# MODULE 3

# Extended Surface Heat Transfer

# EXTENDED SURFACES / FINS

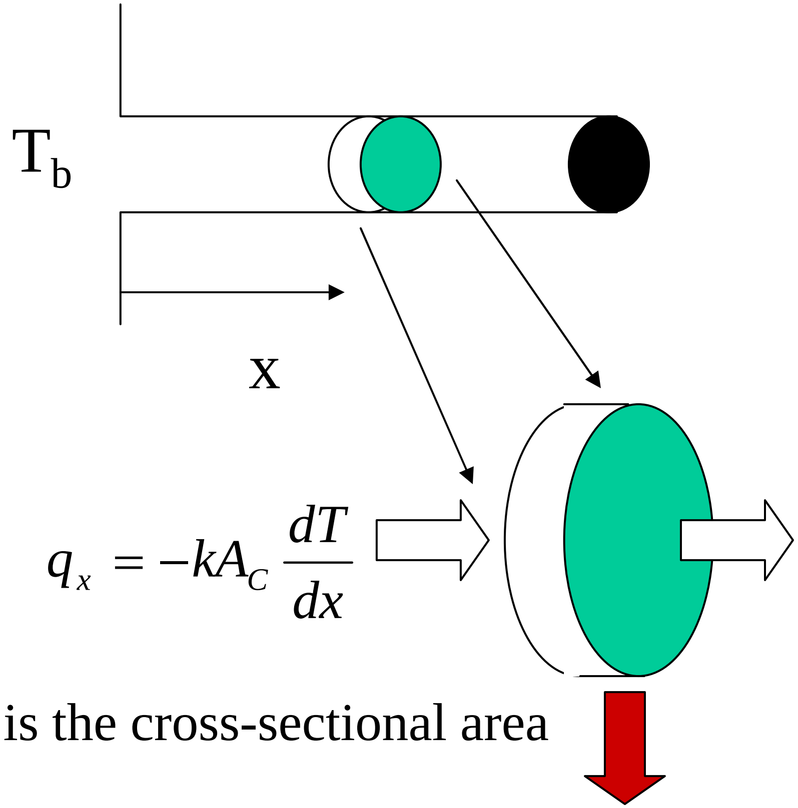
Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$. Therefore, to increase the convective heat transfer, one can

- ❑ Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.

- ❑ Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.

- ❑ Increase the contact surface area A. Example: a heat sink with fins.

# Extended Surface Analysis




P: the fin perimeter

$A_c$: the fin cross-sectional area

$$q_x = -kA_C \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$A_C$ is the cross-sectional area

$dq_{conv} = h(dA_S)(T - T_\infty)$, where $dA_S$ is the surface area of the element

Energy Balance: $q_x = q_{x+dx} + dq_{conv} = q_x + \frac{dq_x}{dx} dx + h dA_S (T - T_\infty)$

$-kA_C \frac{d^2T}{dx^2} dx + hP(T - T_\infty) dx = 0$, if k, $A_C$ are all constants.

# Extended Surface Analysis (contd….)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_C}(T - T_\infty) = 0,$$ A second-order, ordinary differential equation

Define a new variable $\theta(x) = T(x) - T_\infty$, so that

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \text{ where } m^2 = \frac{hP}{kA_C}, \quad (D^2 - m^2)\theta = 0$$

Characteristics equation with two real roots: + m & - m

The general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

To evaluate the two constants $C_1$ and $C_2$, we need to specify two boundary conditions:

The first one is obvious: the base temperature is known as $T(0) = T_b$

The second condition will depend on the end condition of the tip

# Extended Surface Analysis (contd...)

For example: assume the tip is insulated and no heat transfer $d\theta/dx(x=L)=0$

The temperature distribution is given by

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

The fin heat transfer rate is

$$q_f = -kA_C \frac{dT}{dx}(x=0) = \sqrt{hPkA_C}\tanh mL = M \tanh mL$$

These results and other solutions using different end conditions are tabulated in Table 3.4 in HT textbook, p. 118.

