Module 4: Worked out problems

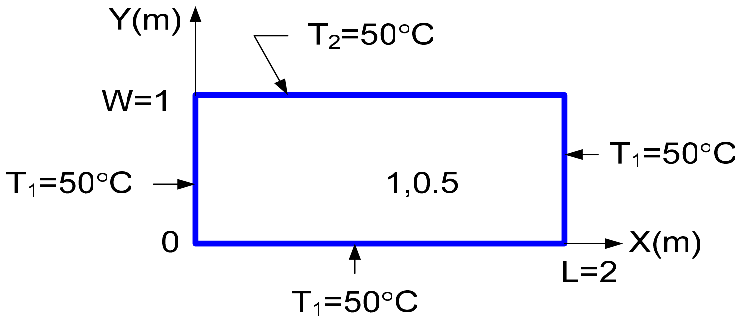
Problem 1:

A two dimensional rectangular plate is subjected to the uniform temperature boundary conditions shown. Using the results of the analytical solution for the heat equation, calculate the temperature at the midpoint (1, 0.5) by considering the first five nonzero terms of the infinite series that must be evaluated. Assess the error from using only the first three terms of the infinite series.

Known: Two-dimensional rectangular plate to prescribed uniform temperature boundary conditions.

Find: temperature at the midpoint using the exact solution considering the first five nonzero terms: Assess the error from using only the first three terms.

Schematic:



Assumptions: (1) Two-dimensional, steady-state conduction, (2) constant properties.

Analysis: From analytical solution, the temperature distribution is

$$\theta(x,y) = \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi\, x}{L}.\frac{\sinh(n\pi\, y/L)}{\sinh(n\pi\, W/L)}$$

Considering now the point (x, y) = (1.0, 0.5) and recognizing x/L =1/2, y/L=1/4 and W/L=1/2, the distribution has the form

$$\theta(1,0.5) = \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi}{L}.\frac{\sinh(n\pi/L)}{\sinh(n\pi/L)}$$

When n is even (2,4,6…..), the corresponding term is zero; hence we need only consider n=1,3,5,7 and 9 as the first five non-zero terms.

$$\theta(1,0.5)=\frac{2}{\pi}\left\{\begin{array}{r}2\sin\left(\frac{\pi}{2}\right)\frac{\sinh\frac{\pi}{4}}{\sinh\frac{\pi}{2}}+\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh\frac{3\pi}{4}}{\sinh\frac{3\pi}{2}}+\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh\frac{5\pi}{4}}{\sinh\frac{5\pi}{2}}+\\ \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh\frac{7\pi}{4}}{\sinh\frac{7\pi}{2}}+\frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh\frac{9\pi}{4}}{\sinh\frac{9\pi}{2}}\end{array}\right\}$$

$$\theta(1,0.5)=\frac{2}{\pi}\left[0.755+0.063+0.008-0.001+0.000\right]=0.445$$

Since $\theta = (T-T_1)/(T_2-T_1)$, it follows that

$T(1, 0.5) = \theta(1, 0.5)(T_2-T_1) + T_1 = 0.445(150-50) + 50 = 94.5°C$

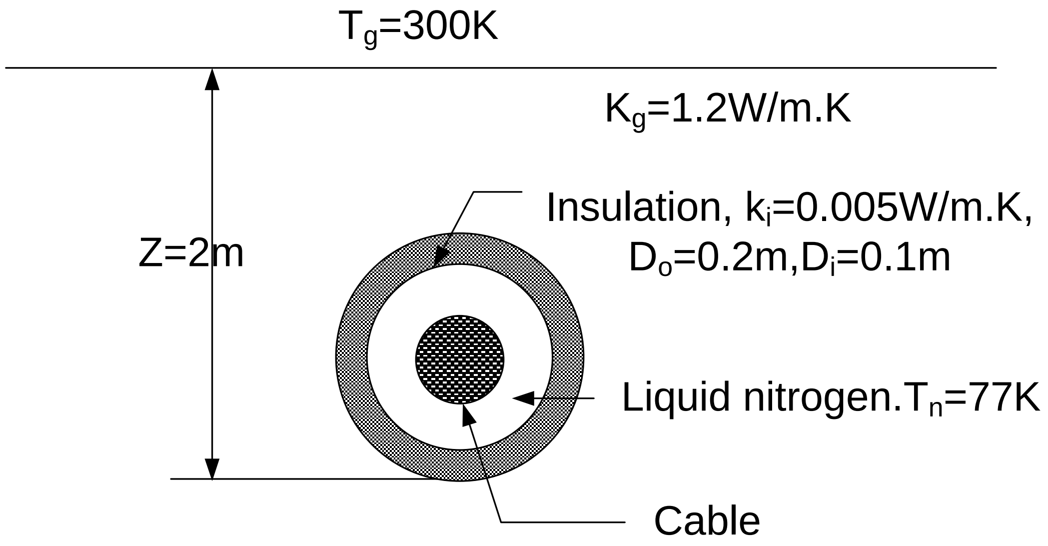
If only the first term of the series, Eq (2) is considered, the result will be $\theta(1, 0.5) = 0.446$ that is, there is less than a 0.2% effect.

