



MODULE 2

One Dimensional Steady State Heat Conduction



Objectives of conduction analysis



To determine the temperature field, $T(x,y,z,t)$, in a body (i.e. how temperature varies with position within the body)

□ $T(x,y,z,t)$ depends on:

- boundary conditions
- initial condition
- material properties (k , c^p , ρ ...)
- geometry of the body (shape, size)

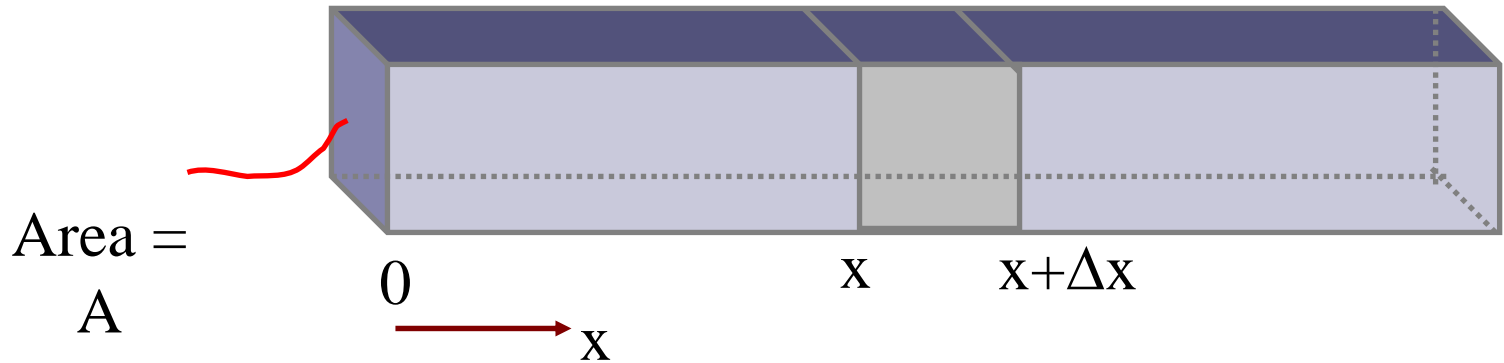
$T(x,y,z)$

□ Why we need $T(x,y,z,t)$?

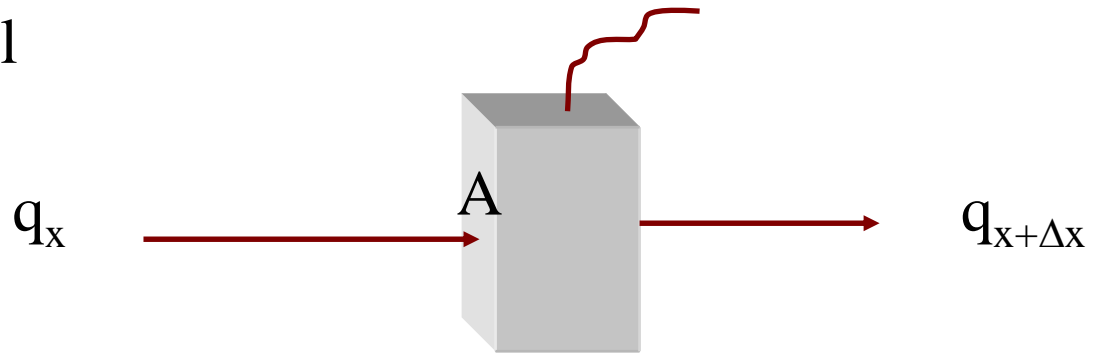
- to compute heat flux at any location (using Fourier's eqn.)
- compute thermal stresses, expansion, deflection due to temp. etc.
- design insulation thickness
- chip temperature calculation
- heat treatment of metals



Unidirectional heat conduction (1D)



Solid bar, insulated on all long sides (1D heat conduction)



\dot{q} = Internal heat generation per unit vol. (W/m^3)



Unidirectional heat conduction (1D)



First Law (energy balance) $(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st}$

$$q_x - q_{x+\Delta x} + A(\Delta x)\dot{q} = \frac{\partial E}{\partial t}$$

$$E = (\rho A \Delta x)u$$

$$\frac{\partial E}{\partial t} = \rho A \Delta x \frac{\partial u}{\partial t} = \rho A \Delta x c \frac{\partial T}{\partial t}$$

$$q_x = -kA \frac{\partial T}{\partial x}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x} \Delta x$$



Unidirectional heat conduction (1D)(contd...)



$$-kA \frac{\partial T}{\partial x} + kA \frac{\partial T}{\partial x} + A \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \Delta x + A \Delta x \dot{q} = \rho A c \Delta x \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

Longitudinal
conduction

Internal heat
generation

Thermal inertia

If k is a constant

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{\rho c}{k} \frac{\partial T}{\partial t} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$



Unidirectional heat conduction (1D)(contd...)

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.



Boundary and Initial conditions:

- ❑ The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.
- ❑ We have set up a differential equation, with T as the dependent variable. The solution will give us $T(x,y,z)$. Solution depends on boundary conditions (BC) and initial conditions (IC).



Boundary and Initial conditions (contd...)