# Temperature distribution for fins different configurations

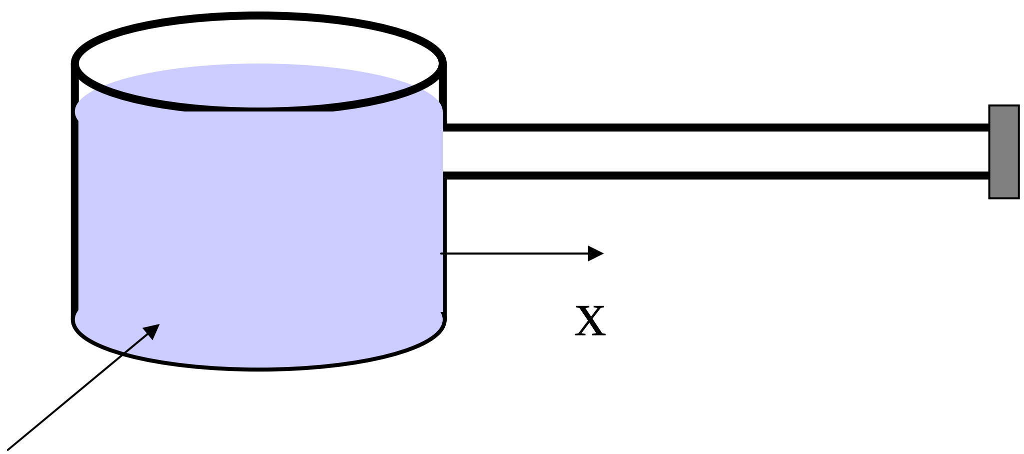
| Case | Tip Condition | Temp. Distribution | Fin heat transfer |
|---|---|---|---|
| A | Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$ | $\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ | $M\theta_o \dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ |
| B | Adiabatic $(d\theta/dx)_{x=L}=0$ | $\dfrac{\cosh m(L-x)}{\cosh mL}$ | $M\theta_0 \tanh mL$ |
| C | Given temperature: $\theta(L)=\theta_L$ | $\dfrac{(\theta_L/\theta_b)\sinh m(L-x) + \sinh m(L-x)}{\sinh mL}$ | $M\theta_0 \dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ |
| D | Infinitely long fin $\theta(L)=0$ | $e^{-mx}$ | $M\ \theta_0$ |

$$\theta \equiv T - T_\infty, \quad m^2 \equiv \frac{hP}{kA_C}$$

$$\theta_b = \theta(0) = T_b - T_\infty, \quad M = \sqrt{hPkA_C}\,\theta_b$$

# Example

- An Aluminum pot is used to boil water as shown below. The handle of the pot is 20-cm long, 3-cm wide, and 0.5-cm thick. The pot is exposed to room air at 25°C, and the convection coefficient is 5 W/m$^2$ °C. Question: can you touch the handle when the water is boiling? (k for aluminum is 237 W/m °C)



# Example (contd...)

We can model the pot handle as an extended surface. Assume that there is no heat transfer at the free end of the handle. The condition matches that specified in the fins Table, case B.

h=5 W/ m$^2$ °C, P=2W+2t=2(0.03+0.005)=0.07(m), k=237 W/m °C, $A_C$=Wt=0.00015(m$^2$), L=0.2(m)

Therefore, $m=(hP/kA_C)^{1/2}=3.138$,

$M=\sqrt{(hPkA_C)}(T_b-T_\infty)=0.111\theta_b=0.111(100-25)=8.325$(W)

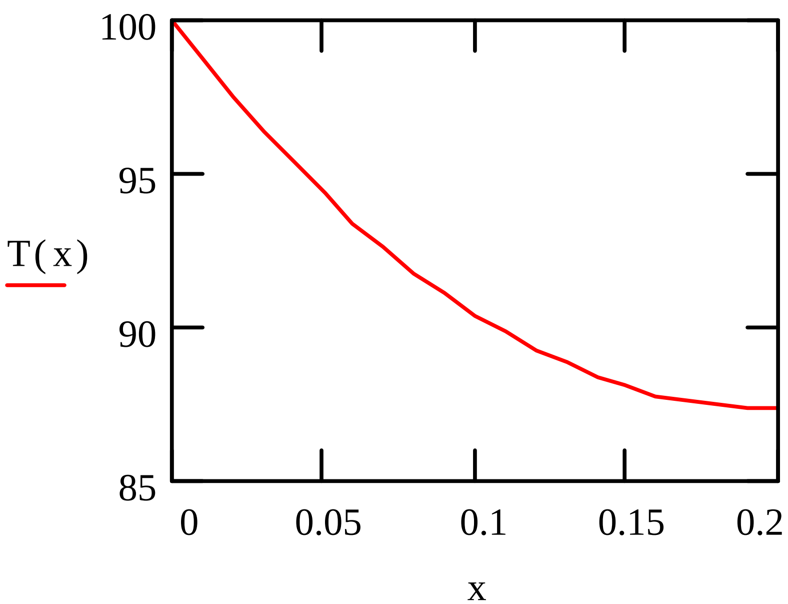
$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T-25}{100-25}=\frac{\cosh[3.138(0.2-x)]}{\cosh(3.138*0.2)},$$

$$T(x)=25+62.32*\cosh[3.138(0.2-x)]$$

# Example (contd…)

Plot the temperature distribution along the pot handle




As shown, temperature drops off very quickly. At the midpoint T(0.1)=90.4°C. At the end T(0.2)=87.3°C.

Therefore, it should not be safe to touch the end of the handle

# Example (contd...)

The total heat transfer through the handle can be calculated also. $q_f$=Mtanh(mL)=8.325*tanh(3.138*0.2)=4.632(W) Very small amount: latent heat of evaporation for water: 2257 kJ/kg. Therefore, the amount of heat loss is just enough to vaporize 0.007 kg of water in one hour.