Problem 2:
A long power transmission cable is buried at a depth (ground to cable centerline distance) of 2m. The cable is encased in a thin walled pipe of 0.1 m diameter, and to render the cable superconducting (essentially zero power dissipation), the space between the cable and pipe is filled with liquid nitrogen at 77 K. If the pipe is covered a super insulator(k_i=0.005W/m.K) of 0.05-m thickness and the surface of the earth (k_g=1.2W/m.K) is at 300K, what is the cooling load in W/m which must be maintained by a cryogenic refrigerator per unit pipe length.

Known: Operating conditions of a buried superconducting cable.

Find: required cooling load.

Schematic:



Assumptions: (1) steady-state conditions, (2) constant properties, (3) two-dimensional conduction in soil, (4) one-dimensional conduction in insulation.

Analysis: The heat rate per unit length is

$$q' = \frac{T_g - T_n}{R'_g + R'_I}$$

$$q' = \frac{T_g - T_n}{[k_g(2\pi/\ln(4z/D_o))]^{-1} + \ln(D_o/D_i)/2\pi k_i}$$

where table 4.1 have been used to evaluate the ground resistance. Hence,

$$q' = \frac{(300-77)K}{[(1.2W/m.K)(2\pi/\ln(8/0.2))]^{-1} + \ln(2)/2\pi \times 0.005\,W/m.K}$$

$$q' = \frac{223K}{(0.489 + 22.064)m.K/W}$$

$$q' = 9.9W/m$$

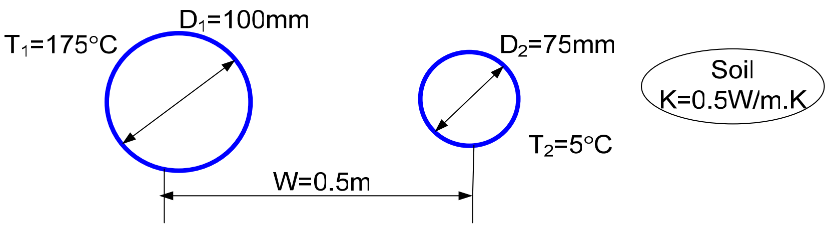
Comments: the heat gain is small and the dominant contribution to the thermal resistance is made by the insulation.

Problem 3:
Two parallel pipelines spaced 0.5 m apart are buried in soil having a thermal conductivity of 0.5W/m.K. the pipes have outer-diameters of 100 and 75 mm with surface temperatures of 175°C and 5°C, respectively. Estimate the heat transfer rate per unit length between the two pipe lines.

Known: Surfaces temperatures of two parallel pipe lines buried in soil.

Find: heat transfer per unit length between the pipe lines.

Schematic:



Assumptions: (1) steady state conditions, (2) two-dimensional conduction, (3) constant properties, (4) pipe lines are buried very deeply approximating burial in an infinite medium, (5) pipe length>> D_1 or D_2 and w>D_1 or D_2

Analysis: the heat transfer rate per length from the hot pipe to the cool pipe is

$$q' = \frac{q}{L} = \frac{S}{L} k(T_1 - T_2)$$

The shape factor S for this configuration is given in table 4.1 as

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$

Substituting numerical values

$$\frac{S}{L} = 2\pi / \cosh^{-1}\frac{4\times(0.5m)^2 - (0.1m)^2 - (0.075m)^2}{2\times 0.1m \times 0.075m} = 2\pi / \cosh^{-1}(65.63)$$

$$\frac{S}{l} = 2\pi / 4.88 = 1.29.$$

hence, the heat rate per unit length is

$$q' = 1.29 \times 0.5W / m.K(175-5)°C = 110W / m$$

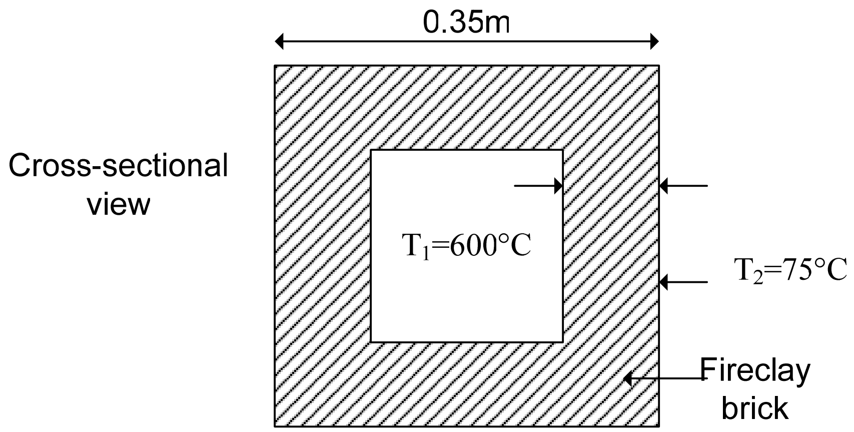
Comments: The heat gain to the cooler pipe line will be larger than 110W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

Problem 4:
A furnace of cubical shape, with external dimensions of 0.35m, is constructed from a refractory brick (fireclay). If the wall thickness is 50mm, the inner surface temperature is 600°C, and the outer surface temperature is 75°C, calculate the heat loss from the furnace.

