Module
2
Stresses in machine elements

Lesson
2
Compound stresses in machine parts

Instructional Objectives

At the end of this lesson, the student should be able to understand

- Elements of force system at a beam section.
- Superposition of axial and bending stresses.
- Transformation of plane stresses; principal stresses
- Combining normal and shear stresses.

2.2.1 Introduction

The elements of a force system acting at a section of a member are axial force, shear force and bending moment and the formulae for these force systems were derived based on the assumption that only a single force element is acting at the section. Figure-2.2.1.1 shows a simply supported beam while figure-2.2.1.2 shows the forces and the moment acting at any cross-section X-X of the beam. The force system can be given as:

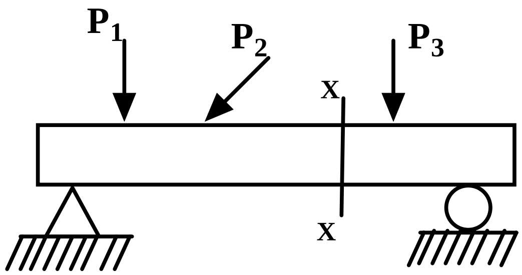
Axial force : $\sigma = \frac{P}{A}$

Bending moment : $\sigma = \frac{My}{I}$

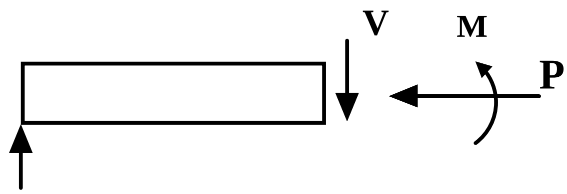
Shearforce : $\tau = \frac{VQ}{It}$

Torque $T = \frac{\tau J}{r}$:

where, σ is the normal stress, τ the shear stress, P the normal load, A the cross-sectional area, M the moment acting at section X-X, V the shear stress acting at section X-X, Q the first moment of area, I the moment of inertia, t the width at which transverse shear is calculated, J the polar moment of inertia and r the radius of the circular cross-section.



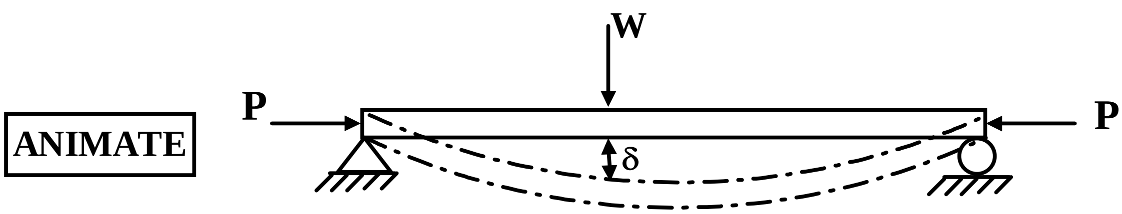
***2.2.1.1F**- A simply supported beam with concentrated loads*



***2.2.1.2F**- Force systems on section XX of figure-2.2.1.1*

Combined effect of these elements at a section may be obtained by the method of superposition provided that the following limitations are tolerated:

(a) Deformation is small (figure-2.2.1.3)