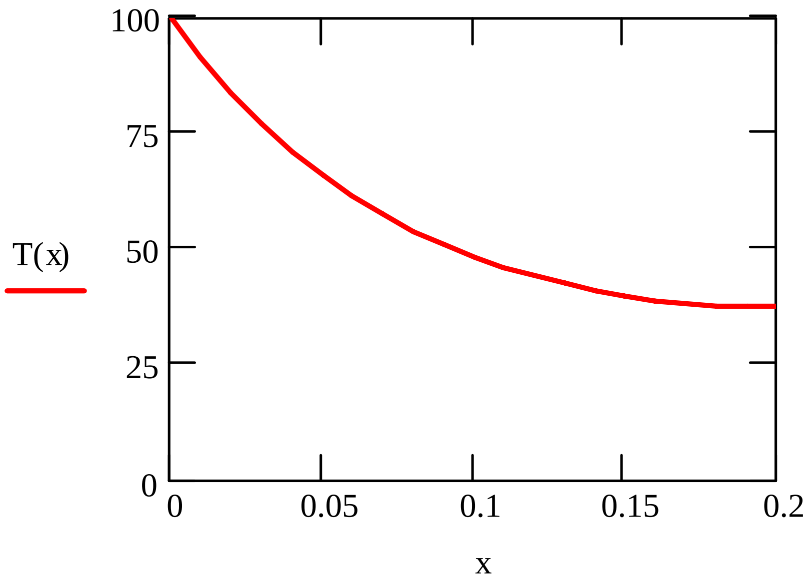
If a stainless steel handle is used instead, what will happen: For a stainless steel, the thermal conductivity k=15 W/m°C. Use the same parameter as before:

$$m = \left(\frac{hP}{kA_C}\right)^{1/2} = 12.47, \quad M = \sqrt{hPkA_C} = 0.0281$$

# Example (contd...)

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$T(x)=25+12.3\cosh[12.47(L-x)]$$




Temperature at the handle (x=0.2 m) is only 37.3 °C, not hot at all. This example illustrates the important role played by the thermal conductivity of the material in terms of conductive heat transfer.

# Fins-2

- If the pot from previous lecture is made of other materials other than the aluminum, what will be the temperature distribution? Try stainless steel (k=15 W/m.K) and copper (385 W/m.K).

Recall: $h=5W/m^2{}^\circ C$, $P=2W+2t=2(0.03+0.005)=0.07(m)$

$A_C=Wt=0.00015(m^2)$, $L=0.2(m)$

Therefore, $m_{ss}=(hP/kA_C)^{1/2}=12.47$, $m_{cu}=2.46$

$M_{ss}=\sqrt{(hPk_{ss}A_C)}\,(T_b-T_\infty)=0.028(100-25)=2.1(W)$

$M_{cu}= \sqrt{(hPk_{ss}A_C)}\,\theta_b=0.142(100-25)=10.66(W)$

For stainless steel, $$\frac{T_{ss}(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T_{ss}-25}{100-25}=\frac{\cosh[12.47(0.2-x)]}{\cosh(12.47*0.2)},$$

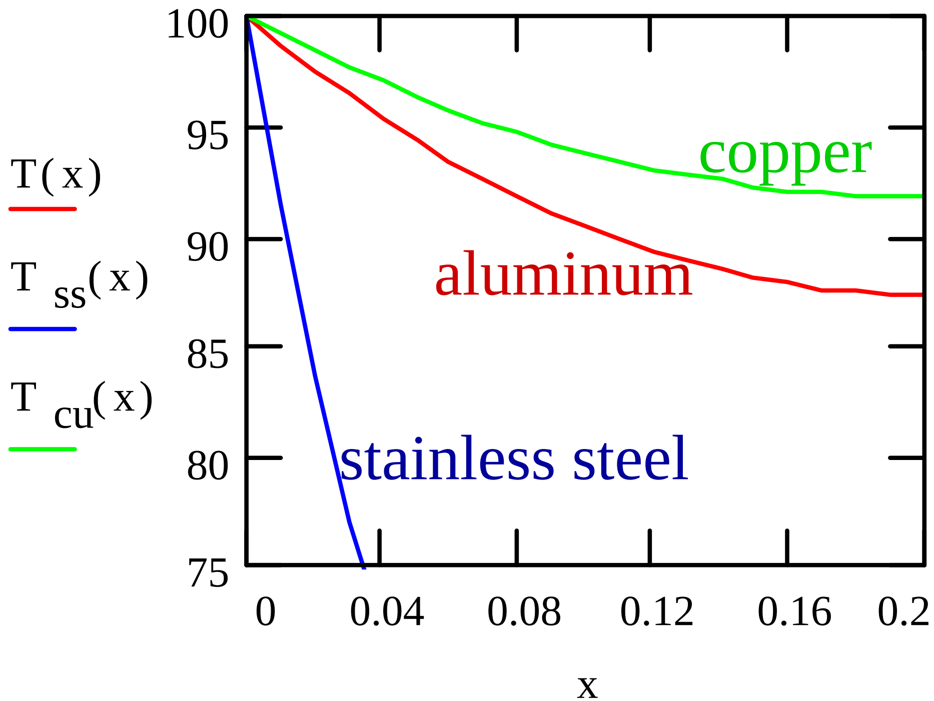
$$T_{ss}(x)=25+12.3*\cosh[12.47(0.2-x)]$$