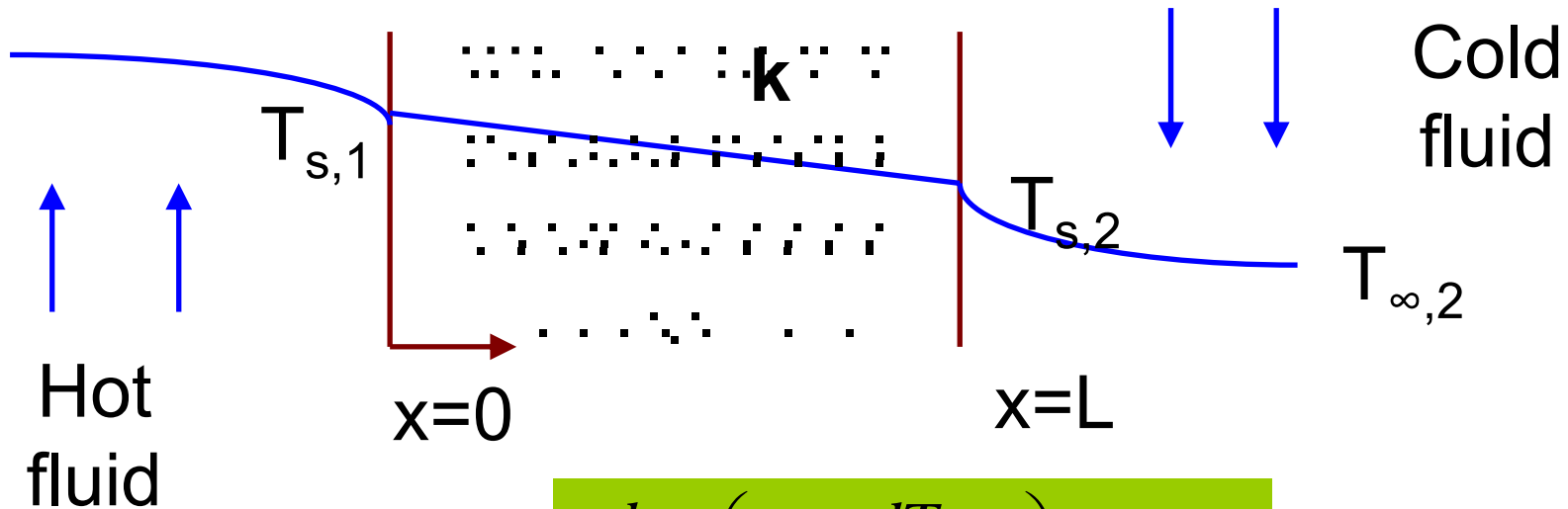
How many BC's and IC's ?

- Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
 - * 1D problem: 2 BC in x-direction
 - * 2D problem: 2 BC in x-direction, 2 in y-direction
 - * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
- Heat equation is first order in time. Hence one IC needed



1- Dimensional Heat Conduction

The Plane Wall :



$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0$$

Const. K; solution is:

$$q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,1} - T_{s,2}) = \frac{T_{s,1} - T_{s,2}}{L / kA}$$

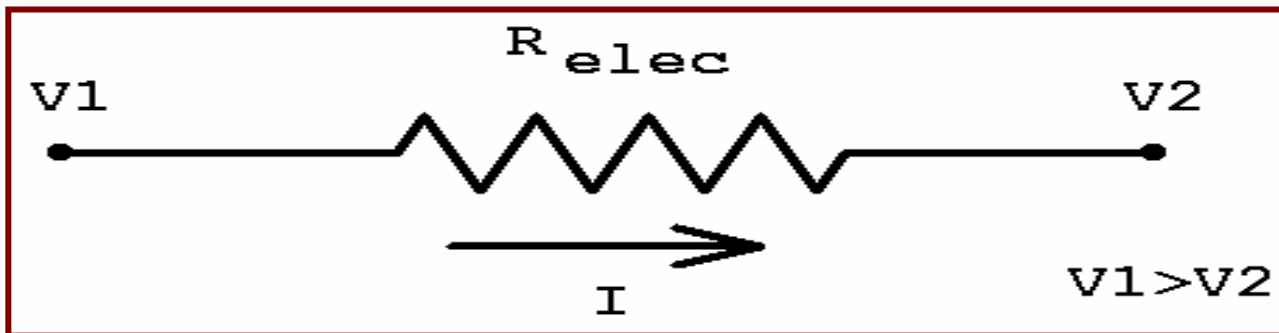


Thermal resistance (electrical analogy)



