Sub Module 2.9

U – Tube manometer

The simplest of the gages that is used for measuring pressure is a U – tube manometer shown in Figure 63. The U tube needs to be vertically oriented and the acceleration due to gravity is assumed to be known. The height 'h' is the measured quantity.

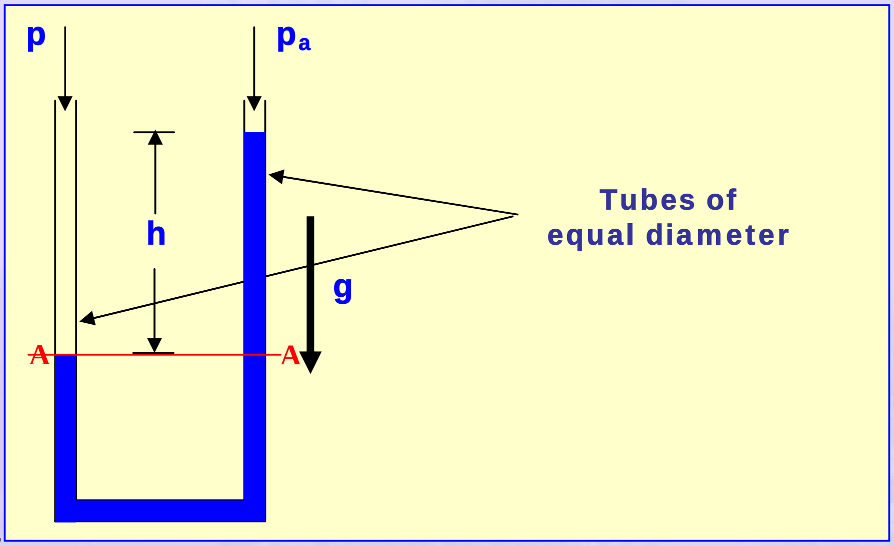


Figure 63 U tube manometer

The pressure to be measured is that of a system that involves a fluid (liquid or a gas) *different* from the manometer liquid. Let the density of the fluid whose pressure being measured be ρ_f and that of the manometer liquid be ρ_f. Equilibrium of the manometer liquid requires that there be the same force in the two limbs across the plane AA. We then have

$$p + \rho_f gh = p_a + \rho_m gh \qquad \textbf{(62)}$$

This may be rearranged to read

$$p - p_a = \left(\rho_m - \rho_f\right) gh \qquad \textbf{(63)}$$

Even though mercury is a common liquid used in manometers, other liquids are also used. A second common liquid is water. When measuring pressures close to the atmospheric pressure in gases, the fluid density may be quite negligible in comparison with the manometer liquid density. One may then use the approximate expression

$$p - p_a \approx \rho_m g h \tag{64}$$

The manometer liquid is chosen based on its density with respect to the density of the fluid whose pressure is being measured and also the pressure difference that needs to be measured. Indeed small pressure differences are measured using water or organic liquids as the manometer liquid such that the height h measured is sufficiently large as to be estimated with sufficient precision.

Example 22

- *a) A U tube manometer employs special oil having a specific gravity of 0.82 as the manometer liquid. One limb of the manometer is exposed to the .atmosphere at a pressure of 740 mm Hg and the difference in column heights is measured as 20 cm± 1 mm when exposed to an air source at 25°C. Calculate the air pressure in Pa and the uncertainty.*
- *b) The above manometer was unfortunately mounted with an angle of 3° with respect to the vertical. What is the error in the indicated pressure due to this, corresponding to the data given above?*

Part a

- *Specific gravity uses the density of water at 25°C as the reference. From table of properties, the density of water at this temperature is 996 kg/m*^{3}*. The density of the manometer liquid is*

$$\begin{aligned}\rho_m &= \rho_{Special\ oil} \\ &= Specific\ gravity\ of\ special\ oil \times Density\ of\ water\ at\ 25°C \\ &= 0.82 \times 996 = 816.7\ kg/m^3\end{aligned}$$

