Sub Module 4.4

Measurement of Force or Acceleration Torque and Power

Introduction:

In many mechanical engineering applications the quantities mentioned above need to be measured. Some of these applications are listed below:

- Force/Stress measurement is important in many engineering applications such as
  - Weighing of an object
  - Dynamics of vehicles
  - Control applications such as deployment of air bag in a vehicle
  - Study of behavior of materials under different types of loads
  - Vibration studies
  - Seismology or monitoring of earthquakes
- Torque measurement
  - Measurement of brake power of an engine
  - Measurement of torque produced by an electric motor
  - Studies on a structural member under torsion
- Power measurement
  - Measurement of brake horse power of an engine
  - Measurement of power produced by an electric generator

As is apparent from the above the measurement of force, torque and power are involved in dynamic systems and hence cover a very wide range of mechanical engineering applications such as power plants, engines, and transport vehicles and so on. Other areas where these quantities are involved are in biological applications, sports medicine, ergonomics and mechanical property measurements of engineering materials. Since the list is very long we cover some of the important applications only in what follows.

A typical example that involves the measurement of torque and power as well as other parameters is shown in Figure 47. The reader is encouraged to study this figure carefully and make a list of all the instruments that are involved in this study. The student will realize that many of the instruments have been already dealt with in the earlier modules but some of them need attention in what follows.

1. Force Measurement

There are many methods of measurement of a force. Some of these are given below:

i. Force may be measured by mechanical balancing using simple elements such as the lever
   a. A platform balance is an example – of course mass is the measured quantity since acceleration is equal to the local acceleration due to gravity
ii. Simplest method is to use a transducer that transforms force to displacement
   a. Example: Spring element
   b. Spring element may be an actual spring or an elastic member that undergoes a strain
      - Strain is measured using a strain gage that was discussed during our discussion on pressure measurement
iii. Force measurement by converting it to hydraulic pressure in a piston cylinder device
   a. The pressure itself is measured using a pressure transducer

iv. Force measurement using a piezoelectric transducer

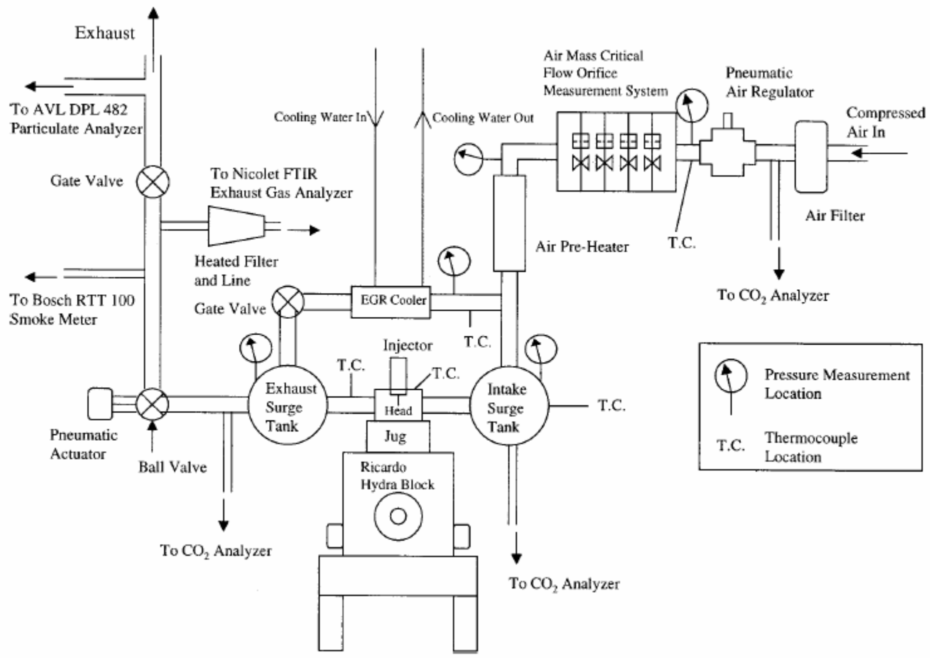


Fig. 1 Test engine cell layout

168 / Vol. 123, JANUARY 2001 Transactions of the ASME

Figure 46 Typical layout used in engine studies

(P. J. Tennison and R. Reitz, An experimental investigation of the effects of common-rail injection system parameters on emissions and performance in a high speed direct injection diesel engine, ASME Journal of Engineering for Gas Turbines and Power, Vol. 123, pp. 167-174, January 2001)

i) Platform balance

The platform balance is basically a weighing machine that uses the acceleration due to gravity to provide forces and uses levers to convert these in to moments that are balanced to ascertain the weight of an unknown sample of material. The working principle of a platform balance may be understood by looking at the cross sectional skeletal view of the balance shown in Figure 47.

The weight W to be measured may be placed anywhere on the platform. The knife edges on which the platform rests share this load as W_1 and W_2 as shown. Let T be the force transmitted by the vertical link as shown in the figure. There are essentially four levers and the appropriate Moment equations are given below.

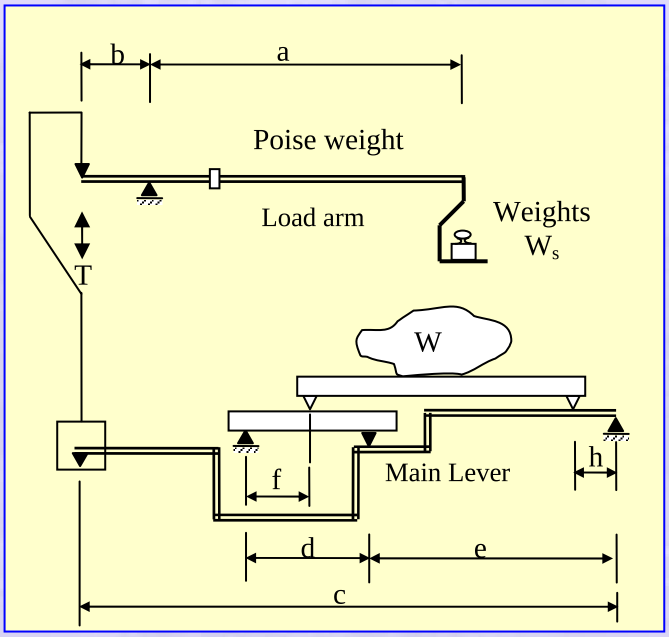


Figure 47 Sectional skeletal view of a platform balance

1) Consider the horizontal load arm. Taking moments about the fixed fulcrum, we have

$$Tb = W_s a$$

2) For the main lever we balance the moments at the fixed fulcrum to get

$$Tc = W_2 h + W_1 \frac{f}{d} e \text{ or } T = \frac{e}{c}\left[W_2 \frac{h}{e} + W_1 \frac{f}{d}\right]$$

3) The ratio $\frac{h}{e}$ is made equal to $\frac{f}{D}$ so that the above becomes

$$T = \frac{h}{c}\left(W_1 + W_2\right) = \frac{h}{c}W$$

4) Thus T the force transmitted through the vertical link is independent of where the load is placed on the platform. From the equations in 1 and 3 we get

$$T = \frac{a}{b}W_s = \frac{h}{c}W \ or \ W = \frac{ac}{bh}W_s \qquad \textbf{(41)}$$

Thus the gage factor for the platform balance is $G = \frac{ac}{bh}$ such that $W = GW_s$. In practice the load arm floats between two stops and the weighing is done by making the arm stay between the two stops indicated by a mark. The main weights are added into the pan (usually hooked on to the end of the arm) and small poise weight is moved along the arm to make fine adjustments. Obviously the poise weight W_p is moved by a unit distance along the arm it is equivalent to and extra weight of $W_s' = \frac{W_p}{a}$ added in the pan. The unit of distance on the arm on which the poise weight slides is marked in this unit!

ii) Force to displacement conversion:

A spring balance is an example where a force may be converted to a displacement based on the spring constant. For a spring element (it need not actually be a spring in the form of a coil of wire) the relationship between force *F* and displacement *x* is linear and given by

$$F = K\,x \qquad \textbf{(42)}$$

where K is the spring constant. Simplest device of this type is in fact the spring balance whose schematic is shown in Figure 48.

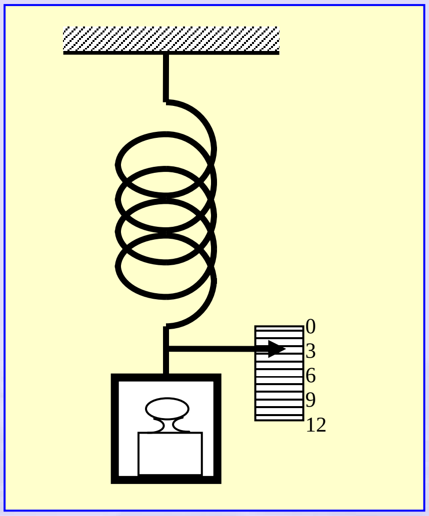


Figure 48 Schematic of a spring balance

The spring is fixed at one end and at the other end hangs a pan. The object to be weighed is placed in the pan and the position of the needle along the graduated scale gives the weight of the object. For a coiled spring like the one shown in the illustration, the spring constant is given by

$$K = \frac{E_s D_w^4}{8 D_m^3 N} \tag{43}$$

In this equation E_s is the shear modulus of the material of the spring, D_w is the diameter of the wire from which the spring is wound, D_m is the mean diameter of the coil and N is the number of coils in the spring.

An elastic element may be used to convert a force to a displacement. Any elastic material follows Hooke's law within its elastic limit and hence is a potential spring element. Several examples are given in Figure 49 along with appropriate expressions for the applicable spring constants. Spring constants involve E, the Young's modulus of the material of the element, the geometric parameters indicated in the figure. In case of an element that undergoes bending the moment of inertia of the cross section is the appropriate geometric parameter. The expressions for spring constant are easily derived and are available in any book on strength of materials.

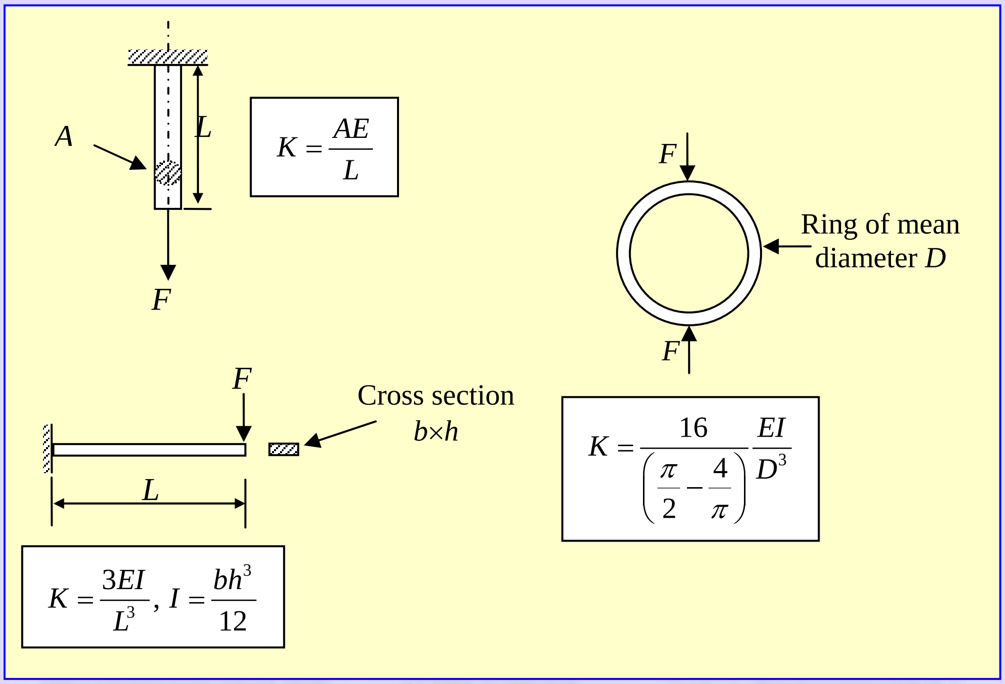


Figure 49 Several configurations for measuring force with the appropriate spring constants: (a) Rod in tension or compression (b) Cantilever beam (c) Thin ring

An example is presented below to get an idea about typical numbers that characterize force transducers.

Example 14

A cantilever beam made of spring steel (Young's modulus 200 GPa) 25 mm long has a width of 2 mm and thickness of 0.8 mm. Determine the spring constant. If all the lengths are subject to measurement uncertainties of 0.5% determine the percent uncertainty in the estimated spring constant. What is the force if the deflection of the free end of the cantilever beam under a force acting there is 3 mm? What is the uncertainty in the estimated force if the deflection itself is measured with an uncertainty of 0.5%?

Sketch below describes the situation. The cantilever beam will bend as shown in the figure.

The given data is written down as:

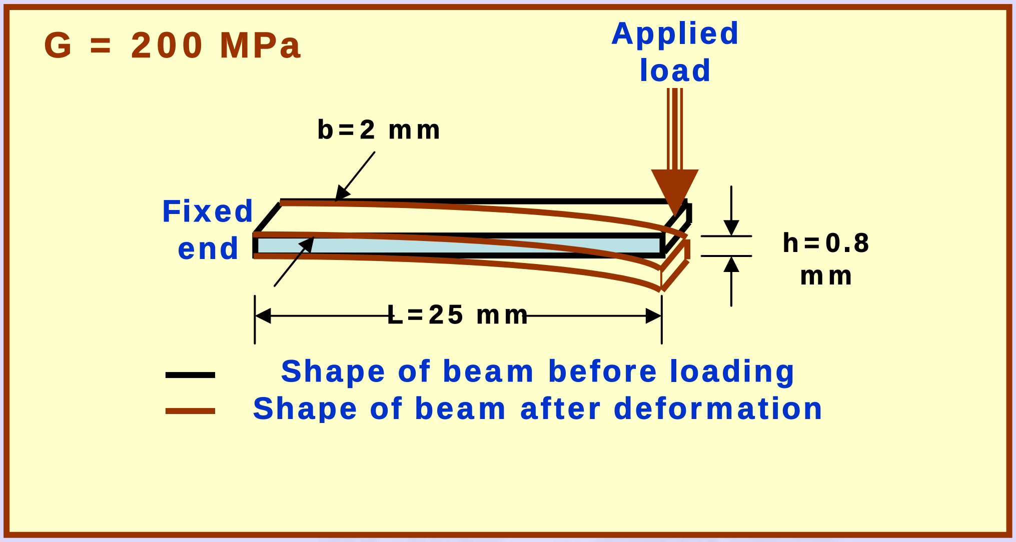
$$E = 200\ GPa = 200\times 10^{9}\ Pa$$
$$\text{Width } b = 2\ mm = 0.002\ m$$
$$\text{Thickness } t = 0.8\ mm = 0.0008\ m$$
$$\text{Length of cantilever beam } L = 25\ mm = 0.025\ m$$

The moment of inertia is calculated using the well known formula

$$I = \frac{bt^3}{12} = \frac{0.002\times 0.0008^3}{12} = 8.5333\times 10^{-14}\, m^4$$

Using the formula for the spring constant given in Figure 49, we have

$$K = \frac{3EI}{L^3} = \frac{3\times 200\times 10^{9}\times 8.5333\times 10^{-14}}{0.025^3} = 3277\ N/m$$



Since all the relevant formulae involve products of parameters raised to some powers, logarithmic differentiation will yield results directly in percentages.

We have:

$$\frac{\Delta I}{I}\% = \sqrt{\left(\frac{\Delta b}{b}\%\right)^2 + \left(3\frac{\Delta t}{t}\%\right)^2} = \sqrt{0.5^2 + (3\times 0.5)^2} = 1.58\%$$

Hence

$$\frac{\Delta K}{K}\% = \sqrt{\left(\frac{\Delta I}{I}\%\right)^2 + \left(3\frac{\Delta L}{L}\%\right)^2} = \sqrt{1.58^2 + (3\times 0.5)^2} = 2.18\%$$

Thus the spring constant may be specified as

$$K = 3277 \pm \frac{2.18}{100}\times 3277 = 3277 \pm 71.4\ N/m$$

For the given deflection under the load of $y = 3\ mm = 0.003\ m$ *the nominal value of the force may be calculated as* $F = Ky = 3277 \times 0.003 = 9.83\ N$. *Further the error may be calculated as*

$$\frac{\Delta F}{F}\% = \pm\sqrt{\left(\frac{\Delta K}{K}\%\right)^2 + \left(\frac{\Delta y}{y}\%\right)^2} = \pm\sqrt{(2.18)^2 + (0.5)^2} = \pm 2.24\%$$

The estimated force may then be specified as

$$F = 9.83 \pm \frac{2.24}{100} \times 9.83 = 9.83 \pm 0.22\ N$$

Note that the deflection may easily be measured with a vernier scale held against the free end of the beam element.

In the case considered in Example 14 the force may be inferred from the displacement measured at the free end where the force is also applied. Alternately one may measure the strain at a suitable location on the beam which itself is related to the applied force. The advantage of this method is that the strain to be measured is converted to an electrical signal which may be recorded and manipulated easily using suitable electronic circuits.

iii) Conversion of force to hydraulic pressure:

While discussing pressure measurement we have discussed a dead weight tester that essentially consisted of a piton cylinder arrangement in which the pressure was converted to hydraulic pressure. It is immediately apparent that this arrangement may be used for measuring a force. If the piston area is accurately known, the pressure in the hydraulic liquid developed by the force acting on the piston may be measured by a pressure transducer. This pressure when multiplied by the piston area gives the force. The pressure may be measured by using several transducers that were discussed earlier.

iv) Piezoelectric force transducer:

A piezoelectric material develops an electrical output when it is compressed by the application of a force. This signal is proportional to the force acting on the material. We shall discuss more fully this later.

2) Measurement of acceleration:

Acceleration measurement is closely related to the measurement of force. Effect of acceleration on a mass is to give rise to a force. This force is directly proportional to the mass, which if known, will give the acceleration when the force is divided by it. Force itself may be measured by the various methods that have discussed above.

Preliminary ideas:

Consider a spring mass system as shown in Figure 50. We shall assume that there is no damping. The spring is attached to a table (or an object whose acceleration is to be measured) as shown with a mass *M* attached to the other end and sitting on the table. When the table is stationary there is no force in the spring and it is in the undeflected position. When the table moves to the right under a steady acceleration *a,* the mass tends to extend the spring because of its inertia. If the initial location of the mass is given by x_i, the location of the mass on the table when the acceleration is applied is *x*. If the spring constant is *K*, then the tension force in the spring is $F = K\left(x - x_i\right)$. The same force is acting on the mass also. Hence the acceleration is given by

$$a = \frac{F}{M} = \frac{K\left(x - x_i\right)}{M} \qquad (44)$$

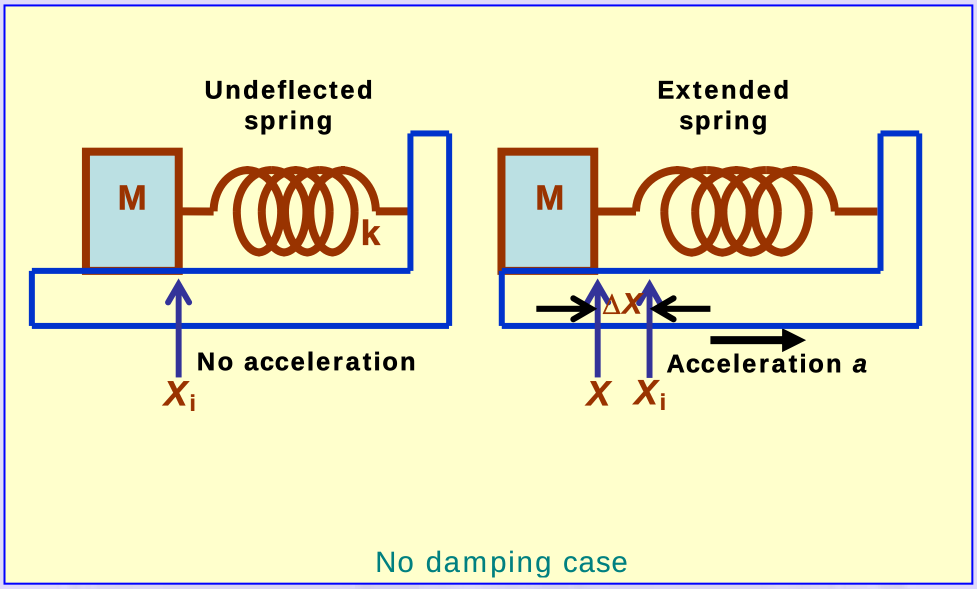


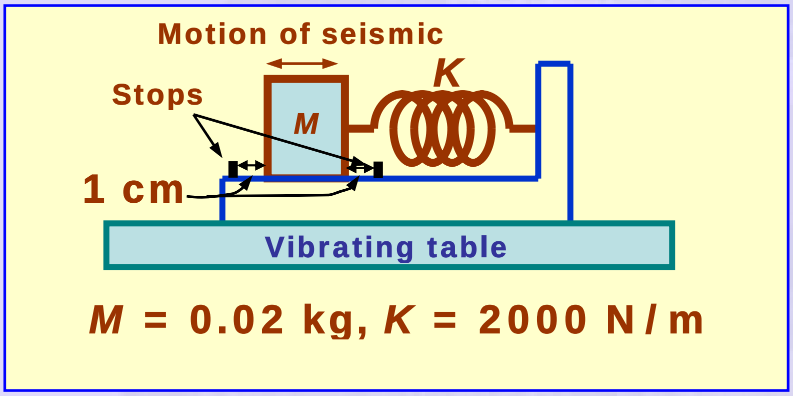
Figure 50 Spring mass system under the influence of an external force

This is, of course, a simplistic approach since it is difficult to achieve the zero damping condition. In practice the acceleration may not also be constant. In deed we may want to measure acceleration during periodic oscillations or vibrations of a system. We consider this later on.

Example 14

An accelerometer has a seismic mass of M = 0.02 kg and a spring of spring constant equal to K = 2000 N/m. Maximum mass displacement is ± 1 cm. What is the maximum acceleration that may be measured? What is the natural frequency of the accelerometer?

The figure explains the concept in this case. Stops are provided so as not to damage the spring during operation by excessive strain.



Thus the maximum acceleration that may be measured corresponds to the maximum allowed displacement of $\Delta X = \pm 1 cm = \pm 0.01 m$*. Thus, using Equation 44,*

$$a_{max} = \frac{k}{M}\Delta X = \pm\frac{2000}{0.02}\times 0.01 = \pm 1000\ m/s^2$$

The corresponding acceleration in terms of standard g is given by

$$a = \pm\frac{1000}{9.8}g = 102g$$

The natural frequency of the accelerometer is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{2000}{0.02}} = 50.33\ Hz$$

Characteristics of a spring – mass – damper system:

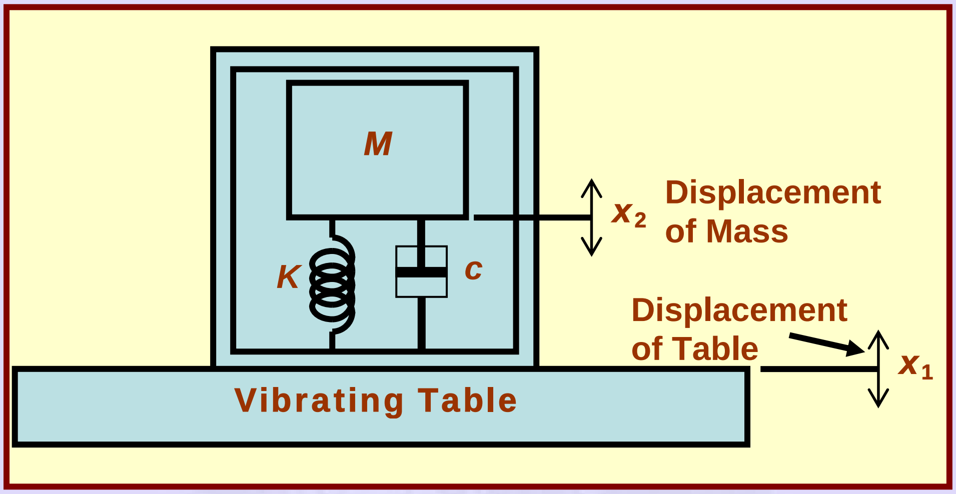


Figure 51 Schematic of a vibration or acceleration measuring system

Consider the dynamics of the system shown in Figure 51. In vibration measurement the vibrating table executes vibrations in the vertical direction and may be represented by a complex wave form. It is however possible to represent it as a Fourier series involving vibrations at a series of frequencies. Let one such component be represented by the input $x_1 = x_0 \cos \omega_1 t$. Here *x* stands for the displacement, *t* stands for time and represents the circular frequency. Force experienced by the mass due to its acceleration (as a response to the input) is $M\frac{d^2x_2}{dt^2}$. Due to the displacement of the spring the mass experiences a force given by $K\left(x_1 - x_2\right)$. The force of damping (linear case – damping is proportional to velocity) $c\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$. For dynamic equilibrium of the system, we have

$$M\frac{d^2x_2}{dt^2} = c\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K\left(x_1 - x_2\right) \qquad \textbf{(45)}$$

Dividing through by *M* and rearranging we get

$$\frac{d^2x_2}{dt^2}+\frac{c}{M}\frac{dx_2}{dt}+\frac{K}{M}x_2=\frac{c}{M}\frac{dx_1}{dt}+\frac{K}{M}x_1 \quad \textbf{(46)}$$

We now substitute the input on the right hand side to get

$$\frac{d^2x_2}{dt^2}+\frac{c}{M}\frac{dx_2}{dt}+\frac{K}{M}x_2=x_0\left(\frac{K}{M}\cos\omega_1 t-\frac{c}{M}\omega_1\sin\omega_1 t\right) \quad \textbf{(47)}$$

This is a second order ordinary differential equation that is reminiscent of the equation we encountered while discussing the transient response of a U tube manometer. The natural frequency of the system is $\omega_n=\sqrt{\frac{K}{M}}$ and the critical damping coefficient is $c_c=2\sqrt{MK}$. The solution to this equation may be worked out easily to get

$$(x_2-x_1)=\underbrace{e^{-\frac{c}{2M}t}\left[A\cos\omega t+B\sin\omega t\right]}_{\text{Damped transient response}}+\underbrace{\frac{M\omega_1^2x_0\cos(\omega_1 t-\phi)}{\sqrt{\left(K-M\omega_1^2\right)^2+\left(c\omega_1\right)^2}}}_{\text{Steady state response}} \quad \textbf{(48)}$$

In the above equation $\omega=\sqrt{\frac{K}{M}-\left(\frac{c}{2M}\right)^2}$ for $\frac{c}{c_c}<1$ and phase lag $\phi=\tan^{-1}\left(\frac{c\omega_1}{K-M\omega_1^2}\right)$. The steady sate response survives for large times by which time the damped transient response would have died down. The transient response also depends on the initial conditions that determine the constants *A* and *B*. We may recast the steady state response and the phase lag as

$$\frac{x_2-x_1}{x_0}=\frac{\left(\omega_1/\omega_n\right)^2}{\sqrt{\left[1-\left(\omega_1/\omega_n\right)^2\right]^2+\left(2c\omega_1/c_c\omega_n\right)^2}}$$

$$\phi=\tan^{-1}\left[\frac{2c\omega_1/c_c\omega_n}{1-\left(\omega_1/\omega_n\right)^2}\right] \quad \textbf{(49)}$$

The solution presented above is the basic theoretical framework on which vibration measuring devices are built. In case we want to measure the acceleration, the input to be followed is the second derivative with respect to time of the displacement given by

$$a(t)=\frac{d^2x_1}{dt^2}=\frac{d^2(x_0\cos\omega_1 t)}{dt^2}=-x_0\omega_1^2\cos\omega_1 t \qquad (50)$$

The amplitude of acceleration is hence equal to $x_0\omega_1^2$ and hence we may rearrange the steady state response as

$$\frac{x_2-x_1}{x_0\omega_1^2}=\frac{1}{\omega_n^2\sqrt{\left[1-(\omega_1/\omega_n)^2\right]^2+(2c\omega_1/c_c\omega_n)^2}} \qquad (51)$$

The above is nothing but the acceleration response of the system. Suitable plots help us making suitable conclusions.

Figure 52 shows the frequency response of a second order system. Note that the ordinate is non-dimensional response normalized with the input amplitude as the normalizing factor. The frequency ratio is used along the abscissa. The subscript 1 on the input frequency has been dropped for convenience.

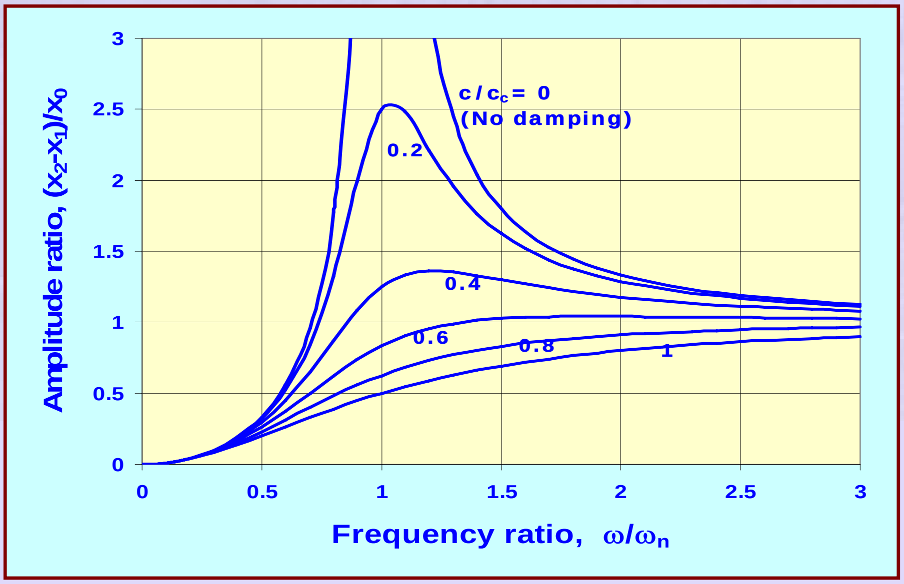


Figure 52 Steady state response of a second order system to periodic input

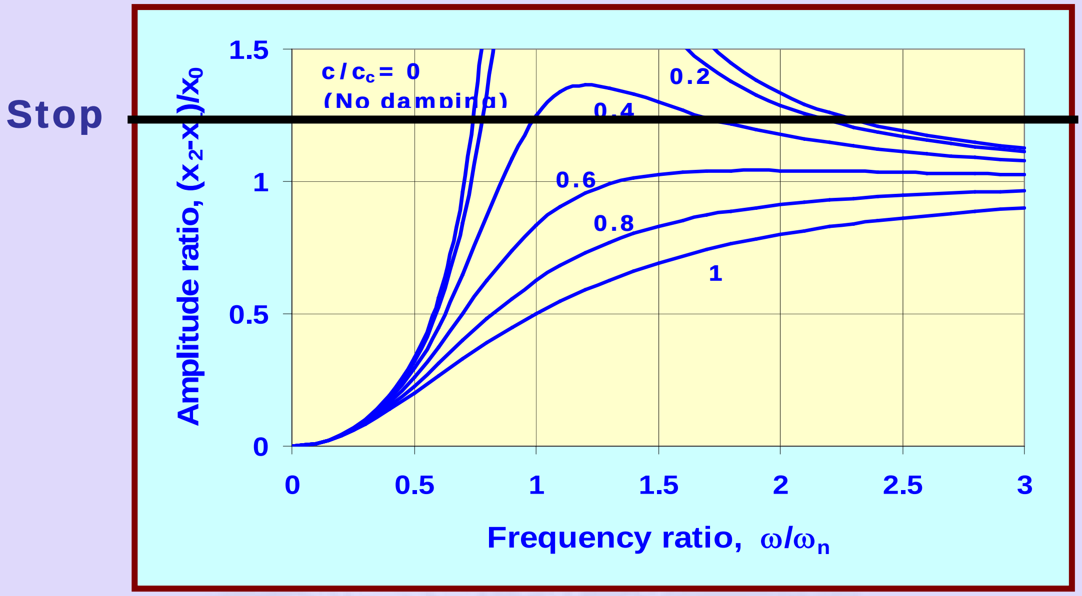


Figure 53 Steady state response of a second order system to periodic input with amplitude limiting stops

Note that the amplitude response of the spring mass damper system will take very large values close to resonance where $\omega_1 = \omega_n$. The second order system may not survive such a situation and one way of protecting the system is to provide amplitude limiting stops as was shown in the figure in Example 14. In Figure 53 the stops are provided such that the amplitude ratio is limited to 25% above the input value.

We now look at the phase relation shown plotted in Figure 54 for a second order system. The output always lags the input and varies from 0° for very low frequencies to 180° for infinite frequency. However, the phase angle varies very slowly for high frequencies.

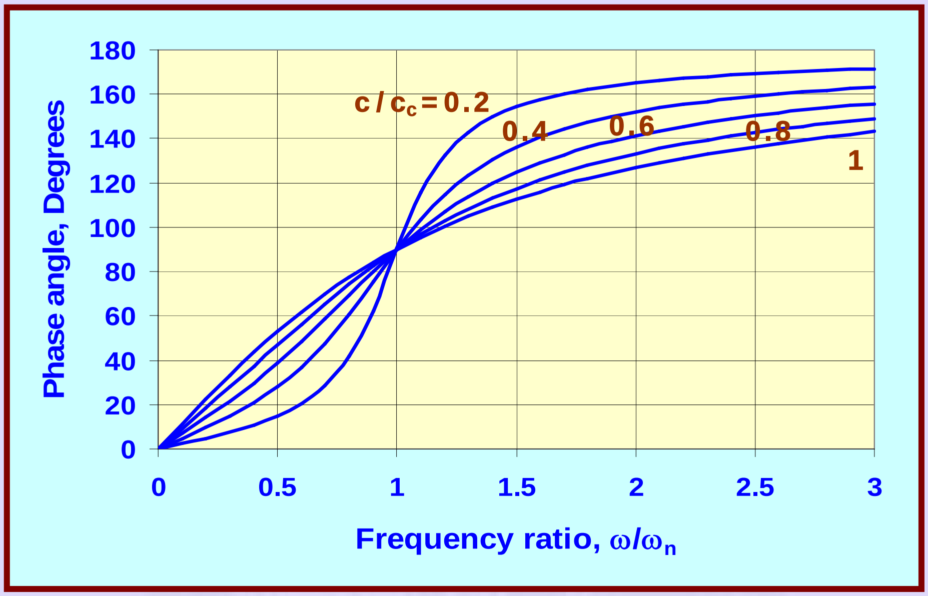


Figure 54 Phase relation for a second order system

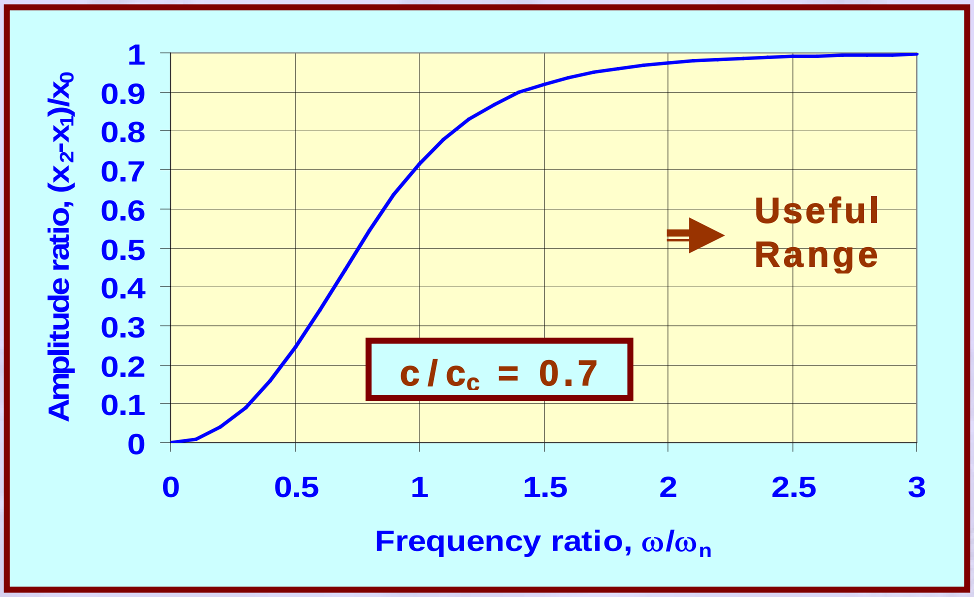


Figure 55 Amplitude response of a system with damping ratio of 0.7

What is now of interest is to find out how to design the system for optimum performance. For this purpose consider the case shown in Figure 55. For the chosen damping ratio of 0.7 the amplitude response is less than or equal to one for all input frequencies. We notice also that the response of the system is good for input frequency much larger than the natural frequency of the system. This indicates that the vibration amplitude is more faithfully given by a spring mass damper device with very small natural frequency, assuming that the frequency response is required at relatively large frequencies. For input frequency more than twice the natural frequency the response is very close to unity. Now let us look at the phase relation for a second order system with the same damping ratio.

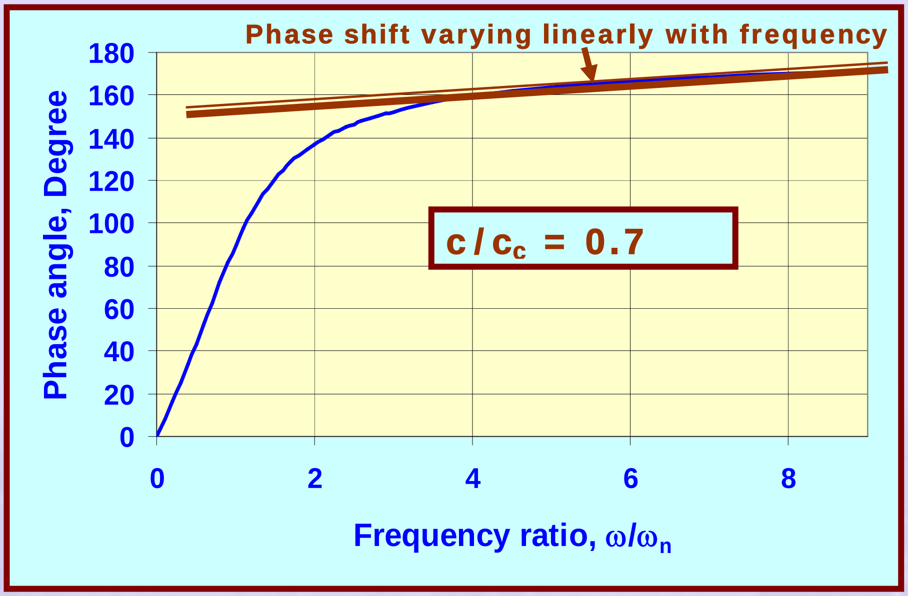


Figure 56 Phase lag of a system with damping ratio of 0.7

It is noted that for input frequency in excess of about 4 times the natural frequency the phase angle varies linearly with the input frequency. This is a very useful property of a second order system as will become clear from the following discussion.

An arbitrary periodic function with a fundamental frequency of ω_1 may be written in the form of a Fourier series given by

$$x_1 = x_{00}\cos\omega_1 t + x_{01}\cos 2\omega_1 t + x_{02}\cos 3\omega_1 t + ... \qquad (52)$$

If ω_1 corresponds to a frequency in the linear phase lag region of the second order system and ϕ is the corresponding phase lag, the phase lag for the higher harmonics are multiples of this phase lag. Thus the fundamental will have a phase lag of ϕ, the next harmonic a phase lag of 2 and so on. Thus the steady state output response will be of the form

$$\text{Output} \propto x_{00}\cos(\omega_1 t-\phi)+x_{01}\cos(2\omega_1 t-2\phi)+x_{02}\cos(3\omega_1 t-3\phi)+... \qquad \textbf{(53)}$$

This simply means that the output is of the form

$$\text{Output} \propto x_{00}\cos(\omega_1 t')+x_{01}\cos(2\omega_1 t')+x_{02}\cos(3\omega_1 t')+... \qquad \textbf{(54)}$$

Thus the output retains the shape of the input.

Now we shall look at the desired characteristics of an accelerometer or an acceleration measuring instrument. We make a plot of the acceleration response of a second order system as shown in Figure 57.

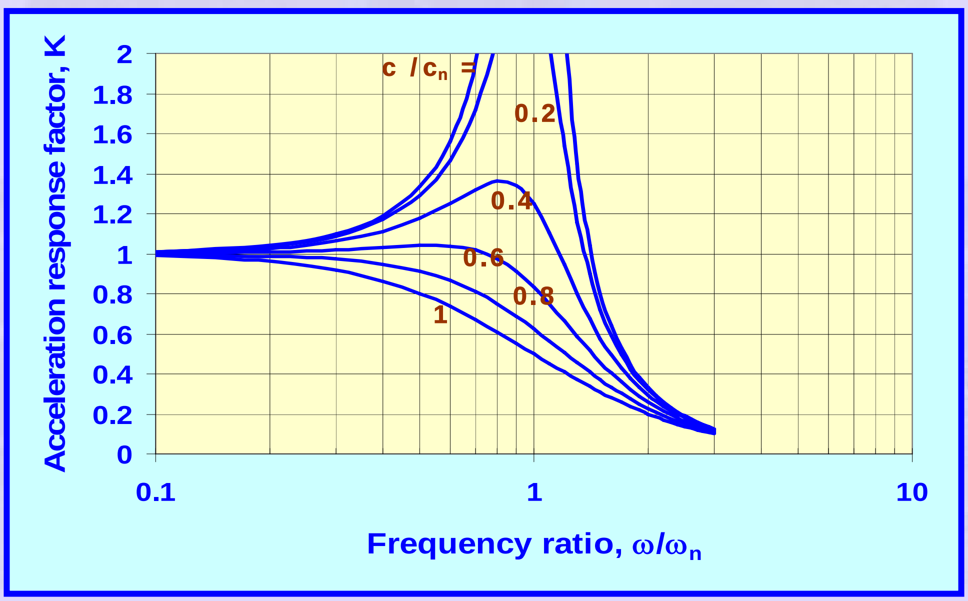


Figure 57 Acceleration response of a second order system

It is clear from this figure that for a faithful acceleration response the measured frequency must be very small compared to the natural frequency of the second order system. Damping ratio does not play a significant role.

In summary we may make the following statements:

1) Displacement measurement of a vibrating system is best done with a transducer that has a very small natural frequency coupled with damping ratio of about 0.7. The transducer has to be made with a large mass with a soft spring.
2) Accelerometer is best designed with a large natural frequency. The transducer should use a small mass with a stiff spring. Damping ratio does not have significant effect on the respose.

Example 15

A big seismic instrument is constructed with M = 100 kg, c/c_c = 0.7 and a spring of spring constant K = 5000 N/m. Calculate the value of linear acceleration that would produce a displacement of 5 mm on the instrument. What is the frequency ratio ω/ω_n such that the displacement ratio is 0.99? What is the useful frequency of operation of this system as an accelerometer?

Displacement is Δx = 5 mm = 0.005 m

Spring constant K = 5000 N/m

The spring force corresponding to the given displacement is

$F = K \quad x = 5000\times0.005 = 25$ N

The seismic mass is M = 100 kg

Hence the linear acceleration is given by a = F/M =25 N/100 kg = 0.25 m/s^2.

Let us represent ω/ω_n by the symbol y. The damping ratio has been specified as 0.7. From the response of a second order system given earlier the condition that needs to be satisfied is

$$Amplitude\ ratio = 0.99 = \frac{y^2}{\left\{\left(1-y^2\right)^2 + (1.4y)^2\right\}^{0.5}} = \frac{y^2}{\left\{\left(1-y^2\right)^2 + 1.96y^2\right\}^{0.5}}$$

We shall substitute $z = y^2$. The equation to be solved then becomes

$$Amplitude\ ratio = 0.99 = \frac{z}{\left\{(1-z)^2 + 1.96z\right\}^{0.5}} \ or\ (1-z)^2 + 1.96z = \left(\frac{z}{0.99}\right)^2$$

$$or\ 1 - 2z + z^2 + 1.96z = \left(\frac{z}{0.99}\right)^2 \ or\ z^2\left(\frac{1}{0.99^2} - 1\right) + 0.04z - 1 = 0$$

$$or\ 0.0203z^2 + 0.04z - 1 = 0$$

The quadratic equation has a meaningful solution z = 6.1022. The positive square root of this gives the desired result $y = \sqrt{6.1022} = 2.47$.

In the present case the natural frequency of the system is given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5000}{100}} = 7.07\ Hz$$

The frequency at which the amplitude ratio is equal to 0.99 is thus given by

$$\omega = 2.47\omega_n = 2.47 \times 7.07 = 17.5\ Hz$$

Again we shall assume that the useful frequency is such that the acceleration response is 0.99 at this cut off frequency. We have

$$Acceleration\ response = 0.99 = \frac{1}{\left\{\left(1-y^2\right)^2 + \left(1.4y\right)^2\right\}^{0.5}} = \frac{1}{\left\{\left(1-y^2\right)^2 + 1.96y^2\right\}^{0.5}}$$

We shall substitute $z = y^2$. *The equation to be solved then becomes*

$$Amplitude\ ratio = 0.99 = \frac{1}{\left\{(1-z)^2 + 1.96z\right\}^{0.5}}\ or\ (1-z)^2 + 1.96z = \left(\frac{1}{0.99}\right)^2$$

$$or\ 1 - 2z + z^2 + 1.96z = \left(\frac{1}{0.99}\right)^2\ or\ z^2 - 0.04z - 0.0203 = 0$$

The quadratic equation has a meaningful solution z = 0.1639. The positive square root of this gives the desired result $y = 0.1639 = 0.405$.

The frequency at which the amplitude ratio is equal to 0.99 is thus given by

$$\omega = 0.405\omega_n = 0.405 \times 7.07 = 2.86\ Hz$$

Earthquake waves tend to have most of their energy at periods (the time from one wave crest to the next) of ten seconds to a few minutes. These correspond to frequencies of $\omega = 2\pi/T = 2\times3.142/60 = 0.105$ *Hz to a maximum of* $\omega = 2\pi/T =$

2×3.142/10 = 0.63 Hz. The accelerometer being considered in this example is eminently suited for this application.

Example 16

A vibration measuring instrument is used to measure the vibration of a machine vibrating according to the relation $x = 0.007\,cos(2\pi t) + 0.0015\,cos(7\pi t)$ *where the amplitude x is in m and t is in s. The vibration measuring instrument has an undamped natural frequency of 0.4 Hz and a damping ratio of 0.7. Will the output be faithful to the input? Explain.*

The natural frequency of the vibration measuring instrument is given by

$$\omega_n = 2\pi \times 0.4 = 0.8\pi\ Hz$$

For the first part of the input the impressed frequency is $\omega_1 = 2\pi$*. Hence the frequency ratio for this part is* $y_1 = \frac{\omega_1}{\omega_n} = \frac{2\pi}{0.8\pi} = 2.5$*. The corresponding amplitude ratio is*

$$Amplitude\ ratio = \frac{y_1^2}{\left\{\left(1-y_1^2\right)^2 + \left(1.4y_1\right)^2\right\}^{0.5}} = \frac{y_1^2}{\left\{\left(1-y_1^2\right)^2 + 1.96y_1^2\right\}^{0.5}}$$

$$= \frac{2.5^2}{\left\{\left(1-2.5^2\right)^2 + 1.96\times 2.5^2\right\}^{0.5}} = 0.9905$$

The phase angle is given by

$$\phi_1 = \tan^{-1}\left\{\frac{2\frac{c}{c_c}y_1}{1-y_1^2}\right\} = \tan^{-1}\left\{\frac{2\times 0.7\times 2.5}{1-2.5^2}\right\} = 146.31° = 2.554 rad$$

For the second part of the input the impressed frequency is $\omega_2 = 7\pi$. *Hence the frequency ratio for this part is* $y_2 = \frac{\omega_2}{\omega_n} = \frac{7\pi}{0.8\pi} = 8.75$. *The corresponding amplitude ratio is*

$$Amplitude\ ratio = \frac{8.75^2}{\left\{\left(1-8.75^2\right)^2 + 1.96\times 8.75^2\right\}^{0.5}} = 1.000$$

The corresponding phase angle is given by

$$\phi_2 = tan^{-1}\left\{\frac{2\times 0.7\times 8.75}{1-8.75^2}\right\} = 170.8° = 2.981\ rad$$

The output of the accelerometer is thus given by

$$\begin{aligned} Output &= 0.9905\times 0.007\ \cos\left(2\pi t - 2.554\right) + 1.000\times 0.0015\ \cos\left(7\pi t - 2.981\right) \\ &= 0.0069\ \cos\left(2\pi t - 2.554\right) + 1.000\times 0.0015\ \cos\left(7\pi t - 2.981\right) \end{aligned}$$

Introduce the notation $\theta = 2\pi t - 2.554$.

The quantity $3.5\theta = 7\pi t - 3.5\times 2.554 = 7\pi t - 8.939$ *may be written as follows:*

$$7\pi t - 2.981 + 2.981 - 8.939 = 7\pi t - 2.981 - 5.958 = 7\pi t - 2.981 - 2\pi + 0.3252$$

Thus the output response may be written as

$$Output = 0.0069\ \cos\left(\theta\right) + 1.000\times 0.0015\ \cos\left(3.5\theta + 0.3252\right)$$

Since the second term has an effective lead with respect to the first term and hence the output does not follow the input faithfully. However, the amplitude part is followed very closely for both the terms.

Peizo-electric accelerometer:

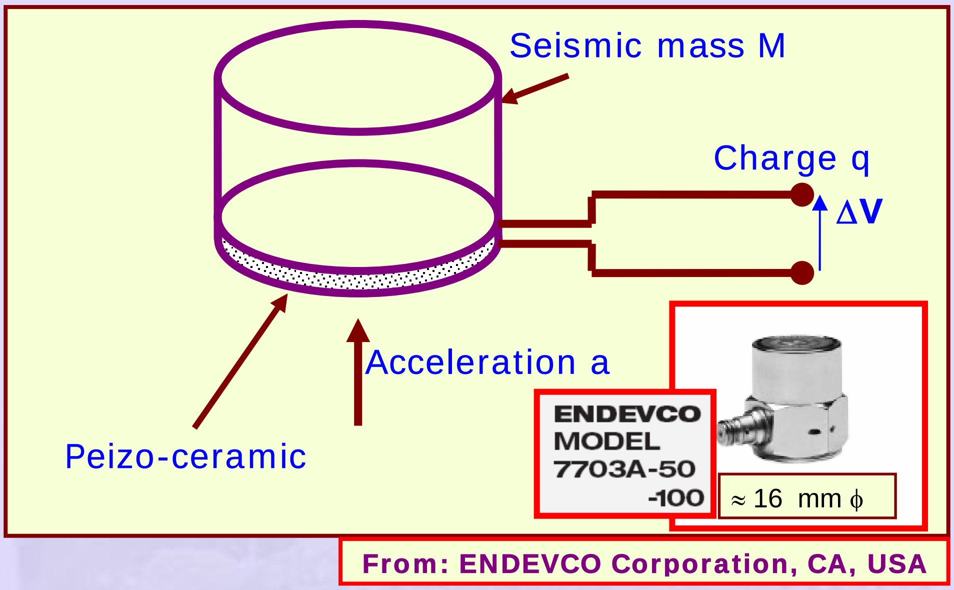


Figure 57 Peizo-electric accelerometer

The principles that were dealt with above help us in the design of peizo-electric accelerometers. These are devices that use a mass mounted on a peizo-ceramic as shown in Figure 57. The entire assembly is mounted on the device whose acceleration is to be measured. The ENDEVCO MODEL 7703A-50-100 series of peizo-electric accelerometers are typical of such accelerometers. The diameter of the accelerometer is about 16 mm. When the mass is subject to acceleration it applies a force on the peizo-electric material. A charge is developed that may be converted to a potential difference by suitable electronics.

The following relations describe the behavior of a peizo-electric transducer.

$$C = \kappa \frac{A}{\delta},\ q = d\,F,\ \Delta V = \frac{q}{C} = \frac{d}{\kappa}\frac{\delta}{A} M\ a \qquad (55)$$

In Equation 55 the various symbols have the following meanings:

C = Capacitance of the peizo-electric element

= Dielectric constant

d = Peizo-electric constant

A =Area of Peizo-ceramic

= Thickness of Peizo-ceramic

M = Seismic mass

q = Charge

V = Potential difference

Charge Sensitiity is defined as $S_q = \frac{q}{a}$ which has a typical value of $50 \times 10^{-12}\ Coulomb/g$. The acceleration is in units of *g*, the acceleration due to gravity. The voltage sensitivity is defined as $S_V = \frac{\Delta V}{a} = \frac{d}{\kappa}\frac{\delta}{A} M$. Peizo transducers are supplied by Brüel & Kjaer and have specifications as shown in Table 7.

Table 7 Typical product data Brüel & Kjaer 8200

Range	**1000 to + 5000 N**
Charge sensitivity	**4×10^{-12} Coulomb/N**
Capacitance	**25×10^{-12} F**
Stiffness	**5×10^{8} N/m**
Resonance frequency with 5 g load mounted on top	**35 kHz**
Effective seismic mass: **Above Peizo-electric element** **Below Peizo-electric element**	 3g 18 g
Peizo-electric material	Quartz
Transducer housing material	SS 316

Diameter	~18 mm
Transducer mounting	**Threaded spigot and tapped hole in the body**
Signal conditioning	Charge amplifier
Useful frequency range	~10 kHz

Laser Doppler Vibrometer

We have earlier discussed the use of laser Doppler for measurement of fluid velocity. It is possible to use the method also for the remote measurement of vibration. Consider the optical arrangement shown in Figure 58. A laser beam is directed towards the object that is vibrating. The reflected laser radiation is split in to two beams that travel different path lengths by the arrangement shown in the figure. The two beams are then combined at the detector. The detector signal contains information about the vibrating object and this may be elucidated using suitable electronics.

Consider the target to vibrate in a direction coinciding with the laser incidence direction. Let the velocity of the target due to its vibratory motion be $U(t)$. The laser beam that is reflected by the vibrating target is split in to two beams by the beam splitter 2. One of these travels to the fixed mirror 1 and reaches the photo detector after reflection off the beam splitter 2. The other beam is reflected by fixed mirror 2 and returns directly to the photo detector. The path lengths covered by the two beams are different. Let l be the extra distance covered by the second beam with respect to the first. The second beam reaches the detector after a time delay of $t_d = \frac{\Delta l}{c}$ where c is the speed of light. Since the speed of light is very large (3×10^8 m/s) the time delay is rather a small quantity, i.e. $\frac{\Delta l}{c} \ll 1$.

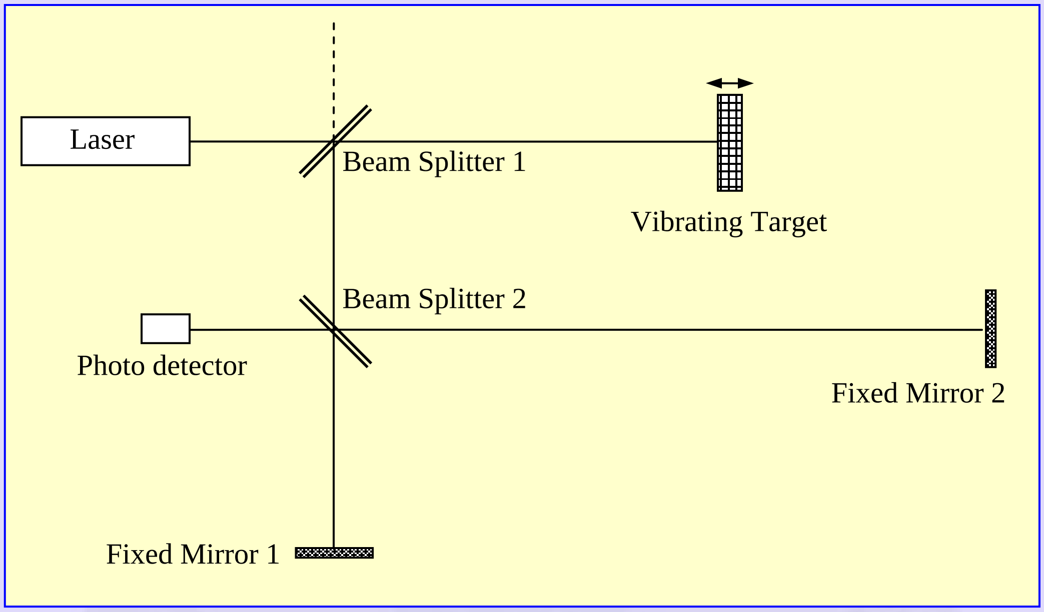


Figure 58 Schematic to explain the working principle of a Laser Doppler Vibrometer

We know that the reflected radiation will have a slightly different frequency than the incident laser frequency and this is due to the Doppler shift we have discussed earlier. If we look carefully we see that the two beams that reach the detector have in fact undergone slightly different Doppler shifts because of the time delay referred to above. When we combine these two beams at the detector we will get a beat frequency given by

$$f_{beat} = \frac{2}{\lambda}\left[U(t) - U\left(t - \frac{\Delta l}{c}\right)\right] \quad \textbf{(56)}$$

The velocity of the target may be represented by a Fourier integral given by

$$U(t) = \int_0^\infty A(\omega)\sin\left(\omega t - \phi(\omega)\right)d\omega \quad \textbf{(57)}$$

Substitute Expression 57 in Expression 56 to get

$$f_{beat} = \frac{2}{\lambda}\int_0^\infty \left[A(\omega)\sin\left(\omega t - \phi(\omega)\right) - A(\omega)\sin\left(\omega\left(t - \frac{\Delta l}{c}\right) - \phi(\omega)\right)\right] d\omega$$

(58)

Noting now that $\frac{\Delta l}{c} \ll 1$the integrand may be approximated, using well known trigonometric identities as

$$\sin\left(\omega t - \phi(\omega)\right) - \sin\left(\omega\left(t - \frac{\Delta l}{c}\right) - \phi(\omega)\right) \approx 2\frac{\omega \Delta l}{c}\cos\left(\omega t - \phi(\omega)\right) \quad \textbf{(59)}$$

Thus the beat frequency is given by

$$f_{beat} = \frac{4}{\lambda}\frac{\Delta l}{c}\int_0^\infty \left[A(\omega)\,\omega\cos\left(\omega t - \phi(\omega)\right)\right] d\omega \quad \textbf{(60)}$$

We notice that $\frac{dU}{dt} = \int_0^\infty A(\omega)\,\omega\cos\left(\omega t - \phi(\omega)\right) d\omega$. With this Expression 60 becomes

$$f_{beat} = \frac{4\Delta l}{c\lambda}\frac{dU}{dt} = \frac{4\Delta l}{c\lambda}a(t) \quad \textbf{(61)}$$

Thus the beat frequency is proportional to the instantaneous acceleration of the vibrating target.

In practice the physical path length needs to be very large within the requirement $\frac{\Delta l}{c} \ll 1$. A method of achieving this is to use a long fiber optic cable in the second path between the beam splitter 2 and the fixed mirror 2. Recently S Rothberg et. al. describe the development of a Laser Doppler Accelerometer (S Rothberg, A Hocknell1 and J Coupland, Developments in laser Doppler accelerometry (LDAc) and comparison with laser Doppler velocimetry, Optics and Lasers in Engineering, V -32, No. 6, 2000, pp - 549-564.).

Fiber Optic Accelerometer

Another interesting recent development is a fiber optic accelerometer described by Lopez – Higuera et. al (Lopez – Higuera et. al. Journal of Light Wave Technology, Vol.15, No.7, pp. 1120-30, July 1997). The schematic of the instrument is shown in Figure 59.

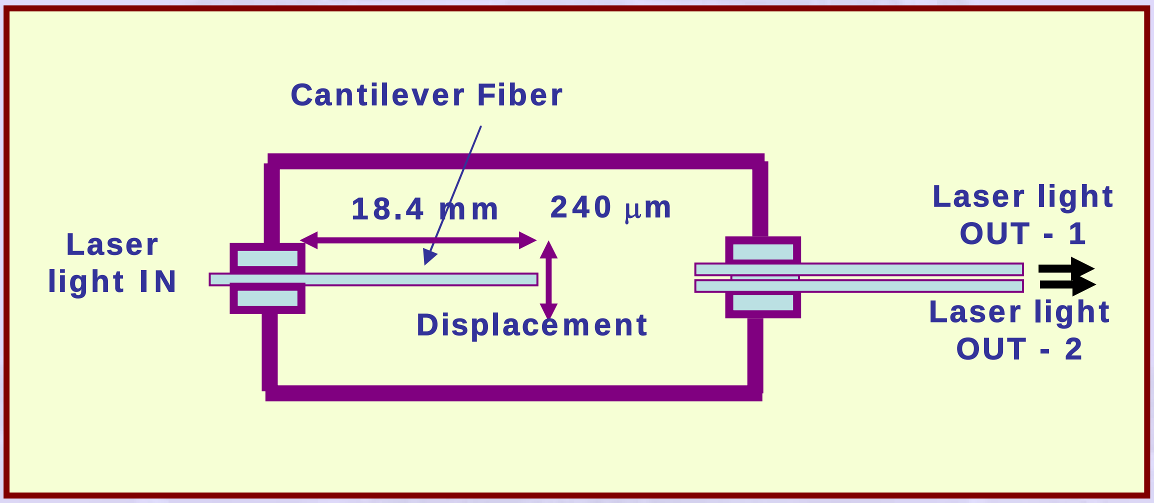


Figure 59 Schematic of a fiber optic accelerometer

In this transducer a laser beam is fed in through a cantilever fiber that vibrates with the probe body that is mounted on a vibrating object. The laser beam emerging out of the cantilever is incident on a fiber optic pair that are rigidly fixed and do not vibrate. The vibrating cantilever modulates the light communicated to the two collecting fibers. The behavior of the fiber optic cantilever is given by the following response function which represents the ratio of the relative displacement and the input acceleration.

$$H(\omega)=\frac{1}{\omega^2}\left[\frac{\cos\left(\alpha\sqrt{\omega}\right)+\cosh\left(\alpha\sqrt{\omega}\right)}{1+\cos\left(\alpha\sqrt{\omega}\right)\cosh\left(\alpha\sqrt{\omega}\right)}-1\right] \quad \textbf{(62)}$$

In the above equation $\alpha=\left[\frac{\rho L^4}{EI}\right]^{\frac{1}{4}}$ with ρ = linear mass density of fiber, *E* = Young's modulus of the fiber material, L = length of cantilever and *I* = transverse moment of inertia. For frequencies significantly lower than about 20% of the natural frequency of the cantilever beam given by $f_n=\frac{3.516}{2\pi\sqrt{\alpha}}$ the response function is a constant to within 2%. Thus the lateral displacement is proportional to the acceleration. In the design presented in the paper the maximum transverse displacement is about 5 μm with reference to the collection optics. The performance figures for this are:

Range: 0.2 – 140 Hz

Sensitivity: 6.943 *V*/*g* at a frequency of 30 Hz which converts to $\sim 700\frac{mV}{m/s^2}$

For more details the student should refer to the paper.

Measurement of torque and power:

Torque and power are important quantities involved in power transmission in rotating machines like engines, turbines, compressors, motors and so on. Torque and power measurements are made by the use of a dynamometer. In a dynamometer the torque and rotational speed are independently measured and the product of these gives the power.

(i) Torque Measurement:

- Brake Arrangement
- Load electrically – Engine drives a generator
- Measure shear stress on the shaft

(ii) Measurement of rotational speed:

- Tachometer – Mechanical Device
- Non contact optical rpm meter

(i) Torque measurement:

Brake drum dynamometer (the Prony brake):

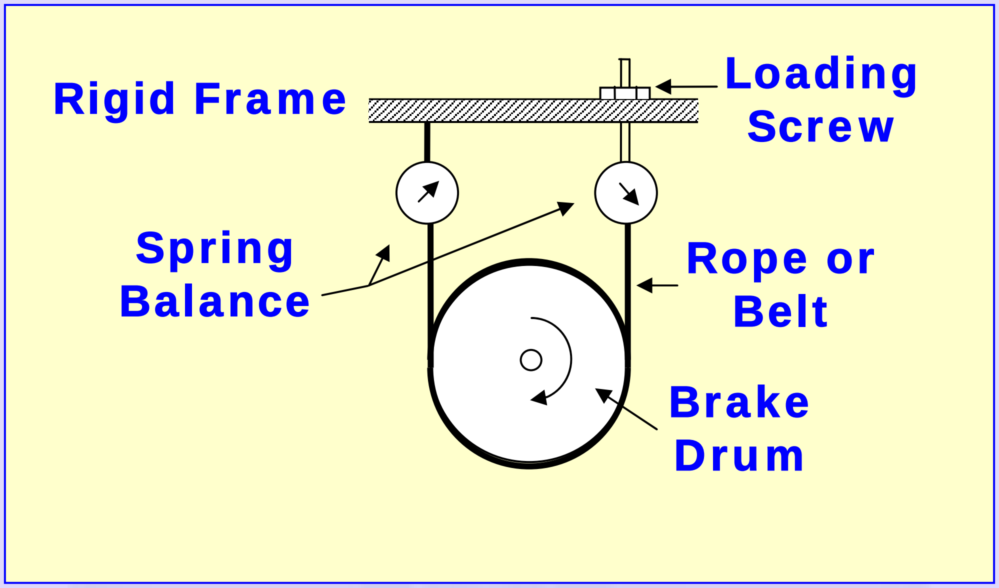


Figure 60 Schematic of a brake drum dynamometer

The brake drum dynamometer is a device by which a known torque can be applied on a rotating shaft that may belong to any of the devices that were mentioned earlier. Schematic of the brake drum dynamometer is shown in Figure 60. A rope or belt is wrapped around the brake drum attached to the shaft. The two ends of the rope or belt are attached to rigid supports with two spring balances as shown. The loading screw may be tightened to increase or loosened to decrease the frictional torque applied on the drum. When the shaft rotates the tension on the two sides will be different. The difference is jus the frictional force applied at the periphery of the brake drum. The product of this difference multiplied by the radius of the drum gives the torque. Alternate way of measuring the torque using essentially the brake drum dynamometer is shown in Figure 61. The torque arm is adjusted to take on the horizontal position by the addition of suitable weights in the pan after adjusting the loading screw to a suitable level of tightness. The torque is given by the product of the torque arm and the weight in the pan.

It is to be noted that the power is absorbed by the brake drum and dissipated as heat. In practice it is necessary to cool the brake drum by passing cold water through tubes embedded in the brake blocks or the brake drum.

Electric generator as a dynamometer:

An electric generator is mounted on the shaft that is driven by the power device whose output power is to be measured as shown in Figure 62. The stator that would normally be fixed is allowed float between two bearings. The loading of the dynamometer is done by passing the current from the generator through a bank of resistors. The power generated is again dissipated as heat by the resistor bank. In practice some cooling arrangement is needed to dissipate this heat. The stator has an arm attached to it which rests on a platform balance as shown in the figure. The stator experiences a torque due to the rotation of the rotor due to electromagnetic forces that is balanced by an equal and opposite torque provided by the reaction force acting at the point where the torque arm rests on the platform balance. The torque on the shaft is given by the reading of the platform balance multiplied by the torque arm.

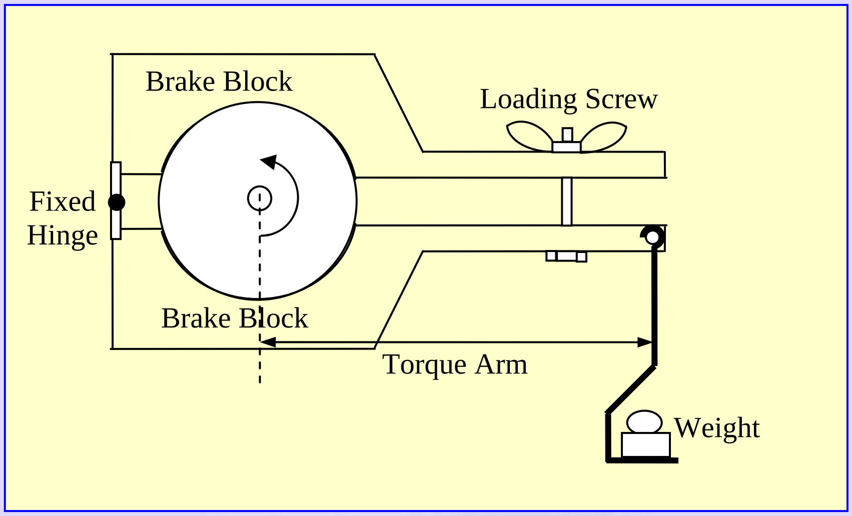


Figure 61 An alternate method of measuring torque with a Prony brake

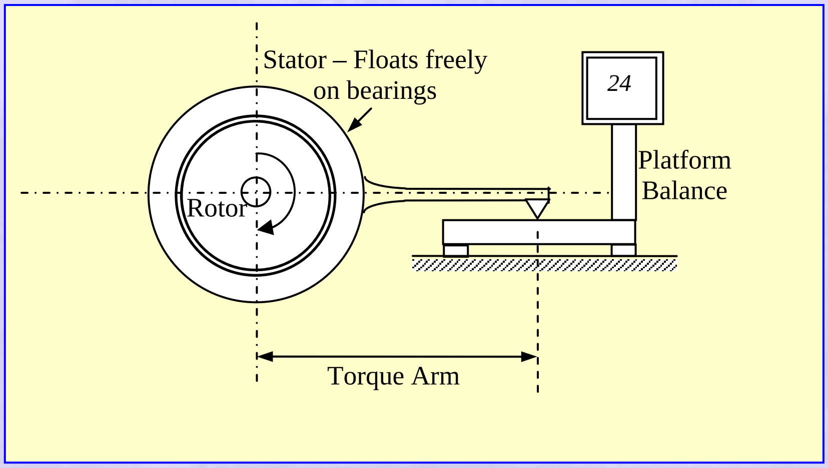


Figure 62 Electric generator used as a dynamometer

Torque may also be measured by measuring the shear stress experienced by the shaft that is being driven. If the shaft is subject to torsion the principal stresses and the shear stress are identical as shown by Figure 63.

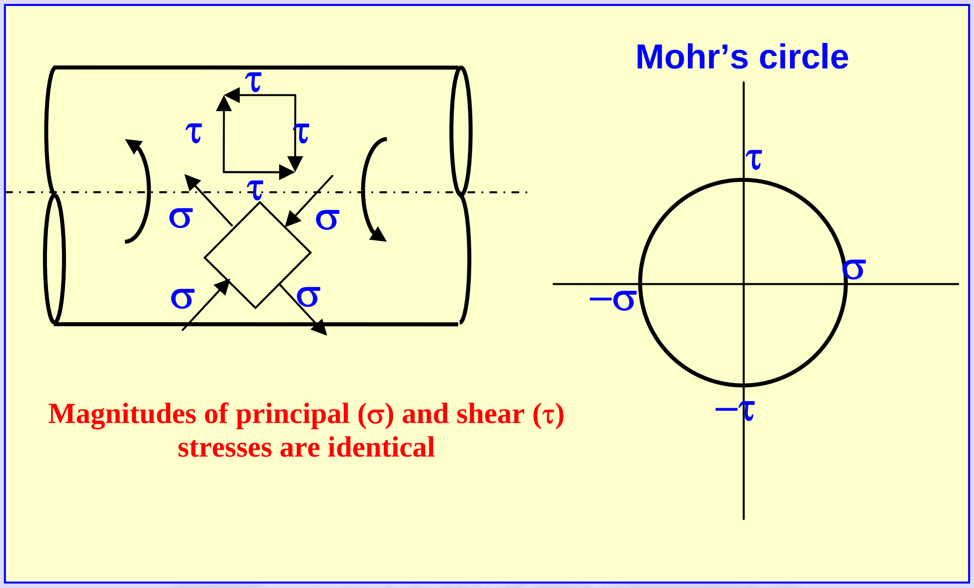


Figure 63 Stresses on the surface of a shaft in torsion

We notice that the principal stresses are tensile and compressive along the two diagonals. If load cells are mounted in the form of a bridge with the arms of the bridge along the edges of a square oriented at 45° to the shaft axis, the imbalance voltage produced by the bridge is a measure of the principal stress that is also equal to the shear stress. The arrangement of the load cells is as shown in Figure 64, sourced from the net. This arrangement improves the gage sensitivity by a factor of 2. We know that $\tau = \sigma = \frac{TR}{J}$ where $J = \frac{\pi R^4}{2}$. Here *T* is the torque, *R* is the radius of the shaft and *J* is the polar moment of inertia. Using the load cell measured value of τ, we may calculate the torque experienced by the shaft. In this method the dynamometer is used only for applying the load and the torque is measured using the load cell readings.

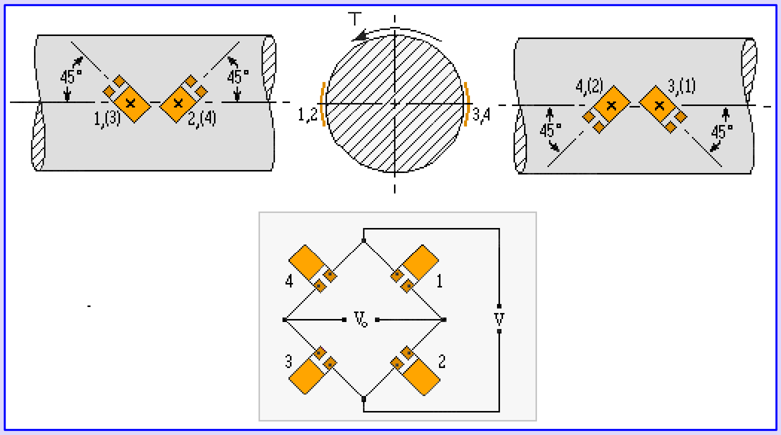


Figure 64 Strain gages for shear measurement
Visit: - http://www.vishay.com

ii) Measurement of rotational speed:

A mentioned earlier the measurement of power requires the measurement of rotational speed, in addition to the measurement of torque. We describe below two ways of doinjg this.

Tachometer – Mechanical Device

This is a mechanical method of measuring the rotational speed of a rotating shaft. The tachometer is mechanically driven by being coupled to the rotating shaft. The rotary motion is either transmitted by friction or by a gear arrangement (as shown in Figure 65). The device consists of a magnet which is rotated by the drive shaft. A speed cup made of aluminum is mounted close to the rotating magnet with an air gap as indicated.

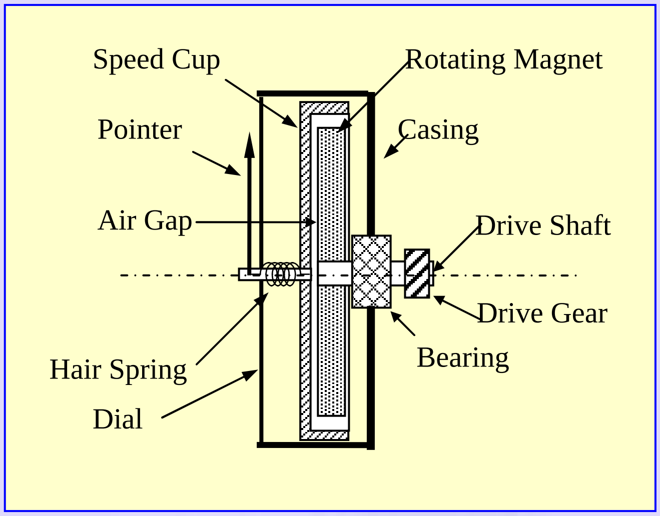


Figure 65 Speedometer or tachometer

The speed cup is restrained by a hair spring and has a pointer attached to its own shaft that moves over a dial. When the magnet rotates due to the rotation of its own shaft the speed cup tends to be dragged along by the moving magnet and hence experiences a torque and tends to rotate along with it. The speed cup moves and takes up a position in which the rotating magnet induced torque is balanced by the restraining torque provided by the spring. Knowing the drive gear speed ratio one may calibrate the angular position on the dial in terms of the rotational speed in rpm.

Non contact optical rpm meter:

This is a non contact method of rotational speed of a shaft. However it requires a wheel with openings to be mounted on the rotating shaft. The optical arrangement is shown in Figure 66. The arrangement is essentially like that was used for chopping a light beam in applications that were considered earlier. The frequency of interruption of the beam is directly proportional to the rpm of the wheel, that is usually the same as the rpm of the shaft to be measured and the number of holes provided along the periphery of the wheel. If there is only one hole the beam is interrupted once every revolution. If there are *n* holes the beam is interrupted *n* times per revolution. The rotational speed of the shaft is thus equal to the frequency of interruptions divided by the number of holes in the wheel.

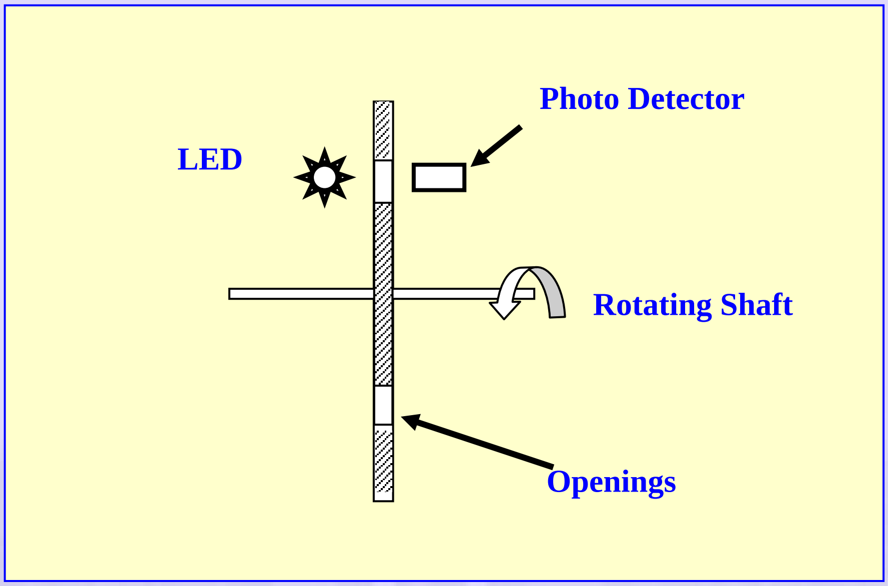


Figure 66 Optical rpm measurement

The LED photo detector wheel assembly is available from suppliers as a unit readily useable for rpm measurement.

Example 17

An engine is expected to develop 5 kW of mechanical output while running at an angular speed of 1200 rpm. A brake drum of 250 mm diameter is available. It is proposed to design a Prony brake dynamometer using a spring balance as the force measuring instrument. The spring balance can measure a maximum force of 100 N. Choose the proper torque arm for the dynamometer.

The schematic of the Prony brake is shown in the figure below.

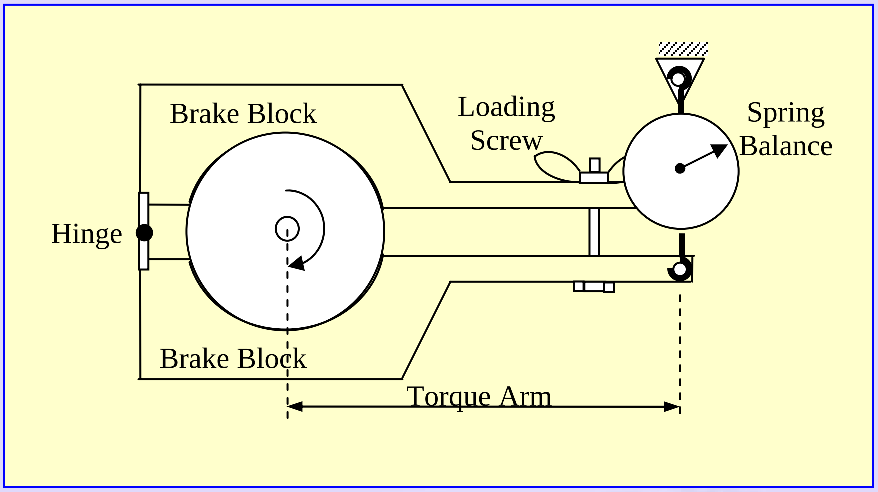
Even though the spring balance is well known in its traditional linear form a dial type spring balance is commonly employed in dynamometer applications. The spring is in the form of a planar coil which will rotate a needle against a dial to indicate the force.

The given data is written down:

Expected power is $P = 5\ kW = 5000\ W$

Rotational speed of the engine $N = 1200\ rpm = \frac{2 \times \pi \times 1200}{60} Hz = 125.66\ Hz$

We shall assume that the force registered by the spring balance is some 90% of the maximum such that $F = 100 \times 0.9 = 90\,N$



We know that the power developed is the product of torque and the angular speed. Hence we get $T = \frac{P}{\omega} = \frac{5000}{125.66} = 39.79\ N\,m$

The required torque arm L may now be obtained as $L = \frac{T}{F} = \frac{39.79}{90} = 0.442\ m$