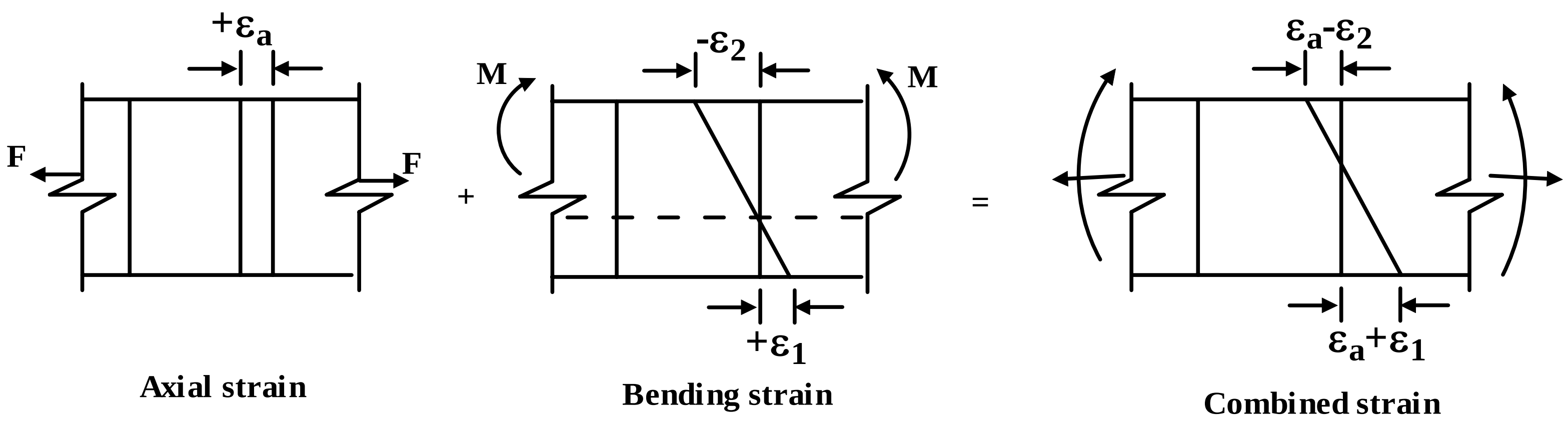
***2.2.1.3A**- Small deflection of a simply supported beam with a concentrated load*

If the deflection is large, another additional moment of Pδ would be developed.

(b) Superposition of strains are more fundamental than stress superposition and the principle applies to both elastic and inelastic cases.

2.2.2 Strain superposition due to combined effect of axial force P and bending moment M.

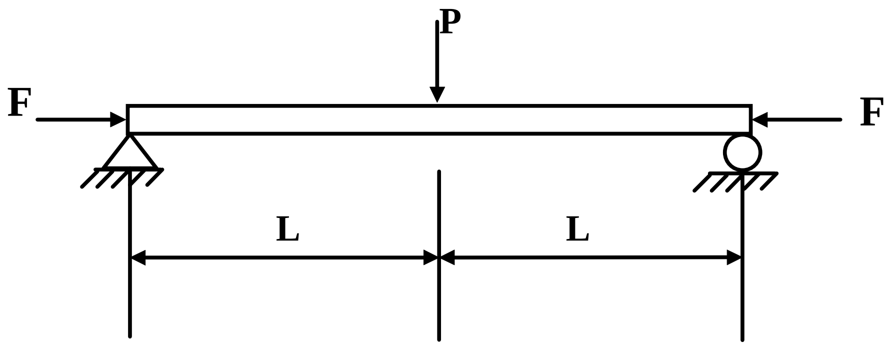
Figure-2.2.2.1 shows the combined action of a tensile axial force and bending moment on a beam with a circular cross-section. At any cross-section of the beam, the axial force produces an axial strain ε_a while the moment M causes a bending strain. If the applied moment causes upward bending such that the strain at the upper most layer is compressive ($-\varepsilon_2$) and that at the lower most layer is tensile ($+\varepsilon_1$), consequently the strains at the lowermost fibre are additive ($\varepsilon_a+\varepsilon_1$) and the strains at the uppermost fibre are subtractive ($\varepsilon_a-\varepsilon_2$). This is demonstrated in figure-2.2.2.1.



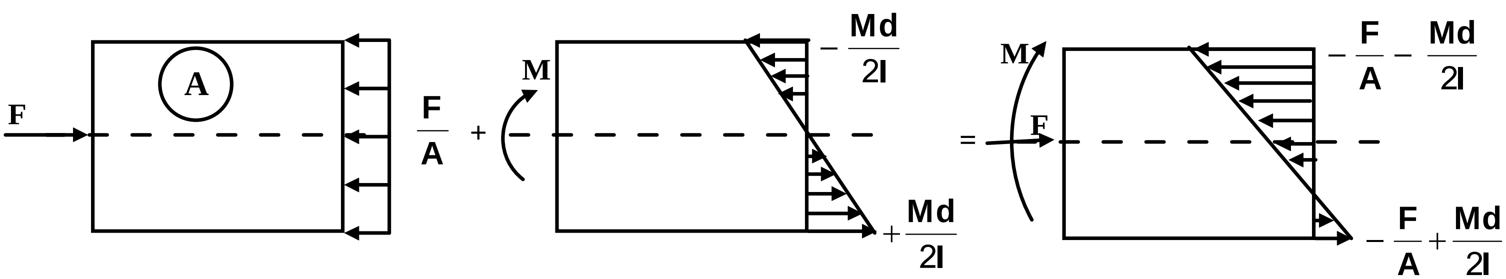
2.2.2.1F- *Superposition of strain due to axial loading and bending moment.*