OHM'S LAW :Flow of Electricity

$$V=IR_{\text{elec}}$$



Voltage Drop = Current flow \times Resistance

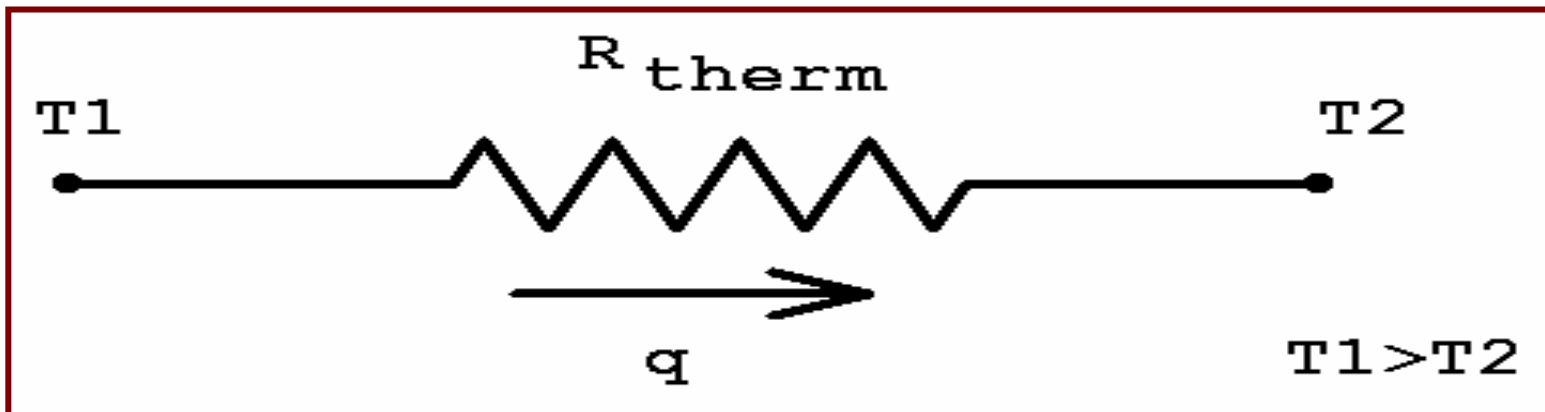


Thermal Analogy to Ohm's Law :



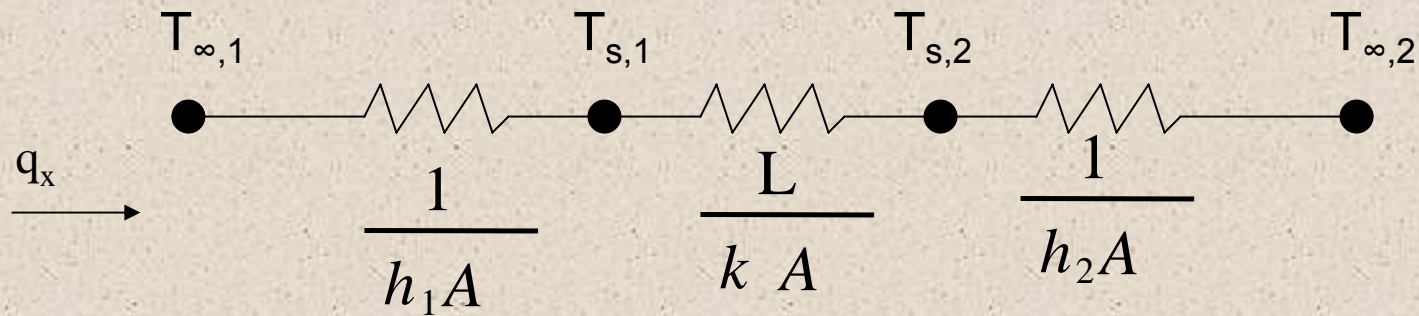
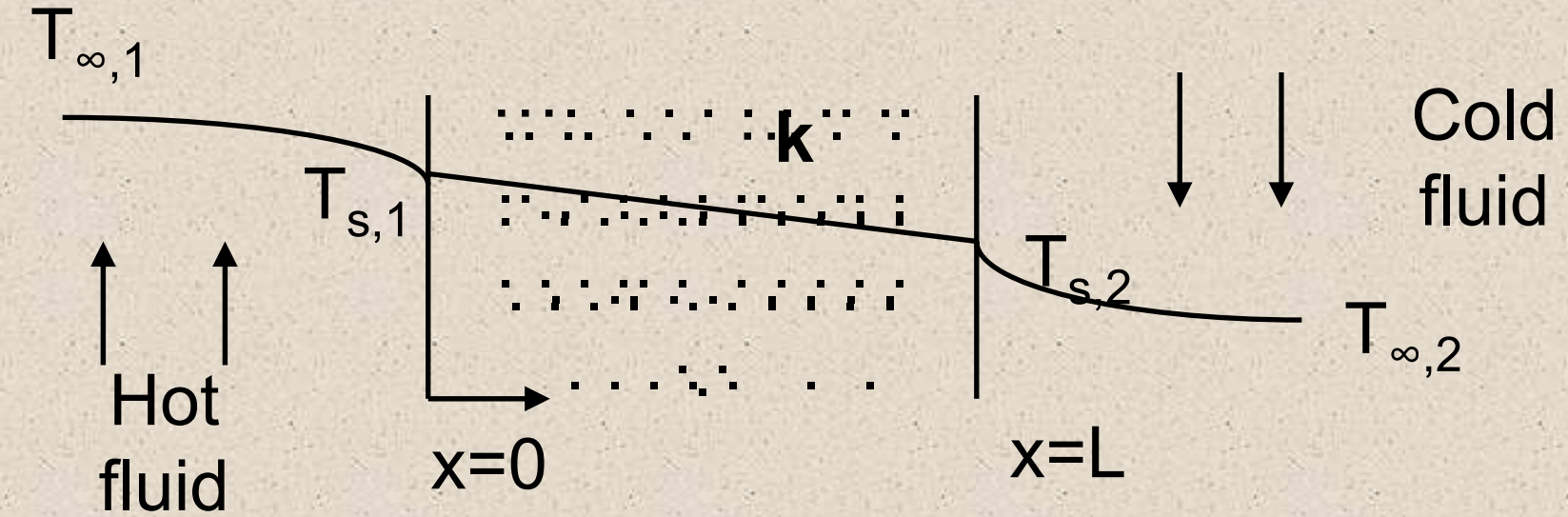
$$\Delta T = qR_{therm}$$