# Fins-2 (contd....)

For copper, $\dfrac{T_{cu}(x) - T_\infty}{T_b - T_\infty} = \dfrac{\theta}{\theta_b} = \dfrac{\cosh m(L-x)}{\cosh mL}$

$$\frac{T_{cu} - 25}{100 - 25} = \frac{\cosh[2.46(0.2 - x)]}{\cosh(2.46 * 0.2)},$$

$$T_{cu}(x) = 25 + 66.76 * \cosh[2.46(0.2 - x)]$$



# Fins-2 (contd...)

- Inside the handle of the stainless steel pot, temperature drops quickly. Temperature at the end of the handle is 37.3°C. This is because the stainless steel has low thermal conductivity and heat can not penetrate easily into the handle.
- Copper has the highest k and, correspondingly, the temperature inside the copper handle distributes more uniformly. Heat easily transfers into the copper handle.
- Question? Which material is most suitable to be used in a heat sink?

# Fins-2 (contd...)

- How do we know the adiabatic tip assumption is good? Try using the convection heat transfer condition at the tip (case A in fins table) We will use the aluminum pot as the example.

h=5 W/m$^2$.K, k=237 W/m.K, *m*=3.138, M=8.325W

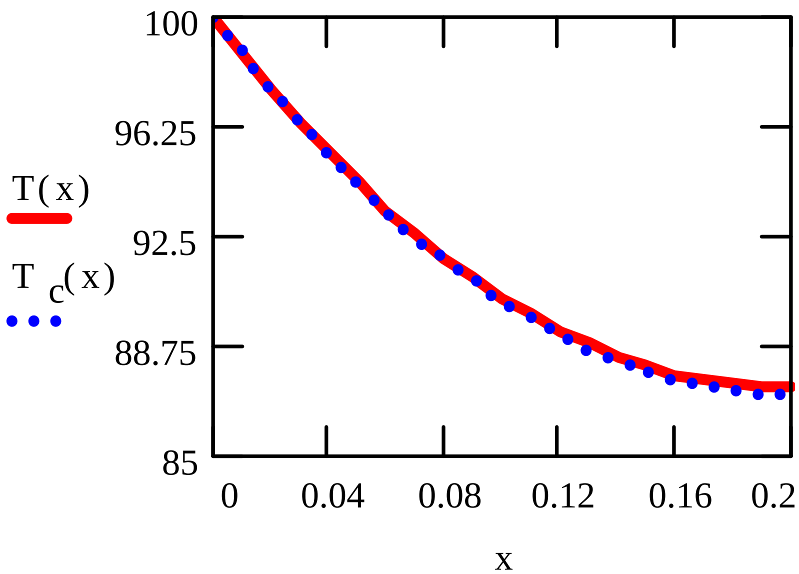
Long equation

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh[m(L-x)] + (h/mk)\sinh[m(L-x)]}{\cosh mL + (h/mk)\sinh mL}$$

$$= \frac{\cosh[3.138(0.2-x)] + 0.00672\sinh[3.138(0.2-x)]}{\cosh(0.6276) + 0.00672\sinh(0.6276)}$$

