Sub Module 1.4

4. Regression analysis:

Now we are ready to consider curve fit or regression analysis. Suitable plot of data will indicate the nature of the trend in data and hence will indicate the nature of the relation between the independent and the dependent variables. A few examples are shown in **Figure 6(a-c)**.

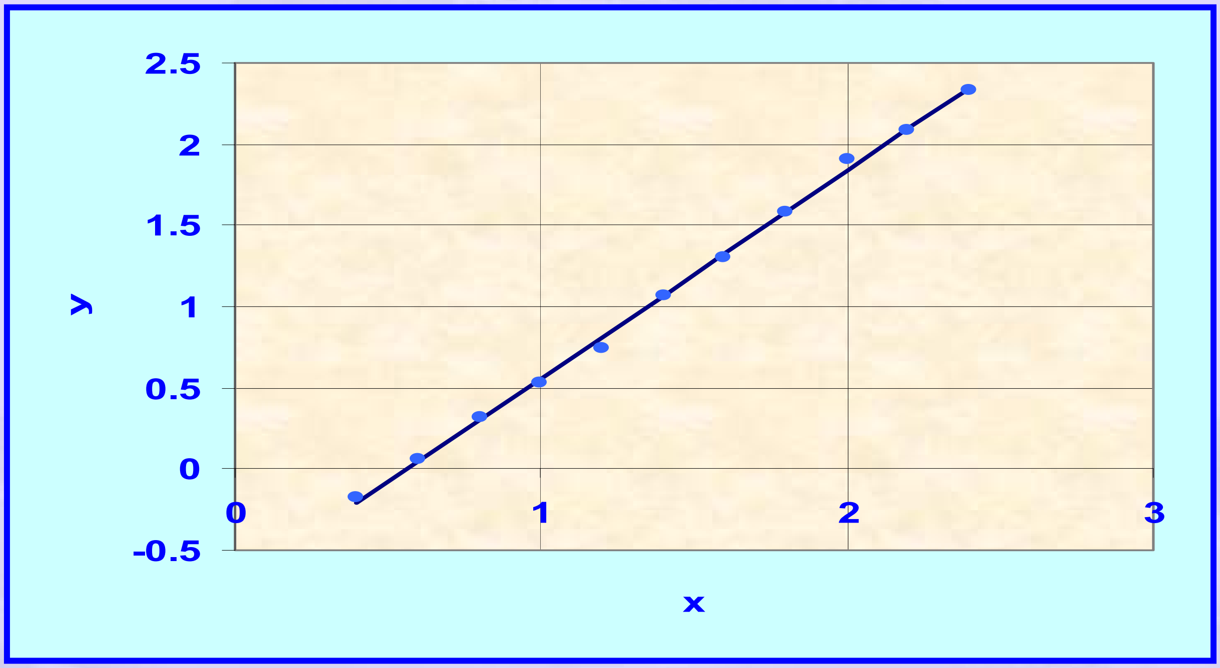

Figure 6 (a) Linear relation between y and x

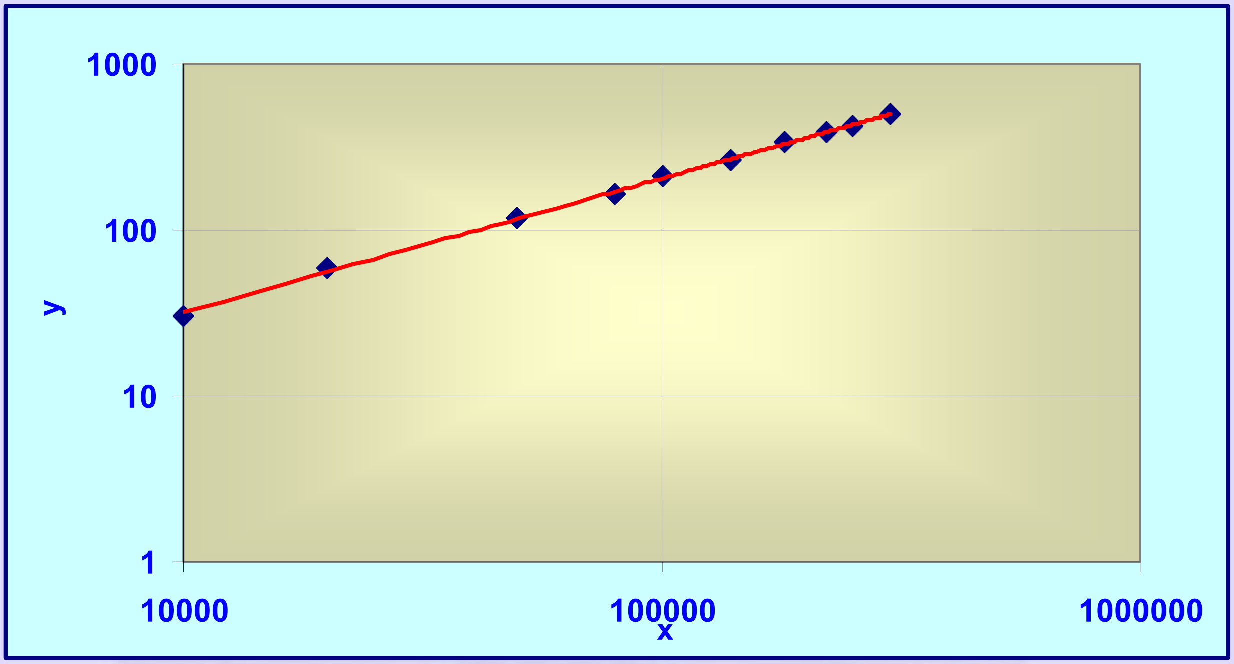

Figure 6(b) Linear relation between log x and log y

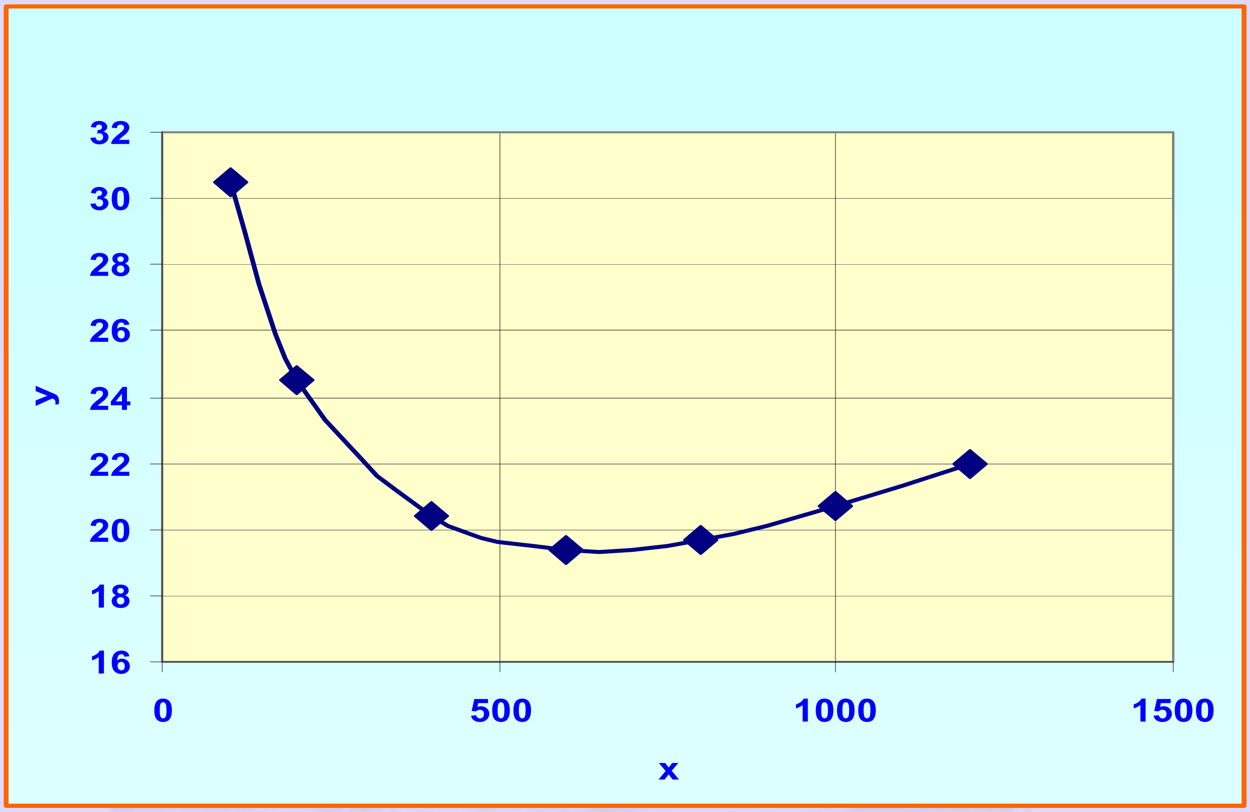

Figure 6(c) Non-linear relation between y and x

The linear graph shown in Figure 6(a) follows a relationship of the form y=ax+b. The linear relationship on the log-log plot shown in Figure 6(b) follows the form $y = ax^b$. The non-linear relationship shown in Figure 6(c) follows a polynomial relationship of the form $y = ax^3 + bx^2 + cx + d$. The parameters a, b, c, d are known as the fit parameters and need to be determined as a part of the regression analysis.

Linear fit is possible in all the cases shown in Table 2.