Temp Drop=Heat Flow×Resistance





1 D Heat Conduction through a Plane Wall



$$\sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \quad (\text{Thermal Resistance})$$



Resistance expressions



THERMAL RESISTANCES

- Conduction

$$R_{\text{cond}} = \Delta x / kA$$

- Convection

$$R_{\text{conv}} = (hA)^{-1}$$

- Fins

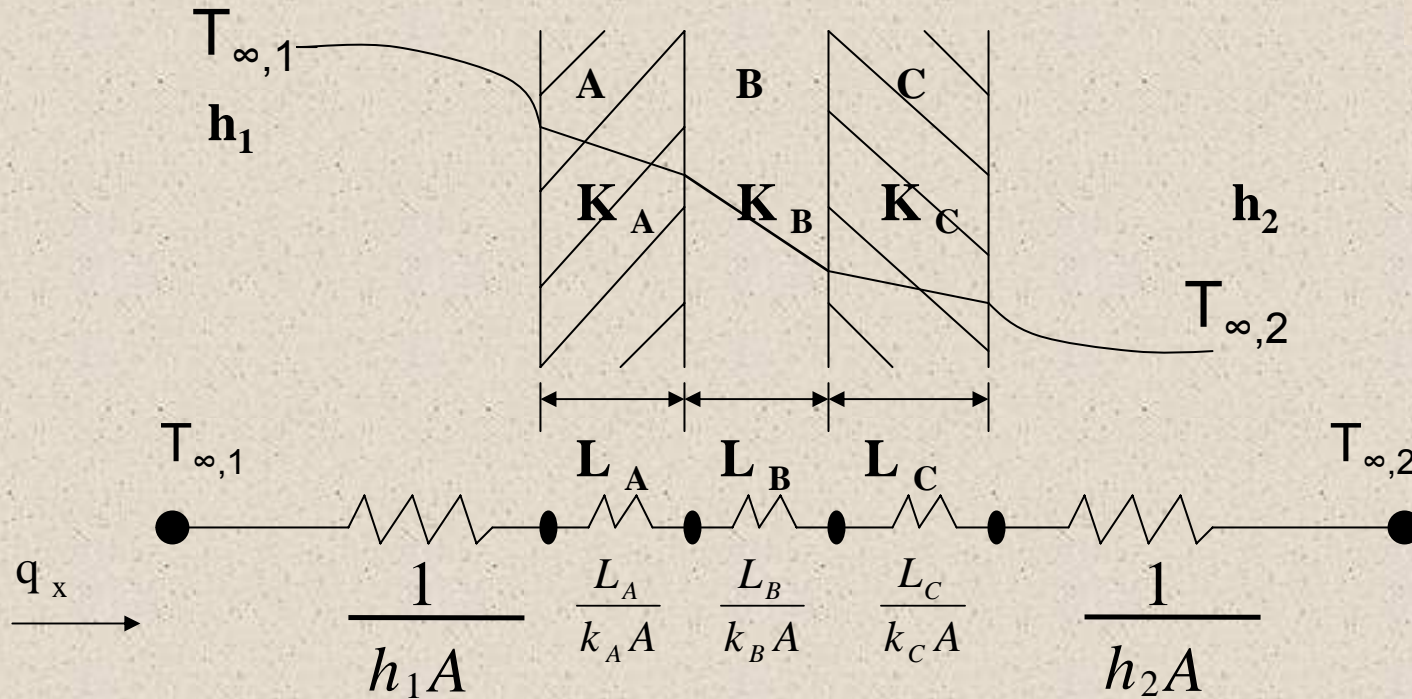
$$R_{\text{fin}} = (h\eta A)^{-1}$$

- Radiation(*aprox*)

$$R_{\text{rad}} = [4A\sigma F(T_1 T_2)^{1.5}]^{-1}$$



Composite Walls :



$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_2 A}} = UA \Delta T$$

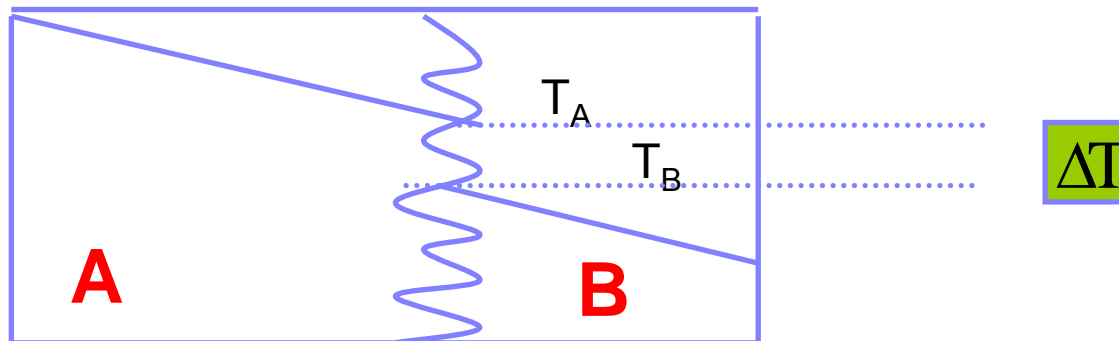
where, $U = \frac{1}{R_{tot} A} =$ Overall heat transfer coefficient



Overall Heat transfer Coefficient

$$U = \frac{1}{R_{\text{total}} A} = \frac{1}{\frac{1}{h_1} + \Sigma \frac{L}{k} + \frac{1}{h_2}}$$