$$T(x) = 25 + 62.09\{\cosh(0.6276 - 3.138x) + 0.00672\sinh(0.6276 - 3.138x)\}$$

# Fins-2 (contd….)




T: adiabatic tip

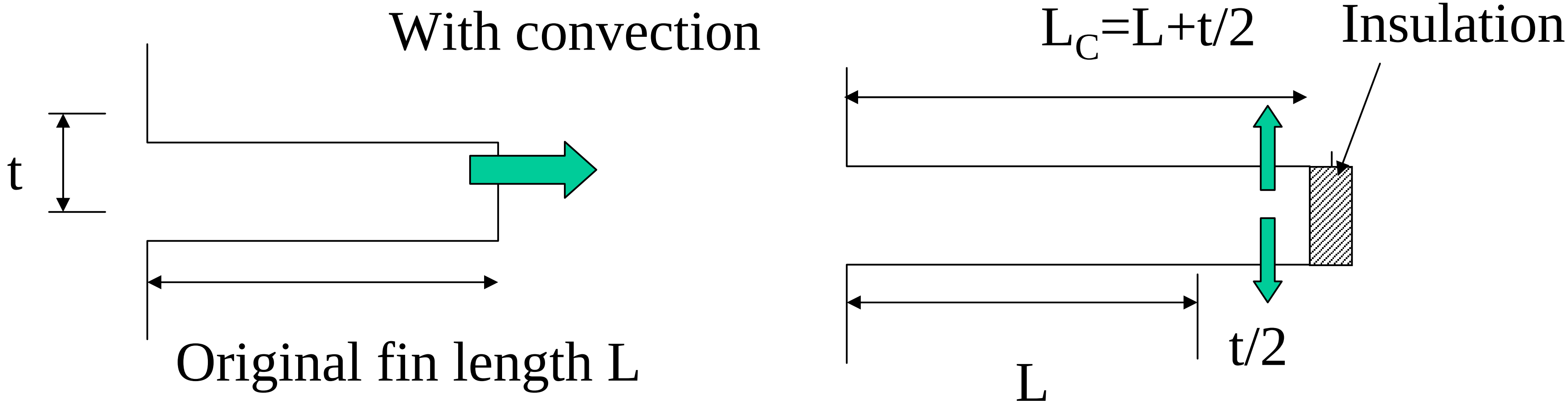
$T_c$: convective tip

T(0.2)=87.32 °C
$T_c$(0.2)=87.09 °C

Note 1: Convective tip case has a slightly lower tip temperature as expected since there is additional heat transfer at the tip.
Note 2: There is no significant difference between these two solutions, therefore, correct choice of boundary condition is not that important here.  However, sometimes correction might be needed to compensate the effect of convective heat transfer at the end. (especially for thick fins)

# Fins-2 (contd...)

- In some situations, it might be necessary to include the convective heat transfer at the tip. However, one would like to avoid using the long equation as described in case A, fins table. The alternative is to use case B instead and accounts for the convective heat transfer at the tip by extending the fin length L to $L_C = L + (t/2)$.




Then apply the adiabatic condition at the tip of the extended fin as shown above.

# Fins-2 (contd...)

Use the same example: aluminum pot handle, $m$=3.138, the length will need to be corrected to

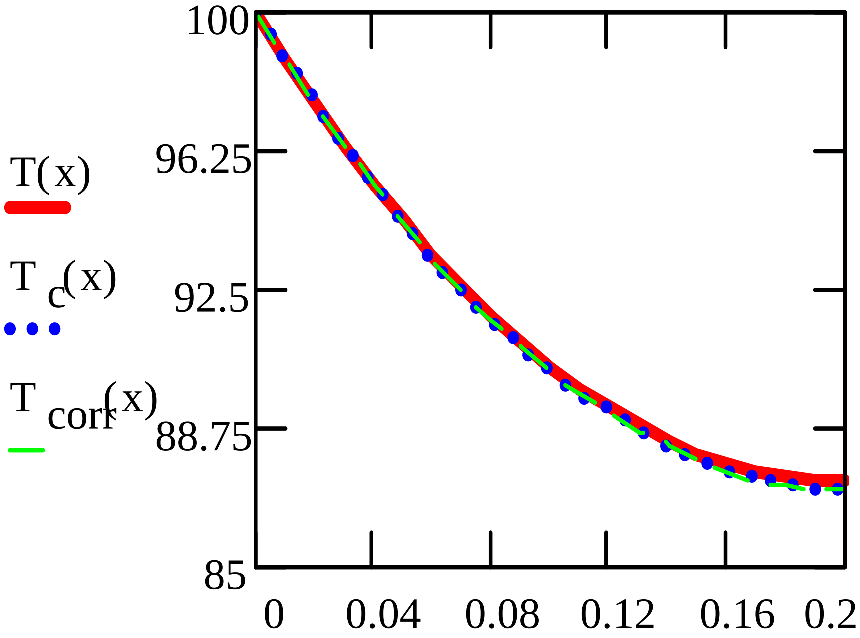
$L_C$=l+(t/2)=0.2+0.0025=0.2025(m)

$$\frac{T_{corr}(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L_c - x)}{\cosh mL_c}$$

$$\frac{T_{corr} - 25}{100 - 25} = \frac{\cosh[3.138(0.2025 - x)]}{\cosh(3.138 * 0.2025)},$$

$$T_{corr}(x) = 25 + 62.05 * \cosh[3.138(0.2025 - x)]$$

# Fins-2 (contd...)




$T(0.2)=87.32$ °C
$T_c(0.2)=87.09$ °C
$T_{corr}(0.2025)=87.05$ °C

slight improvement over the uncorrected solution

# Correction Length

- The correction length can be determined by using the formula: $L_c = L + (A_c/P)$, where $A_c$ is the cross-sectional area and P is the perimeter of the fin at the tip.