Known: Cubical furnace, 350mm external dimensions, with 50mm thick walls.

Find: The heat loss, q(W).

Schematic:



Assumptions: (1) steady-state conditions, (2) two-dimensional conduction, (3) constant properties.

Properties: From table of properties, fireclay brick ($\bar{T} = (T_1 + T_2)/2 = 610K$) : $k \approx 1.1W/m.K$

Analysis: using relations for the shape factor from table 4.1,

Plane walls (6) $$S_W = \frac{A}{L} = \frac{0.25 \times 0.25m^2}{0.05m} = 1.25m$$

Edges (12) $$S_E = 0.54D = 0.52 \times 0.25m = 0.14m$$

Corners (8) $$S_C = 0.15L = 0.15 \times 0.05m = 0.008m$$

The heat rate in terms of the shape factor is

$$q = kS(T_1 - T_2) = k(6S_W + 12S_E + 8S_C)(T_1 - T_2)$$

$$q = 1.1\frac{W}{m.K}(6\times 1.25m + 12\times 0.14m + 0.15\times 0.008m)(600 - 75)°C$$

$$q = 5.30kW$$

Comments: Be sure to note that the restriction for S_E and S_C has been met.

Problem 5:

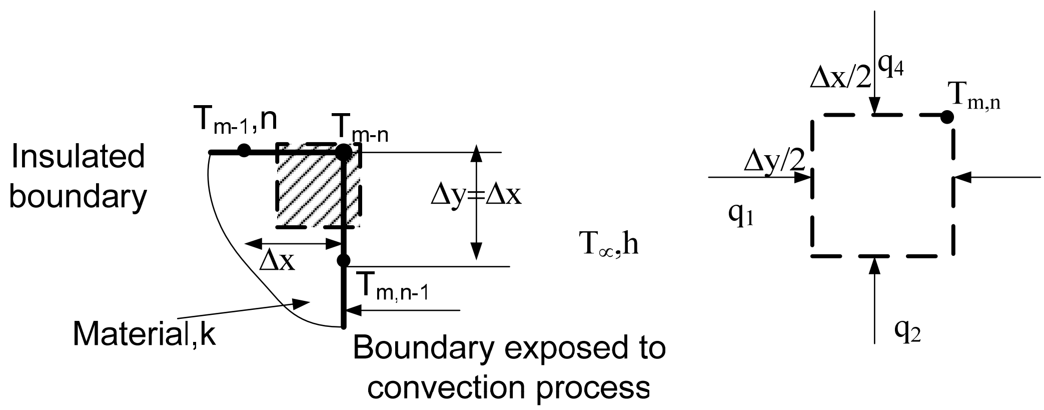
Consider nodal configuration 4 of table 4.2. Derive the finite-difference equation under steady-state conditions for the following situations.

(a) The upper boundary of the external corner is perfectly insulated and the side boundary is subjected to the convection process (T_∞, h)
(b) Both boundaries of external corner are perfectly insulated.

Known: External corner of a two-dimensional system whose boundaries are subjected to prescribed conditions.

Find: finite-difference equations for these situations: (a) upper boundary is perfectly insulated and side boundary is subjected to a convection process, (b) both boundaries are perfectly insulated.

Schematic:



Assumptions: (1) steady-state conditions, (2) two-dimensional conduction, (3) constant properties, (4) no internal generation.

Analysis: Consider the nodal point configuration shown in schematic and also as case 4, table 4.2. the control volume about the nodes shaded area above of unit thickness normal to the page has dimensions, (Δx/2)(Δy/2).1. The heat transfer processes at the surface of the CV are identified as q_1, q2….perform an energy balance wherein the processes are expressed using the appropriate rate equations.

With the upper boundary insulated and the side boundary and the side boundary subjected to a convection process, the energy balance has the form

$$\dot{E}_{in} - E_{out} = 0 \qquad q_1 + q_2 + q_3 + q_4 = 0$$

$$k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left(\frac{\Delta y}{2}.1\right)(T_\infty - T_{m,n}) + 0 = 0$$

Letting Δx=Δy, and regrouping, find

$$T_{m,n-1} + T_{m-1,n} + \frac{h\Delta x}{k} T_{\infty} - 2\left(\frac{1}{2}\frac{h\Delta x}{k} + 1\right) T_{m,n} = 0$$

with both boundaries insulated, the energy balance of Eq(2) would have q3=q4=0. the same result would be obtained by letting h=0 in the finite difference equation, Eq(3). The result is

$$T_{m,n-1} + T_{m-1,n} - 2T_{m,n} = 0$$

Comments: Note the convenience resulting formulating the energy balance by assuming that all the heat flow is into the node.