Contact Resistance :

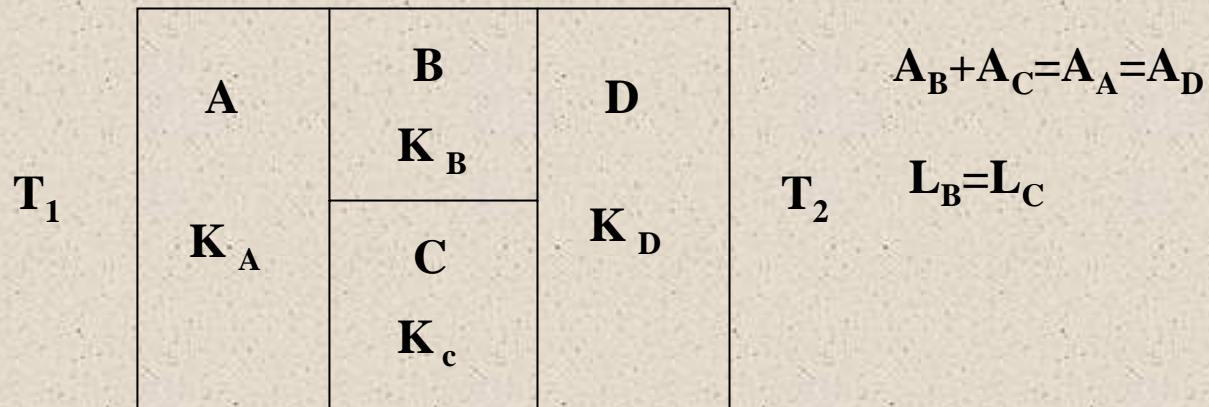


$$R_{t, c} = \frac{\Delta T}{q_x}$$



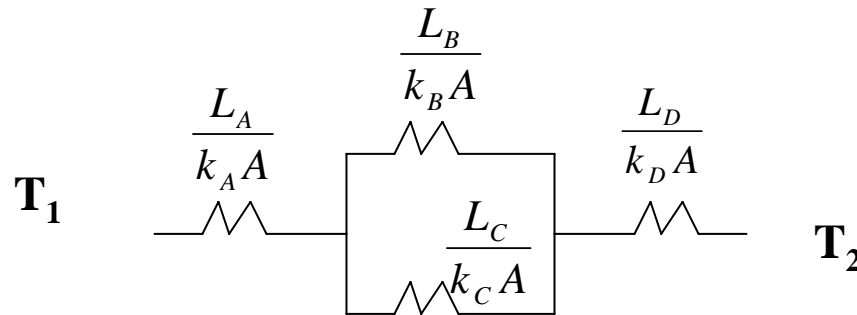
$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-Parallel :





Series-Parallel (contd...)



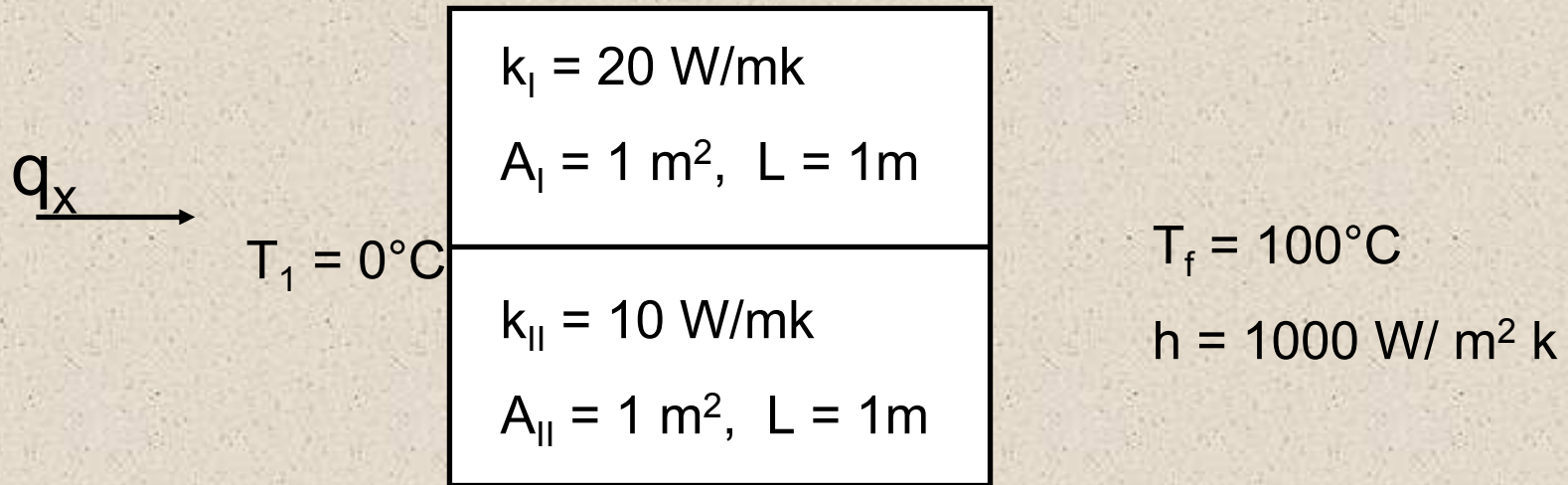
Assumptions :

- (1) Face between B and C is insulated.
- (2) Uniform temperature at any face normal to X.