- *Air density is calculated using the ideal gas relation. The gas constant for air is taken as* R_g *= 287 J/kg K. The air temperature is T = 273+25=298 K. The air pressure is converted to Pa as*

$$p_a = \frac{740}{760} \times 1.013 \times 10^5 = 98634.2\ Pa$$

- *Air density is thus given by*

$$\rho_f = \frac{p_a}{R_g T} = \frac{98634.2}{287 \times 298} = 1.15\ kg/m^3$$

- *The column height is given to be h = 20 cm = 0.2 m. The measured pressure differential is then given by*

$$p - p_a = (816.7 - 1.15) \times 9.8 \times 0.2 = 1598.48\ Pa$$

- *The uncertainty calculation is straight forward. It is the same as the % uncertainty in the column height i.e.* $\Delta h\% = \pm\frac{0.001}{0.2} \times 100 = \pm 0.5$. *The error in the measured pressure difference is*

$$\Delta\left(p-p_a\right)=\pm\frac{0.5}{100}\times 1598.48=\pm 7.99\approx 8\ Pa$$

Part b

- *It is clear that the manometer liquid height difference is given by* $h'=h\cos 3°=0.2\times 0.9986=0.1997\ m$*. Indicated pressure difference is then given by*

$$p-p_a=\left(816.7-1.15\right)\times 9.8\times 0.1997=1596.29\,Pa$$

- *Note that there is thus a systematic error of* $1596.29-1598.48=-2.19\ Pa$ *because of mounting error. This is about 25% of the error due to the error in the measurement of h.*

Well type manometer

Sometimes a well type manometer is used. Schematic of a well type manometer is shown in Figure 64.

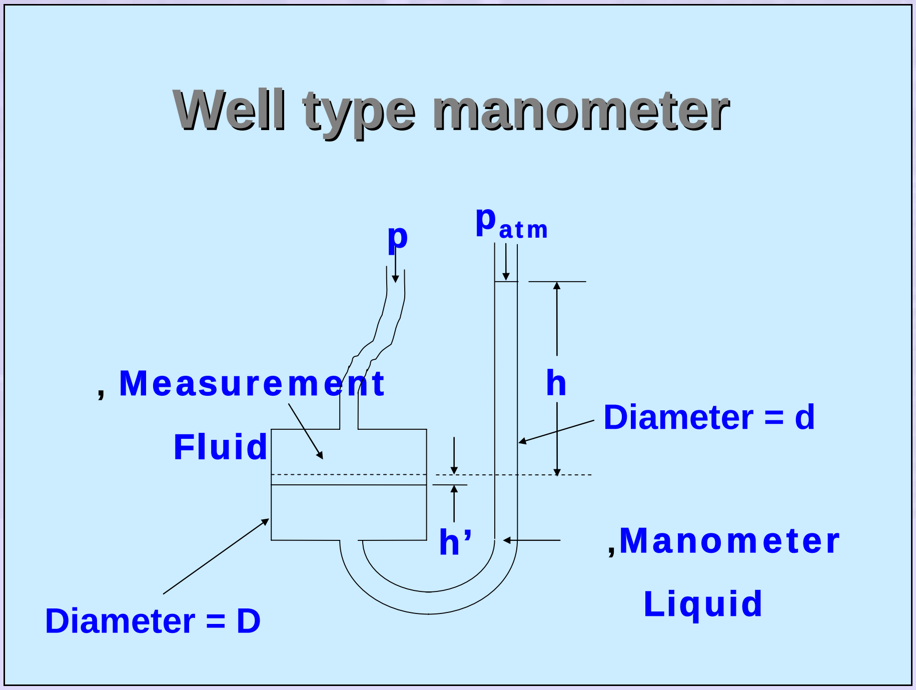


Figure 64 Schematic of a well type manometer

The dashed line indicates the datum with reference to which the manometer height is measured. The advantage of the well type design is that relatively large pressure differences may be measured with enough manometer liquid being available for doing so! We assume that the manometer liquid is incompressible and hence the following holds:

$$h' A = ha \qquad \textbf{(65)}$$

This expression simply states that there is no change in the volume of the fluid and hence the mass. Here 'A' is the well cross section area given by $A = \frac{\pi D^2}{4}$ while 'a' is the tube cross section area given by $A = \frac{\pi d^2}{4}$. Equation 62 is recast for this case as

$$p + \rho_f g(h + h') = p_a + \rho_m g(h + h') \qquad \textbf{(66)}$$

Using Equation 65 in Equation 66, after some rearrangement we get

$$p - p_a = \left(\rho_m - \rho_f\right) g \left(1 + \frac{a}{A}\right) h \qquad \textbf{(67)}$$

If the area ratio $\frac{a}{A}$ is very *small* compared to unity, we may use the approximate formula that is identical with Equation 63.

In case the measured pressure difference is small one may use an inclined well type manometer shown in Figure 65.

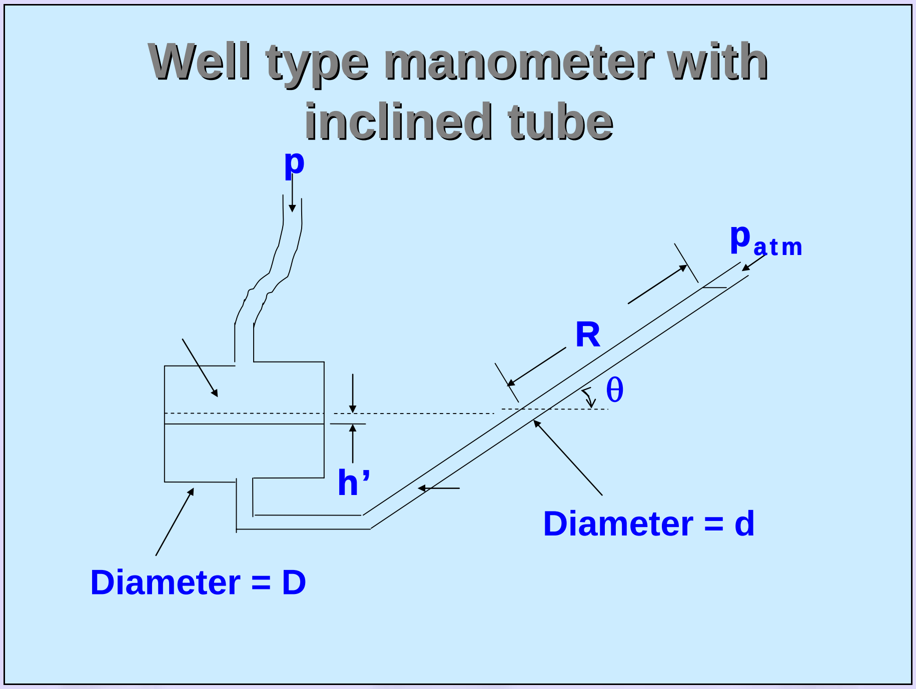


Figure 65 Well type inclined tube manometer

Incompressibility of the manometer liquid requires that $Ah' = aL$. The manometer height is now given by $R\sin\theta + h' = R\left(\sin\theta + \frac{a}{A}\right)$. Equation 67 is replaced by

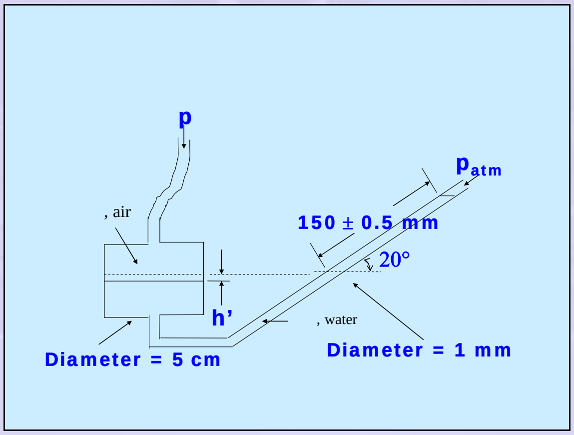
$$p - p_a = \left(\rho_m - \rho_f\right) g \left(\sin\theta + \frac{a}{A}\right) R \quad \textbf{(68)}$$

It is clear that the inclination of the tube *amplifies* (recall the mechanical advantage of an inclined plane) the measured quantity and hence improves the precision of the measurement.

Example 23

- a) In an inclined tube manometer the manometer fluid is water at 20°C while the fluid whose pressure is to be measured is air. The angle of the inclined tube is 20°. The well is a cylinder of diameter 0.05 m while the tube has a diameter of 0.001 m. The manometer reading is given to be 150 mm. Determine the pressure differential in mm water and Pa. What is the error in per cent if the density of air is neglected?
- b) Determine the error in the measured pressure differential if the reading of the manometer is within ±0.5 mm and the density of water has an error of ±0.2%. Assume that all other parameters have no errors in them. Neglect air density in this part of the question.

- Figure below indicates the numerical values specified in this problem.



Part a

- From the data shown in the Figure we calculate the area ratio as

$$\frac{a}{A}=\left(\frac{d}{D}\right)^2=\left(\frac{0.001}{D0.05}\right)^2=\frac{1}{2500}=0.0004$$

- Water density at the indicated temperature of 20°C is read off a table of properties as 997.6 kg/m^3. The fluid is air at the same temperature and its density is obtained using ideal gas law. We use $p_a=1.013\times10^5\ Pa,\ T=273+20=297\,K,\ R_g=287\ J/kgK$

to get

$$\rho_f = \frac{p_a}{R_g T} = \frac{1.013 \times 10^5}{287 \times 297} = 1.205\ kg/m^3$$

- The density of air has been calculated at the atmospheric temperature since it is at a pressure very close to it! The indicated pressure difference is then given by

$$\begin{aligned} p - p_a &= (997.6 - 1.205) \times 9.81 \times (\sin 20 + 0.0004) \times 0.15 \\ &= 500.88\ Pa \end{aligned}$$

- This may be converted to mm of water column by dividing the above by $\rho_m g$. Thus

$$p - p_a = \frac{500.88}{997.6 \times 9.8} \times 1000 = 51.2\ mm\ water$$

- If we neglect the density of air in the above, we get

$$\begin{aligned} p - p_a &= 997.6 \times 9.81 \times (\sin 20 + 0.0004) \times 0.15 \\ &= 501.49\ Pa \end{aligned}$$

- The percentage error is given by

$$\Delta p\% = \frac{501.49 - 500.88}{500.88} \times 100 = 0.12$$

<u>Part b</u>

- The influence coefficients are now calculated. We have

$$\frac{\partial (p - p_a)}{\partial \rho_f} = g\left(\sin\theta + \frac{a}{A}\right)R = 9.87 \times (\sin 20 + 0.0004) \times 0.15 = 0.5058$$

$$\frac{\partial (p - p_a)}{\partial R} = \rho_f g\left(\sin\theta + \frac{a}{A}\right) = 997.6 \times 9.87 \times (\sin 20 + 0.0004) = 3363.7$$

- The errors have been specified as

$$\Delta R = \pm 0.5\ mm = \pm 0.0005\ m,\ \Delta\rho_f = \pm\frac{0.2}{100} \times 997.6 = \pm 1.995\ kg/m^3$$

- Use of error propagation formula yields the error in measured pressure as

$$\Delta p = \pm\sqrt{(0.5058 \times 1.995)^2 + (3363.7 \times 0.0005)^2} = \pm 1.96\ Pa$$

Dynamic response of a U tube manometer

In many applications the pressure difference to be measured may vary with time. The response time of the measuring instrument and the connecting tubes decide the response time. We make below a simple analysis of a U tube manometer subject to a step change in input.

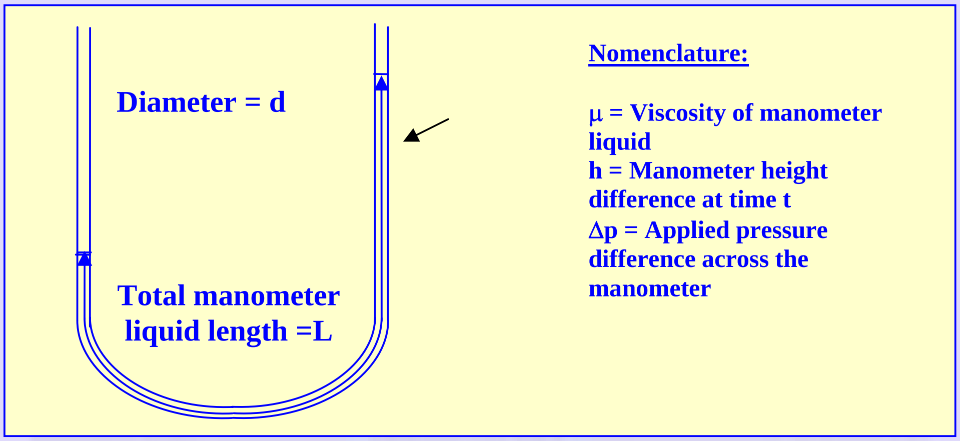


Figure 66 Nomenclature for the transient analysis

Because the manometer liquid is assumed to be incompressible the total length remains fixed at L. We assume that the manometer is initially in the equilibrium position and the pressure difference Δp is applied across it. The liquid column will move and will be as shown in Figure 66 at time t > 0. The forces that are acting on the length L of the manometer liquid are:

1. Force due to acceleration of the liquid given by $F_a = \left(\rho_m AL\right)\frac{d^2h}{dt^2}$
2. Force supporting the change in h $F_s = A\Delta p$
3. Forces opposing the change:
   a. Weight of column of liquid $W = \left(\rho_m Ah\right)g$
   b. Fluid friction due to viscosity of the liquid.

The viscous force opposing the motion is calculated based on the assumption of fully developed Hagen-Poiseuelle flow. The velocity of the liquid column is expected to be small and the laminar assumption is thus valid. We know from

Fluid Mechanics that the pressure gradient and the mean velocity are related as

$$u_m = \frac{dh}{dt} = -\frac{d^2}{32\mu}\frac{dp_f}{dh} = -\frac{d^2}{32\mu}\frac{\Delta p_f}{L}$$

where Δp_f is the pressure drop due to friction. We define fluid resistance R as the ratio of frictional (viscous) pressure drop (potential difference) to the mass flow rate (current). We note that the mass flow rate is given by

$$\begin{aligned}\dot{m} &= \rho_m A u_m \\ &= -\rho_m \frac{\pi d^2}{4}\left(\frac{d^2}{32\mu}\right)\frac{\Delta p_f}{L} = -\rho_m\left(\frac{\pi d^4}{128\mu}\right)\frac{\Delta p_f}{L}\end{aligned}$$

Hence the fluid resistance due to friction is given by $R = -\frac{\Delta p_f}{\dot{m}} = \frac{128\mu L}{\pi\rho_m d^4}$. Note that the resistance involves only the geometric parameters and the liquid properties. The frictional force opposing the motion is thus given by $F_f = \dot{m}RA$.

Note that the mass flow rate is itself given by $\dot{m} = \rho_m A u_m = \rho_m A\frac{dh}{dt}$. Hence the frictional force opposing the motion is $F_f = \rho_m A^2 R\frac{dh}{dt}$.

We may now apply Newton's law as $F_a = F_s - W - F_f$. Introducing the expressions given above for the various terms, we get

$$\rho_m AL\frac{d^2h}{dt^2} = A\Delta p - \rho_m Agh - \rho_m RA^2\frac{dh}{dt} \tag{69}$$

We may rearrange this equation as

$$\frac{L}{g}\frac{d^2h}{dt^2} + \frac{RA}{g}\frac{dh}{dt} + h = \frac{\Delta p}{\rho_m g} \tag{70}$$

This is a second order ordinary differential equation that resembles the equation governing a spring mass dashpot system that is familiar to us from mechanics. The system is thus inherently a second order system. We define a characteristic time given by $\tau = \sqrt{\frac{L}{g}}$, damping ratio $\zeta = \frac{RA}{2g\tau}$ to recast Equation 70 in the standard form

$$\tau^2 \frac{d^2h}{dt^2} + 2\zeta\tau\frac{dh}{dt} + h = \frac{\Delta p}{\rho_m g} \quad \textbf{(71)}$$

The above equation may easily be solved by standard methods. The response of the system is shown in Figure 67 for three different cases. The system is under-damped if $\zeta < 1$, critically damped if $\zeta = 1$ and over-damped if $\zeta > 1$. When the system is under-damped the output shows oscillatory behaviour, the output shows an overshoot (a value more than the input) and the output settles down slowly. In the other two cases the response is monotonic, as shown in the figure. In the over-damped case the response grows slowly to eventually reach the full value.