Table 2

y=ax+b	Linear fit	Plots as a straight line on a linear graph sheet
$y = ax^b$	Power law fit	Plots as a straight line on a log-log graph
$y = ae^{bx}$	Exponential fit	Plots as a straight line on a semi-log graph

Linear regression:

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), ...(x_n, y_n)$ be a set of ordered pairs of data. It is expected that there is a linear relation between **y** and **x**. Thus, if we plot the data on a linear graph sheet as in Figure 2(a) the trend of the data should be well represented by a straight line. We notice that the straight line shown in the figure does not pass through ***any*** of the data points shown by full symbols. There is a deviation between the data and the line and this deviation is sometimes positive, sometimes negative, sometimes large and sometimes small. If we look at the value given by the straight line as a local mean then the deviations are distributed with respect to the local mean as a normal distribution. If all data are obtained with equal care one may expect the deviations at various data points to follow the same distribution and hence the least square principle may be applied as under:

$$\text{Minimise } s^2 = \frac{\sum_1^n [y_i - y_f]^2}{n} = \frac{\sum_1^n [y_i - (ax_i + b)]^2}{n} \qquad \textbf{(26)}$$

where $y_f = ax + b$ is the desired linear fit to data. We see that s^2 is the variance of the data with respect to the fit and minimization will yield the proper choice of the mean line represented by the proper parameters **a** and **b**. The minimization requires that

$$\frac{\partial s^2}{\partial a} = -\frac{1}{n}\sum_1^n 2\left[y_i - \left(ax_i + b\right)\right]x_i = 0; \frac{\partial s^2}{\partial a} = -\frac{1}{n}\sum_1^n 2\left[y_i - \left(ax_i + b\right)\right] = 0 \qquad \textbf{(27)}$$

These equations may be rearranged as two simultaneous equations for *a* and *b* as given below:

$$\left(\sum x_i^2\right)a + \left(\sum x_i\right)b = \sum x_i y_i$$

$$\left(\sum x_i\right)a + nb = \sum y_i \qquad \textbf{(28)}$$

These are known as normal equations. The summation is from *i=1* to *n* and is not indicated explicitly. The solution to these two equations may be obtained easily by the use of Kramer's rule.

$$a = \frac{\begin{vmatrix} \sum y_i & \sum x_i \\ \sum x_i y_i & \sum x_i^2 \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}}, b = \frac{\begin{vmatrix} n & \sum y_i \\ \sum x_i & \sum x_i y_i \end{vmatrix}}{\begin{vmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{vmatrix}} \qquad \textbf{(29)}$$

We now introduce the following definitions:

$$\bar{x} = \frac{\sum x_i}{n}, \bar{y} = \frac{\sum x_i}{n}, \sigma_x^2 = \frac{\sum x_i^2}{n} - \bar{x}^2, \sigma_y^2 = \frac{\sum y_i^2}{n} - \bar{y}^2 \text{ and } \sigma_{xy} = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y} \qquad \textbf{(30)}$$

The last of the quantities defined in (30) is known as the covariance. All the other quantities are already familiar to us from statistical analysis. With these definitions the slope of the line fit a may be written as

$$a = \frac{\sigma_{xy}}{\sigma_x^2} \qquad \textbf{(31)}$$

The latter of the expressions in (28) may be solved for the fit line intercept *b* as

$$b = \frac{\bar{y} - a}{\bar{x}} \qquad \textbf{(32)}$$

In fact the last equation indicates that the regression line passes through the point $\left(\bar{x}, \bar{y}\right)$. The fit line may be represented in the alternate form $Y_f = aX$ where $Y_f = y_f - \bar{y}$ and $\bar{X} = x - \bar{x}$.

Example 6

The following data is expected to follow a relation of the form y=ax+b. Determine the fit parameters by linear regression.

x	0.9	2.3	3.3	4.5	5.7	6.7
y	1.1	1.6	2.6	3.2	4	5

It is convenient to make a table as shown below. The data given are in columns 2 and 3. The other quantities needed to calculate the fit parameters are in the other columns.

Data No.	x	y	x^2	y^2	$x\,y$
1	0.9	1.1	0.8100	1.2100	0.9900
2	2.3	1.6	5.2900	2.5600	3.6800
3	3.3	2.6	10.8900	6.7600	8.5800
4	4.5	3.2	20.2500	10.2400	14.4000
5	5.7	4	32.4900	16.0000	22.8000
6	6.7	5	44.8900	25.0000	33.5000
Column Sum:	23.4	17.5	114.6200	61.7700	83.9500
Column Mean	3.9	2.9167	19.1033	10.2950	13.9917
σ_x^2	3.8933	Slope of the fit line is: a =			0.6721
σ_y^2	1.7881	The intercept is: b =			0.2955

Sums are calculated column-wise and are shown in row 8. Various means are then in row 9. The variances are in rows 10, 11 and column 2. The regression parameters are then calculated using the results of the analysis presented earlier.

The regression line is thus given by $y_f = 0.6721x + 0.2955$. The data and the fit are compared in the following table.

x	*y*	y_f
0.9	1.1	0.9
2.3	1.6	1.8
3.3	2.6	2.5
4.5	3.2	3.3
5.7	4	4.1
6.7	5	4.8

That the fit is a good representation of the data is indicated by the proximity of the respective values in the second and third columns. The plot shown in Figure 7 is further proof of this.