Example:

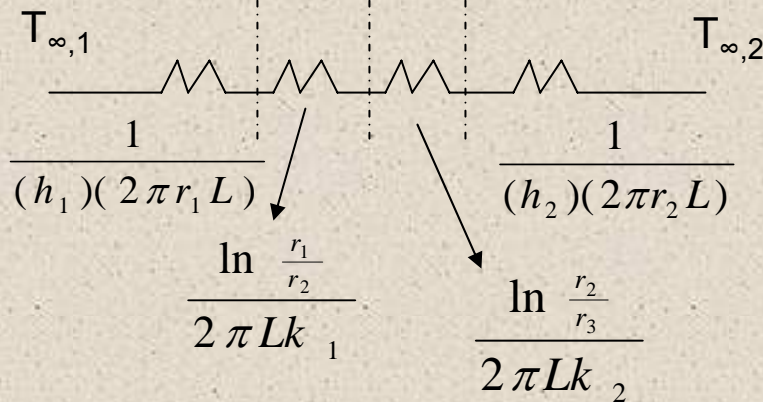
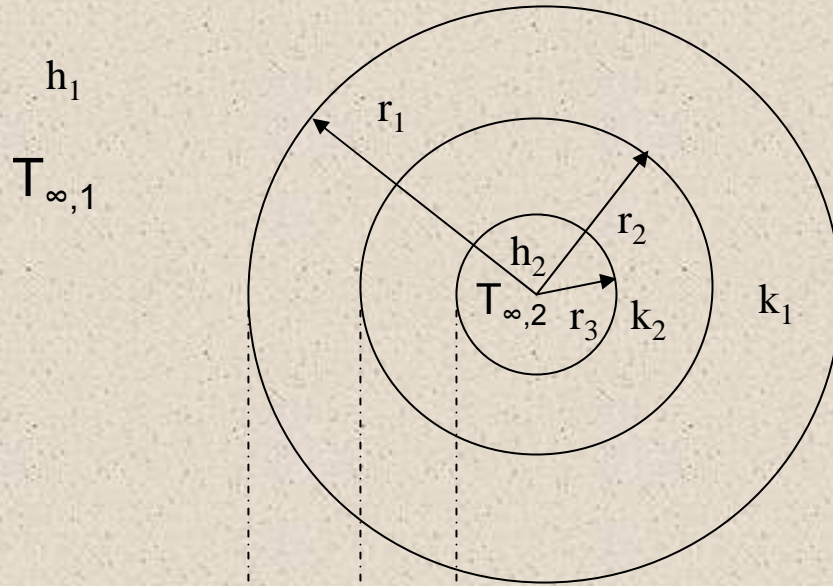
Consider a composite plane wall as shown:



Develop an approximate solution for the rate of heat transfer through the wall.



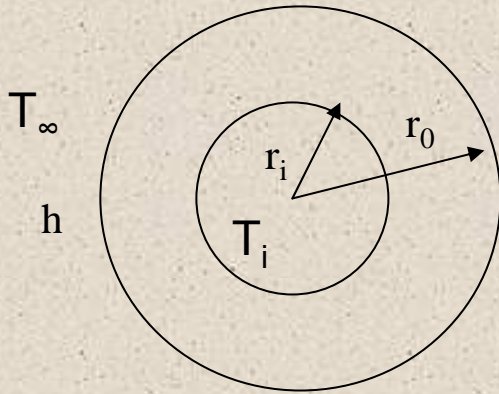
1 D Conduction(Radial conduction in a composite cylinder)



$$q_r = \frac{T_{\infty,2} - T_{\infty,1}}{\sum R_t}$$



Critical Insulation Thickness :



Insulation Thickness : $r_o - r_i$

$$R_{tot} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL} + \frac{1}{(2\pi r_o L)h}$$

Objective : decrease q , increases R_{tot}

Vary r_o ; as r_o increases ,first term increases, second term decreases.



Critical Insulation Thickness (contd...)



Maximum – Minimum problem

$$\text{Set } \frac{dR_{tot}}{dr_0} = 0$$

$$\frac{1}{2\pi kr_0 L} - \frac{1}{2\pi hLr_0^2} = 0$$

$$r_0 = \frac{k}{h}$$

Max or Min. ?

$$\text{Take : } \frac{d^2 R_{tot}}{dr_0^2} = 0 \quad \text{at} \quad r_0 = \frac{k}{h}$$

$$\frac{d^2 R_{tot}}{dr_0^2} = \frac{-1}{2\pi kr_0^2 L} + \frac{1}{\pi r_0^2 hL} \Bigg|_{r_0 = \frac{k}{h}}$$

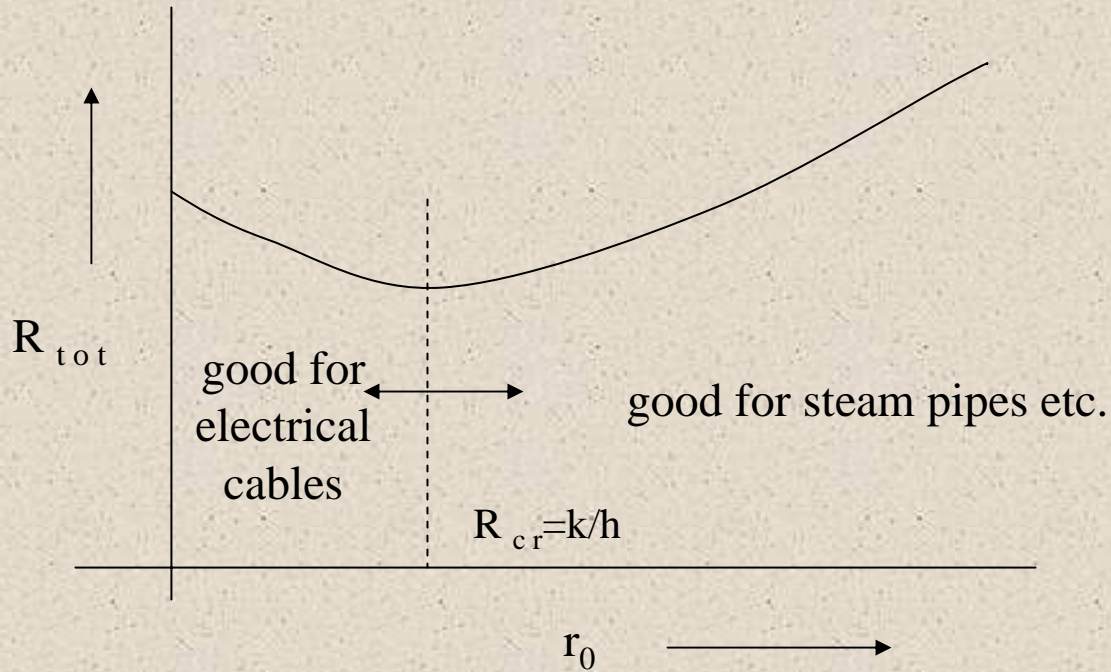
$$= \frac{h^2}{2\pi Lk^3} > 0$$



Critical Insulation Thickness (contd...)

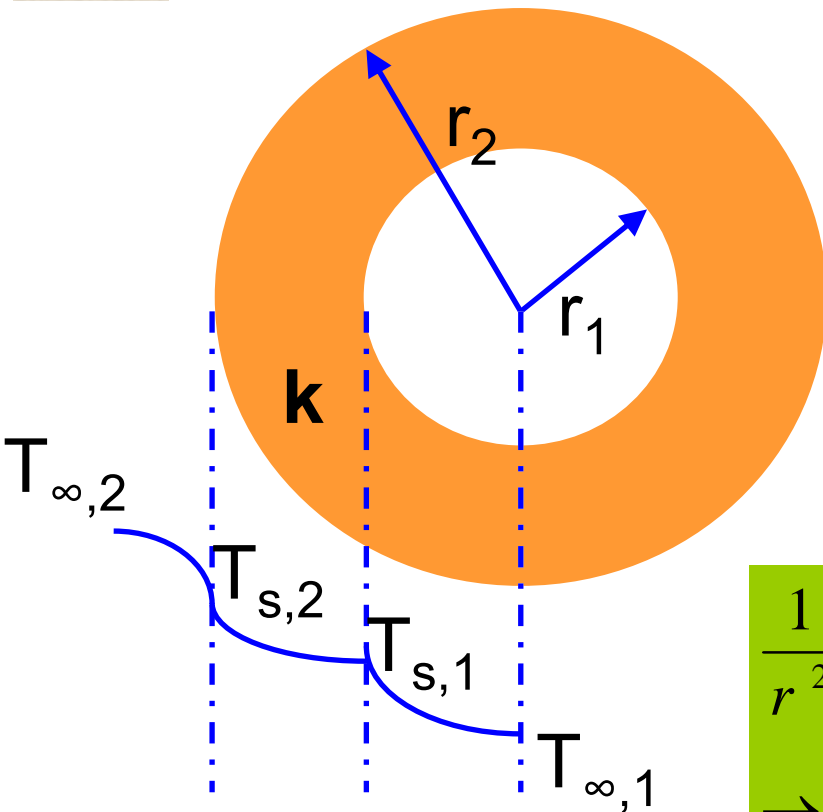


Minimum q at $r_0 = (k/h) = r_{cr}$ (critical radius)





1D Conduction in Sphere



Inside Solid:

$$\frac{1}{r^2} \frac{d}{dr} \left(kr^2 \frac{dT}{dr} \right) = 0$$

$$\rightarrow T(r) = T_{s,1} - \{T_{s,1} - T_{s,2}\} \left[\frac{1 - (r_1/r)}{1 - (r_1/r_2)} \right]$$

$$\rightarrow q_r = -kA \frac{dT}{dr} = \frac{4\pi k (T_{s,1} - T_{s,2})}{(1/r_1 - 1/r_2)}$$

$$\rightarrow R_{t,cond} = \frac{1/r_1 - 1/r_2}{4\pi k}$$



Conduction with Thermal Energy Generation



$$\dot{q} = \frac{\dot{E}}{V} = \text{Energy generation per unit volume}$$

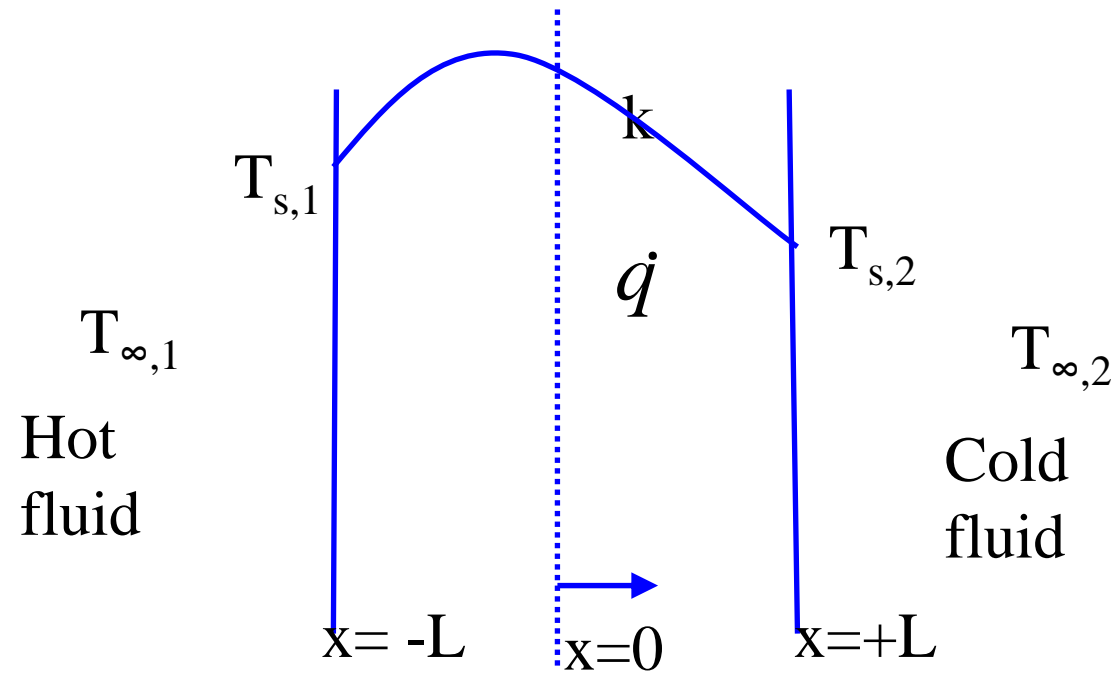
- Applications:**
- * current carrying conductors
 - * chemically reacting systems
 - * nuclear reactors



Conduction with Thermal Energy Generation



The Plane Wall :



Assumptions:
1D, steady state,
constant k ,
uniform \dot{q}



Conduction With Thermal Energy Generation (contd...)



$$\frac{d^2 T}{dx^2} + \frac{q}{k} = 0$$

Boundary cond .: $x = -L, \quad T = T_{s,1}$

$x = +L, \quad T = T_{s,2}$

Solution : $T = -\frac{q}{2k} x^2 + C_1 x + C_2$



Conduction with Thermal Energy Generation (cont..)



Use boundary conditions to find C_1 and C_2

$$\text{Final solution: } T = \frac{qL^2}{2k} \left(1 - \frac{x^2}{L^2} \right) + \frac{T_{s,2} - T_{s,1}}{2} \frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

No more linear

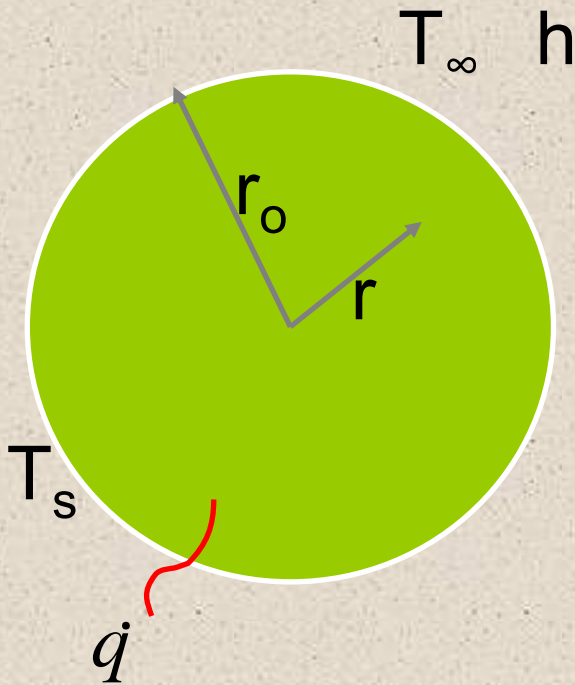
Heat flux: $q_x'' = -k \frac{dT}{dx}$

Derive the expression and show that it is no more independent of x

Hence thermal resistance concept is not correct to use when there is internal heat generation



Cylinder with heat source



Assumptions:
1D, steady state, constant k, uniform \dot{q}

Start with 1D heat equation in cylindrical co-ordinates:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$



Cylinder With Heat Source

$$\text{Boundary cond.: } r = r_0, \quad T = T_s$$

$$r = 0, \quad \frac{dT}{dr} = 0$$

$$\text{Solution: } T(r) = \frac{\dot{q}}{4k} r_0^2 \left(1 - \frac{r^2}{r_0^2} \right) + T_s$$

T_s may not be known. Instead, T_∞ and h may be specified.

Exercise: Eliminate T_s , using T_∞ and h .



Cylinder with heat source (contd...)



Example:

A current of 200A is passed through a stainless steel wire having a thermal conductivity $K=19\text{W/mK}$, diameter 3mm, and electrical resistivity $R = 0.99 \Omega$. The length of the wire is 1m. The wire is submerged in a liquid at 110°C , and the heat transfer coefficient is $4\text{W/m}^2\text{K}$. Calculate the centre temperature of the wire at steady state condition.

Solution: to be worked out in class