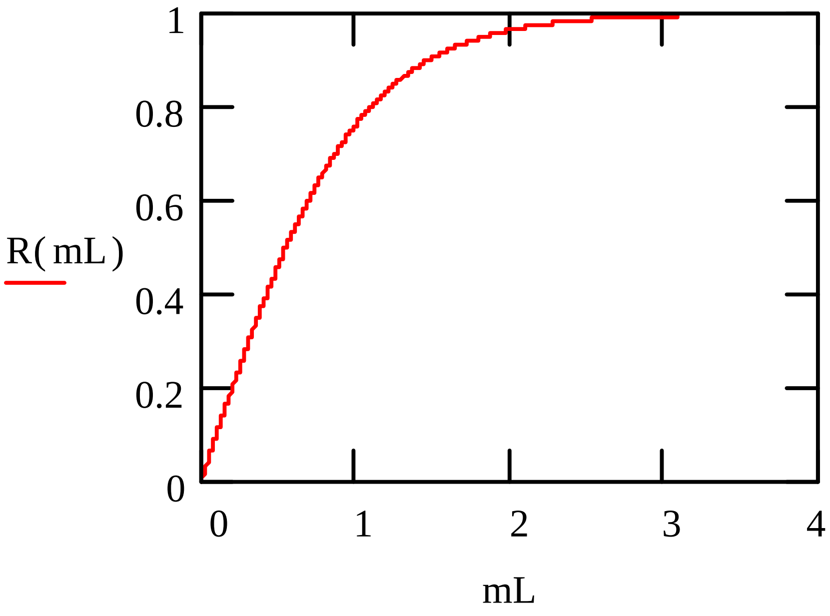
- Thin rectangular fin: $A_c = Wt$, $P = 2(W+t) \approx 2W$, since $t \ll W$
$L_c = L + (A_c/P) = L + (Wt/2W) = L + (t/2)$

- Cylindrical fin: $A_c = (\pi/4)D^2$, $P = \pi D$, $L_c = L + (A_c/P) = L + (D/4)$

- Square fin: $A_c = W^2$, $P = 4W$,
$L_c = L + (A_c/P) = L + (W^2/4W) = L + (W/4)$

# Optimal Length of a Fin

❑ In general, the longer the fin, the higher the heat transfer. However, a long fin means more material and increased size and cost. Question: how do we determine the optimal fin length? Use the rectangular fin as an example:




$q_f = M \tanh mL$, for an adiabatic tip fin

$(q_f)_\infty = M$, for an infinitely long fin

Their ratio: $R(mL) = \frac{q_f}{(q_f)_\infty} = \tanh mL$

Note: heat transfer increases with mL as expected. Initially the rate of change is large and slows down drastically when mL> 2.

R(1)=0.762, means any increase beyond mL=1 will increase no more than 23.8% of the fin heat transfer.

# Temperature Distribution

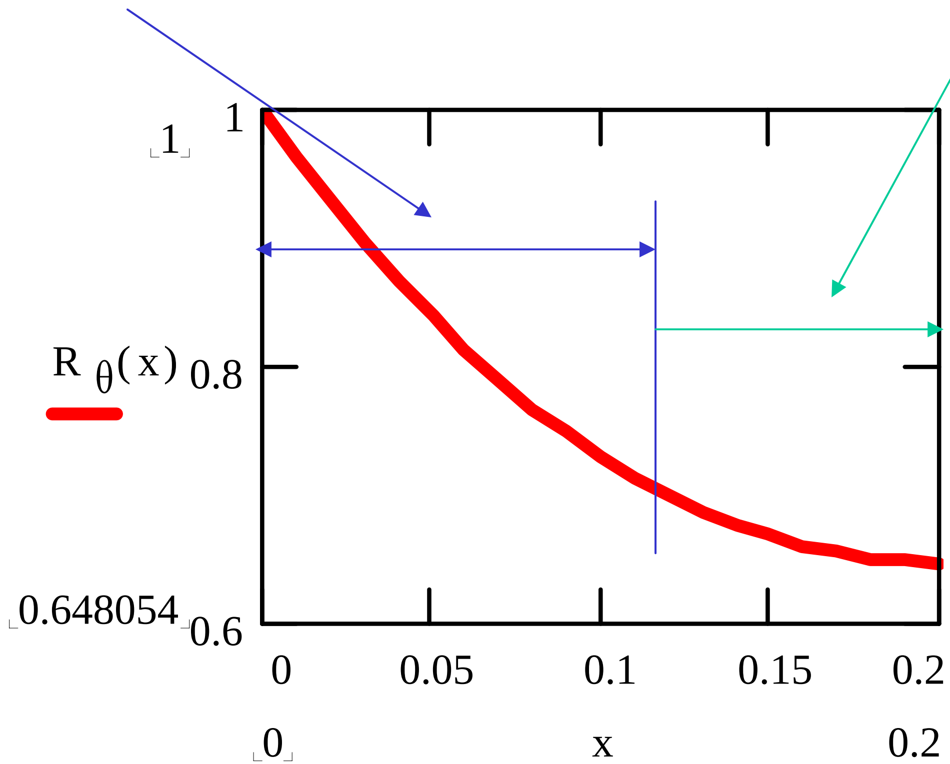
For an adiabatic tip fin case:

$$R_\theta = \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

➢ Use *m*=5, and L=0.2 as an example:

High ΔT, good fin heat transfer

Low ΔT, poor fin heat transfer



# Correction Length for a Fin with a Non-adiabatic Tip

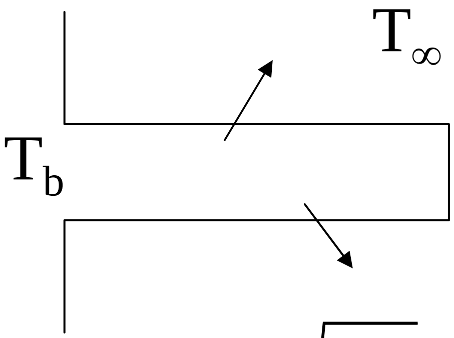
- The correction length can be determined by using the formula: $L_c = L + (A_c/P)$, where $A_c$ is the cross-sectional area and P is the perimeter of the fin at the tip.

- Thin rectangular fin: $A_c = Wt$, $P = 2(W+t) \approx 2W$, since $t << W$
$L_c = L + (A_c/P) = L + (Wt/2W) = L + (t/2)$

- Cylindrical fin: $A_c = (\pi/4)D^2$, $P = \pi D$, $L_c = L + (A_c/P) = L + (D/4)$

- Square fin: $A_c = W^2$, $P = 4W$,
$L_c = L + (A_c/P) = L + (W^2/4W) = L + (W/4)$

# Fin Design




Total heat loss: $q_f$=Mtanh(mL) for an adiabatic fin, or $q_f$=Mtanh($mL_C$) if there is convective heat transfer at the tip

where $m=\sqrt{\frac{hP}{kA_c}}$, and $M=\sqrt{hPkA_C}\,\theta_b = \sqrt{hPkA_C}\,(T_b - T_\infty)$

Use the thermal resistance concept:

$$q_f = \sqrt{hPkA_C}\,\tanh(mL)(T_b - T_\infty) = \frac{(T_b - T_\infty)}{R_{t,f}}$$

where $R_{t,f}$ is the thermal resistance of the fin.

For a fin with an adiabatic tip, the fin resistance can be expressed as

$$R_{t,f} = \frac{(T_b - T_\infty)}{q_f} = \frac{1}{\sqrt{hPkA_C}\,[\tanh(mL)]}$$

# Fin Effectiveness

How effective a fin can enhance heat transfer is charact
fin effectiveness $\varepsilon_f$: Ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{\sqrt{\mathrm{hPkA_C}}\tanh(mL)}{hA_C} = \sqrt{\frac{kP}{hA_C}}\tanh(mL)$$

If the fin is long enough, mL>2, tanh(mL) $\rightarrow 1$,

it can be considered an infinite fin (case D of table3.4)

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_C}} = \sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

In order to enhance heat transfer, $\varepsilon_f > 1$.

However, $\varepsilon_f \geq 2$ will be considered justifiable

If $\varepsilon_f$ <1 then we have an insulator instead of a heat fin

# Fin Effectiveness (contd...)

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_C}} = \sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

- To increase $\varepsilon_f$, the fin's material should have higher thermal conductivity, k.
- It seems to be counterintuitive that the lower convection coefficient, h, the higher $\varepsilon_f$. But it is not because if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low. Observation: If fins are to be used on surfaces separating gas and liquid. Fins are usually placed on the gas side. (Why?)

# Fin Effectiveness (contd...)

- P/AC should be as high as possible. Use a square fin with a dimension of W by W as an example: P=4W, AC=W2, P/AC=(4/W). The smaller W, the higher the P/AC, and the higher $\varepsilon$f.
- Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

# Fin Effectiveness (contd...)

The effectiveness of a fin can also be characterized as

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}}$$

It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than that of the resistance due only to convection.

# Fin Efficiency

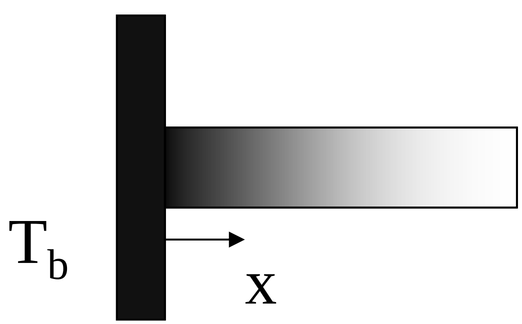
Define Fin efficiency: $\eta_{\mathrm{f}} = \dfrac{q_f}{q_{\max}}$

where $q_{\max}$ represents an idealized situation such that the fin is made up of material with infinite thermal conductivity. Therefore, the fin should be at the same temperature as the temperature of the base.

$$q_{\max} = hA_f(T_b - T_\infty)$$

# Fin Efficiency (contd…)

$T(x)<T_b$ for heat transfer to take place

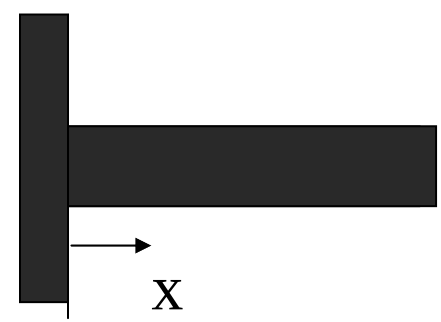



Total fin heat transfer $q_f$

*Real situation*

For infinite k

$T(x)=T_{b,}$ the heat transfer is maximum




Ideal heat transfer $q_{max}$

*Ideal situation*

# Fin Efficiency (cont.)

Use an adiabatic rectangular fin as an example:

$$\eta_f = \frac{q_f}{q_{\max}} = \frac{M \tanh mL}{hA_f(T_b - T_\infty)} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL}{hPL(T_b - T_\infty)}$$

$$= \frac{\tanh mL}{\sqrt{\frac{hP}{kA_c}}\,L} = \frac{\tanh mL}{mL} \quad \text{(see Table 3.5 for } \eta_f \text{ of common fins)}$$

The fin heat transfer: $q_f = \eta_f q_{\max} = \eta_f hA_f(T_b - T_\infty)$

$$q_f = \frac{T_b - T_\infty}{1/(\eta_f hA_f)} = \frac{T_b - T_\infty}{R_{t,f}}, \text{ where } R_{t,f} = \frac{1}{\eta_f hA_f}$$

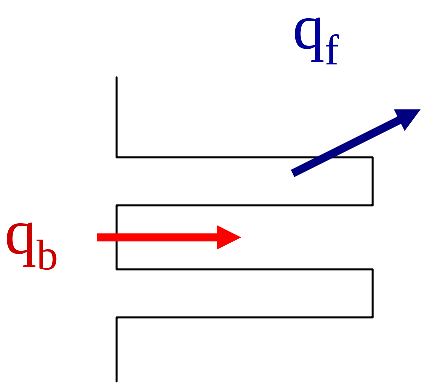
Thermal resistance for a single fin.

As compared to convective heat transfer: $R_{t,b} = \dfrac{1}{hA_b}$

In order to have a lower resistance as that is required to enhance heat transfer: $R_{t,b} > R_{t,f}$ or $A_b < \eta_f A_f$

# Overall Fin Efficiency

Overall fin efficiency for an array of fins:




Define terms: $A_b$: base area exposed to coolant

$A_f$: surface area of a single fin

$A_t$: total area including base area and total finned surface, $A_t=A_b+NA_f$

N: total number of fins

# Overall Fin Efficiency (contd…)

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$

$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$

$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

# Heat Transfer from a Fin Array

$$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}} \text{ where } R_{t,O} = \frac{1}{hA_t\eta_O}$$

Compare to heat transfer without fins

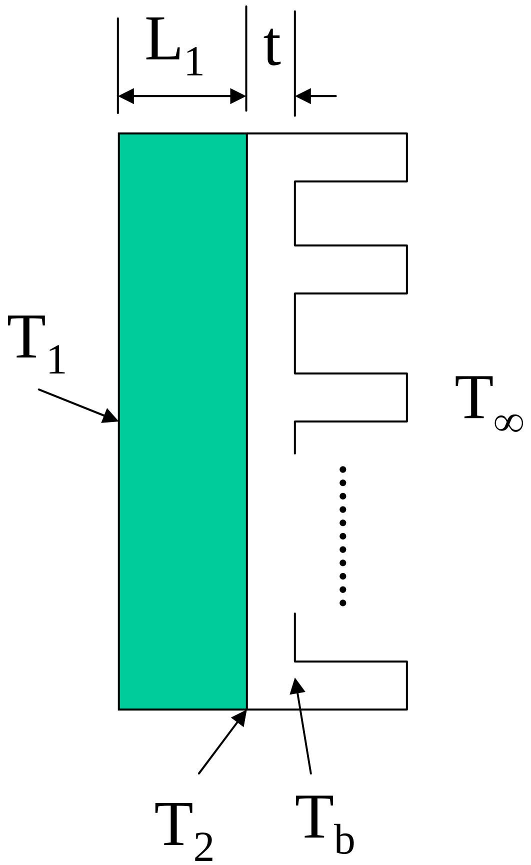
$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

To enhance heat transfer $A_t\eta_O >> A$

That is, to increase the effective area $\eta_O A_t$.

# Thermal Resistance Concept




$$A = A_b + NA_{b,f}$$

$$R_b = t/(k_b A)$$


$$R_1 = L_1/(k_1 A)$$

$$R_{t,O} = 1/(hA_t\eta_O)$$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,O}}$$