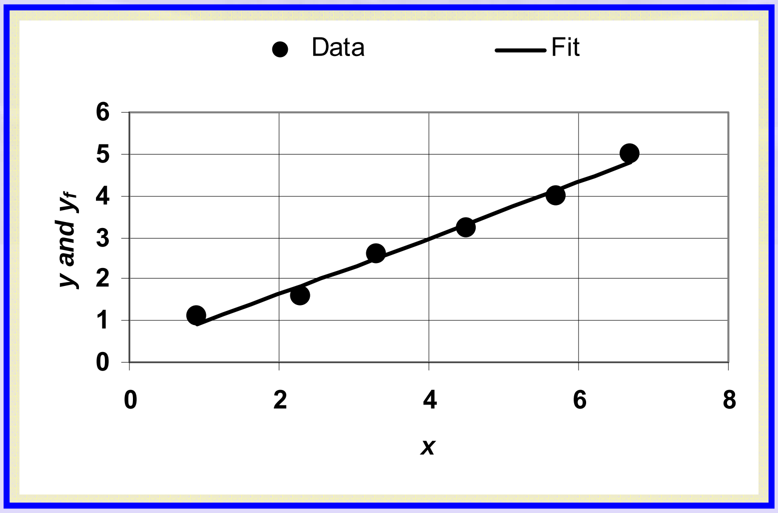

Figure 7 Comparison of data with the fit

Goodness of fit and the correlation coefficient:

A measure of how good the regression line as a representation of the data is deduced now. In fact it is possible to fit two lines to data by (a) treating x as the independent variable and y as the dependent variable or by (b) treating y as the independent variable and x as the dependent variable. The former has been done above. The latter is described by a relation of the form $x = a'y + b'$. The procedure followed earlier can be followed through to get the following (the reader is expected to show these results):

$$a' = \frac{\sigma_{xy}}{\sigma_y^2}, b' = \bar{x} - a'\bar{y} \quad \textbf{(33)}$$

The second fit line may be recast in the form

$$y' = \frac{1}{a'}x - \frac{b'}{a'} \quad \textbf{(34)}$$

The slope of this line is $\frac{1}{a'}$ which is not the same, in general, as a, the slope of the first regression line. If the two slopes are the same the two regression lines coincide. Otherwise the two lines are distinct. The ratio of the slopes of the two lines is a measure of how good the form of the fit is to the data. In view of this we introduce the correlation coefficient ρ defined through the relation

$$\rho^2 = \frac{\text{slope of first Regression line}}{\text{slope of second Regression line}} = aa' = \frac{\sigma_{xy}^2}{\sigma_x^2 \sigma_y^2} \quad \textbf{(35)}$$

Or

$$\rho = \pm \frac{\sigma_{xy}}{\sigma_x \sigma_y} \quad \textbf{(36)}$$

The sign of the correlation coefficient is determined by the sign of the covariance. If the regression line has a negative slope the correlation coefficient is negative while it is positive if the regression line has a positive slope. The correlation is said to be **perfect** if $\rho = \pm 1$. The correlation is poor if $\rho \approx 0$. Absolute value of the correlation coefficient should be greater than 0.5 to indicate that y and x are related!

In Example 6 the correlation coefficient is positive. The pertinent parameters are $\sigma_x^2 = 3.8933, \sigma_y^2 = 1.7811 \text{ and } \sigma_{xy} = 2.6167$. With these the correlation coefficient is $\rho = \frac{2.6167}{\sqrt{3.8933 \times 1.7811}} = 0.992$. Since the correlation coefficient is close to unity the fit represents the data very closely (Figure 7 has already indicated this).

Polynomial regression:

Sometimes the data may show a non-linear behavior that may be modeled by a polynomial relation. Consider a quadratic fit as an example. Let the fit equation be given by $y_f = ax^2 + bx + c$. The variance of the data with respect to the fit is again minimized with respect to the three fit parameters a, b, c to get three normal equations. These are solved for the fit parameters. Thus we have

$$s^2 = \frac{\sum\left[y_i - \left(ax^2 + bx + c\right)\right]^2}{n} \qquad \textbf{(37)}$$

Least square principle requires

$$\frac{\partial s^2}{\partial a} = \frac{2}{n}\sum\left[y_i - \left(ax^2 + bx + c\right)\right]x_i^2 = 0$$
$$\frac{\partial s^2}{\partial b} = \frac{2}{n}\sum\left[y_i - \left(ax^2 + bx + c\right)\right]x_i = 0 \qquad \textbf{(38)}$$
$$\frac{\partial s^2}{\partial c} = \frac{2}{n}\sum\left[y_i - \left(ax^2 + bx + c\right)\right] = 0$$

These may be rewritten as

$$a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 = \sum x_i^2 y_i$$
$$a\sum x_i^3 + b\sum x_i^2 + c\sum x_i = \sum x_i y_i \qquad \textbf{(39)}$$
$$a\sum x_i^2 + b\sum x_i + nc = \sum y_i$$

Normal equations (39) are easily solved for the three fit parameters to complete the regression analysis.

Goodness of fit and the index of correlation:

In the case of a non-linear fit we define a quantity known as the index of correlation to determine the goodness of the fit. The fit is termed good if the variance of the deviates is much less than the variance of the y's. Thus we require the index of correlation defined below to be close to ±1 for the fit to be considered good.

$$\rho = \pm\sqrt{1 - \frac{s^2}{\sigma_y^2}} = \pm\sqrt{1 - \frac{\sum\left[y - y_f\right]^2}{\sum\left[y - \bar{y}\right]^2}} \qquad \textbf{(40)}$$

It can be shown that the index of correlation is identical to the correlation coefficient for a linear fit. The index of correlation compares the scatter of the data with respect to its own mean as compared to the scatter of the data with respect to the regression curve.

Example 7

The friction factor Reynolds number product fRe for laminar flow in a rectangular duct is a function of the aspect ratio $A = \frac{h}{w}$ where h is the height and w is the width of the rectangle. The following table gives the available data:

A	0	0.05	0.10	0.125	0.167	0.25	0.4	0.5	0.75	1
fRe	96	89.81	84.68	82.34	78.81	72.93	65.47	62.19	57.87	56.91

Make a suitable fit to data.

A plot of the given data indicates that a quadratic fit may be appropriate. For the purpose of the following analysis we represent the aspect ratio as *x* and the fRe product as *y*. We seek a fit to data of the form $y_f = ax^2 + bx + c$. The following tabulation helps in the regression analysis.

No.	*x*	*y*	x^2	x^3	x^4	*x y*	x^2y
1	0	96	0	0	0	0	0
2	0.05	89.81	0.0025	0.000125	6.25E-06	4.4905	0.224525
3	0.1	84.68	0.01	0.001	0.0001	8.468	0.8468
4	0.125	82.34	0.015625	0.001953	0.000244	10.2925	1.286563
5	0.167	78.81	0.027889	0.004657	0.000778	13.16127	2.197932
6	0.25	72.93	0.0625	0.015625	0.003906	18.2325	4.558125
7	0.4	65.47	0.16	0.064	0.0256	26.188	10.4752
8	0.5	62.19	0.25	0.125	0.0625	31.095	15.5475
9	0.75	57.89	0.5625	0.421875	0.316406	43.4175	32.56313
10	1	56.91	1	1	1	56.91	56.91
sum	3.342	747.03	2.091014	1.634236	1.409541	212.2553	124.6098

The three normal equations are then given by

$$1.409541a + 1.634236b + 2.091014c = 124.6098$$
$$1.634236a + 2.091014b + 3.342c = 212.2553$$
$$2.091014a + 3.342b + 10c = 747.03$$

These three simultaneous equations are solved to get the three fit parameters as a=58.354, b=-94.432, c=94.06

The following table helps in comparing the data with the fit.

x	y	y_f	$s^2=(y-y_f)^2$	s_y^2
0	96	94.06	3.763026	453.5622
0.05	89.81	89.48	0.105978	228.2214
0.1	84.68	85.20	0.270956	99.54053
0.125	82.34	83.17	0.685562	58.32377
0.167	78.81	79.91	1.226588	16.86745
0.25	72.93	74.09	1.367455	3.143529
0.4	65.47	65.62	0.023763	85.24829
0.5	62.19	61.43	0.573285	156.5752
0.75	57.89	56.06	3.346951	282.677
1	56.91	57.98	1.150145	316.5908
Sum	747.03	747.03	12.51371	1700.75
Mean	74.703		1.251371	170.075

The table also shows how the index of correlation is calculated. The column sums and column means required are given the last two rows of the table. Note that calculation of σ_y^2 requires sums of the form $\left[y-\bar{y}\right]^2$ where $\bar{y}$ is available as the last entry in column 2. The index of correlation uses the mean values of columns 4 and 5 given by $\sigma_y^2 = 170.075$ and $s^2 = 1.251371$. The index of correlation is thus equal to $\rho = \sqrt{1-\frac{1.251371}{170.075}} = -0.963.$ The negative sign indicates that y decreases when x increases. The index of correlation is close to -1 and hence the fit represents the data very well. A plot of the data along with the fit given in Figure 8 also indicates this. The standard error of the fit is given by $s = \sqrt{1.251371} = \pm 1.12$.

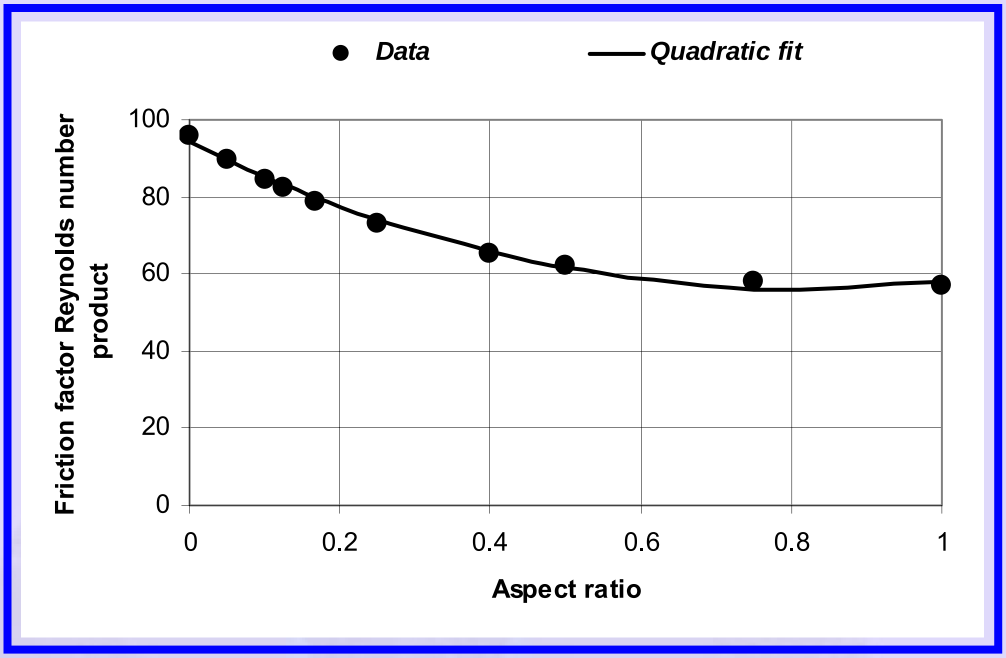

Figure 8 Comparison of the data with the quadratic fit

In the above we have considered cases that involved one independent variable and one dependent variable. Sometimes the dependent variable may be a function of more than one variable. For example, the relation of the form $\mathrm{Nu} = a\,\mathrm{Re}^b\,\mathrm{Pr}^c$ is a common type of relationship between the Nusselt number (*Nu*, dependent variable) and Reynolds (Re) and Prandtl (Pr) numbers both of which are independent variables. By taking logarithms, we see that $\log(\mathrm{Nu}) = \log(a) + b\log(\mathrm{Re}) + c\log(\mathrm{Pr})$. It is thus seen that the relationship is linear when logarithms of the dependent and independent variables are used to describe the fit. Also the relationship may be expressed in the form z=ax+by+c, where z is the dependent variable, x and y are independent variables and a, b, c are the fit parameters. The least square method may be used to determine the fit parameters. Let the data be available for set of n x, y values. The quantity to be minimized is given by

$$s^2 = \sum_i \left[z_i - \left(ax + bx + c \right) \right]^2 \quad \textbf{(41)}$$

The normal equations are obtained by the usual process of setting the first partial derivatives with respect to the fit parameters to zero.

$$\begin{aligned} a\sum x_i^2 + b\sum x_i y_i + c\sum x_i &= \sum x_i z_i \\ a\sum x_i y_i + b\sum y_i^2 + c\sum y_i &= \sum y_i z_i \\ a\sum x_i + b\sum y_i + nc &= \sum z_i \end{aligned} \quad \textbf{(42)}$$

These equations are solved simultaneously to get the three fit parameters.

Example 8

The following table gives the variation of z with x and y. Obtain a multiple linear fit to the data and comment on the goodness of the fit.

No.	x	y	z
1	0.1	0.2	0.426
2	0.3	0.35	0.539
3	0.559	0.5	0.651
4	0.847	0.65	0.786
5	1.156	0.8	0.892
6	1.48	0.95	1.058
7	1.817	1.1	1.185
8	2.168	1.25	1.33
9	2.525	1.4	1.474
10	2.893	1.55	1.634

The calculation procedure follows that given previously. Several sums are required and these are tabulated below.

No.	x	y	z	x^2	x y	y^2	x z	y z
1	0.1	0.2	0.426	0.01	0.02	0.04	0.0426	0.0852
2	0.3	0.35	0.539	0.09	0.105	0.1225	0.1617	0.18865
3	0.559	0.5	0.651	0.312481	0.2795	0.25	0.363909	0.3255
4	0.847	0.65	0.786	0.717409	0.55055	0.4225	0.665742	0.5109
5	1.156	0.8	0.892	1.336336	0.9248	0.64	1.031152	0.7136
6	1.48	0.95	1.058	2.1904	1.406	0.9025	1.56584	1.0051
7	1.817	1.1	1.185	3.301489	1.9987	1.21	2.153145	1.3035
8	2.168	1.25	1.33	4.700224	2.71	1.5625	2.88344	1.6625
9	2.525	1.4	1.474	6.375625	3.535	1.96	3.72185	2.0636
10	2.893	1.55	1.634	8.369449	4.48415	2.4025	4.727162	2.5327
Sum	13.845	8.75	9.975	27.40341	16.0137	9.5125	17.31654	10.39125

The last row contains the sums required and the normal equations are easily written down as under:

$$27.40341a + 16.0137b + 13.845c = 17.31654$$
$$16.0137a + 9.5125b + 8.75c = 10.39125$$
$$13.845a + 8.75b + 10c = 9.975$$

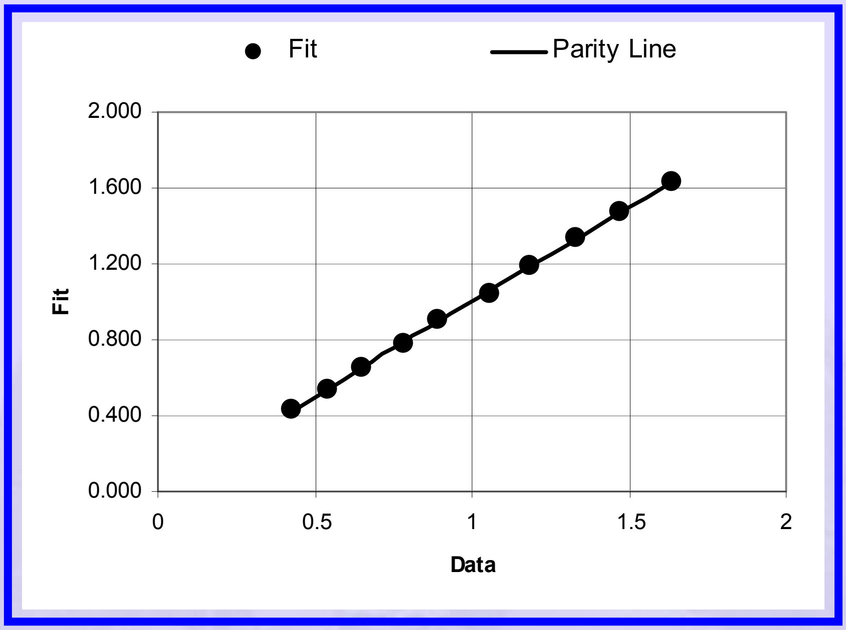

Figure 9 Parity plot showing the goodness of the fit

These are solved to get the fit parameters as a=0.285, b=0.297, c=0.343. The data and the fit may be compared by making a parity plot as shown in Figure 9. The parity plot is a plot of given data (z) along the abscissa and the fit (z_f) along the ordinate. The parity line is a line of equality between the two. The departure of the data from the parity line is an indication of the quality of the fit. The above figure indicates that the fit is indeed very good. When the data is a function of more than one independent variable it is not always possible to make plots between independent and dependent variables. In such a case the parity plot is a way out.

We may also calculate the index of correlation as an indicator of the quality of the fit. This calculation is left to the reader!

General non-linear fit:

The fit equation may sometimes have to be chosen as a non-linear relation that is not either a polynomial or in a form that may be reduced to the linear form. In such a case the parameter estimation is more involved and requires the use of a search method to determine the best parameter set that minimizes the sum of the squares of the residual. The method is described in some detail below.

Let us represent the fit relation in the form $y_f = f(x; a_1, a_2, ...a_m)$ where the dependent variable is *x* and $a_1 - a_m$ are m fit parameters to be determined by the regression analysis. As before we assume that n sets of x, y values are available. Consider the sum of the squares of the residual given by

$$s^2(a_1..a_m) = S(a_1..a_m) = \sum_i \left[y_i - f(x; a_1, a_2, ...a_m) \right]^2 \qquad \textbf{(43)}$$

In general it is not possible to set the partial derivatives with respect to the parameters to zero to obtain the normal equations and thus obtain the fit parameters. In view of this let us look at what is happening to the sum of squares near a starting parameter set $a_1^0, a_2^0, ...a_m^0$. The sum of squares is evaluated using this parameter set in equation **(43)** to get $S^0 = S\left(a_1^0, a_2^0, ...a_m^0\right)$. Perturb each of the a's individually to get $S\left(a_1^0 + \Delta a_1, a_2^0..a_m^0\right), S\left(a_1^0 + \Delta a_1, a_2^0..a_m^0\right), S\left(a_1^0 + \Delta a_1, a_2^0..a_m^0\right), .. S\left(a_1^0, a_2^0, ..a_j^0 + \Delta a_j..a_m^0\right)$ $S\left(a_1^0, a_2^0, ..a_j^0 + \Delta a_j..a_m^0\right), S\left(a_1^0, a_2^0, ..a_m^0 + a_m^0\right)$. Using these we may estimate the partial derivatives by the use of finite difference approximation as $\left.\frac{\partial S}{\partial a_j}\right|_{\left(a_1^0, a_2^0,..a_m^0\right)} = \frac{S\left(a_1^0, a_2^0,..a_j^0 + \Delta a_j..a_m^0\right) - S\left(a_1^0, a_2^0, a_j^0..a_m^0\right)}{\Delta a_j}$. There are *m* such partial derivatives and they are all likely to be non-zero (if they are all zero we are already at the optimum point where the sum of squares is possibly a minimum). The gradient vector is then given by the components $\left(\frac{\partial S}{\partial a_1}, \frac{\partial S}{\partial a_2},..\frac{\partial S}{\partial a_j}..\frac{\partial S}{\partial a_m}\right)$. The magnitude of this vector is obtained by summing the squares of all the partial derivatives and then taking the square root of this sum.

$$\left|\text{grad S}\right| = \sqrt{\sum_{j=1}^{m}\left(\frac{\partial S}{\partial a_j}\right)^2} \qquad \textbf{(44)}$$

We divide each of the partial derivatives occurring in the gradient vector by the magnitude of the gradient vector thus calculated to get the components of a unit vector that is aligned with the gradient vector. Thus

$$\frac{\left(\frac{\partial S}{\partial a_1}\right)}{\left|\text{grad S}\right|}, \frac{\left(\frac{\partial S}{\partial a_2}\right)}{\left|\text{grad S}\right|},..\frac{\left(\frac{\partial S}{\partial a_j}\right)}{\left|\text{grad S}\right|},..\frac{\left(\frac{\partial S}{\partial a_m}\right)}{\left|\text{grad S}\right|} \qquad \textbf{(45)}$$

We now take a specific fraction **(α, small)** of each of these components to define a step along a direction opposite the gradient vector to get

$$\begin{gathered} a_1^1 = a_1^0 - \alpha\frac{\left(\frac{\partial S}{\partial a_1}\right)}{\left|\text{grad S}\right|}, a_2^1 = a_2^0 - \alpha\frac{\left(\frac{\partial S}{\partial a_2}\right)}{\left|\text{grad S}\right|},.. \\ a_j^1 = a_j^0 - \alpha\frac{\left(\frac{\partial S}{\partial a_j}\right)}{\left|\text{grad S}\right|},..a_m^1 = a_m^0 - \alpha\frac{\left(\frac{\partial S}{\partial a_m}\right)}{\left|\text{grad S}\right|} \end{gathered} \qquad \textbf{(46)}$$

The calculation above is redone with the new values of the parameter set.

The calculation is continued till the magnitude of the gradient reaches **zero** or **acceptably small value** at which the calculation stops and the parameter set is assumed to have satisfied the least square principle. An example will make this procedure clear.

Example 9

The data given in the following table is expected to follow a relation of the form $y_f = ae^{bx} + cx$. Determine the fit parameters by general non-linear regression.

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1,8
y	1.196	1.379	1.581	1.79	2.013	2.279	2.545	2.842	3.173	3.5

The sum of squares of the residual is given by $S = \sum_{i=1}^{10}\left[y_i - \left(ae^{bx_i} + cx_i\right)\right]^2$. The partial derivatives needed are obtained analytically

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{10} 2\left[y_i - \left(ae^{bx_i} + cx_i\right)\right]e^{bx_i},\ \frac{\partial S}{\partial b} = \sum_{i=1}^{10} 2\left[y_i - \left(ae^{bx_i} + cx_i\right)\right]\left(-ax_i e^{bx_i}\right),$$
$$\frac{\partial S}{\partial a} = \sum_{i=1}^{10} 2\left[y_i - \left(ae^{bx_i} + cx_i\right)\right]x_i \quad \textbf{(47)}$$

The above means that the partial derivatives may be computed once the starting set of parameters is known or assumed. We start the calculation with the initial parameter seta=1,b=0.2,c=0.1. The value of S turns out to be 11.673 for this set of parameter values. Using (47) the partial derivatives are obtained respectively as-24.023,-30.681,-23.003.The magnitude of the gradient vector is then given by $\left[(-24.023)^2,(-24.023)^2,(-24.023)^2\right]^{0.5}=45.249$. The components of the unit vector u_a^0,u_b^0,u_c^0 are then given by $-\frac{24.023}{45.249},-\frac{-30.681}{45.249},-\frac{-23.003}{45.249}$ or- 0.531,-0.678,-0.508. We shall choose α value of 0.02 to get the next trial values for the parameters as

$$a^1=a^0-\alpha u_a^0=1-(0.02\times-0.531)=1.011$$
$$b^1=b^0-\alpha u_b^0=1-(0.02\times-0.678)=0.214$$
$$c^1=c^0-\alpha u_c^0=1-(0.02\times-0.508)=0.11$$

The S value for this parameter set turns out to be 10.759. The calculations may be repeated as above. The results are summarized below.

α	a	b	c	$\lvert\text{grad S}\rvert$	S
0.02	1	0.2	0.1	45.25	11.67
	1.011	0.214	0.11	44.29	10.76
	1.022	0.228	0.12	43.25	9.87
..	..	..	..	..	..
	1.219	0.505	0.265	0.848	0.0286
0.005	1.21004	0.5984	0.2665	0.2831	0.001265
0.001	1.209288	0.509251	0.266293	0.108248	0.00108

It is clear from the table that a large number of trials are involved in the regression analysis. The value of α needs to be reduced as we approach the optimum set. The final set of parameters for the present case is given by a=1.2093,b=0.5093,c=0.2663

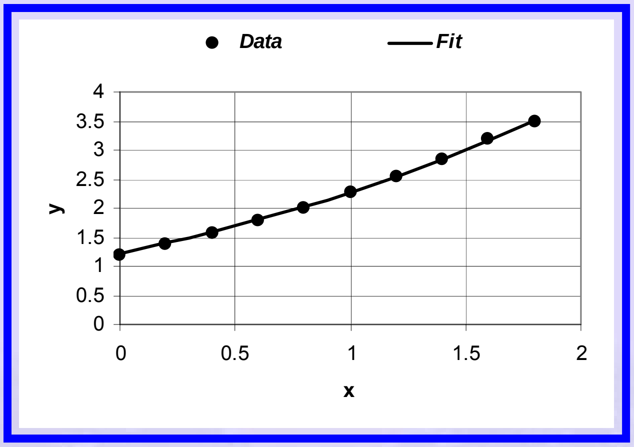

Figure 10 Comparison of data with the fit

That the regression analysis has indeed converged to the proper fit parameters is seen from the excellent agreement between the data and the fit shown in Figure 10. The reader is left to determine the index of correlation and the standard error of the fit.