained in one plane by the beam web and in a plane normal to this by some skin.

Occasionally, however, this out-of-plane (normal to the web) supporting member is not inherently present and must be provided. A typical example is the inner flange of a fuselage bulkhead or frame (the outer flange is stabilized, usually, by the outer skin). The inner flange will usually be subject to a compression load over much of its length for some design condition. Since the flange will have little stiffness of its own, its L/P will be small and it will buckle at very low compressive stress levels as a long column. It is then necessary to provide supporting members, as shown in Fig. D3.33, to reduce the unsupported length and bring the buckling stress up to an efficiently high level, nearer to the local crippling strength of the flange.

The supporting member will not be subject to any appreciable load but it must have sufficient stiffness to prevent column buckling. The required stiffness criteria will not be discussed here but the reader should consult Ref. (1) or similar textbooks to obtain such criteria. Since the supporting members are sized by stiffness rather than strength requirements they can usually consist of light tubes or intercostals. Their weight is usually less than would result from beefing-up the bulkhead flanges for out-of-plane strength. The stiffness of the supporting member must, of course, include the end fitting attaching it to the flange.

D3.8 Structural Skin Panel Details.

The general principles involving the design of structural skin or floor panels are covered in Chapters C10 and C11. Chapter C11 concerning buckling panels, in particular, should be thoroughly understood by the designer. In addition to this information, several details of design are presented below.

a) Rectangular holes
b) Recessed panels
c) Installation of long axial members on panels
d) Spot welding sheet metal doublers
e) Tension skin splices
(a) Rectangular Holes.

It is frequently necessary to cut rectangular holes into load-carrying panels to provide doors or to mount equipment. These holes must, of course, be framed as discussed in Art. D3.6. Equally important from a fatigue standpoint is that the internal corner radii not be too small or cracks will eventually start there. As an arbitrary design requirement it is suggested that the corner radii be maintained at \( R = 0.10 \) inch or larger with the normally obtained finish. In those cases where a smaller radius is absolutely necessary, it is suggested that the lower limit of \( R \) be maintained, and that an f-40 finish be specified around the edge of the panel at the corner.

(b) Recessed Panels.

It is sometimes necessary to recess a structural panel locally in order to mount equipment, as shown in Fig. D3.34.

![Fig. D3.34](image)

The recessed panel can continue to carry its shear load but there will be out-of-plane "kick" loads. These are resisted by the framing members "a" and "b" as shown in the figure and carried over to some beam or frame that can redistribute them into the main side panels. If the recess must be very deep and its size is not too large, it may be better to omit the panel and simply frame the resulting hole. This is more likely to be the case if highly buckling skins are involved. The recessed panel cannot, of course, carry tension stresses, only shear. The hole in the basic skin at the recess must not have sharp corners, as discussed in (a) above.

(c) Installation of Axial Members on Skin Panels.

Whenever a local axial member, meaning one lying in the direction of the main bending stresses of the overall structure (i.e. fore-and-aft in a fuselage), is installed, care should be observed. This member will tend to become "effective". That is, it will develop axial stresses as it strains along with the skin, particularly in tension. Most of the load picked up will enter through the fasteners near the ends of the members. The member tends to strain the same amount as the panel it is attached to though it never is as much due to the flexibility of the fasteners. Thus the total load will be \( P = f_{\text{skin}} x \) member as an upper limit, but only 60% to 90% of the skin axial stress is normally developed in the member. The larger the member and the greater its length the more axial load it will develop. Most of this load can be considered to enter through the outer (end) 20% of the fasteners at each end. High loads and relatively large bearing stresses will thus be present here. When these are present in the skin panel along with high skin tension stresses, the fatigue life of the skin panel will suffer. Therefore to keep these induced fastener loads lower, axial members (other than primary structure) should be kept as short as is practical. If their area is large then their ends should be tapered. There are methods for evaluating these effects more specifically, but they are beyond the scope of this article. The possible deleterious effect should be anticipated as it can sometimes cause cracks in panels.

Whenever a member is installed on a buckling (diagonal tension) shear panel it must be strong enough to prevent the possibility of "forced crippling" failure due to the action of the buckling skin. The reader should consult Chapter CII in this respect for design criteria.

(d) Spot Welded Doubler.

Frequently some bay or area of a skin or floor panel must be made thicker than other bays because of higher local shear flow. For example, a skin of .040 thickness might have to be made .064 in a bay because of higher loads as shown in Fig. D3.35.

![Fig. D3.35](image)

One common way of achieving this, for large panels, is to spot weld (or sometimes to bond) an .025 doubler to the .040 skin. The spot welds should be put in at a close spacing to make the combination act as unit rather than as two separate skins, which would
be weaker. It is considered good practice to use rivets along the edge of the doubler, particularly if the panel is of a tension field design with its accompanying buckles.

(e) Tension Skin Splices.

Skin splices should be kept to a minimum number, but they are common in aircraft structures. The splices of major structural skins should be given due consideration since they always contain stress concentrations and limit the fatigue life.

There are two factors present in tension splices which reduce the fatigue life:

a) There is the basic tension stress in the skin being spliced and the stress concentration due to the holes.

b) There is also the bearing stress in the holes, due to the fastener splice loads, which worsens the situation. That is, the combination of tension and high bearing stresses in the skins is worse than tension stresses only.