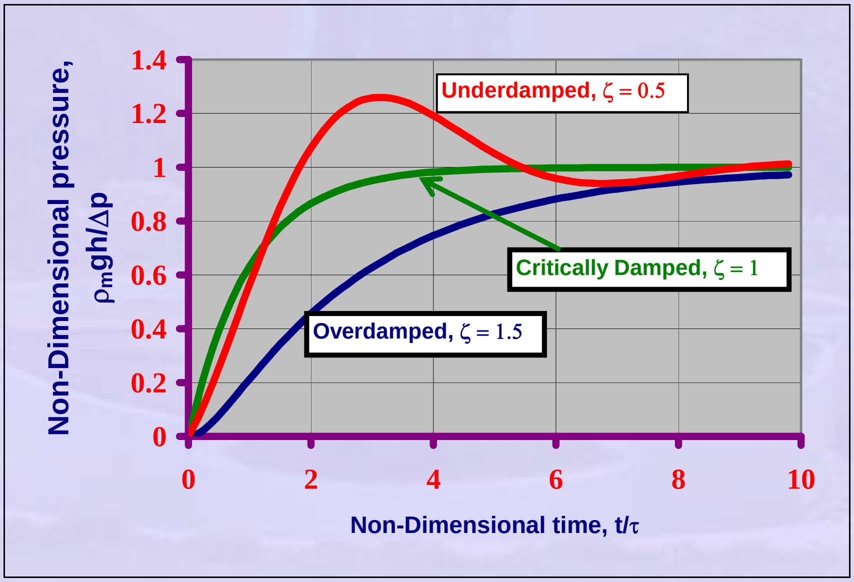


Figure 67 Response of U tube manometer to step input

Example 24

- A U tube manometer uses mercury as the manometer liquid having a density of $\rho_m = 13580\ kg/m^3$ and kinematic viscosity of $\nu = 1.1\times10^{-7}\ m^2/s$. The total length of the liquid is L = 0.6 m. The tube diameter is 2 mm. Determine the characteristic time and the damping ratio for this installation.
- Redo the above with water as the manometer liquid. The density and kinematic viscosity of water are 996 kg/m^3 and 10^{-6} m^2/s respectively.

Part a

- The given data is: $L = 0.6\ m, g = 9.81\ m/s^2$ and hence the characteristic time is

$$\tau = \sqrt{\frac{L}{g}} = \sqrt{\frac{0.6}{9.81}} = 0.247\ s$$

- The liquid viscous resistance is calculated as

$$R = \frac{128\nu L}{\pi d^4} = \frac{128\times1.1\times10^{-7}\times0.6}{\pi\times0.002^4} = 168067.6\,\frac{Pa}{kg/s}$$

- The area of cross section of the tube is

$$A = \frac{\pi d^2}{4} = \frac{\pi\times0.002^2}{4} = 3.142\times10^{-6}\ m^2$$

- The damping ratio is then calculated as

$$\zeta = \frac{RA}{2g\tau} = \frac{168067.6\times3.142\times10^{-6}}{2\times9.81\times0.247} = 0.109$$

- The system is under-damped

Part b

- When the manometer liquid is changed to water, only the resistance and the damping ratio change. The characteristic time remains the same at 0.247 s. The calculations may be repeated to get the following:

$$R = \frac{128\nu L}{\pi d^4} = \frac{128\times 10^{-6}\times 0.6}{\pi \times 0.002^4} = 1527887.5\,\frac{Pa}{kg/s}$$

$$\zeta = \frac{RA}{2g\tau} = \frac{1527887.5\times 3.142\times 10^{-6}}{2\times 9.81\times 0.247} = 0.9905$$

- Note that the system is very nearly critically damped.

This example shows how liquid properties affect the dynamic response