2.2.3 Superposition of stresses due to axial force and bending moment

In linear elasticity, stresses of same kind may be superposed in homogeneous and isotropic materials. One such example (figure-2.2.3.1) is a simply supported beam with a central vertical load P and an axial compressive load F. At any section a compressive stress of $\frac{4F}{\pi d^2}$ and a bending stress of My/I are produced. Here d is the diameter of the circular bar, I the second moment of area and the moment is $PL/2$ where the beam length is 2L. Total stresses at the upper and lower most fibres in any beam cross-section are $-\left(\frac{32\mathbf{M}}{2\pi\mathbf{d}^3}+\frac{4\mathbf{F}}{\pi\mathbf{d}^2}\right)$ and $\left(\frac{32\mathbf{M}}{2\pi\mathbf{d}^3}-\frac{4\mathbf{F}}{\pi\mathbf{d}^2}\right)$ respectively. This is illustrated in figure-2.2.3.2



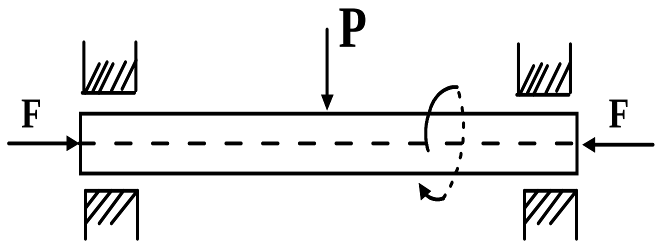
2.2.3.1F- *A simply supported beam with an axial and transverse loading.*



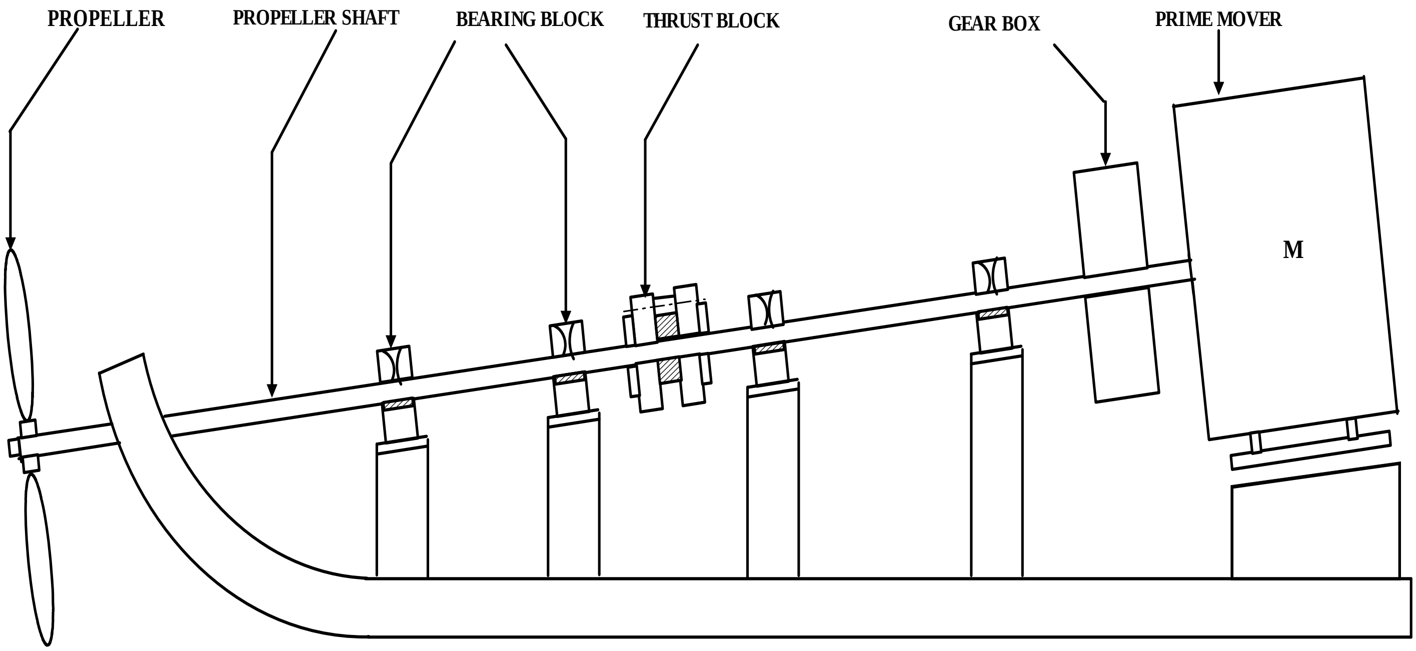
2.2.3.2F- *Combined stresses due to axial loading and bending moment.*

2.2.4 Superposition of stresses due to axial force, bending moment and torsion

Until now, we have been discussing the methods of compounding stresses of same kind for example, axial and bending stresses both of which are normal stresses. However, in many cases members on machine elements are subjected to both normal and shear stresses, for example, a shaft subjected to torsion, bending and axial force. This is shown in figure-2.2.4.1. A typical example of this type of loading is seen in a ship's propeller shafts. Figure-2.2.4.2 gives a schematic view of a propulsion system. In such cases normal and shearing stresses need to be compounded.



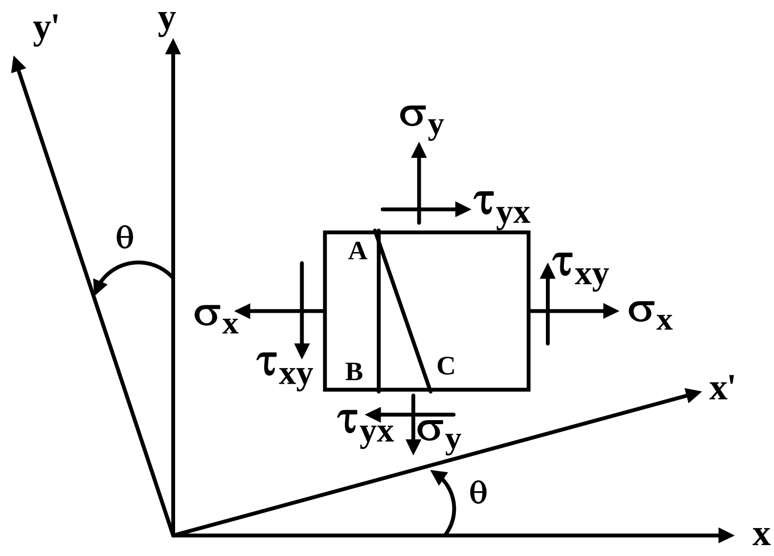
***2.2.4.1F**- A simply supported shaft subjected to axial force bending moment and torsion.*



2.2.4.2F- A schematic diagram of a typical marine propulsion shafting

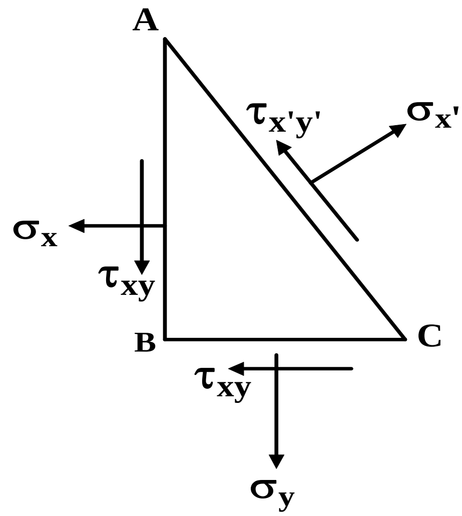
2.2.5 Transformation of plane stresses

Consider a state of general plane stress in x-y co-ordinate system. We now wish to transform this to another stress system in, say, x′- y′ co-ordinates, which is inclined at an angle θ. This is shown in figure-2.2.5.1.



2.2.5.1F*- Transformation of stresses from x-y to x′-y′ co-ordinate system.*

A two dimensional stress field acting on the faces of a cubic element is shown in figure-2.2.5.2. In plane stress assumptions, the non-zero stresses are σ_x, σ_y and $\tau_{xy}=\tau_{yx}$.We may now isolate an element ABC such that the plane AC is inclined at an angle θ and the stresses on the inclined face are σ'_x and τ'_{xy} .



2.2.5.2F*- Stresses on an isolated triangular element*

Considering the force equilibrium in x-direction we may write

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

This may be reduced to

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \; ---(1)$$

Similarly, force equilibrium in y-direction gives

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta ------- (2)$$

Since plane AC can assume any arbitrary inclination, a stationary value of σ_x' is given by

$$\frac{d\sigma_{x'}}{d\theta} = 0$$

This gives

$$\tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \; -------(3)$$

This equation has two roots and let the two values of θ be θ_1 and $(\theta_1 + 90^o)$. Therefore these two planes are the planes of maximum and minimum normal stresses.

Now if we set $\tau_{x'y'} = 0$ we get the values of θ corresponding to planes of zero shear stress.

This also gives

$$\tan 2\theta = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

And this is same as equation (3) indicating that at the planes of maximum and minimum stresses no shearing stress occurs. These planes are known as **Principal planes** and stresses acting on these planes are known as **Principal stresses.** From equation (1) and (3) the principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad ------(4)$$

In the same way, condition for maximum shear stress is obtained from

$$\frac{d}{d\theta}(\tau_{x'y'}) = 0$$

$$\tan 2\theta = -\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \; ---------(5)$$

This also gives two values of θ say θ_2 and (θ_2+90^o), at which shear stress is maximum or minimum. Combining equations (2) and (5) the two values of maximum shear stresses are given by

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \text{ ----------(6)}$$

One important thing to note here is that values of tan2θ_2 is negative reciprocal of tan2θ_1 and thus θ_1 and θ_2 are 45^o apart. This means that principal planes and planes of maximum shear stresses are 45^o apart. It also follows that although no shear stress exists at the principal planes, normal stresses may act at the planes of maximum shear stresses.

2.2.6 An example

Consider an element with the following stress system (figure-2.2.6.1)
σ_x=-10 MPa, σ_y = +20 MPa, τ = -20 MPa.
We need to find the principal stresses and show their senses on a properly oriented element.

Solution:
The principal stresses are

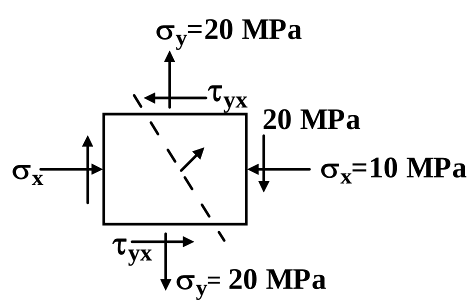
$$\sigma_{1,2} = \frac{-10+20}{2} \pm \sqrt{\left(\frac{-10-20}{2}\right)^2 + (-20)^2}$$

This gives -20 MPa and 30 MPa

The principal planes are given by

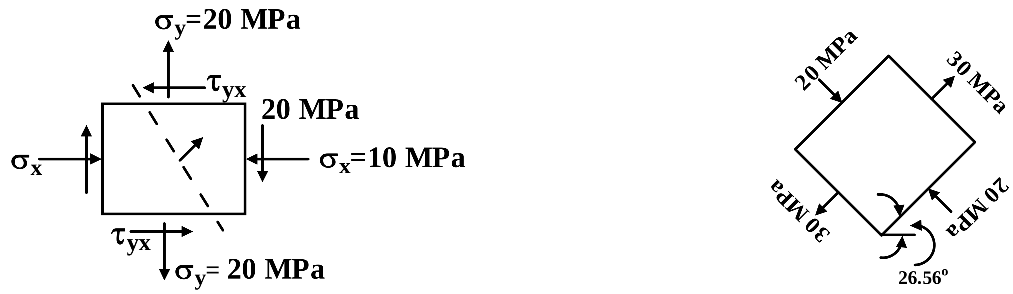
$$\tan 2\theta_1 = \frac{-20}{(-10-20)/2}$$
$$= 1.33$$

The two values are 26.56^o and 116.56^o



***2.2.6.1F**- A 2-D element with normal and shear stresses.*

The oriented element to show the principal stresses is shown in figure-2.2.6.2.



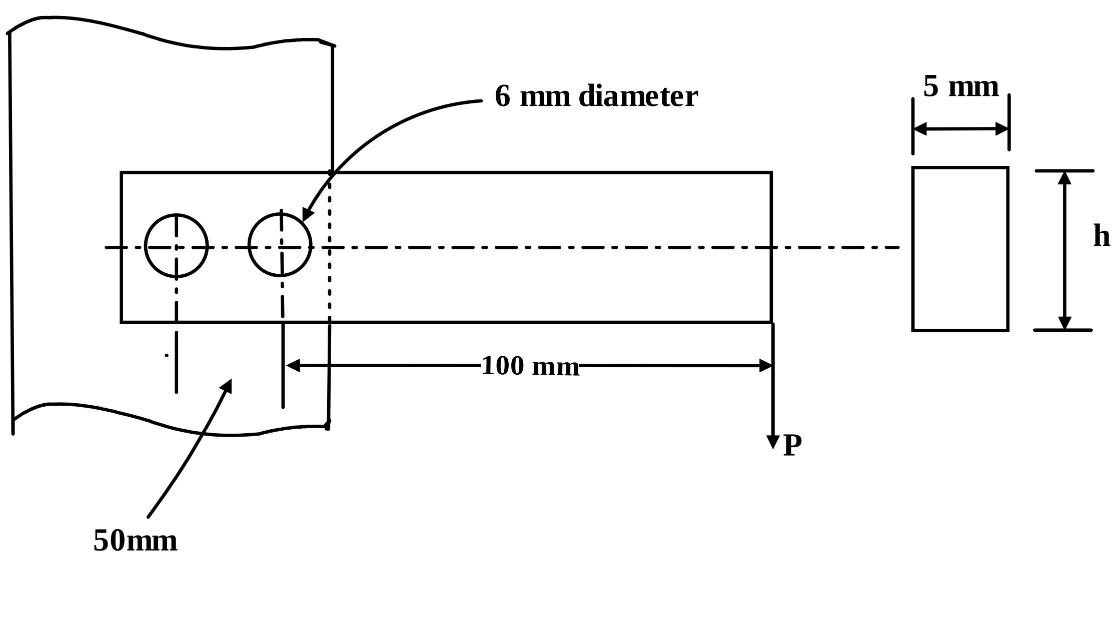
***2.2.6.2F**- Orientation of the loaded element in the left to show the principal stresses.*

2.2.7 Problems with Answers

Q.1: A 5mm thick steel bar is fastened to a ground plate by two 6 mm diameter pins as shown in figure-2.2.7.1. If the load P at the free end of the steel bar is 5 KN, find

(a) The shear stress in each pin

(b) The direct bearing stress in each pin.



2.2.7.1F

A.1:

Due to the application of force P the bar will tend to rotate about point 'O' causing shear and bearing stresses in the pins A and B. This is shown in figure-2.2.7.2F. Let the forces at pins A and B be F_A and F_B and equating moments about 'O' ,

$$5x10^3x0.125 = (F_A+F_B)x\ 0.025 \qquad (1)$$

Also, from force balance, $F_A+P = F_B$ (2)

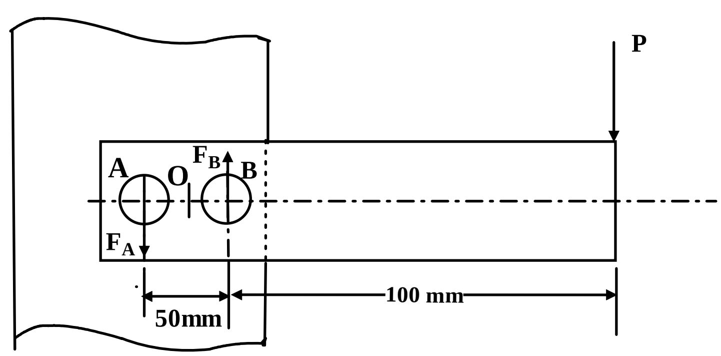
Solving equations-1 and 2 we have, F_A =10 KN and F_B = 15 KN.

(a) Shear stress in pin A = $\dfrac{10x10^3}{\left(\dfrac{\pi x0.006^2}{4}\right)} = 354\,MPa$

Shear stress in pin B = $\dfrac{15x10^3}{\left(\dfrac{\pi x0.006^2}{4}\right)} = 530.5\,MPa$

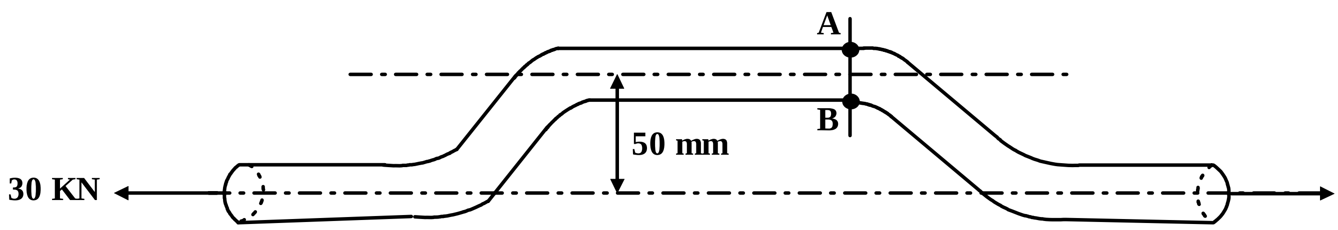
(b) Bearing stress in pin A = $\dfrac{10x10^3}{(0.006x0.005)} = 333\,MPa$

Bearing stress in pin B = $\dfrac{15x10^3}{(0.006x0.005)} = 500\,MPa$



2.2.7.2F

Q.2: A 100 mm diameter off-set link is transmitting an axial pull of 30 KN as shown in the figure- 2.2.7.3. Find the stresses at points A and B.



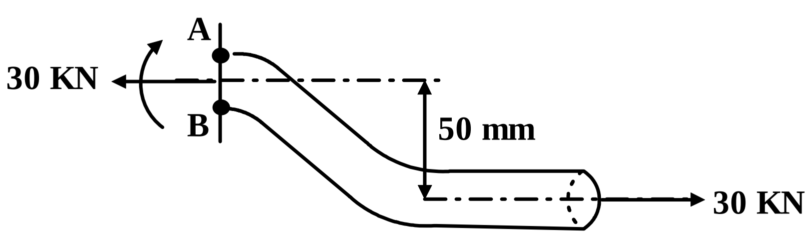
2.2.7.3F

A.2:

The force system at section AB is shown in figure-2.2.7.4.

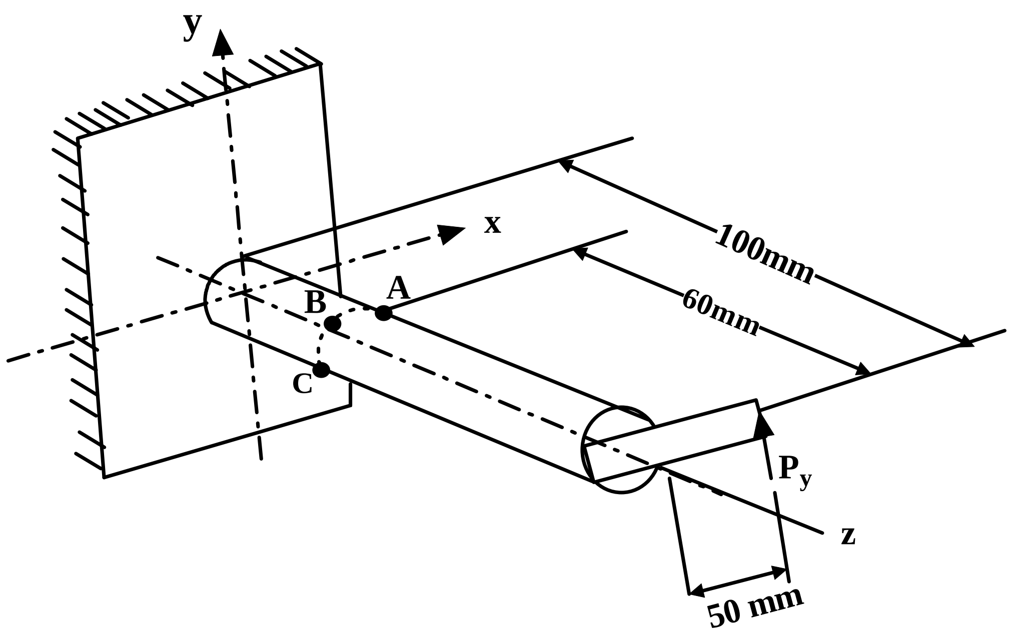
$$\sigma_A = -\frac{30x10^3 x0.05x0.05}{\frac{\pi}{64}(0.1)^4} + \frac{30x10^3}{\frac{\pi}{4}(0.1)^2} = -11.46\,MPa$$

$$\sigma_B = \frac{30x10^3 x0.05x0.05}{\frac{\pi}{64}(0.1)^4} + \frac{30x10^3}{\frac{\pi}{4}(0.1)^2} = 19.1\,MPa$$



2.2.7.4F

Q.3: A vertical load P_y = 20 KN is applied at the free end of a cylindrical bar of radius 50 mm as shown in figure-2.2.7.5. Determine the principal and maximum shear stresses at the points A, B and C.

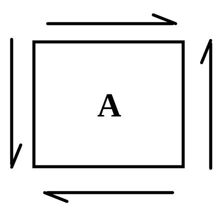


2.2.7.5F

A.3:

At section ABC a bending moment of 1.2 KN-m and a torque of 1KN-m act.On elements A and C there is no bending stress. Only torsional shear stress acts and

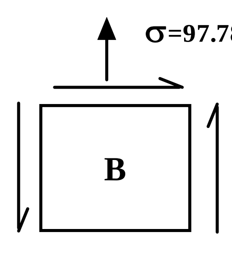
$$\tau = \frac{16T}{\pi d^3} = 40.7 \text{ MPa}$$



On element B both bending (compressive) and torsional shear stress act.

$$\sigma_B = \frac{32M}{\pi d^3} = 97.78\,\text{MPa}$$

$$\tau = 40.7\,\text{MPa}$$



Principal stresses at B = $\left(\frac{97.78}{2} \pm \sqrt{\left(\frac{97.78}{2}\right)^2 + (40.7)^2}\right)$

$$\sigma_{B1} = 112.5\text{MPa};\quad \sigma_{B2} = -14.72\text{MPa}$$

Maximum shear stress at B = $\sqrt{\left(\frac{97.78}{2}\right)^2 + (40.7)^2} = 63.61 \text{ MPa}$

Q.4: A propeller shaft for a launch transmits 75 KW at 150 rpm and is subjected to a maximum bending moment of 1KN-m and an axial thrust of 70 KN. Find the shaft diameter based on maximum principal stress if the shear strength of the shaft material is limited to 100 MPa.

A.4:

$$\text{Torque, T} = \frac{75\text{x}10^3}{\left(\frac{2\pi\text{x}150}{60}\right)} = 4775 \text{ Nm; then, } \tau = \frac{24.3}{d^3}\text{KPa}$$

$$\text{Maximum bending moment} = 1\text{KNm; then, } \sigma_b = \frac{10.19}{d^3}\text{KPa}$$

$$\text{Axial force} = 70\text{ KN; then, } \sigma = \frac{70}{\frac{\pi d^2}{4}}\text{KPa} = \frac{89.12}{d^2}\text{KPa}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{89.12}{2d^2} - \frac{10.19}{2d^3}\right)^2 + \left(\frac{24.3}{d^3}\right)^2} = 100\text{x}10^3$$

Solving we get the value of shaft diameter d = 63.4 mm.

2.2.8 Summary of this Lesson

The stresses developed at a section within a loaded body and methods of superposing similar stresses have been discussed. Methods of combining normal and shear stresses using transformation of plane stresses have been illustrated. Formulations for principal stresses and maximum shear stresses have been derived and typical examples are solved.