To keep the fatigue life as large as possible, the following practices should be observed:

a) When more than one row of fasteners per side is required, as is the usual case:

1. Do not "stagger" the fastener pattern but keep the holes in line, as shown in Fig. D3.36. This gives a lower stress concentration than staggered holes. It is an interesting fact that two or more holes in line in the direction of the load gives a lower stress concentration factor than does a single hole.

2. Avoid using more than 3 rows of attachments per side.

b) When possible use a double shear splice (a splice plate on each side of the skins). This is frequently seen in centerline splices of wing skins. When the skins are machined or chem-milled they can be left thicker at the splice to reduce the tension and bearing stresses locally, as shown in Fig. D3.37.

c) Maintain a fastener spacing in each row and between rows of approximately four times the fastener diameter (the rows should be kept as close together as is practicable).

![Fig. D3.36](image)

**Good Practice (Holes in Line)**

![Fig. D3.37](image)

**Splice Bolts**

**Cross-Section of a Double Shear Tension Skin Splice With Skins Machined Thicker at the Splice Area. (Angle Shown Due to Wing Dihedral and Thickness Taper.)**

D3.9 Additional Important Structural Details.

The following list of details is presented with a minimum amount of comment as representative of "good practice". In addition to the following list, the details listed in Chapter 829 "Fatigue Analysis and Fail Safe Design" should be observed.

1. Avoid mixing hole-filling and non-hole-filling fasteners in the same pattern (i.e. aluminum and hi-Shear type rivets or steel bolts). When this cannot be avoided, ream the holes for the non-hole-filling fasteners to insure their picking up load better than a plain drilled hole, with its "slop", would produce.

2. When using less than four non-hole-filling fasteners in a pattern, use reamed holes to insure a better distribution of load among the fasteners.

3. When using fasteners in thin sheets where the value of D/t (fastener diameter to sheet thickness is greater than 5.5) determine allowables from tests or use
conservative extrapolations. The thin sheet tends to "buckle" around the hole at relatively low bearing stresses.

4. Do not use hi-shear type fasteners in joints where the bottom skin is dimpled.

5. Maintain an arbitrary margin of safety of .15 in shear joints for fastener patterns to allow for uneven distribution of loads.

6. Do not use spot welds to attach buckling skins to their supporting frames unless a "one-shot" structure, such as a missile, is involved. Even in these cases do not use spot welds on either side at the joggled area of a joggled member; use rivets at the joggle.

7. Do not use a "long string" of fasteners in a splice. In such cases the end fasteners will load up first and yield early. Three, or at most 4, fasteners per side is the upper limit unless a carefully tapered, thoroughly analyzed splice is used. Such a design can be fashioned but is beyond the scope of this article.

8. Carefully insure against feather edges in all fitting design. Re-entrant surface intersections must have their edges rounded or else fatigue cracks will inevitably begin in such places. Any angle between surfaces less than 70° can be considered a feather-edge. These commonly occur in design if not watched, in drilling and other machining operation call-outs.

9. The permanent buckling data presented in Chapter A11 applies to structures similar to fuselages where the sub-structure rings are closely spaced as compared to the radius of curvature of the skin. That is, the ratio of skin support spacing to radius of curvature is small, considerably less than one. When the spacing of the sub-structure becomes larger, permanent buckling should be considered to occur at approximately the same shear stress that produces initial buckling; that is when the above ratio approaches unity.

10. Avoid the use of "open section" members when torsion is present. Open section members are extremely flexible compared to closed sections as can be seen from Chapter A6. For a given torsional stiffness, an open section member will be far heavier than a closed one. In the example below, member "b" is 65 times as stiff as member "a" for pure torsion applied at each end (see also the example in Chapter A6). For even a closed section to operate efficiently the torsion should be able of being distributed into all sides of the closed section. This may require a "bulkhead" type end fitting, or a much thicker section locally where the torsion is put in. The action is similar to a fuselage bulkhead distributing a twist loading into the skins of the fuselage or a wing rib distributing an applied twist into the skins and spar webs.

11. Compressive buckling does not necessarily mean failure. It means failure only if there is no other member to keep taking additional load. Shear buckling simply indicates that additional load must be carried as diagonal tension as discussed in Chapter C11. Of course, the members supporting the web must be able to withstand the ensuing so-called "secondary" effects.

When the compression skins of a fuselage or wing buckle they will carry no additional compression load but the stringers and flanges are still capable of this, as discussed in Chapter A19. The buckled skins can, however, carry additional shear load through tension field action. Thus to achieve a light efficient design the designer should have a thorough understanding of the factors involved after members have buckled as covered in other chapters of this book.

12. Probably the most single important item regarding detail structural design is the matter of equilibrium. If the designer will show the load equilibrium for every part of his assembly, most errors will be prevented. The majority of all structural strength problems occur simply because the laws of statics have not been observed, and it is usually in the smaller detail parts that the time is not taken to do this. It takes a considerable amount of experience to safely substitute the "eyeball" for the slide rule and data book. The beginning detail designer and many others who have been at it for a while simply do not have this experience
in all areas of structural design. The only safe course in such circumstances is always to show the member loads in static balance, for every part of a structural assembly. When this is done it will also give the designer a better feel as to how the structure is actually deflecting under load. This can be of significant help in anticipating problems where members are joined together and therefore must push, pull or pry on each other when loaded.

REFERENCES: